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Title	Stationary bands in three-dimensional Minkowski space
Author(s)	López, Rafael
Citation	Osaka Journal of Mathematics. 2009, 46(1), p. 1-20
Version Type	VoR
URL	https://doi.org/10.18910/12260
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Osaka J. Math.
46 (2009), 1–20

STATIONARY BANDS IN THREE-DIMENSIONAL MINKOWSKI SPACE

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(Received January 26, 2007, revised October 29, 2007)

Abstract

In this paper we consider a free boundary problem for spacelike surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ whose energy functional involves the area of a surface and a timelike potential. The critical points of this energy for any volume-preserving admissible variation are spacelike surfaces supported in a plane and whose mean curvature is a linear function of the time coordinate. In this paper, we consider those surfaces that are invariant in a parallel coordinate to the support plane. We call these surfaces stationary bands. We establish existence of such surfaces and we investigate their qualitative properties. Finally, we give estimates of its size in terms of the initial data.

1. Introduction and statement of results

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2$, where $x = (x_1, x_2, x_3)$ are the canonical coordinates in $\mathbb{R}^3$. Let Π be a spacelike plane, which we shall assume horizontal, and consider a potential energy Y that, up to constant multiple, measures at each point the distance to Π. We are interested in the following

> *Variational problem.* Find spacelike compact surfaces with maximal surface area whose boundaries are supported on Π and which enclose a fixed volume of the ambient space. We assume the effect of the potential Y.

The plane Π is called the support plane. If we search solutions up to the first order, we are interested in surfaces S that are critical points of the corresponding energy functional for any volume-preserving perturbation of the surface. The energy E of the system involves the surface area $|S|$, the area $|\Omega|$ of the planar domain $\Omega \subset \Pi$ bounded by the boundary ∂S of S and the potential defined by Y. Then

$$E = |S| - \cosh(\beta)|\Omega| + \int_G Y\, dS,$$

2000 Mathematics Subject Classification. 53C50, 53C42, 53A10.

Partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucia grant no. P06-FQM-01642.

where β is a constant and G is the bounded region by $S \cup \Omega$. Consider then an admissible variation of S to our problem, that is, a one parameter differentiable family of surfaces S_t indexed by a parameter t, with $S_0 = S$, and supported all in Π, that is, $\partial S_t \subset \Pi$. We assume that S_t is a volume-preserving variation of S and we denote by $E(t)$ the corresponding energy of the surface S_t. We look for those surfaces S that are critical points of the energy for all admissible variations, that is,

$$\left.\frac{d}{dt}\right|_{t=0} E(t) = 0.$$

In such case, we say that S is a *stationary surface*. According to the principle of virtual works, stationary surfaces are characterized by the following:

Theorem 1.1. *A spacelike surface S in $\mathbb{L}^3$ is stationary if and only if the following two conditions hold*:
1. *The mean curvature H of S is a linear function of the distance to Π*:

$$2H(x) = \kappa x_3(x) + \lambda \quad (\textit{Laplace equation}), \tag{1}$$

where κ is a constant called the capillary constant, and λ is a constant to be determined by the volume constraint.
2. *The surface S intersects Π at a constant hyperbolic angle β along ∂S* (*Young condition*).

We refer to [2, 3, 5, 10] for more details. In absence of the potential Y, the constant κ is zero and S is a spacelike surface with constant mean curvature or, more briefly, a CMC spacelike surface. Some results on CMC spacelike surfaces have been obtained in [1, 2]. Recently, the author has considered the case that H is a linear function of the x_3 coordinate in a series of articles. When the surface is compact, then it must be rotational symmetric with respect to an orthogonal axis to the support plane ([10]). Rotational stationary surfaces have been described in [11, 12].

The present paper continues this work with non-compact surfaces S that are invariant with respect to the x_2-coordinate. We say then that S is a *band*. Our motivation for studying stationary surfaces under this condition has its origin in the theory of CMC spacelike surfaces. The simplest examples of bands with constant mean curvature are hyperbolic cylinders: up to isometry of the ambient space, they are defined by $\mathcal{H}_a = \{((1/m)\sinh(x_1), x_2, (1/m)\cosh(x_1));\ -a < x_1 < a,\ x_2 \in \mathbb{R}\}$, $m > 0$ and whose mean curvature is $H = m/2$. A hyperbolic cylinder is also the graph of the function $y(x_1, x_2) = \sqrt{x_1^2 + 1/m^2}$ defined on the strip $\Omega_a = \{(x_1, x_2, 0);\ -a < x_1 < a\} \subset \Pi$. Hyperbolic cylinders and hyperbolic planes are used in the theory of CMC spacelike surfaces as barrier surfaces, as it can be seen in [15]. See also Section 5 below. Finally, maximal bands ($H = 0$) with singularities in $\mathbb{L}^3$ have been studied in the literature [8, 14].

The purpose of this paper is to establish results on existence and certain qualitative features of stationary bands. We begin by proving:

> *Given real numbers κ, λ and β, there exists a stationary band S supported on a strip of a plane which is parallel to Π that satisfies the Laplace equation and makes a hyperbolic angle β with the support plane along its boundary* (Theorems 4.3 *and* 6.5). *Moreover, the surface can extend to be an entire surface.*

For this, Equation (1) is reduced to an ordinary differential equation of second order and we analyze the existence of solutions. This is carried out in Section 3. The qualitative properties of the shapes of a stationary band depend on the sign of κ. Following the terminology of the Euclidean case, we call a *sessile* or *pendent* stationary band if $\kappa > 0$ or $\kappa < 0$, respectively. In Sections 4 and 5 we study the case $\kappa > 0$. We prove:

> *A sessile stationary band is a convex surface and asymptotic to a lightlike cylinder at infinity* (Theorem 4.1 *and* Corollary 4.2).

Next, we study properties of monotonicity on the parameter κ and we compute the size of the surface in terms of given data. We omit the statements and we refer to Section 5 for details. Finally in Section 6 we study pendent stationary bands and we describe the shapes of such surfaces:

> *A pendent stationary band is invariant by a group of translations in an orthogonal direction to the rulings of the surface. The x_3-coordinate of the surface is a periodic function and the surface extends to be a graph on Π* (Theorem 6.1).

As we have pointed out, stationary surfaces generalize the family of CMC spacelike surfaces. Constant mean curvature spacelike surfaces in a Lorentzian space are well known from the physical point of view because of their role in different problems in general relativity. See for instance [6, 13] and references therein. On the other hand, the Laplace equation (1) is of elliptic type (see also Equation (3) below) and well-known in the theory of partial differential equations. In this sense, the interest of this equation in the Lorentzian ambient space has appeared in [4]. Finally, in connection with the Euclidean ambient space, we recall that physical liquid drops are modeled by surfaces whose mean curvature is a linear function of its height, as it occurs to the stationary surfaces of $\mathbb{L}^3$. In Euclidean space there exists an extensive literature for such surfaces. We refer to the Finn's book [7], which contains an exhaustive bibliography, and more recently, [9].

2. Preliminaries

A nonzero vector $v \in \mathbb{L}^3$ is called spacelike or timelike if $\langle v, v\rangle > 0$ or $\langle v, v\rangle < 0$, respectively. Let S be a (connected) surface and let $x\colon S \to \mathbb{L}^3$ be an immersion of S into $\mathbb{L}^3$. The immersion is said to be spacelike if any tangent vector to S is spacelike. Then the scalar product $\langle\ ,\ \rangle$ induces a Riemannian metric on S. Observe that $\vec{e}_3 = (0,0,1)$ is a unit timelike vector field that is globally defined on $\mathbb{L}^3$, which determines a time-orientation on the space $\mathbb{L}^3$. This allows us to choose a unique unit normal vector field N on S which is in the same time-orientation as $\vec{e}_3$, and hence S is oriented by N. In this article all spacelike surfaces will be oriented according to this choice of N. Because the support plane in our variational problem is horizontal, the hyperbolic angle β between S and Π along its boundary is given by $\langle N, \vec{e}_3\rangle = -\cosh\beta$.

The notions of the first and second fundamental form for spacelike immersions are defined in the same way as in Euclidean space, namely,

$$\mathrm{I} = \sum_{ij} g_{ij}\, dx_i\, dx_j, \quad \text{and} \quad \mathrm{II} = \sum_{ij} h_{ij}\, dx_i\, dx_j,$$

respectively, where $g_{ij} = \langle \partial_i x, \partial_j x\rangle$ is the induced metric on S by x and $h_{ij} = -\langle \partial_i N, \partial_j x\rangle$. Then the mean curvature H of x is given by

$$2H = \operatorname{trace}(\mathrm{I}^{-1}\mathrm{II}) = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{\det(g_{ij})}. \tag{2}$$

Locally, if we write S as the graph of a smooth function $u = u(x_1, x_2)$ defined over a domain Ω, the spacelike condition implies $|\nabla u| < 1$. According to the choice of the time orientation,

$$N = \frac{(\nabla u, 1)}{\sqrt{1 - |\nabla u|^2}}$$

and the mean curvature H of S at each point $(x, u(x))$ satisfies the equation

$$(1 - |\nabla u|^2)\Delta u + \sum u_i u_j u_{ij} = 2H(1 - |\nabla u|^2)^{3/2}.$$

This equation is of quasilinear elliptic type and it can alternatively be written in divergence form

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) = 2H. \tag{3}$$

In particular, if u and v are two functions which are solutions of the same equation (3), the difference function $w = u - v$ satisfies an elliptic linear equation $Lw = 0$ and one can apply the Hopf maximum principle. Then we obtain uniqueness of solutions for each given boundary data.

We now consider the type of surfaces which are interesting in this work. A cylindrical surface S is a ruled surface generated by a one-parameter family of straight-lines $\{\alpha(s) + t\vec{w};\ t \in \mathbb{R}\}$, parametrized by the parameter s, where $\alpha(s)$, $s \in I$, is a regular curve contained in a plane P and $\vec{w}$ is a given vector which is not parallel to P. The curve α is called a directrix of S and the lines are called the rulings of S. The shape of a cylindrical surface is completely determined then by the geometry of α. Assuming that S is a spacelike surface, then both $\alpha'(s)$ and $\vec{w}$ are spacelike vectors. For example, the directrix of a hyperbolic cylinder is a (spacelike) hyperbola in a vertical plane P and $\vec{w}$ is a horizontal vector orthogonal to P.

If we impose the stationary condition to a cylindrical surface, then we prove that the rulings of the surface must be horizontal, and then, α is an embedded curve. Exactly:

Proposition 2.1. *Let S be a spacelike cylindrical surface in $\mathbb{L}^3$. If the mean curvature of S is a linear function of the time coordinate with $\kappa \neq 0$, then the rulings are horizontal.*

Proof. We parametrize S as

$$\mathbf{x}(s, t) = \alpha(s) + t\vec{w}, \quad s \in I,\ t \in \mathbb{R},$$

where α is the directrix of S parametrized by the length arc. The computation of the mean curvature H of S according to (2) gives

$$2H = \frac{\langle \alpha'(s) \times \vec{w}, \alpha''(s)\rangle}{(1 - \langle \alpha'(s), \vec{w}\rangle^2)^{3/2}}, \tag{4}$$

where $\times$ is the cross product in $\mathbb{L}^3$. In particular, the mean curvature function depends only on the s-variable. Therefore, if H satisfies the relation (1), that is,

$$H = \kappa x_3 \circ (\alpha(s) + t\vec{w}) + \lambda = \kappa x_3 \circ \alpha(s) + \lambda + \kappa(x_3 \circ \vec{w})t,$$

we infer that $x_3 \circ \vec{w} = 0$, and then, $\vec{w}$ is a horizontal vector. □

As a consequence of Proposition 2.1, we can choose the plane P containing the directrix to be vertical, with the rulings being horizontal straight-lines. On the other hand, because α is a spacelike curve in a vertical plane, α is globally the graph of a certain function u defined on an interval of any horizontal line of P. Hence, we conclude:

Corollary 2.2. *Any cylindrical surface in $\mathbb{L}^3$ that satisfies the Laplace equation* (1) *is a band.*

DEFINITION 2.3. A stationary band in $\mathbb{L}^3$ is a cylindrical surface that satisfies the Laplace equation (1).

3. Stationary bands: existence and symmetries

In this section, we write our variational problem in terms of the theory of ordinary differential equations, strictly speaking, the Laplace equation (1) is reduced to an ordinary differential equation of second order. The purpose of this section is to establish results of existence of solutions of the corresponding boundary value problem together with properties of symmetries of the solutions. Let S be a stationary band of $\mathbb{L}^3$. We have then

$$\kappa x_3 + \lambda = \langle \alpha' \times \vec{w}, \alpha'' \rangle = C_\alpha,$$

where C_α denotes the curvature of α. In addition, the angle between such a surface and the given horizontal support plane is constant.

If $\kappa = 0$ in the Laplace equation (1), the surface has constant mean curvature $H = \lambda/2$. Then the curvature C_α is constant, namely, $C_\alpha = \lambda$. Therefore α is a straight-line or a spacelike hyperbola of $\mathbb{L}^3$ and the corresponding surfaces are planes or hyperbolic cylinders respectively. Assuming that $\kappa \neq 0$, we do a change of variables to get $\lambda = 0$ in the Laplace equation. For this, it suffices with the change of the immersion x by $x + (\lambda/\kappa)\vec{e}_3$. Then the new surface is a stationary band with mean curvature $H(x) = \kappa x_3$. The support plane Π has been changed to another horizontal plane, namely, $\{x_3 = \lambda/\kappa\}$. However, the contact angle of x along its boundary is β again. It also follows that the shape of the original surface S is independent of the constraint λ in (1). Throughout this work, we shall consider the case that $\kappa \neq 0$ and $\lambda = 0$ in the Laplace equation (1). Then the problem of existence of our variational problem is expressed as follows:

> *Variational problem.* Let β and $\kappa \neq 0$ be two real numbers and let Π be a horizontal plane. Does there exist a stationary band S supported on Π such that: i) the mean curvature in each point $x \in S$ is $H(x) = \kappa x_3$ and; ii) the hyperbolic angle between S and Π along ∂S is β?

Let S be a stationary band given as the graph of a function u defined on a strip Ω_a. We parametrize S by $S = \{r, x_2, u(r)); -a < r < a, x_2 \in \mathbb{R}\}$. According to the choice of the orientation on S, the hyperbolic angle β between S and Π along ∂S is

$$\cosh \beta = -\langle N_S, \vec{e}_3 \rangle = -\left\langle \frac{(u_{x_1}, u_{x_2}, 1)}{\sqrt{1 - u'^2}}, (0, 0, 1) \right\rangle = \frac{1}{\sqrt{1 - u'^2}} \quad \text{at} \quad |r| = a.$$

Laplace and the Young equations are written, respectively, as

$$\frac{u''(r)}{(1 - u'(r)^2)^{3/2}} = \kappa u(r), \quad -a < r < a, \tag{5}$$

$$u'(\pm a) = \pm \tanh \beta, \quad u(a) = u(-a). \tag{6}$$

In order to attack the problem (5)–(6), we first begin with the next initial value problem

$$\frac{u''(r)}{(1-u'(r)^2)^{3/2}} = \kappa u(r), \quad r > 0, \tag{7}$$

$$u(0) = u_0, \quad u'(0) = 0, \tag{8}$$

where u_0 is a real number.

Theorem 3.1. *Given u_0, there exists a unique solution of* (7)–(8). *The solution $u = u(r; u_0, \kappa)$ continuously depends on the parameters u_0, κ. The maximal interval of definition of u is $\mathbb{R}$.*

Proof. Put $v = u'/\sqrt{1-u'^2}$. Then the problem (7)–(8) becomes equivalent to a pair of differential equations

$$u' = \frac{v}{\sqrt{1+v^2}}, \quad u(0) = u_0, \tag{9}$$

$$v' = \kappa u, \quad v(0) = 0. \tag{10}$$

The solution u that we look for is then defined by

$$u(r) = u_0 + \int_0^r \frac{v(t)}{\sqrt{1+v(t)^2}}\, dt. \tag{11}$$

Then standard existence theorems of ordinary differential equations assures local existence and uniqueness of (9)–(10) as well as the continuity with respect to the parameters u_0 and κ. We study the maximal domain of the solution. On the contrary, suppose that $[0, R)$ is the maximal interval of the solution u, with $R < \infty$. By (10) and (11),

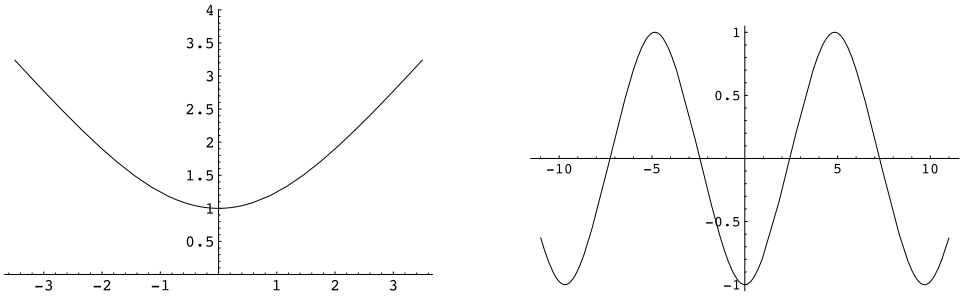
$$|u(r)| < |u_0| + r, \quad |v(r)| \le |\kappa| r \left(|u_0| + \frac{r}{2} \right).$$

Thus, and using (9)–(10), the limits of u' and v' at $r = R$ are finite, which would imply that we can extend the solutions (u, v) beyond $r = R$: a contradiction. □

Corollary 3.2. *A stationary band of $\mathbb{L}^3$ supported on a horizontal plane can be extended to a graph defined in Π, that is, it is an entire spacelike surface of $\mathbb{L}^3$.*

In Fig. 1 we have the graphs of two solutions of (7)–(8) for different initial values. A first integration of (7)–(8) is obtained by multiplying both sides in (7) by u':

$$u^2 = u_0^2 + \frac{2}{\kappa}\left(\frac{1}{\sqrt{1-u'^2}} - 1 \right). \tag{12}$$



(a) (b)

Fig. 1. (a): the solution $u(r;1,0.5)$; (b): the solution $u(r;-1,-0.5)$

Denote by $\psi = \psi(r)$ the hyperbolic angle between the directrix $\alpha(r) = (r, u(r))$ and the horizontal direction. Put

$$\sinh\psi = \frac{u'}{\sqrt{1-u'^2}}, \quad \cosh\psi = \frac{1}{\sqrt{1-u'^2}}. \tag{13}$$

Then the Euler-Lagrange equation (7) takes the form

$$(\sinh\psi)' = \kappa u, \tag{14}$$

and so,

$$\sinh\psi = \kappa \int_0^r u(t)\,dt. \tag{15}$$

The identity (14) actually corresponds with the mean curvature equation in its divergence form (3). Using (12) and (13), we have

$$u^2 = u_0^2 + \frac{2}{\kappa}(\cosh\psi - 1). \tag{16}$$

We study the symmetries of a stationary band, excluding the trivial symmetries with respect to each orthogonal plane to the rulings.

Theorem 3.3 (Symmetry). *Let u be a solution of* (7).

1. *If $u'(r_0) = 0$, the graph of u is symmetric with respect to the vertical line $\{r = r_0\}$.*
2. *If $u(r_0) = 0$, the graph of u is symmetric with respect to the point $(r_0, 0)$.*

Proof. In both cases, we can assume that $r_0 = 0$. We prove the first statement. The functions $u(r)$ and $u(-r)$ are solutions of the same equation (7) and with the same initial conditions at $r = 0$, namely, u_0 and $u'(0) = 0$. Then the uniqueness of

solutions gives $u(r) = u(-r)$. The proof of the second statement is similar in showing $u(r) = -u(-r)$: now both functions are solutions of (7) with initial conditions $u(0) = 0$ and $u'(0)$. □

Finally, we establish a result that says that we can take the signs of u_0 and κ to be the same. As a direct consequence of the uniqueness of solutions, we have:

Proposition 3.4. *Let $u = u(r; u_0, \kappa)$ be a solution of* (7)–(8). *Then $u(r; u_0, \kappa) = -u(r; -u_0, \kappa)$.*

Then it is possible to choose u_0 to have the same sign as κ. This will be assumed throughout the text.

4. Stationary bands: the case $\kappa > 0$

This section is devoted to studying the qualitative properties of the shape of a sessile stationary band. Assume $\kappa > 0$. Recall that $u_0 > 0$. The geometry of the directrix is described by the next:

Theorem 4.1 (Sessile case). *Let $u = u(r; u_0)$ be a solution of* (7)–(8). *Then the function u has exactly a minimum at $r = 0$ with*

$$\lim_{r\to\infty} u(r) = \infty, \qquad \lim_{r\to\infty} u'(r) = 1.$$

Moreover, u is convex with

$$\lim_{r\to\infty} u''(r) = 0.$$

Proof. Since $u'(0) = 0$, we restrict ourselves to studying u for $r \geq 0$ (Theorem 3.3). Since the integrand in (15) is positive near to $r = 0$, $\sin\psi > 0$, and so, $u'(r) > 0$. This means that u is increasing near $r = 0$. If r_o is the first point where $u'(r_o) = 0$, then (12) implies that $u(r_o) = u(0)$: a contradiction. Thus, $u'(r) > 0$ for any r and this proves that u is strictly increasing and $r = 0$ is the unique minimum. On the other hand, at $r = 0$, $u''(0) = \kappa u_0 > 0$, which implies that u is convex around $r = 0$. Since $u(r) > u_0 > 0$, Equation (7) implies that u'' does not have zeroes, that is, u is a convex function.

Because $u(r) \to \infty$ as $r \to \infty$, it follows from (12) that $\cosh\psi(r) \to \infty$, that is, $u'(r) \to 1$ as $r \to \infty$. Finally, from (7) and (16),

$$0 \leq u'' = \frac{\kappa}{\cosh^3\psi}\sqrt{u_0^2 + \frac{2}{\kappa}(\cosh\psi - 1)} \to 0,$$

as $r \to \infty$. □

In Fig. 1 (a), we show the graph of the directrix of a sessile stationary band.

As a consequence of Theorem 4.1, a sessile stationary band has the lowest height if $u_0 > 0$ (or the highest height if $u_0 < 0$) and this height is reached at the ruling $\{(0, x_2, u_0); x_2 \in \mathbb{R}\}$. The fact that $u'(r) \to 1$ as $r \to \infty$ can be written as follows:

Corollary 4.2. *Any sessile stationary band of $\mathbb{L}^3$ is asymptotic at infinity to a lightlike cylinder of $\mathbb{L}^3$.*

Recall that, up to isometry, a lightlike cylinder in $\mathbb{L}^3$ is the surface defined as $\{(x_1, x_2, x_3) \in \mathbb{L}^3;\ x_1^2 - x_3^2 = 0\}$. Since u is an increasing function on $(0, \infty)$, we bound the integrand in (15) by $u_0 < u(t) < u(r)$ obtaining

$$\kappa u_0 < \frac{\sinh \psi(r)}{r} < \kappa u(r). \tag{17}$$

Moreover,

$$\lim_{r \to 0} \frac{\sinh \psi(r)}{r} = \kappa u_0. \tag{18}$$

We establish the existence of the original variational problem introduced in this work.

Theorem 4.3 (Existence). *Let Ω_a be a strip of the (x_1, x_2)-plane, $a > 0$. Given constants $\kappa > 0$ and β, there exists a stationary band on Ω_a whose directrix is defined by a function $u = u(r; u_0)$, that makes a contact hyperbolic angle β with the support plane $\{x_3 = u(a)\}$.*

Proof. If $\beta = 0$, we take $S = \{x_3 = 0\}$. Without loss of generality, we now assume $\beta > 0$. The problem is equivalent to search a solution of (5)–(6). For this, we take the initial value problem (7)–(8), with $u_0 > 0$. The problem then reduces to find $u_0 > 0$ such that $u'(a; u_0) = \tanh \beta$. By the continuity of the parameters, $\lim_{u_0 \to 0} u'(a; u_0) = 0$. On the other hand, by using (17), we have

$$u'(a; u_0) = \tanh \psi(a) \geq \frac{\kappa u_0 a}{\sqrt{1 + \kappa^2 u_0^2 a^2}} \to 1$$

as $u_0 \to \infty$. As a consequence, and by continuity again, we can find $u_0 > 0$ such that $u'(a; u_0) = \tanh \beta$. □

The uniqueness of solutions will be derived at the end of this section: see Corollary 4.8. After the existence of solution of (5)–(6), we show certain results about the monotonicity of the solutions with respect to the parameters κ and u_0 of the differential equation. First, for the capillary constant κ, we prove:

Theorem 4.4. *Let $\kappa_1, \kappa_2 > 0$. Denote $u_i = u_i(r; u_0, \kappa_i)$, $i = 1, 2$, two solutions of* (7)–(8) *with the same initial condition u_0. If $\kappa_1 < \kappa_2$, then $u_1(r) < u_2(r)$ for any $r \neq 0$ and $u_1'(r) < u_2'(r)$ for $r > 0$.*

Proof. Denote by $\psi^{(i)}$ the angle functions defined by (13) for each function u_i. By (15), we know that

$$\sinh \psi^{(2)}(r) - \sinh \psi^{(1)}(r) = \int_0^r (\kappa_2 u_2(t) - \kappa_1 u_1(t))\, dt. \tag{19}$$

At $r = 0$, the integrand is positive and so, $\psi^{(2)}(r) > \psi^{(1)}(r)$ on some interval $(0, \epsilon)$. Then $u_2'(r) > u_1'(r)$ and because $u_2(0) = u_1(0)$, we have $u_2(r) > u_1(r)$ in $(0, \epsilon)$. We prove that $u_2'(r) > u_1'(r)$ holds for any $r > 0$. If $r_0 > 0$ is the first point where $u_2'(r_0) = u_1'(r_0)$, then $u_2''(r_0) \leq u_1''(r_0)$ and $u_2(r_0) > u_1(r_0)$. But (7) gives

$$\frac{u_2''(r_0)}{(1 - u_2'(r_0)^2)^{3/2}} = C_{u_2}(r_0) = \kappa_2 u_2(r_0) > \kappa_1 u_1(r_0) = C_{u_1}(r_0) = \frac{u_1''(r_0)}{(1 - u_1'(r_0)^2)^{3/2}}.$$

This contradiction implies that $u_2'(r) > u_1'(r)$ for any $r > 0$ and then, $u_2(r) > u_1(r)$. □

We now return to the boundary value problem (5)–(6). We prove that for fixed β, the solution and its derivative with respect to r are monotone functions on κ.

Theorem 4.5 (Monotonicity with respect to κ). *Let $\kappa_1, \kappa_2 > 0$. Denote by $u_i = u_i(r)$, $i = 1, 2$, two solutions of* (5)–(6) *for $\kappa = \kappa_i$ with $u_i(0) > 0$. If $\kappa_1 < \kappa_2$, then*
1. *$u_1(r) > u_2(r)$ for $0 \leq r \leq a$.*
2. *$u_1'(r) > u_2'(r)$ for $0 < r < a$.*

Proof. Put $v_i = \sinh \psi^{(i)}$. For each $0 \leq r_0 < r < a$, Equation (15) yields

$$v_i(r) - v_i(r_0) = \int_{r_0}^r \kappa_i u_i(t)\, dt.$$

Then

$$v_2(r) - v_1(r) = v_2(r_0) - v_1(r_0) + \int_{r_0}^r (\kappa_2 u_2(t) - \kappa_1 u_1(t))\, dt. \tag{20}$$

Because $u_1'(a) = u_2'(a)$,

$$v_1(r_0) - v_2(r_0) = \int_{r_0}^a (\kappa_2 u_2(t) - \kappa_1 u_1(t))\, dt. \tag{21}$$

Claim 1. *If $\kappa_2 u_2(r_0) \geq \kappa_1 u_1(r_0)$, then $v_2(r_0) < v_1(r_0)$.*

On the contrary case, that is, if $v_1(r_0) \le v_2(r_0)$, then $0 < u_1'(r_0) < u_2'(r_0)$. The fact that $\kappa_1 < \kappa_2$ implies $\kappa_1 u_1'(r_0) < \kappa_2 u_2'(r_0)$. Hence $\kappa_1 u_1 < \kappa_2 u_2$ on a certain interval $(r_0, r_0+\delta)$. Let $r_1 \in (r_0, a]$ be the largest number where such inequality holds. In view of (20), for each $r_0 < s \le r_1$, $v_2(s) > v_1(s)$. Thus $u_2'(s) > u_1'(s)$ and $\kappa_2 u_2' > \kappa_1 u_1'$. This implies $\kappa_2 u_2 > \kappa_1 u_1$ for each $r_0 < s \le r_1$. Since r_1 is maximal, $r_1 = a$. We put now $s = a$ in (21) and we obtain

$$0 \ge v_1(r_0) - v_2(r_0) = \int_{r_0}^{a} (\kappa_2 u_2(t) - \kappa_1 u_1(t))\, dt > 0,$$

which is a contradiction. This proves the Claim.

Let us prove the Theorem and we begin with the item 2. Assume there exists r_0, $0 < r_0 < a$, such that $u_1'(r_0) \le u_2'(r_0)$. Then $v_1(r_0) \le v_2(r_0)$. By the Claim, $\kappa_2 u_2(r_0) < \kappa_1 u_1(r_0)$, and so, (7) implies $v_2'(r_0) < v_1'(r_0)$. For a certain neighborhood on the left of r_0, we obtain then

$$0 \le v_2(r_0) - v_1(r_0) < v_2(r) - v_1(r)$$

which yields $v_2(r) > v_1(r)$. Because $v_1(0) = 0 = v_2(0)$, there exists a last number r_1, $0 \le r_1 < r_0$, such that $v_2 > v_1$ in the interval (r_1, r_0) and $v_2(r_1) = v_1(r_1)$. The Claim implies now $\kappa_2 u_2(r) < \kappa_1 u_1(r)$, for $r_1 < r \le r_0$. But (21) yields $v_2(r) < v_1(r)$ and that is a contradiction. Consequently, $u_2' < u_1'$ in $(0, a)$.

Let us prove the item 1. Because $u_2' < u_1'$, $v_2 < v_1$. If r is close to 0, $\kappa_i u_i(r) r \ge \kappa_i u_i(0) r$, it follows from (18) that

$$\lim_{r \to 0} \frac{v_i(r)}{r} = \kappa_i u_i(0).$$

Because $v_1 > v_2$, we infer then $u_2(0) < u_1(0)$. Since $u_2' < u_1'$, an integration leads to $u_2 < u_1$ on the interval $[0, a]$. □

Given a capillary constant κ, we would like to control the dependence of solutions $u(r; u_0)$ with respect to the initial condition u_0. We will obtain monotonicity, that is, if $u_0 < v_0$, then $u(r; u_0) < u(r; v_0)$ for any r. Moreover, we can estimate the distance between the two solutions.

Theorem 4.6. *Fix $\kappa > 0$. If $\delta > 0$, then $u(r; u_0 + \delta) - \delta > u(r; u_0)$ for any $r \ne 0$.*

Proof. By symmetry, it is sufficient to show the inequality for $r > 0$. Define the function $u_\delta = u(r; u_0 + \delta)$, and let ψ^δ be the corresponding hyperbolic angle, see (13). It follows from (15) that

$$\sinh \psi^\delta - \sinh \psi = \kappa \int_0^r (u_\delta(t) - u(t))\, dt. \tag{22}$$

Since the integrand is positive at $r = 0$, there exists $\epsilon > 0$ such that

$$\sinh \psi^{\delta}(r) - \sinh \psi(r) > 0, \quad \text{in} \quad (0, \epsilon).$$

Because

$$\sinh \psi^{\delta}(0) - \sinh \psi(0) = 0,$$

we have the inequality $\psi^{\delta} > \psi$ in the interval $(0, \epsilon)$. In addition,

$$(u_{\delta}(r) - u(r))' = \tanh \psi^{\delta}(r) - \tanh \psi(r) > 0.$$

Therefore the function $u_{\delta} - u$ is strictly increasing on r. So, $u_{\delta}(r) - \delta > u(r)$. Let $r_0 > \epsilon$ be the first point where $u_{\delta}(r_0) - \delta = u(r_0)$. Again (22) yields $\sinh \psi^{\delta}(r_0) - \sinh \psi(r_0) > 0$ and $(u_{\delta} - u)'(r_0) \leq 0$. But this implies that $\psi^{\delta}(r_0) \leq \psi(r_0)$, which it is a contradiction. As conclusion, $u_{\delta} - \delta > u$ in $(0, \infty)$ and this shows the result. □

Corollary 4.7. *Let S_1, S_2 be two sessile stationary bands with the same capillary constant κ. Let h_i be the lowest heights of S_i, $i = 1, 2$. If $0 < h_1 < h_2$, we can move S_2 by translations until it touches S_1 in such way that S_2 lies completely above S_1.*

Corollary 4.8 (Uniqueness). *The solution obtained in* Theorem 4.3 *is unique.*

Proof. By contradiction, assume that S_1 and S_2 are two different stationary bands on Ω_a and with the same Young condition. By the symmetries of solution of (5) and Theorem 4.1, S_1 and S_2 are determined by functions $u_1 = u(r; u_0)$ and $u_2 = u(r; v_0)$ respectively, solutions of (7)–(8) on the same strip Ω_a and with $u_1'(a) = u_2'(a)$. Without loss of generality, we assume that $0 < u_0 < v_0$. The proof of Theorem 4.6 says that $u_2'(r) > u_1'(r)$ in some interval $(0, \epsilon)$. Actually, we now prove that this inequality holds for any $r > 0$. If r_0 is the first point where $u_1'(r_0) = u_2'(r_0)$, then $u_2''(r_0) \leq u_1''(r_0)$ and $u_2 > u_1$ on $[0, r_0]$. But Equation (7) implies that

$$\frac{u_2''(r_0)}{(1 - u_2'(r_0)^2)^{3/2}} = \kappa u_2(r_0) > \kappa u_1(r_0) = \frac{u_1''(r_0)}{(1 - u_1'(r_0)^2)^{3/2}}.$$

This contradiction implies that $u_2'(r) > u_1'(r)$ for any $r > 0$. But then it is impossible that $u_2'(a) = u_1'(a)$. □

This section ends with a result of foliations of the ambient space $\mathbb{L}^3$ by stationary bands.

Corollary 4.9. *Fix $\kappa > 0$. Then the Lorentz-Minkowski space $\mathbb{L}^3$ can be foliated by a one-parameter family of sessile stationary bands, for the same capillary constant κ. The foliations are given by stationary bands that are entire spacelike surfaces whose profile curves are $\{u(r; u_0); u_0 \in \mathbb{R}\}$ and u_0 is the parameter of the foliation.*

Proof. Let (a, b) be a point in the (x_1, x_3)-plane. We have to show that there exists a unique u_0 such that $b = u(a; u_0)$. If $b = 0$, we take $u_0 = 0$. We assume that $b > 0$ (the reasoning is similar if $b < 0$). By the dependence of solutions of (7)–(8) and because $u(r; u_0) \geq u_0$ (assuming $u_0 > 0$), we have

$$\lim_{u_0 \to 0} u(a; u_0) = 0, \qquad \lim_{u_0 \to \infty} u(a; u_0) = \infty.$$

Then we employ again the dependence of the parameter u_0 to assure the existence of u_0 such $u(a; u_0) = b$. The uniqueness of u_0 is given by Theorem 4.6. □

REMARK 4.10. It is worth to point out that in Relativity, there is an interest of finding real-valued functions on a given spacetime, all of whose level sets provide a global time coordinate. Consequently, Corollary 4.9 says that we can find a foliation of the ambient space $\mathbb{L}^3$ whose leaves are sessile stationary bands tending to lightlike cylinders at infinity.

5. Sessile stationary bands: estimates

This section is devoted to obtaining estimates of the size for a sessile stationary band. Strictly speaking, we will give bounds of the height of the solutions of our variational problem in terms of the lowest height u_0, the hyperbolic angle of contact β or the width $2a$ of the strip Ω_a.

Fix $\kappa > 0$. Let S be a stationary band that is a graph on Ω_a and given by a solution $u = u(r; u_0)$ of (7)–(8). Suppose $u'(a) = \tanh \beta$. A first control of $u(a)$ comes from the identity (12):

$$u(a) = \sqrt{u_0^2 + \frac{2}{\kappa}(\cosh \beta - 1)}. \tag{23}$$

The estimates that we shall obtain are a consequence of the comparison of our stationary bands with hyperbolic cylinders. Consider y_1 and y_2 two hyperbolas defined on $[0, a]$ and given by

$$y_1(r) = u_0 - \mu_1 + \sqrt{r^2 + \mu_1^2}, \quad \mu_1 = \frac{1}{\kappa u_0}, \tag{24}$$

$$y_2(r) = u_0 - \mu_2 + \sqrt{r^2 + \mu_2^2}, \quad \mu_2 = \frac{a}{\sinh \psi(a)}. \tag{25}$$

Let Σ_1 and Σ_2 be the hyperbolic cylinders whose directrix are y_1 and y_2, respectively. Both surfaces have constant mean curvature:

$$H_1 = \frac{C_{y_1}}{2} = \frac{\kappa u_0}{2}, \quad H_2 = \frac{C_{y_2}}{2} = \frac{\sinh \psi(a)}{2a}.$$

The functions y_1 and y_2 agree with u at $r = 0$. On the other hand, the mean curvature of S at the point $(0, x_2, u_0)$ agrees with that of Σ_1 and $u'(a) = y_2'(a)$.

Lemma 5.1. *The surface S lies between Σ_1 and Σ_2.*

Proof. In order to prove the result, it is sufficient to show $y_1 < u < y_2$ on the interval $(0, a]$. Denote by C_u the curvature of the graph of u. At $r = 0$, $C_u(0) = \kappa u_0 = C_{y_1}(0)$, but C_u is increasing on r since both κ and u' are positive. Because $u(0) = y_1(0)$, we conclude that $y_1(r) < u(r)$ in $0 < r < a$. We now prove the inequality $u < y_2$. Since the curve y_2 has constant curvature, inequalities (17) yield

$$C_{y_2}(0) = C_{y_2}(a) = \frac{\sinh \psi(a)}{a} > \kappa u_0 = C_u(0).$$

Then at $r = 0$, $C_u(0) < C_{y_2}(0)$. Since $y_2(0) = u(0)$, it follows that there exists $\delta > 0$ such that $u(r) < y_2(r)$ for $0 < r < \delta$. We assume that δ is the least upper bound of such values. By contradiction, suppose that $\delta < a$. As $y_2(\delta) = u(\delta)$ and $y_2'(\delta) \le u'(\delta)$, $\psi^{(2)}(\delta) \le \psi(\delta)$ and thus

$$\int_0^\delta \frac{d}{dr}(\sinh \psi(r) - \sinh \psi^{(2)}(r))\, dr = \sinh \psi(\delta) - \sinh \psi^{(2)}(\delta) := \alpha(\delta) \ge 0. \tag{26}$$

Hence there exists $\bar{r} \in (0, \delta)$ such that

$$C_u(\bar{r}) = (\sinh \psi)'(\bar{r}) > (\sinh \psi^{(2)})'(\bar{r}) = C_{y_2}(\bar{r}).$$

As $C_u(r)$ is increasing, $C_u(r) > C_{y_2}(r)$ for $r \in (\bar{r}, a)$. In particular, and using $u'(a) = y_2'(a)$,

$$0 < \int_\delta^a (C_u(r) - C_{y_2}(r))\, dr = \int_\delta^a \frac{d}{dr}(\sinh \psi(r) - \sinh \psi^{(2)}(r))\, dr = -\alpha(\delta),$$

which contradicts (26). □

As a consequence of Lemma 5.1 and putting $y_1(a) < u(a) < y_2(a)$, we have:

Theorem 5.2. *Let $\kappa > 0$ and let u be a solution of the problem* (5)–(6) *given by $u = u(r; u_0)$. If u_0 is the lowest height of u, then*

$$u_0 - \frac{1}{\kappa u_0} + \sqrt{a^2 + \frac{1}{\kappa^2 u_0^2}} < u(a) < u_0 + a\frac{\cosh \beta - 1}{\sinh \beta}.$$

We point out that the upper bound for $u(a)$ obtained in Theorem 5.2 does not depend on κ but only on u_0 and β. Other source to control the shape of u comes from the integration of u. We know from (15) and Lemma 5.1 that

$$\kappa \int_0^a y_1(t)\,dt < \sinh \psi(a) < \kappa \int_0^a y_2(t)\,dt. \tag{27}$$

Actually, these inequalities inform us about the volume per unit of length that encloses each one of the three surfaces with the support plane $\{x_3 = u(a)\}$. The difference with the estimate obtained in Theorem 5.2 is that we now obtain a control of the value $\psi(a) = \beta$. For the integrals involving y_i, we write

$$\int_0^a \left(\sqrt{r^2+m^2}+c\right) dr := F(c, m) = ac + \frac{a}{2}\sqrt{a^2+m^2} + \frac{m^2}{2}\log\left(\frac{a+\sqrt{a^2+m^2}}{m}\right).$$

Then inequalities (27) are written as $\kappa F(u_0-\mu_1, \mu_1) < \sinh\beta < \kappa F(u_0-\mu_2, \mu_2)$.

Theorem 5.3. *Fix $\kappa > 0$ and let $u = u(r; u_0)$ be a solution of the problem* (5)–(6). *Then*

$$\frac{\sinh\beta}{a\kappa} + \frac{a}{\sinh\beta} - \frac{a\coth\beta}{2} - \frac{a\beta}{2\sinh^2\beta} < u_0 < \frac{\sinh\beta}{a\kappa}. \tag{28}$$

Proof. The left inequality in (28) is a consequence of $\sinh\beta < \kappa F(u_0-\mu_2, \mu_2)$. The right inequality comes by comparing the slopes of y_1 and u at the point $r = a$, that is, $y_1'(a) < u'(a)$. □

We obtain a new estimate of the solution u. For this, let us move down the hyperbola y_2 until it meets u at $(a, u(a))$. We denote by y_3 the new position of y_2.

Lemma 5.4. *The function y_3 satisfies $y_3 < u$ on the interval* $[0, a)$.

Proof. With a similar argument as in Lemma 5.1, we compare the curvatures of u and y_3. By (17), we have

$$C_u(a) = \kappa u(a) > \frac{\sinh\psi(a)}{a} = C_{y_3}(a).$$

Thus, around the point $r = a$, $y_3 < u$. By contradiction, assume that there is $\delta \in (0, a)$ such that $y_3(r) < u(r)$ for $r \in (\delta, a)$ and $y_3(\delta) = u(\delta)$. Since $u'(\delta) \geq y_3'(\delta)$, then $\psi^{(3)}(\delta) \leq \psi(\delta)$. This implies

$$\int_\delta^a (C_{y_3}(r) - C_u(r))\,dr = \sinh\psi(\delta) - \sinh\psi^{(3)}(\delta) \geq 0. \tag{29}$$

Then there would be $\bar{r} \in (\delta, a)$ such that $C_{y_3}(\bar{r}) - C_u(\bar{r}) > 0$. Because $C_u(r)$ is increasing on r, $C_u(r) < C_{y_3}(r)$ on $(0, \bar{r})$ and hence the same inequality holds also throughout $(0, \delta) \subset (0, \bar{r})$. Thus

$$0 < \int_0^\delta (C_{y_3}(r) - C_u(r))\, dr = \sinh \psi^{(3)}(\delta) - \sinh \psi(\delta) \leq 0$$

by (29). This contradiction shows the result. □

As conclusion, we have the estimates:

$$y_3(r) < u(r), \quad 0 \leq r < a.$$

$$F(u(a) - a \coth \beta, \mu_2) < \frac{\sinh \beta}{\kappa}.$$

Both inequalities give the next

Theorem 5.5. *With the same notation as in* Theorem 5.3, *we have*

$$u(a) - a \coth \beta + \sqrt{r^2 + \frac{a^2}{\sinh^2 \beta}} < u(r), \quad 0 \leq r < a, \tag{30}$$

$$u(a) < \frac{\sinh \beta}{\kappa a} + \frac{a}{2} \coth \beta - \frac{a\beta}{2 \sinh^2 \beta}. \tag{31}$$

6. Stationary bands: the case $\kappa < 0$

In this section we study stationary bands when $\kappa < 0$. We assume that $u_0 < 0$.

Theorem 6.1. *Let $u(r; u_0)$ be a solution of the problem* (7)–(8). *Then u is a periodic function that vanishes in an infinite discrete set of points. The inflections of u are their zeroes. Moreover, $u_0 \leq u(r) \leq -u_0$, attaining both values at exactly the only critical points of u.*

Proof. From (15), u' is positive near $r = 0$ and then, u is strictly increasing on some interval $[0, \epsilon)$. As a consequence of (7), u is convex around $r = 0$ with $u'' > 0$ provided $u < 0$. This implies that u must vanish at some point $r = r_o$. From (12), u' vanishes when $u = \pm u_0$ and from (7), the inflections agree with the zeroes of u. By Theorem 3.3, we obtain the result. □

Corollary 6.2. *Let S be a pendent stationary band. Then S is invariant by a group of horizontal translations orthogonal to the rulings.*

In Fig. 1 (b), we show the graph of the directrix of a pendent stationary band. Using (12) again and Theorem 6.1, we have

Corollary 6.3. *Let $\kappa < 0$. Then the maximum slope of a solution $u(r; u_0)$ of* (7)–(8) *occurs at each zero of u and its value is*

$$u'(r_o) = \frac{-u_0}{2 - \kappa u_0^2}\sqrt{\kappa^2 u_0^2 - 4\kappa}.$$

We show the existence of pendent stationary bands in the variational problem. We need the following

Lemma 6.4. *Consider $u = u(r; u_0)$ a solution of* (7)–(8) *and denote r_o the first zero of u. Then*

$$\sqrt{\frac{-2}{\kappa}} < \sqrt{u_0^2 - \frac{2}{\kappa}} < r_o. \tag{32}$$

Proof. We consider the hyperbola y_4 defined by

$$y_4(r) = \sqrt{r^2 + \left(\frac{1}{\kappa u_0}\right)^2} + u_0 - \frac{1}{\kappa u_0}.$$

Using the same argument as in (17), the function u is negative in the interval $(0, r_o)$ and then, $\kappa r u(r) < \sinh\psi(r) < \kappa r u_0$. It follows that $u'(r) < y_4'(r)$. Since $y_4(0) = u(0)$, $u(r) < y_4(r)$. Because y_4 meets the r-axis at the point $\sqrt{u_0^2 - 2/\kappa}$, a comparison between y_4 and u gives the desired estimates. □

Theorem 6.5 (Existence). *Let Ω_a be a strip of the (x_1, x_2)-plane, $a > 0$. Given constants $\kappa < 0$ and β, there exists a stationary band that is a graph on Ω_a whose directrix is defined by a function $u = u(r; u_0)$, that makes a contact hyperbolic angle β with the support plane $\{x_3 = u(a)\}$.*

Proof. If $\beta = 0$, we take $S = \{x_3 = 0\}$. Without loss of generality, we now assume $\beta > 0$. The problem is equivalent to searching a solution of (5)–(6). For this, we take the initial value problem (7)–(8), with $u_0 < 0$. The problem is reduced to finding $u_0 < 0$ such that $u'(a; u_0) = \tanh\beta$. We will search the solution in such way that u is negative in its domain. We know by the continuity of parameters that $\lim_{u_0 \to 0} u'(a; u_0) = 0$.

On the other hand, we show that $\lim_{u_0 \to -\infty} u'(a; u_0) = 1$. For this, we know that if $|u_0|$ is sufficiently big, then $a < \sqrt{u_0^2 - 2/\kappa} < r_o$, where r_o is the first zero of $u(r; u_0)$. It follows from the proof of Lemma 6.4 that $u(r; u_0) < y_4(r)$, for $0 < r < a$. Since both functions are negative, we have from (15) that

$$\sinh\psi(a) = \kappa \int_0^a u(t)\,dt > \kappa \int_0^a y_4(t)\,dt = F\left(u_0 - \frac{1}{\kappa u_0}, \frac{1}{\kappa u_0}\right) \to +\infty,$$

as $u_0 \to -\infty$. Thus $u'(a; u_0) = \tanh \psi(a) \to 1$, as $u_0 \to -\infty$. By the continuity of parameters, we conclude the existence of a number $u_0 < 0$ with the desired condition of Theorem 6.5. □

Finally, we establish some estimates of the solution u for the problem (5)–(6). By the periodicity of u, we restrict ourselves to the interval $[0, r_o]$ and that $a \leq r_o$ (for example, this condition holds if $a < \sqrt{-2/\kappa}$, see Lemma 6.4). We use the function y_4. Then $u(r) < y_4(r)$ for $0 < r \leq a$ and

$$\int_0^a u(t)\,dt < \int_0^a y_4(t)\,dt.$$

A similar argument as in Theorems 5.2 concludes

$$u(a) < u_0 - \frac{1}{\kappa u_0} + \sqrt{a^2 + \frac{1}{\kappa^2 u_0^2}}.$$

For pendent stationary bands we do not have a result of monotonicity with respect to the parameters, such as it was done in Theorems 4.4 and 4.6 for the case $\kappa > 0$. This is due to the periodicity of solutions. At this state, we can only assure the following results of monotonicity on a certain interval $I = (-\epsilon, \epsilon)$ around $r = 0$:

1. If $\kappa_1 < \kappa_2$, then $u(r; u_0, \kappa_1) > u(r; u_0, \kappa_2)$, $r \in I \setminus \{0\}$.
2. Let $\delta > 0$. Then $u(r; u_0 - \delta) + \delta > u(r; u_0)$, $r \in I \setminus \{0\}$.

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