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PROPAGATION OF WAVE FRONT SETS OF SOLUTIONS OF THE CAUCHY PROBLEM FOR HYPERBOLIC EQUATIONS IN GEVREY CLASSES

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Introduction. In the recent paper [18] the second author has constructed the fundamental solution of the Cauchy problem for hyperbolic equations in Gevrey classes, and investigated the propagation of wave front sets of their solutions in Gevrey classes by assuming the constant multiplicities of their characteristic roots. The purpose of the present paper is to study the propagation of wave front sets in Gevrey classes for solutions of hyperbolic equations with characteristic roots of variable multiplicities and to give a similar result to the one for the C^∞-case obtained by Kumano-go and the second author [10]. Main results of the present paper are announced in [15] and [19].

Let $\mathcal{L}$ be an $l\times l$ hyperbolic system of the form

$$(1)\qquad \mathcal{L}=D_t-\begin{bmatrix}\lambda_1(t,X,D_x) & & 0\\ & \ddots & \\ 0 & & \lambda_l(t,X,D_x)\end{bmatrix}+(b_{jk}(t,X,D_x)) \qquad \text{on}\quad [0,T]\times R^n_x$$

with real symbols $\lambda_j(t,x,\xi)$ in $G^{(\kappa)}([0,T];S^1_{G(\kappa)})$ and symbols $b_{jk}(t,x,\xi)$ in $G^{(\kappa)}([0,T];S^\sigma_{G(\kappa)})$ $(0\leqq\sigma<1/\kappa)$. Here, for $\kappa>1$ and a real m we denote by $G^{(\kappa)}([0,T];S^m_{G(\kappa)})$ a class of symbols $p(t,x,\xi)$ of pseudo-differential operators satisfying for any multi-indices α, β and non-negative integer γ

$$(2)\qquad |\partial_t^\gamma\partial_\xi^\alpha\partial_x^\beta p(t,x,\xi)|\leqq CM^{-(|\alpha|+|\beta|+\gamma)}(\alpha!\,\beta!\,\gamma!)^\kappa\langle\xi\rangle^{m-|\alpha|} \qquad \text{for}\quad (t,x,\xi)\in[0,T]\times R^n_x\times R^n_\xi\,,$$

with constants C and M (>0) independent of α, β and γ. Throughout the present paper we assume the symbols λ_j are positively homogeneous in ξ (for $|\xi|\geqq1$), that is, λ_j satisfy

$$\lambda_j(t,x,\theta\xi)=\theta\lambda_j(t,x,\xi)\qquad \textit{for}\quad \theta\geqq1\quad \textit{and}\quad |\xi|\geqq1\,.$$

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Let $\mathcal{D}_{L^2}^{\{\kappa\}'}$ denote a class of ultradistributions defined by [6], that is,

$$\mathcal{D}_{L^2}^{\{\kappa\}'} = \operatorname{proj}\lim_{\varepsilon\downarrow 0} \mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}'} .$$

Here, for $\varepsilon>0$, $\mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}'}$ is a dual space of the Hilbert space

$$\mathcal{D}_{L^2,\varepsilon}^{\{\kappa\}} = \{u(x)\in L^2;\ \exp(\varepsilon\langle\xi\rangle^{1/\kappa})\,\hat{u}(\xi)\in L^2\}$$

and $\hat{u}(\xi)$ is the Fourier transform of $u(x)$ (see [20]). If $u\in\mathcal{D}_{L^2}^{\{\kappa\}'}$ and $\kappa_1\geqq\kappa$ we denote by $\mathrm{WF}_{G(\kappa_1)}(u)$ the wave front set of u in the Gevrey class of order κ_1 defined as follows:

DEFINITION. Let (x^o, ξ^o) be a point in $T^*(R^n)\backslash 0$ and let $u\in\mathcal{D}_{L^2}^{\{\kappa\}'}$. The point (x^o, ξ^o) does not belong to $\mathrm{WF}_{G(\kappa_1)}(u)$ for $\kappa_1\geqq\kappa$ if there exists a symbol $a(x, \xi)$ in $S^0_{G(\kappa)}$ (see Definition 1.1-ii) in Section 1) with $a(x^o, \theta\xi^o)\neq 0$ $(\theta\geqq 1)$ such that $f(x)\equiv a(X, D_x)u$ belongs to $\gamma^{(\kappa_1)}(R^n_x)$, that is, it satisfies

$$|\partial_x^\alpha f(x)|\leqq CM^{-|\alpha|}\alpha!^{\kappa_1} \qquad \textit{for all} \quad x\in R^n_x .$$

This definition is equivalent to that of Hörmander [2] if $u\in\mathcal{E}'$ (see Theorem 3 of [20]).

Consider the Cauchy problem

$$\mathcal{L}U(t) = 0 \quad (t\in[0, T]), \quad U(0) = G\in\mathcal{D}_{L^2}^{\{\kappa\}'} , \tag{7}$$

for $\mathcal{L}$ of the form (1) with $\lambda_j\in G^{(\kappa)}([0, T];\ S^1_{G(\kappa)})$ and $b_{jk}\in G^{(\kappa)}([0, T];\ S^\sigma_{G(\kappa)})$ for $0\leqq\sigma<1/\kappa$. Then we obtain the following:

Theorem 1 (cf. Theorem 3.4 of [10]). *Assume $\lambda_j(t, x, \xi)$ are homogeneous for $|\xi|\geqq 1$. Then, for any initial data $G\in\mathcal{D}_{L^2}^{\{\kappa\}'}$ there exists a unique solution $U(t)$ of (7) in $\mathcal{B}^\infty([0, T];\ \mathcal{D}_{L^2}^{\{\kappa\}'})$ and it satisfies*

$$\mathrm{WF}_{G(\kappa_1)}(U(t))\subset\Gamma(t;\ \mathrm{WF}_{G(\kappa_1)}(G)) \quad (0<t\leqq T) \tag{8}$$

for any κ_1 satisfying $\kappa\leqq\kappa_1<1/\sigma$.

The theorem of this type in the C^∞-case was given in [10] and [11]. In (8) the set $\Gamma(t_0;\ V)$, for a fixed $t_0\in(0, T]$ and a conic set V in $T^*(R^n_x)\backslash 0$, is defined as follows: First, we define $\Gamma^\nu_\varepsilon(t_0;\ V)$ $(\varepsilon>0,\ \nu=0, 1, 2, \cdots)$ as the conic hull of the set of end points (at $t=t_0$) of all ε-admissible trajectories of, at most, step ν issuing from $(y, \eta)\in V$ for large $|\eta|$. Then, the set $\Gamma(t_0;\ V)$ is defined by

$$\Gamma(t_0;\ V) = \bigcap_{\varepsilon>0}\bigcup_{\nu=0}^{\infty}\Gamma^\nu_\varepsilon(t_0;\ V_\varepsilon) , \tag{9}$$

where V_ε denotes an ε-conic neighborhood of V defined by

$$V_\varepsilon = \{(x, \xi);\ |x-y| \leqq \varepsilon,\ |\xi/|\xi|-\eta/|\eta|| \leqq \varepsilon,\ (y, \eta) \in V\}\ .$$

Roughly speaking, the ε-admissible trajectory is an ε-appoximation of the so-called broken null-bicharacteristic flow (see its precise definition in §4). The estimate (8) seems to be loose apparently because the limiting curve of ε-admissible trajectories ($\varepsilon \downarrow 0$) is not always broken null-bicharacteristic flow (see last sections in [4] and [5]). However, a result about the optimality of (8) was shown by the first author [14].

Next, we consider an application of Theorem 1 to the Cauchy problem

$$\text{(10)} \qquad \begin{cases} Lu = 0 \quad (0 \leqq t \leqq T)\,, \\ \partial_t^j u(0) = g_j \in \mathcal{D}_{L^2}^{\{\kappa\}'} \quad (j = 0, \cdots, m-1) \end{cases}$$

for a single hyperbolic operator

$$\text{(11)} \qquad L = D_t^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leqq m-j} a_{j,\alpha}(t, x)\, D_x^\alpha D_t^j \quad \text{on} \quad [0, T] \times R_x^n$$

with coefficients $a_{j,\alpha}(t, x)$ in a Gevrey class $\gamma^{(\kappa)}([0, T] \times R_x^n)$, that is, they satisfy

$$|\partial_t^\gamma \partial_x^\beta a_{j,\alpha}(t, x)| \leqq CM^{-(\gamma+|\beta|)}(\beta!\, \gamma!)^\kappa \quad on \quad [0, T] \times R_x^n\,.$$

As shown in [18] the problem (10) can be reduced to the equivalent Cauchy problem (7) with $\sigma = (r-q)/r$ and is $\gamma^{(\kappa)}$-well-posed for $1 \leqq \kappa < 1/\sigma$ (cf. [12]) if there exist regularly hyperbolic differential operators $L_1, L_2, \cdots, L_r$ with coefficients in $\gamma^{(\kappa)}([0, T] \times R_x^n)$ such that L has a form

$$\text{(12)} \qquad L = L_1 L_2 \cdots L_r + \sum_{j=0}^{m-q} a_j(t, X, D_x)\, D_t^j$$

with $a_j(t, x, \xi)$ in $G^{(\kappa)}([0, T]; S_{G(\kappa)}^{m-q-j})$ and $1 \leqq q \leqq r$. From this reduction it follows that for any $t \in (0, T]$

$$\text{(13)} \qquad \mathrm{WF}_{G(\kappa_1)}(u(t)) \subset \Gamma(t; \bigcup_{j=0}^{m-1} \mathrm{WF}_{G(\kappa_1)}(g_j)) \qquad for \quad \kappa \leqq \kappa_1 < r/(r-q)\,.$$

In the present paper, we shall consider the reduction to an equivalent problem (7) from the problem (10) for a hyperbolic operator of more general form than (12), which is inspired by the work of Komatsu [7] (see (5.1) and Theorem 2 in §5). In the case that the maximal multiplicity of characteristic roots of L is at most three, we also clarify the conditions of lower order terms of (11) in order that the problem (10) is reduced to an equivalent problem (7) of a hyperbolic system (1) with a given σ (<1) (see Theorem 3 in §5). We remark that the hyperbolic operator L has always the form (12) with $q=1$ if characteristic roots of L belong to $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$ and we admit L_j to be pseudo-differential cperators with respect to x.

For the hyperbolic operator L without any conditions on lower order terms and without assuming the smoothness of characteristic roots, Wakabayashi [21] has recently investigated the propagation of wave front sets for solutions of (10) in Gevrey class of order κ_1 satisfying $\kappa \leqq \kappa_1 < r/(r-1)$, where r is the maximal multiplicity of characteristic roots. The method of [21] is based on the construction of a parametrix of L, as in Bronshtein [1], and on the notion of "flows" K_z^+ in $T^*(R_t^1 \times R_x^n)$ emanating from a point $z \in T^*(R_t^1 \times R_x^n)$. When characteristic roots of hyperbolic operator L are smooth, that is, they belong to $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$, for any $t_0 \in (0, T]$ and any closed set V in $T^*(R_x^n)$, it follows that

$$(14) \qquad \Gamma(t_0, V) = \{\pi(K_z^+ \cap \{t = t_0\}); z \in \pi^{-1}(V) \cap \{t = 0\} \cap p^{-1}(0)\} .$$

Here $p = p(t, x, \tau, \xi)$ is the principal symbol of L and π is the natural projection from $T^*(R_t^1 \times R_x^n)$ to $T^*(R_x^n)$ (see Theorem 4 in §6, cf. Theorem 4.4 in [22]). So, our result (8) is the same as the one in [21] in the case that characteristic roots of L are smooth.

The plan of the present paper is as follows: In §§1–4 we prove Theorem 1. §§1–3 are devoted to preparatory lemmas and in §4 we complete the proof of Theorem 1 with the precise definition of $\Gamma(t_0; V)$. In §5, we show a method of the reduction of the form (1), and give, as an application of Theorem 1, a result on the propagation of wave front sets for the Cauchy problem (10). In §6 we show the equivalence of the estimate given by the flows K_z^+ of Wakabayashi [21], [22] and the one given by the set $\Gamma(t_0; V)$ of Kumano-go, Taniguchi and Tozaki [10], [11].

More precisely, we shall state the main idea of §1–3. In §1 we separate the symbol of the multi-product of Fourier integral operators to the sum of a main symbol and a regularizer and give the precise estimate for the part of the regularizer. To obtain this estimate we represent each factor of the multi-product to the sum of symbols depending on a parameter ζ. Then, we can use the similar discussions as in [17]. In §2 we estimate the part of the main symbol of the multi-product of Fourier integral operators which is given by the oscillatory integral of the multiple symbol. In [18], to estimate this we transform the multi-product of Fourier integral operators to the multi-product of pseudo-differential operators multiplied by a Fourier integral operator, using the decomposition $I_{\phi*}RI_\phi$ of the identity operator. In the present paper, since we estimate main symbols represented by oscillatory integrals of multiple symbols we use the transformation of oscillatory integrals which corresponds to the one by means of the decomposition $I = I_{\phi*}RI_\phi$. In §3, we give a method of the integration by parts for the symbol represented by an iterated integral of Volterra type. To show the corresponding estimate for the C^∞-case, in [10] we have estimated the iterated integral after we have simplified the multi-product of Fourier integral operators to one Fourier integral operator with multi-phase. But in the Gevrey case we can not employ this method since we use the equations of the critical points $X_\nu^j=\nabla_\xi\phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j)$, $\Xi_\nu^j=\nabla_x \phi_{j+1}(t_j, t_{j+1}; X_\nu^j, \Xi_\nu^{j+1})$, $(j=1, \cdots, \nu)$ to obtain the simplified symbol of a $(\nu+1)$-multi-product and we have no uniform estimate in the Gevrey class for the solutions of the equations of the critical points. Here the uniform estimate means the estimate independent of ν. So, we use the integration by parts for the iterated integral of Volterra type before simplifying the multiple symbol. It should be noted that to perform this method we must treat the oscillatory integral of the multiple symbol instead of the multi-product of Fourier integral operators and so we estimate in §2 the simplified symbol derived from the multiple symbol.

1. Fourier integral operators in Gevrey classes

First we recall symbol classes introduced in [18] and [20], which are subclasses of a symbol class S^m studied in [9]. In what follows we tacitly use the notation in [9] and [20] and assume that the constant κ is always larger than 1.

DEFINITION 1.1. i) We say that a symbol $p(x, \xi)$ $(\in S^m)$ belongs to a class $S^m_{G(\kappa,\infty)}$ if for any multi-index α there exists a constant C_α such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-|\beta|} \beta!^\kappa \langle\xi\rangle^{m-|\alpha|} \tag{1.1}$$

holds for a constant M independent of α and β.

ii) We say that a symbol $p(x, \xi)$ $(\in S^m)$ belongs to a class $S^m_{G(\kappa)}$ if

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq CM^{-(|\alpha|+|\beta|)}(\alpha!\,\beta!)^\kappa \langle\xi\rangle^{m-|\alpha|} \tag{1.2}$$

holds for constants C and M independent of α and β.

iii) We say that a symbol $p(x, \xi)$ $(\in S^{-\infty})$ belongs to a class $\mathcal{R}_{G(\kappa)}$ if for any α there exists a constant C_α such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-|\beta|}\beta!^\kappa \exp(-\varepsilon\langle\xi\rangle^{1/\kappa}) \tag{1.3}$$

holds for positive constants M and ε independent of α and β.

REMARK. The inequality (1.3) is equivalent to the condition that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-(|\beta|+N)}(\beta!\,N!)^\kappa \langle\xi\rangle^{-|\alpha|-N}$$

holds for any integer N with a constant M independent of N, α and β.

Following Definition 1.1 of [18], for a $\tau\in[0, 1)$ we define a class $\mathcal{P}_{G(\kappa)}(\tau)$

of phase functions of Fourier integral operators as follows: We say that a real valued function $\phi(x, \xi)$ is a phase function belonging to a class $\mathcal{P}_{G(\kappa)}(\tau)$ if $J(x, \xi)\equiv\phi(x, \xi)-x\cdot\xi$ satisfieds

$$(1.4)\qquad \begin{cases} \sum_{|\alpha|+|\beta|\leqq 2} |J_{(\beta)}^{(\alpha)}(x, \xi)|/\langle\xi\rangle^{1-|\alpha|}\leqq\tau\,, \\ |J_{(\beta)}^{(\alpha)}(x, \xi)|\leqq\tau M^{-(|\alpha|+|\beta|)}(\alpha!\,\beta!)^{\kappa}\langle\xi\rangle^{1-|\alpha|} \end{cases}$$

for a constant M independent of α and β. We put $\mathcal{P}_{G(\kappa)}=\bigcup_{0\leqq\tau<1}\mathcal{P}_{G(\kappa)}(\tau)$.

Let $\phi(x, \xi)$ be a phase function in $\mathcal{P}_{G(\kappa)}$. Then, a Fourier integral operator $P_\phi=p_\phi(X, D_x)$ with phase function $\phi(x, \xi)$ and a symbol $\sigma(P_\phi)=p(x, \xi)$ in $S^m_{G(\kappa,\infty)}$ is defined by

$$(1.5)\qquad P_\phi u(x) = O_s-\iint e^{i(\phi(x,\xi)-x'\cdot\xi)}\,p(x, \xi)\,u(x')\,dx'\,d\!\!^-\xi \quad \text{for} \quad u\in\mathcal{S}\,,$$

where $d\!\!^-\xi=(2\pi)^{-n}d\xi$, $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions on R^n_x and the right hand side of (1.5) is the oscillatory integral defined in Chapter 1 of [9]. We denote the set of such Fourier integral operators by $S^m_{G(\kappa,\infty),\phi}$. If $\phi=x\cdot\xi$, the set $S^m_{G(\kappa,\infty),\phi}$ is the one of pseudo-differential operators. In this case we write it simply by $S^m_{G(\kappa,\infty)}$. Since $S^m_{G(\kappa)}$ and $\mathcal{R}_{G(\kappa)}$ are subclasses of $S^m_{G(\kappa,\infty)}$, we similarly denote by $S^m_{G(\kappa),\phi}$ and $\mathcal{R}_{G(\kappa),\phi}$ the corresponding classes of Fourier integral operators, which are represented by the formula (1.5). We can identify $\mathcal{R}_{G(\kappa),\phi}$ with the set $\mathcal{R}_{G(\kappa)}$ of pseudo-differential operators because it follows from (1.4) that if $p(x, \xi)$ belongs to $\mathcal{R}_{G(\kappa)}$ then $e^{iJ(x,\xi)}p(x, \xi)$ so does. Here and in what follows, we use the same notation $\mathcal{R}_{G(\kappa)}$ for the class of pseudo-differential operators with symbols in $\mathcal{R}_{G(\kappa)}$ because no confusion occurs between the class of symbols and the one of operators.

As proved in [20] we have

Proposition 1.2 (see Theorem 2 and Lemma 2.1 of [20]).

i) *Let $p(x, \xi)$ belong to $S^m_{G(\kappa,\infty)}$ and let $\phi(x, \xi)$ belong to $\mathcal{P}_{G(\kappa)}$. Then the Fourier integral operator $P_\phi=p_\phi(X, D_x)$ maps $\mathcal{D}^{\{\kappa\}'}_{L^2}$ into itself.*

ii) *Let R be a pseudo-differential operator in $\mathcal{R}_{G(\kappa)}$ and let u belong to $\mathcal{D}^{\{\kappa\}'}_{L^2}$. Then we have $Ru\in\gamma^{(\kappa)}(R^n_x)$.*

Following §2 of [18] we denote for $\phi\in\mathcal{P}_{G(\kappa)}$

$$L^m_{G(\kappa)}(\phi) = \{p_\phi(X, D_x) = p^0_\phi(X, D_x)+\tilde{p}_\phi(X, D_x)\,; \\ p^0(x, \xi)\in S^m_{G(\kappa)}\,,\quad \tilde{p}(x, \xi)\in\mathcal{R}_{G(\kappa)}\}\,,$$

that is, symbolically $L^m_{G(\kappa)}(\phi)=S^m_{G(\kappa),\phi}+\mathcal{R}^m_{G(\kappa),\phi}$. In what follows we often say that $p^0(x, \xi)$ is a main symbol of $p_\phi(X, D_x)$. If $\phi(x, \xi)=x\cdot\xi$ we denote $L^m_{G(\kappa)}(\phi)$ simply by $L^m_{G(\kappa)}$.

For a sequence $\{\phi_j\}$ of phase functions $\phi_j(x, \xi)\in\mathscr{P}_{G(\kappa)}(\tau_j)$, we consider multi-products

$$P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}} \tag{1.6}$$

of Fourier integral operators P_{j,ϕ_j} in $L^{\sigma}_{G(\kappa)}(\phi_j)$ with $\sigma\geqq 0$. As in §2 of [18] we assume the following:

(A–1) If we set $J_j(x, \xi)=\phi_j(x, \xi)-x\cdot\xi$, $\{J_j/\tau_j\}$ is bounded in $S^1_{G(\kappa)}$ and an inequality $\sum_{j=1}^{\infty}\tau_j\leqq\tau^o$ holds for a small constant τ^o.

(A–2) If we write $P_{j,\phi_j}\equiv p_{j,\phi_j}(X, D_x)=p^o_{j,\phi_j}(X, D_x)+\tilde{p}_{j,\phi_j}(X, D_x)\in S^{\sigma}_{G(\kappa),\phi_j}+\mathscr{R}_{G(\kappa),\phi_j}$ the set $\{p^o_j(x, \xi)\}$ is bounded in $S^{\sigma}_{G(\kappa)}$ and the set $\{\tilde{p}_j(x, \xi)\}$ is bounded in $\mathscr{R}_{G(\kappa)}$.

REMARK. Concerning the bounded set in $S^m_{G(\kappa)}$ or $\mathscr{R}_{G(\kappa)}$, see remarks after Definition 1.1 in [18].

We assume τ^o in (A–1) small enough so that Proposition 2.4 in [18] and Lemmas 1.4–1.6 below hold. Then, a multi-product $\Phi_{\nu+1}\equiv\Phi_{\nu+1}(x, \xi)=\phi_1\#\phi_2\#\cdots\#\phi_{\nu+1}(x, \xi)$ of phase functions $\phi_1(x, \xi)$, $\phi_2(x, \xi)$, $\cdots$, $\phi_{\nu+1}(x, \xi)$ is defined by

$$\Phi_{\nu+1}(x, \xi)=\sum_{j=1}^{\nu}(\phi_j(X^{j-1}_\nu, \Xi^j_\nu)-X^j_\nu\cdot\Xi^j_\nu)+\phi_{\nu+1}(X^\nu_\nu, \xi)\quad (X^0_\nu=x)$$

and it belongs to $\mathscr{P}_{G(\kappa)}$, where $\{X^j_\nu, \Xi^j_\nu\}^\nu_{j=1}\equiv\{X^j_\nu, \Xi^j_\nu\}^\nu_{j=1}(x, \xi)$ is a solution of

$$\begin{cases} x^j=\nabla_\xi\,\phi_j(x^{j-1}, \xi^j)\,, \\ \xi^j=\nabla_x\,\phi_{j+1}(x^j, \xi^{j+1}) \end{cases} \tag{1.7}$$
$$(j=1, \cdots, \nu;\quad x^0=x\,,\quad \xi^{\nu+1}=\xi)\,.\qquad \text{(cf. [11])}.$$

Recall that the multi-product (1.6) is a Fourier integral operator in $L^{(\nu+1)\sigma}_{G(\kappa)}(\Phi_{\nu+1})$ with the above phase function $\Phi_{\nu+1}(x, \xi)$ and its symbol $q_{\nu+1}(x, \xi)=\sigma(P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}})$ is written as

$$\begin{aligned} &q_{\nu+1}(x, \xi)\\ &=O_s\text{-}\int\cdots\int\exp[i(\psi_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)-\Phi_{\nu+1}(x, \xi))]\\ &\quad\times\prod_{j=1}^{\nu+1}p_j(x^{j-1}, \xi^j)\,d\tilde{x}^\nu\,đ\tilde{\xi}^\nu\quad(x^0=x,\ \xi^{\nu+1}=\xi)\,, \end{aligned} \tag{1.8}$$

where $\tilde{x}^\nu=(x^1, x^2, \cdots, x^\nu)$, $\tilde{\xi}^\nu=(\xi^1, \xi^2, \cdots, \xi^\nu)$, $d\tilde{x}^\nu=dx^1\cdots dx^\nu$, $đ\tilde{\xi}^\nu=đ\xi^1\cdots đ\xi^\nu$ and

$$\Psi_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)=\sum_{j=1}^{\nu}(\phi_j(x^{j-1}, \xi^j)-x^j\cdot\xi^j))+\phi_{\nu+1}(x^\nu, \xi) \tag{1.9}$$
$$(x^0=x)\,.$$

In the above the right hand side of (1.8) is an oscillatory integral, whose welldefinedness will be proved in the next section in a more general form. The aim of this section is to find the main symbol of $q_{\nu+1}(x, \xi)$ for (1.8).

Let $\chi(\xi)$ be a function in $\gamma^{(\kappa)}(R^n_\xi)$ satisfying

$$(1.10)\qquad 0\leqq\chi(\xi)\leqq 1\,,\quad \chi(\xi)=1\ (|\xi|\leqq 2/5)\,,\quad \chi(\xi)=0\quad (|\xi|\geqq 1/2)\,.$$

For a main symbol $p^o_j(x, \xi)\in S^o_{G(\kappa)}$ of P_{j,ϕ_j} and a parameter $\zeta\in R^n$, we set

$$(1.11)\qquad \begin{cases} p^+_j(x, \xi; \zeta)=\chi(4(\xi-\zeta)/\langle\zeta\rangle)\, p^o_j(x, \xi)\,, \\ p^-_j(x, \xi; \zeta)=p^o_j(x, \xi)-p^+_j(x, \xi; \zeta) \end{cases}$$

and consider $P^{\pm}_{j,\phi_j}(\zeta)=p^{\pm}_{j,\phi_j}(X, D_x; \zeta)$ Fourier integral operators with a parameter ζ. Set for $k=1, \cdots, \nu+1$

$$(1.12)\qquad \begin{cases} \tilde{q}^k_{\nu+1}(x, \xi)=\sigma((P^o_{1,\phi_1}\cdots P^o_{k-1_1,\phi_{k-1}})\,\tilde{P}_{k,\phi_k} \\ \qquad\qquad \times(P_{k+1,\phi_{k+1}}\cdots P_{\nu+1,\phi_{\nu+1}}))\,, \\ q^k_{\nu+1}(x, \xi; \zeta)=\sigma(P^+_{1,\phi_1}(\zeta)\cdots P^+_{k-1,\phi_{k-1}}(\zeta)P^-_{k,\phi_k}(\zeta) \\ \qquad\qquad \times P^o_{k+1,\phi_{k+1}}\cdots P^o_{\nu+1,\phi_{\nu+1}}) \end{cases}$$

and set

$$q^o_{\nu+1}(x, \xi; \zeta)=\sigma(P^+_{1,\phi_1}(\zeta)\cdots P^+_{\nu+1,\phi_{\nu+1}}(\zeta))\,.$$

Then, for any fixed ζ we get the decomposition

$$(1.13)\qquad q_{\nu+1}(x, \xi)=q^o_{\nu+1}(x, \xi; \zeta)+\sum_{k=1}^{\nu+1} q^k_{\nu+1}(x, \xi; \zeta)+\sum_{k=1}^{\nu+1}\tilde{q}^k_{\nu+1}(x, \xi)\,.$$

In (1.13) we set $\zeta=\xi$, where ξ is the fiber variable of the simplified symbol of (1.6). Then, we have

Lemma 1.3. *The symbol* $r_{\nu+1}(x, \xi)\equiv q_{\nu+1}(x, \xi)-q^o_{\nu+1}(x, \xi; \xi)$ *belongs to* $\mathcal{R}_{G(\kappa)}$ *and it satisfies for any* α *and* β

$$(1.14)\qquad |r_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)|\leqq C_\alpha A^\nu M^{-|\beta|}\,\beta!^\kappa\nu!^{\sigma\kappa}\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})$$

with positive constants ε, A *and* M *independent of* α, β *and* ν. *Here,* C_α *is a conttant independent of* β *and* ν.

Together with Lemma 2.1 in the following section $q^o_{\nu+1}(x, \xi; \xi)$ is a main symbol of (1.6).

REMARK 1. The constants ε, A, M and C_α in (1.14) are determined only by the dimension n, σ, τ^o and constants C, M, C_α, ε in (1.2), (1.3) and (1.4) for $p^o_j(x, \xi)$, $\tilde{p}_j(x, \xi)$ and $\phi_j(x, \xi)$.

REMARK 2. The estimate (1.14) still holds even if we replace $\Phi_{\nu+1}(x, \xi)$

by $x\cdot\xi$ in the corresponding formula (1.8) for the multi-products (1.12). This follows from (1.4) (see the discussion above Proposition 1.2).

We begin the proof of Lemma 1.3 with the estimation of the third term of (1.13). By means of Theorem 2.1 and Proposition 2.2 (and its Remark 2) in [18] it is clear that $\tilde{q}^k_{\nu+1}(x, \xi)$ satisfies the same inequalities as (1.14) because $\langle\xi\rangle^{(\nu+1)\sigma}\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\leqq A_1^\nu\,\nu!^{\sigma\kappa}\exp(-\varepsilon\langle\xi\rangle^{1/\kappa}/2)$ for a suitable constant A_1. So, for the proof of the lemma it suffices to show

$$(1.15)\qquad |\partial_\xi^\alpha\,\partial_x^\beta\,q^k_{\nu+1}(x, \xi; \xi)|\leqq C_\alpha A^\nu M^{-|\beta|}\,\beta!^\kappa\,\nu!^{\sigma\kappa}\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\,.$$

To this end, we prepare the following three lemmas which are versions of propositions in §2 of [18]. Let $p(x, \xi; \zeta)$ be a symbol in $S^m_{G(\kappa)}$ with a parameter $\zeta\in R^n$ such that for any α, α' and β we have

$$(1.16)\qquad |\partial_\xi^\alpha\,\partial_\zeta^{\alpha'}\partial_x^\beta p|\leqq CM^{-(|\alpha|+|\alpha'|+|\beta|)}(\alpha!\,\alpha'!\,\beta!)^\kappa\langle\xi\rangle^{m-|\alpha|-|\alpha'|}\,.$$

Lemma 1.4 (cf. Proposition 2.2 in [18]). *Let $P_1(\zeta)\equiv p_1(X, D_x; \zeta)$ be a pseudo-differential operator with a symbol $p_1(x, \xi; \zeta)\in S^m_{G(\kappa)}$ satisfying* (1.16) *and set $P_{2,\phi}\equiv p_{2,\phi}(X, D_x)$ be a Fourier integral operator with phase function $\phi(x, \xi)$ in $\mathcal{P}_{G(\kappa)}$ and symbol $p_2(x, \xi)$ in $S^{m'}_{G(\kappa)}$. Then, we have the following:*

i) *The product $P_1(\zeta)P_{2,\phi}$ belongs to $L^{m+m'}_{G(\kappa)}(\phi)$ and has the form*

$$(1.17)\qquad P_1(\zeta)\,P_{2,\phi}=q^0_\phi(X, D_x; \zeta)+\tilde{q}_\phi(X, D_x; \zeta)$$

with symbols $q^0(x, \xi; \zeta)$ *and* $\tilde{q}(x, \xi; \zeta)$ *satisfying*

$$(1.18)\qquad |\partial_\xi^\alpha\,\partial_\zeta^{\alpha'}\,\partial_x^\beta\,q^0(x, \xi; \zeta)|\leqq C_1M_1^{-(|\alpha|+|\alpha'|+|\beta|)}(\alpha!\,\alpha'!\,\beta!)^\kappa \times\langle\xi\rangle^{m+m'|\alpha|+|\alpha'|}\,,$$

$$(1.19)\qquad |\partial_\xi^\alpha\,\partial_\zeta^{\alpha'}\,\partial_x^\beta\,\tilde{q}(x, \xi; \zeta)|\leqq C_{\alpha,\alpha'}M_1^{-|\beta|}\,\beta!^\kappa\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})$$

for constants C_1, M_1, $C_{\alpha,\alpha'}$ and a positive constant ε.

ii) *Let the symbol $p_1(x, \xi; \zeta)$ satisfy for a $\delta>0$*

$$(1.20)\qquad p_1(x, \xi; \zeta)=0\qquad if\quad |\xi-\zeta|\leqq\delta\langle\zeta\rangle\,.$$

Then, there exists a positive constant $\tau^0\equiv\tau^0(\delta)$ such that for $\phi(x, \xi)\in\mathcal{P}_{G(\kappa)}(\tau^0)$ the decomposition (1.17) *still holds with* (1.18)–(1.19) *and the main symbol $q^0(x, \xi; \zeta)$ of* (1.17) *satisfies* (1.20) *with δ replaced by $\delta'>0$ depending only on δ.*

iii) *For the product $P_{2,\phi}\,P_1(\zeta)$ we have the same statements as* i) *and* ii).

REMARK. The constants C_1, M_1 and $C_{\alpha,\alpha'}$ are determined by the constants τ and M in (1.4) for $\phi(x, \xi)$, the constants C and M in (1.16) for $p_1(x, \xi; \zeta)$ and those in (1.2) for $p_2(x, \xi)$.

Proof. The first statement easily follows from the proof of Proposition

2.2 in [18]. In order to show ii) we recall that the main symbol of the product $P_1(\zeta)\,P_{2,\phi}$ is defined by the first term of the right hand side of (4.4) in [18], that is,

$$q^o(x, \xi'; \zeta) = O_s\text{-}\iint e^{i\psi}\, p_1(x, \xi; \zeta)\, \chi((\xi-\xi')/\langle\xi'\rangle) \times p_2(x', \xi')\, dx'\, d\!\!\!{}^-\xi\,, \tag{1.21}$$

where $\psi = x\cdot\xi - x'\cdot\xi + \phi(x', \xi') - \phi(x, \xi')$ and χ is a function satisfying (1.10). In (1.21) we replace $\chi(\xi)$ by $\chi(\xi/\theta)$ for a small $\theta>0$, that is,

$$q^o(x, \xi'; \zeta) = O_s\text{-}\iint e^{i\psi}\, p_1(x, \xi; \zeta)\, \chi((\xi-\xi')/(\theta\langle\xi'\rangle)) \times p_2(x', \xi')\, dx'\, d\!\!\!{}^-\xi\,. \tag{1.21$'$}$$

Then, we can prove (1.20) for $q^o(x, \xi; \zeta)$ if we take θ small enough. This exchange is harmless for the proof of (1.18) and (1.19) if we use $\theta > 5\tau/2$ when we prove (1.19). The proof of iii) is similar to those of i) and ii). Q.E.D.

Let I_ϕ (resp. I_{ϕ^*}) denote the Fourier (resp. the conjugate Fourier) integral operator with symbol 1.

Lemma 1.5. *Assume that $P_\phi(\zeta) \equiv p_\phi(X, D_x; \zeta)$ is a Fourier integral operator with a symbol $p(x, \xi; \zeta) \in S^m_{G(\kappa)}$ satisfying* (1.16). *Then, we have*

$$P_\phi(\zeta)\, I_{\phi^*} \in L^m_{G(\kappa)}\,, \quad I_{\phi^*}\, P_\phi(\zeta) \in L^m_{G(\kappa)} \tag{1.22}$$

and about the symbols of the product operators we have the corresponding results to (1.17)–(1.19) *with $m+m'$ replaced by m in* (1.18). *If $p(x, \xi; \zeta)$ satisfies* (1.20), *then there exists a positive $\tau^o \equiv \tau^o(\delta)$ satisfying the following property: If $\phi(x, \xi) \in \mathcal{P}_{G(\kappa)}(\tau^o)$ then, adding to* (1.17)–(1.19), *the main symbols of $P_\phi(\zeta)\, I_{\phi^*}$ and $I_{\phi^*}\, P_\phi(\zeta)$ satisfy* (1.10) *with δ replaced by δ'* $(0<\delta'<\delta)$.

REMARK. We have the similar statement as in the remark of Lemma 1.4.

Proof. The formula (1.22) is the same as (2.10) in [18]. For the proof of the last statement we replace $\chi(\xi) \in \gamma^{(\kappa)}(R^n_\xi)$ by $\chi(\xi/\theta)$ with a sufficiently small $\theta>0$ when we proceed the proof of Proposition 2.3 in [18]. Then, by means of the inequality (2.3)-a) in [17] we obtain the desired main symbol keeping the properties (1.17)–(1.19). Q.E.D.

Lemma 1.6. *Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_{G(\kappa)}(\tau_j)$, $j=1, 2$, $\tau_1+\tau_2 \leqq \tau^o$ for a sufficiently small $\tau^o>0$, and let $P_{\phi_2}(\zeta) \in S^m_{G(\kappa), \phi_2}$ be the same as in Lemma* 1.5. *Then there exists a pseudo-differential operator $P'(\zeta) \equiv p'(X, D_x; \zeta) = p'^o(X, D_x; \zeta) + \tilde{p}'(X, D_x; \zeta)$ in $L^m_{G(\kappa)}$ such that*

$$I_{\phi_1}\, P_{\phi_2}(\zeta) = P'(\zeta)\, I_{\phi_1 \# \phi_2} \tag{1.23}$$

and $p'^o(x, \xi; \zeta)$ *and* $\tilde{p}'(x, \xi; \zeta)$ *satisfy* (1.18) (*with* $m+m'$ *replaced by* m) *and* (1.19), *respectively. Furthermore, if* $p(x, \xi; \zeta)$ *satisfies* (1.20) *the main symbol* $p'^o(x, \xi; \zeta)$ *also satisfies* (1.20) *with* δ *replaced by* δ' $(0<\delta'<\delta)$, *provided that we take* τ^0 *sufficiently small corresponding to* δ.

REMARK. We have the similar assertion as in the remark of Lemma 1.4.

Proof. The formula (1.23) follows from Propositions 2.2, 2.3, 2.5 and Corollary 2.8 in [18], as in the proof of Lemma 2.10 in [17]. For the proof of (1.20) for $p'^o(x, \xi; \zeta)$ we use Lemmas 1.4 and 1.5 repeatedly. Then, we get the lemma. Q.E.D.

Now, we are prepared to prove (1.15).

Proof of (1.15). It follows from Lemma 1.6 that there exist pseudo-differential operators $P'_j(\zeta)=p'_j(X, D_x; \zeta)$ $(j=1, \cdots, k)$ and $P'_j=p'_j(X, D_x)$ $(j=k+1, \cdots, \nu+1)$ in $L^{\sigma}_{G(\kappa)}$ such that

$$(1.24)\qquad \begin{cases} I_{\Phi_{j-1}} P^+_{j,\phi_j}(\zeta) = P'_j(\zeta)\, I_{\Phi_j}\,, & j=1, \cdots, k-1 \quad (\Phi_0 = x\cdot\xi)\,, \\ I_{\Phi_{k-1}} P^-_{k,\phi_k}(\zeta) = P'_k(\zeta)\, I_{\Phi_k}\,, & \\ I_{\Phi_{j-1}} P_{j,\phi_j} = P'_j\, I_{\Phi_j}\,, & j=k+1, \cdots, \nu+1\,. \end{cases}$$

As in the last paragraph of §2 in [18] it follows that

$$\begin{aligned} &P^+_{1,\phi_1}(\zeta)\cdots P^+_{k-1,\phi_{k-1}}(\zeta)\, P^-_{k,\phi_k}(\zeta)\, P_{k+1,\phi_{k+1}}\cdots P_{\nu+1,\phi_{\nu+1}} \\ &\quad = P'_1(\zeta)\cdots P'_{k-1}(\zeta)\, P'_k(\zeta)\, P'_{k+1}\cdots P'_{\nu+1}\, I_{\Phi_{\nu+1}}\,. \end{aligned}$$

If we apply Theorem 2.6 of [18] to the multi-product $Q'_{\nu+1,k}(\zeta)=P'_1(\zeta)\cdots P'_{k-1}(\zeta)\cdot P'_k(\zeta)\, P'_{k+1}\cdots P'_{\nu+1}$ of pseudo-differential operators we have

$$Q'_{\nu+1,k}(\zeta) = q^{k,o}_{\nu+1}(X, D_x; \zeta)+q^{\nu,\infty}_{\nu+1}(X, D_x; \zeta)\in L^{(\nu+1)\sigma}_{G(\kappa)}\,.$$

As in the proof of Lemmas 1.4–1.6 we exchange $\chi(\xi)$ by $\chi(\xi/\theta)$ for a sufficiently small $\theta>0$ in the proof of Proposition 5.1 in [18]. Then, in view of (1.16), the symbols $q^{k,o}_{\nu+1}(x, \xi; \zeta)$ and $q^{k,\infty}_{\nu+1}(x, \xi; \zeta)$ satisfy

$$(1.25)\qquad \begin{cases} |\partial^{\alpha}_{\xi}\, \partial^{\alpha'}_{\zeta}\, \partial^{\beta}_{x}\, q^{k,o}_{\nu+1}(x, \xi; \zeta)| \leqq \tilde{A}^{\nu} M^{-(|\alpha|+|\alpha'|+|\beta|)}(\alpha!\, \alpha'!\, \beta!)^{\kappa} \\ \qquad\qquad\qquad\qquad \times\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|-|\alpha'|}\,, \\ |\partial^{\alpha}_{\xi}\, \partial^{\alpha'}_{\zeta}\, \partial^{\beta}_{x}\, q^{k,\infty}_{\nu+1}(x, \xi; \zeta)| \leqq C_{\alpha,\alpha'} \tilde{A}^{\nu} M^{-|\beta|}\, \beta!^{\kappa} \nu!^{\sigma\kappa} \exp(-\varepsilon\langle\xi\rangle^{1/\kappa}) \end{cases}$$

and, in addition, the main symbol $q^{k,o}_{\nu+1}(x, \xi; \zeta)$ satisfies (1.20) for a $\delta'>0$ independent of ν. Here, we used the main symbol of $P'_k(\zeta)$ satisfies (1.20) for a small $\delta>0$. Finally we use Lemma 1.4. Then the main symbol of $q^{k,o}_{\nu+1}(X, D_x; \zeta)\, I_{\Phi_{\nu+1}}$ vanishes when $\zeta=\xi$. Noting (2.7) of [18] and the remarks of Proposition 2.2 of [18] we obtain (1.15) by (1.25). This concludes the proof of Lemma 1.3. Q.E.D.

2. Multiple symbols and lemmas

For x^j, $\xi^j \in R^n$ $(j=1, \cdots, \nu)$ we write $\tilde{x}^\nu=(x^1, \cdots, x^\nu)$ and $\tilde{\xi}^\nu=(\xi^1, \cdots, \xi^\nu)$. We consider a multiple symbol $p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi) \in C^\infty$ satisfying for any α, β, $\tilde{\alpha}^\nu=(\alpha^1, \cdots, \alpha^\nu)$ and $\tilde{\beta}^\nu=(\beta^1, \cdots, \beta^\nu)$

$$\begin{aligned}
&|\partial_x^\beta \partial_{\tilde{\xi}^\nu}^{\tilde{\alpha}^\nu} \partial_{\tilde{x}^\nu}^{\tilde{\beta}^\nu} \partial_\xi^\alpha p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)| \\
&\qquad \leqq C_0 M^{-(|\alpha|+|\beta|+|\tilde{\alpha}^\nu|+|\tilde{\beta}^\nu|)}(\alpha!\,\beta!\,\tilde{\alpha}^\nu!\,\tilde{\beta}^\nu!)^\kappa \prod_{j=1}^{\nu+1} \langle \xi^j \rangle^{m_j-|\alpha^j|} \\
&\qquad\qquad (\alpha^{\nu+1}=\alpha\,, \quad \xi^{\nu+1}=\xi)
\end{aligned} \tag{2.1}$$

for positive constants C_0 and M independent of ν. We consider a simplified symbol $q_{\nu+1}(x, \xi)$ defined by

$$\begin{aligned}
q_{\nu+1}(x, \xi) = O_s\text{-}\int\cdots\int &\exp[i(\Psi_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)-\Phi_{\nu+1}(x, \xi))] \\
&\times p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)\, d\tilde{x}^\nu d\tilde{\xi}^\nu \\
&\qquad (d\tilde{x}^\nu = dx^1\cdots dx^\nu\,, \quad d\tilde{\xi}^\nu = d\xi^1\cdots d\xi^\nu)\,,
\end{aligned} \tag{2.2}$$

where $\Psi_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ is defined by (1.9) for a sequence $\{\phi_j\}$ of phase functions $\phi_j(x, \xi)$ satisfying (A–1) in the preceding section. The integral of the right hand side of (2.2) means the oscillatory integral, that is,

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\int\cdots\int &\exp[i(\Psi_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)-\Phi_{\nu+1}(x, \xi))] \\
&\times \prod_{j=1}^{\nu} \chi(\varepsilon x^j)\,\chi(\varepsilon\xi^j)\, p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)\, d\tilde{x}^\nu d\tilde{\xi}^\nu
\end{aligned} \tag{2.3}$$

with $\chi \in C^\infty$ satisfying (1.10). We shall show this limit is well-defined. Set $\tilde{p}_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)=p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi) \prod_{j=1}^{\nu+1} \exp(iJ_j(x^{j-1}, \xi^j))\cdot\exp(-i\Phi_{\nu+1}(x, \xi))$ $(x^0=x,\ \xi^{\nu+1}=\xi)$ with $J_j(x, \xi)=\phi_j(x, \xi)-x\cdot\xi$. Then, we have

$$|\partial_{\tilde{\xi}^\nu}^{\tilde{\alpha}^\nu} \partial_{\tilde{x}^\nu}^{\tilde{\beta}^\nu} \tilde{p}_{\nu+1}| \leqq C_{\tilde{\alpha}^\nu, \tilde{\beta}^\nu} \langle \xi^1 \rangle^{m_1} \prod_{j=1}^{\nu} \langle \xi^{j+1} \rangle^{m_{j+1}+|\beta^j|}\,. \tag{2.4}$$

Set $n_0=[n/2]+1$ and define integers l_j $(j=1, 2, \cdots)$ inductively by $l_1=[(m_1+n)/2]+1$, $l_j=[(m_1+\cdots+m_j+n)/2]+l_1+\cdots+l_{j-1}$. Then, it follows from the integration by parts that the limit (2.3) equals

$$\begin{aligned}
\int\cdots\int &\exp[i(\sum_{j=1}^{\nu}(x^{j-1}-x^j)\cdot\xi^j+x^\nu\cdot\xi)] \\
&\times \prod_{j=1}^{\nu} \{(1+|x^{j-1}-x^j|^2)^{-n_0}(1-\Delta_{\xi^j})^{n_0}\} \\
&\cdot[\prod_{j=1}^{\nu}(1+|\xi^j-\xi^{j+1}|^2)^{-l_j}(1-\Delta_{x^j})^{l_j} \\
&\cdot p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)\, d\tilde{x}^\nu d\tilde{\xi}^\nu \quad (x^0=x\,, \quad \xi^{\nu+1}=\xi)\,,
\end{aligned}$$

which is well-defined by means of (2.4). Therefore, we can exchange the order of integration and differentiate the right hand side of (2.2) under the integral sign (that is, we obtain the Fubini theorem and Lebesgue's convergence theorem for the oscillatory integral, see §6 of Chapter 1 of [9]).

Set $p_{\nu+1}(x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)=\prod_{j=1}^{\nu+1} p_j(x^{j-1}, \xi^j)$ $(x^0=x,\ \xi^{\nu+1}=\xi)$ for a sequence $\{p_j\}$ of symbols $p_j(x, \xi)$ satisfying (A–2) in the preceding section. Then, $p_{\nu+1}(x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)$ satisfies (2.1). Taking the decomposition (1.13) with $\zeta=\xi$ and Lemma 1.3 into account, we may investigate only $q^{0}_{\nu+1}(x, \xi; \xi)$ in (1.13). So, in what follows we may assume

$$|\xi^j-\xi|\leqq\langle\xi\rangle/8 \qquad on \quad \operatorname{supp} p_{\nu+1} \tag{2.5}$$

Then, the condition (2.1) is reduced to

$$\begin{aligned} &|\partial_x^{\beta}\partial_{\tilde{\xi}^{\nu}}^{\tilde{\alpha}^{\nu}}\partial_{\tilde{x}^{\nu}}^{\tilde{\beta}^{\nu}}\partial_{\xi}^{\alpha} p_{\nu+1}(x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)| \\ &\leqq C_0 M^{-(|\alpha|+|\beta|+|\tilde{\alpha}^{\nu}|+|\tilde{\beta}^{\nu}|)}(\alpha!\,\beta!\,\tilde{\alpha}^{\nu}!\,\tilde{\beta}^{\nu}!)^{\kappa}\langle\xi\rangle^{m-|\alpha|-|\tilde{\alpha}^{\nu}|}\,. \end{aligned} \tag{2.6}$$

with $m=\sum_{j=1}^{\nu+1} m_j$. So, till the end of Section 3 we always assume (2.5) and (2.6).

Lemma 2.1. *Assume* (2.5), (2.6) *and* (A–1) *in* §1. *Then, for* $q_{\nu+1}(x, \xi)$ *defined by* (2.2) *we have*

$$|q_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)|\leqq C_0 A^{\nu} M_1^{-(|\alpha|+|\beta|)}(\alpha!\,\beta!)^{\kappa}\langle\xi\rangle^{m-|\alpha|} \tag{2.7}$$

for any α *and* β, *where* A *and* M_1 *are independent of* ν.

REMARK. For the multiple symbol satisfying only (2.1) we can obtain the conclusion of the lemma modulo the regularizer satisfying (1.14). The proof, however, is fairly long. So, in this paper we restrict ourselves to prove the lemma in the above form.

For any $k\in\{1, \cdots, \nu\}$ we write $\tilde{x}^{\nu,k}=(x^1, \cdots, x^{k-1}, x^{k+1}, \cdots, x^{\nu})$ and set

$$\begin{aligned} \Phi_{\nu+1,k}(x, \xi^k, x^k, \xi) = (\phi_1\#\cdots\#\phi_k)\,(x, \xi^k)-x^k\cdot\xi^k \\ +(\phi_{k+1}\#\cdots\#\phi_{\nu+1})\,(x^k, \xi)\,. \end{aligned} \tag{2.8}$$

Then, we have

Lemma 2.2. *Assume* (2.5), (2.6) *and* (A–1). *Then, the symbol* $q_{\nu+1,k}(x, \xi^k, x^k, \xi)$, $k=1, \cdots, \nu$, *defined by*

$$\begin{aligned} &q_{\nu+1,k}(x, \xi^k, x^k, \xi) \\ &\quad = O_s\text{-}\int\cdots\int \exp\,[i(\Psi_{\nu+1}(x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)-\Phi_{\nu+1,k}(x, \xi^k, x^k, \xi))] \\ &\qquad\qquad \times p_{\nu+1}(x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)\, d\tilde{x}^{\nu,k} d\tilde{\xi}^{\nu,k} \end{aligned} \tag{2.9}$$

satisfy

$$(2.10)\qquad \begin{aligned}&|\partial_\xi^\alpha \partial_{\xi^k}^{\alpha^k} \partial_x^\beta \partial_{x^k}^{\beta^k} q_{\nu+1,k}(x, \xi^k, x^k, \xi)| \\ &\qquad \leqq C_0 A^{\nu-1} M_1^{-(|\alpha|+|\alpha^k|+|\beta|+|\beta^k|)}(\alpha!\,\alpha^k!\,\beta!\,\beta^k!)^\kappa \\ &\qquad\qquad \times\langle\xi\rangle^{m-|\alpha|-|\alpha^k|}\end{aligned}$$

for constants A and M_1 independent of ν and k.

REMARK. From the definition (2.9) it follows immediately that if the support of $p_{\nu+1}$ with respect to (x^k, ξ^k) is contained in a subset Ω of $R^{2n}_{(x^k,\xi^k)}$ then the support of $q_{\nu+1,k}$ with respect to (x^k, ξ^k) is also contained in the same subset Ω.

In the following we prove Lemma 2.1 and Lemma 2.2. For the proof we employ the following, which is the case $\nu=1$ in Lemma 2.1 if we set $\zeta=\xi$.

Lemma 2.3. *Let $\phi_j \in \mathcal{P}_{G(\kappa)}(\tau_j)$, $j=1, 2$ $(\tau_1+\tau_2 \leqq 1/4)$ and set $\psi(x, \xi', x', \zeta) = \phi_1(x, \xi')-x'\cdot\xi'+\phi_2(x', \zeta)-\Phi_2(x, \zeta)$, where $\Phi_2(x, \xi)=\phi_1 \# \phi_2(x, \xi)$. Let $p(x, \xi', x', \zeta; \xi) \in C^\infty$ satisfy*

$$(2.11)\qquad \begin{aligned}&|\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_\zeta^{\alpha''} \partial_x^\beta \partial_{x'}^{\beta'} p| \leqq C_0 M^{-(|\alpha|+|\alpha'|+|\alpha''|+|\beta|+|\beta'|)} \\ &\qquad \times(\alpha!\,\alpha'!\,\alpha''!\,\beta!\,\beta'!)^\kappa \langle\xi\rangle^{m-|\kappa|-|\alpha'|-|\alpha''|}\end{aligned}$$

and assume

$$(2.12)\qquad |\xi'-\xi| \leqq \langle\xi\rangle/8\,, \quad |\zeta-\xi| \leqq \langle\xi\rangle/4 \qquad \text{on} \quad \operatorname{supp} p\,.$$

Set

$$(2.13)\qquad q(x, \zeta; \xi) = O_s\text{-}\iint e^{i\psi(x,\xi',x',\zeta)}\, p(x, \xi', x', \zeta; \xi)\, dx'\, đ\xi'\,.$$

Then, there exist constants A and M_1 depending only on M such that

$$(2.14)\qquad \begin{aligned}&|\partial_\xi^\alpha \partial_\zeta^{\alpha'} \partial_x^\beta q(x, \zeta; \xi)| \leqq C_0 A M_1^{-(|\alpha|+|\alpha'|+|\beta|)}(\alpha!\,\alpha'!\,\beta!)^\kappa \\ &\qquad \times\langle\xi\rangle^{m-|\alpha|-|\alpha'|}\end{aligned}$$

for any α, α' and β.

Proof. Consider (2.13) instead of (4.11) in [18]. Then, we can prove (2.14) in the almost same way as in the proof of Proposition 2.5 in [18]. We need not consider the part corresponding to the estimation of $p_\infty(x, \xi')$ there because of (2.12). We can begin with the step (II) of the proof of Proposition 2.5 in [18] by replacing $\chi_0(\xi, \xi')$ by $p(x, \xi', x', \zeta; \xi)$. Hence, we get (2.14) in view of (4.25) and (4.33) in [18]. Q.E.D.

Proof of Lemma 2.1. We divide the proof into two steps.

(I) In this step we shall show

$$q_{\nu+1}(x, \xi)\, e^{i\Phi_{\nu+1}(x,\xi)}$$
$$= O_s\text{-}\int\cdots\int \exp\big(i\big(\sum_{j=1}^{\nu} (x^{j-1}-x^j)\cdot\xi^j+\Phi_{\nu+1}(x^\nu, \xi)\big)\big) \times p'_{\nu+1}(x, \tilde\xi^\nu, \tilde x^\nu, \xi)\, d\tilde x^\nu d\tilde\xi^\nu \quad (x^0 = x)\,, \tag{2.15}$$

where the symbol $p'_{\nu+1}(x, \tilde\xi^\nu, \tilde x^\nu, \xi)$ satisfies

$$c^{-1}\langle\xi\rangle \leqq \langle\xi^j\rangle \leqq c\langle\xi\rangle \qquad on \quad \operatorname{supp} p'_{\nu+1} \tag{2.16}$$

and (2.6) with C_0 and M replaced by C_0A^ν and M_1. Here, the constants A and M_1 are independent of ν. For simplicity we consider the case $\nu=2$ for a while. Since it follows that

$$e^{i\phi_1(x,\xi')} = O_s\text{-}\iint e^{iz\cdot(\xi'-\zeta)+i\phi_1(x,\zeta)}\, \chi(4(\zeta-\xi')/\langle\xi\rangle)\, dz\, d\zeta$$

for $\chi(\xi)\in\gamma^{(\kappa)}(R^n_\xi)$ satisfying (1.10), we get

$$\begin{aligned} &I_1(x, \xi^2, x^2, \xi)\\ &\equiv O_s\text{-}\iint e^{i(\phi_1(x,\xi^1)-x^1\cdot\xi^1+\phi_2(x^1,\xi^2))}\, p_3(x, \tilde\xi^2, \tilde x^2, \xi)\, dx^1 d\xi^1\\ &= O_s\text{-}\iiiint e^{i(\phi_1(x,\zeta^1)-z^1\cdot(\zeta^1-\xi^1)-x^1\cdot\xi^1+\phi_2(x^1,\xi^2))}\\ &\quad\times\chi(4(\zeta^1-\xi^1)/\langle\xi\rangle)\, p_3(x, \tilde\xi^2, \tilde x^2, \xi)\, dx^1 d\xi^1 dz^1 d\zeta^1\,. \end{aligned}$$

Next, we take changes of variables as follows: First we change the variable z^1 to y^1 as

$$z^1 = \tilde\nabla_\xi\phi_1(y^1; \zeta^1, \xi^1) \equiv \int_0^1 \nabla_\xi\phi_1(y^1, \xi^1+\theta(\zeta^1-\xi^1))\, d\theta \tag{2.17}$$

and then the variable ζ^1 to η^1 as

$$\eta^1 = \tilde\nabla_x\phi_1(x, y^1; \zeta^1) \equiv \int_0^1 \nabla_x\phi_1(y^1+\theta(x-y^1), \zeta^1)\, d\theta\,. \tag{2.18}$$

Then, we have

$$\begin{aligned} \phi_1(x, \zeta^1)-z^1\cdot(\zeta^1-\xi^1) &= \phi_1(x, \zeta^1)-\tilde\nabla_\xi\phi_1(y^1; \zeta^1, \xi^1)\cdot(\zeta^1-\xi^1)\\ &= \phi_1(x, \zeta^1)-\phi_1(y^1, \zeta^1)+\phi_1(y^1, \xi^1)\\ &= (x-y^1)\cdot\tilde\nabla_x\phi_1(x, y^1; \zeta^1)+\phi_1(y^1, \xi^1)\\ &= (x-y^1)\cdot\eta^1+\phi_1(y^1, \xi^1) \end{aligned}$$

and

$$dz^1 d\zeta^1 = r_1(x, \xi^1, y^1, \tilde\nabla_x\phi_1^{-1}(x, y^1; \eta^1))\, dy^1 d\eta^1\,,$$

where $\zeta=\tilde\nabla_x\phi_1^{-1}(x, y; \eta)$ is the inverse function of $\eta=\tilde\nabla_x\phi_1(x, y; \zeta)$ and

$$r_1(x, \xi, y, \zeta) = \det \frac{\partial}{\partial x} \tilde{\nabla}_\xi \phi_1(y; \zeta, \xi) \cdot (\det \frac{\partial}{\partial \xi} \tilde{\nabla}_x \phi_1(x, y; \zeta))^{-1}.$$

In the above we denote $\frac{\partial}{\partial x} f(x)$, for a vector $f={}^t(f_1, \cdots, f_n)$ of functions $f_j(x)$, as $\frac{\partial}{\partial x} f(x) = \left(\partial f_j/\partial x_k \begin{smallmatrix} j \downarrow 1, \cdots, n \\ k \to 1, \cdots, n \end{smallmatrix}\right)$.

Now, we set

$$\tilde{r}_1(x, \xi', y, \zeta; \xi) = \chi(4(\zeta-\xi')/\langle\xi\rangle)\, r_1(x, \xi', y, \zeta) \tag{2.19}$$

and set

$$\begin{aligned} &q_3'(x, \zeta^1, y^1, \xi^2, x^2, \xi; \zeta') \\ &\quad = O_s\text{-}\iint e^{i(\phi_1(y^1, \xi^1) - x^1 \cdot \xi^1 + \phi_2(x^1, \zeta') - \Phi_2(y^1, \zeta'))} \\ &\qquad \times \tilde{r}_1(x, \xi^1, y^1, \zeta^1; \xi)\, p_3(x, \tilde{\xi}^2, \tilde{x}^2, \xi)\, dx^1 d\xi^1 \end{aligned} \tag{2.20}$$

with $\Phi_2 = \phi_1 \# \phi_2$. Then, by the change of variables (2.17) and (2.18), we have

$$\begin{aligned} I_1(x, \xi^2, x^2, \xi) &= O_s\text{-}\iint e^{i((x-y^1)\cdot\eta^1 + \Phi_2(y^1, \xi^2))} \\ &\times q_3'(x, \tilde{\nabla}_x \phi_1^{-1}(x, y^1; \eta^1), y^1, \xi^2, x^2, \xi; \xi^2)\, dy^1 d\eta^1. \end{aligned}$$

By the same way we can obtain

$$\begin{aligned} I_2(x, \zeta^1, y^1, \xi) &\equiv O_s\text{-}\iint e^{i(\Phi_2(y^1, \xi^2) - x^2\cdot\xi^2 + \phi_3(x^2, \xi))} \\ &\times q_3'(x, \zeta^1, y^1, \xi^2, x^2, \xi; \xi^2)\, dx^2 d\xi^2 \\ &= O_s\text{-}\iiiint e^{i(\Phi_2(y^1, \zeta^2) - z^2)\cdot(\zeta^2 - \xi^2) - x^2\cdot\xi^2 + \phi_3(x^2, \xi))} \\ &\times \chi(4(\zeta^2 - \xi^2)/\langle\xi\rangle))\, q_3'(x, \zeta^1, y^1, \xi^2, x^2, \xi; \zeta^2)\, dx^2 d\xi^2 dz^2 d\zeta^2 \\ &= O_s\text{-}\iint e^{i((y^1 - y^2)\cdot\eta^2 + \Phi_3(y^2, \xi))} \\ &\times \tilde{q}_3(x, \zeta^1, y^1, \tilde{\nabla}_x \Phi_2^{-1}(y^1, y^2; \eta^2), y^2, \xi; \xi)\, dy^2 d\eta^2, \end{aligned}$$

where

$$\begin{aligned} &\tilde{q}_3(x, \zeta^1, y^1, \zeta^2, y^2, \xi; \zeta') \\ &\quad = O_s\text{-}\iint e^{i(\Phi_2(y^2, \xi^2) - x^2\cdot\xi^2 + \phi_3(x^2, \zeta') - \Phi_3(y^2, \zeta'))} \\ &\qquad \times \tilde{r}_2(y^1, \xi^2, y^2, \zeta^2; \xi)\, q_3'(x, \zeta^1, y^1, \xi^2, x^2, \xi; \zeta^2)\, dx^2 d\xi^2. \end{aligned} \tag{2.21}$$

Here, $\Phi_3 = \Phi_2 \# \phi_3 (= \phi_1 \# \phi_2 \# \phi_3)$ and $\tilde{r}_2(x, \xi', y, \zeta; \xi)$ is defined by the same way as $\tilde{r}_1$ with ϕ_1 replaced by Φ_2. Hence, we obtain

$$\begin{aligned} &q_3(x, \xi)\, e^{i\Phi_3(x, \xi)} \\ &\quad = O_s\text{-}\iint I_1(x, \xi^2, x^2, \xi)\, e^{i(-x^2\cdot\xi^2 + \phi_3(x^2, \xi))}\, dx^2 d\xi^2 \end{aligned}$$

$$= O_s-\iint e^{i(x-y^1)\cdot\eta^1} I_2(x, \tilde{\nabla}_x \phi_1^{-1}(x, y^1; \eta^1), y^1, \xi)\, dy^1 d\eta^1$$
$$= O_s-\iiiint \exp[i(\sum_{j=1}^{2}(y^{j-1}-y^j)\cdot\eta^j+\Phi_3(y^2, \xi))]$$
$$\times p_3'(x, \tilde{\eta}^2, \tilde{y}^2, \xi)\, d\tilde{y}^2 d\tilde{\eta}^2,$$

where $p_3'(x, \tilde{\eta}^2, \tilde{y}^2, \xi)=\tilde{q}_3(x, \tilde{\nabla}_x\phi_1^{-1}(x, y^1, \eta^1), y^1, \tilde{\nabla}_x\Phi_2^{-1}(y^1, y^2; \eta^2), y^2, \xi; \xi)$. It follows from (2.20) and (2.21) that

$$p_3'(x, \tilde{\eta}^2, \tilde{y}^2, \xi)$$
$$= p_3''(x, \tilde{\nabla}_x\phi_1^{-1}(x, y^1, \eta^1), y^1, \tilde{\nabla}_x\Phi_2^{-1}(y^1, y^2; \eta^2), y^2, \xi)$$

if we set

$$p_3''(x, \tilde{\zeta}^2, \tilde{y}^2, \xi)$$
$$= O_s-\iint e^{i\psi_2(y^2,\xi^2,x^2,\xi)}\, \tilde{r}_2(y^1, \xi^2, y^2, \zeta^2; \xi)\, dx^2 d\xi^2$$
$$\cdot O_s-\iint e^{i\psi_1(y^1,\xi^1,x^1,\zeta^2)}\, \tilde{r}_1(x, \xi^1, y^1, \zeta^1; \xi)\, dx^1 d\xi^1$$
$$\cdot p_3(x, \tilde{\xi}^2, \tilde{x}^2, \xi).$$

Here $\psi_j(y, \xi, x, \zeta)=\Phi_j(y, \xi)-x\cdot\xi+\phi_{j+1}(x, \zeta)-\Phi_{j+1}(y, \zeta)$ $(j=1, \cdots, \nu, \Phi_1=\phi_1, \Phi_j=\phi_1\#\cdots\#\phi_j)$.

Now, we consider the case for a general ν. Then, repeating the above method we can prove

$$q_{\nu+1}(x, \xi)\, e^{i\Phi_{\nu+1}(x,\xi)}$$
$$= O_s-\int\cdots\int \exp(i(\sum_{j=1}^{\nu}(y^{j-1}-y^j)\cdot\eta^j+\Phi_{\nu+1}(y^\nu, \xi))) \tag{2.22}$$
$$\times p_{\nu+1}'(x, \tilde{\eta}^\nu, \tilde{y}^\nu, \xi)\, d\tilde{y}^\nu d\tilde{\eta}^\nu$$

for

$$p_{\nu+1}'(x, \tilde{\eta}^\nu, \tilde{y}^\nu, \xi) = p_{\nu+1}''(x, \tilde{\zeta}^\nu, \tilde{y}^\nu, \xi) \tag{2.23}$$
$$\textit{with}\quad \zeta^j = \tilde{\nabla}_x\Phi_j^{-1}(y^{j-1}, y^j; \eta^j) \quad (j=1, \cdots, \nu, \quad \Phi_1=\phi_1).$$

Here, $p_{\nu+1}''(x, \tilde{\zeta}^\nu, \tilde{y}^\nu, \xi)$ is defined by

$$p_{\nu+1}''(x, \tilde{\zeta}^\nu, \tilde{y}^\nu, \xi)$$
$$= O_s-\iint e^{i\psi_\nu(y^\nu,\xi^\nu,x^\nu,\xi)}\, \tilde{r}_\nu(y^{\nu-1}, \xi^\nu, y^\nu, \zeta^\nu; \xi)\, dx^\nu d\xi^\nu$$
$$\cdot O_s-\iint e^{i\psi_{\nu-1}(y^{\nu-1},\xi^{\nu-1},x^{\nu-1},\zeta^\nu)} \tag{2.24}$$
$$\times \tilde{r}_{\nu-1}(y^{\nu-2}, \xi^{\nu-1}, y^{\nu-1}, \zeta^{\nu-1}; \xi)\, dx^{\nu-1} d\xi^{\nu-1}$$
$$\cdots\cdots\cdots$$

$$\cdot O_s\text{-}\iint e^{i\psi_1(y^1,\xi^1,x^1,\zeta^2)}\,\tilde{r}_1(x,\xi^1,y^1,\zeta^1;\xi)\,dx^1 d\xi^1$$
$$\cdot p_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)$$
$$=\prod_{j=1}^{\nu} O_s\text{-}\iint e^{i\psi_j(y^j,\xi^j,x^j,\zeta^{j+1})}$$
$$\times\tilde{r}_j(y^{j-1},\xi^j,y^j,\zeta^j;\xi)\,dx^j d\xi^j\,p_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)$$
$$(y^0=x,\quad \zeta^{\nu+1}=\xi)\,.$$

Since (2.22) is nothing but (2.15), it remains to prove (2.6) for $p'_{\nu+1}$ and (2.16). From (2.19), (2.5) for $p_{\nu+1}$ and (2.24) we can prove

$$|\zeta^j-\xi|\leqq\langle\xi\rangle/4\quad j=1,\cdots,\nu\,, \tag{2.25}$$
on the support of the integrand of $p''_{\nu+1}$.

This property and (2.3)-a) in [17] implies (2.16). For the proof of (2.6) for $p'_{\nu+1}$ we set for a while

$$\tilde{p}''_{\nu+1}(x,\tilde{\zeta}^\nu,\tilde{\zeta}'^\nu,\tilde{y}^\nu,\tilde{y}'^\nu,\xi)=\prod_{j=1}^{\nu} O_s\text{-}\iint e^{i\psi_j(y^j,\xi^j,x^j,\zeta^{j+1})}$$
$$\times\tilde{r}_j(y'^{j-1},\xi^j,y'^j,\zeta'^j;\xi)\,dx^j d\xi^j\,p_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)\quad(y'^0=x)$$

and we estimate this under

$$|\zeta^j-\xi|\leqq\langle\xi\rangle/4\,,\quad |\zeta'^j-\xi|\leqq\langle\xi\rangle/4 \tag{2.26}$$

noting (2.25). Then, applying Lemma 2.3 to each oscillatory integral

$$O_s\text{-}\iint e^{i\psi_j(y^j,\xi^j,x^j,\zeta^{j+1})}\,\tilde{r}_j(y'^{j-1},\xi^j,y'^j,\zeta'^j)$$
$$\times p_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)\,dx^j d\xi^j$$

regarding y'^{j-1}, y'^j, ζ'^j, $\tilde{x}^{\nu,j}$ and $\tilde{\xi}^{\nu,j}$ as parameters, we get for ζ^j and ζ'^j satisfying (2.26)

$$|\partial_\xi^\alpha\partial_{\tilde{\zeta}^\nu}^{\tilde{\alpha}^\nu}\partial_{\tilde{\zeta}'^\nu}^{\tilde{\alpha}'^\nu}\partial_x^\beta\partial_{\tilde{y}^\nu}^{\tilde{\beta}^\nu}\partial_{\tilde{y}'^\nu}^{\tilde{\beta}'^\nu}\tilde{p}''_{\nu+1}(x,\tilde{\zeta}^\nu,\tilde{\zeta}'^\nu,\tilde{y}^\nu,\tilde{y}'^\nu,\xi)|$$
$$\leqq C_0A_1^\nu M_1^{-(|\alpha|+|\tilde{\alpha}^\nu|+|\tilde{\alpha}'^\nu|+|\beta|+|\tilde{\beta}^\nu|+|\tilde{\beta}'^\nu|)}$$
$$\times(\alpha!\,\tilde{\alpha}^\nu!\,\tilde{\alpha}'^\nu!\,\beta!\,\tilde{\beta}^\nu!\,\tilde{\beta}'^\nu!)^\kappa$$
$$\times\langle\xi\rangle^{m-|\alpha|-|\tilde{\alpha}^\nu|-|\tilde{\alpha}'^\nu|}$$

with constants A_1 and M_1 independent of ν. This implies (2.6) for $p'_{\nu+1}$ since we have (2.23) and $p''_{\nu+1}(x,\tilde{\zeta}^\nu,\tilde{y}^\nu,\xi)=\tilde{p}''_{\nu+1}(x,\tilde{\zeta}^\nu,\tilde{\zeta}^\nu,\tilde{y}^\nu,\tilde{y}^\nu,\xi)$.

(II) Take the change of variables

$$\begin{cases} x^j=y^j+x\,,\\ \xi^j=\eta^j+\tilde{\nabla}_x\Phi_{\nu+1}(x,x^\nu;\xi)\end{cases} \tag{2.27}$$

in the integral of the right hand side of (2.15). Then, since we have

$$\begin{aligned}\sum_{j=1}^{\nu}(x^{j-1}-x^j)\cdot\xi^j+\Phi_{\nu+1}(x^\nu,\xi)-\Phi_{\nu+1}(x,\xi)\\ =\sum_{j=1}^{\nu}(x^{j-1}-x^j)\cdot\xi^j-(x-x^\nu)\cdot\tilde{\nabla}_x\Phi_{\nu+1}(x,x^\nu,\xi)\\ =\sum_{j=1}^{\nu}(x^{j-1}-x^j)\cdot(\xi^j-\tilde{\nabla}_x\Phi_{\nu+1}(x,x^\nu;\xi))\\ =-\sum_{j=1}^{\nu}(y^j-y^{j-1})\cdot\eta^j \quad (x^0=x,\ y^0=0),\end{aligned}$$

$q_{\nu+1}(x,\xi)$ in (2.15) is written as

$$q_{\nu+1}(x,\xi)=O_s-\int\cdots\int\exp\Big(-i\sum_{j=1}^{\nu}(y^j-y^{j-1})\cdot\eta^j\Big)\times\tilde{p}_{\nu+1}(x,\tilde{\eta}^\nu,\tilde{y}^\nu,\xi)\,d\tilde{y}^\nu d\tilde{\eta}^\nu, \tag{2.28}$$

where $\tilde{p}_{\nu+1}(x,\tilde{\eta}^\nu,\tilde{y}^\nu,\xi)$ is defined from $p'_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)$ by the change of variables (2.27). Since $p'_{\nu+1}$ satisfies (2.6) (with C_0 replaced by C_0A^ν) and (2.16) we may use only the step (II) in the proof of Proposition 5.1 in [18] if we consider (2.28) instead of (5.2) in [18]. In fact, if τ^0 in (A–1) is small enough, we have $c'^{-1}\langle\xi\rangle\leqq\langle\xi+\eta^j\rangle\leqq c'\langle\xi\rangle$ on supp $\tilde{p}_{\nu+1}$ (with c' independent of ν) on account of (2.16) above and (2.3)-a) in [17]. Thus the proof is completed.

Q.E.D.

The proof of Lemma 2.2 is carried out by the same way as in that of Lemma 2.1 if we note $c^{-1}\langle\xi\rangle\leqq\langle\xi^k\rangle\leqq c\langle\xi\rangle$ on supp $p_{\nu+1}$.

In the rest of this section we shall give another fundamental lemma by means of Lemma 2.2. Let $\{X_\nu^j,\Xi_\nu^j\}_{\nu=1}^{j}(x,\xi)$ denote the solution of

$$\begin{cases} x^j=\nabla_\xi\phi_j(x^{j-1},\xi^j),\\ \xi^j=\nabla_x\phi_{j+1}(x^j,\xi^{j+1}),\quad j=1,\cdots,\nu\quad(x^0=x,\ \xi^{\nu+1}=\xi).\end{cases}$$

We remark that $\{X_\nu^j,\Xi_\nu^j/\langle\xi\rangle\}_{j,\nu}$ are bounded in $S^0_{G(\kappa)}$ (see Proposition 2.4 in [18] and its proof). For a $\delta>0$ and $k=1,\cdots,\nu$ we set

$$\begin{aligned}\chi_{\delta,k}&\equiv\chi_{\delta,k}(x^k,\xi^k;x,\xi)\\ &=\chi((X_\nu^k(x,\xi)-x^k)/\delta)\,\chi((\Xi_\nu^k(x,\xi)-\xi^k)/(\delta\langle\xi\rangle))\end{aligned}$$

for $\chi\in\gamma^{(\kappa)}(R^n_\xi)$ satisfying (1.10). Set

$$\begin{aligned}&p^0_{\nu+1,\delta}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)\\ &\quad=\prod_{j=1}^{\nu}\chi_{\delta,j}(x^j,\xi^j;x,\xi)\,p_{\nu+1}(x,\tilde{\xi}^\nu,\tilde{x}^\nu,\xi)\end{aligned} \tag{2.29}$$

and set for $k=1,\cdots,\nu$

$$(2.30)\qquad p^k_{\nu+1,\delta}(x, \bar{\xi}^\nu, \tilde{x}^\nu, \xi) = \chi_{\delta,1}\cdots\chi_{\delta,k-1}(1-\chi_{\delta,k})\,p_{\nu+1}\,.$$

Then, we get $p_{\nu+1}=p^o_{\nu+1,\delta}+\sum_{k=1}^{\nu} p^k_{\nu+1,\delta}$.

Lemma 2.4. *Let $p_{\nu+1}(x, \bar{\xi}^\nu, \tilde{x}^\nu, \xi)$ satisfy* (2.5) *and* (2.6). *Let $r_{\nu+1,\delta}(x, \xi)$ denote the symbol defined by* (2.2) *with $p_{\nu+1}$ replaced by $p_{\nu+1}-p^o_{\nu+1,\delta}$. Then, for any δ there exist positive constants ε, A_1 and M_1 independent of ν such that for any α and β we have*

$$(2.31)\qquad |\partial^\alpha_\xi\partial^\beta_x r_{\nu+1,\delta}(x, \xi)| \leqq C_0 A_1{}^\nu M_1^{-(|\alpha|+|\beta|)}\alpha!\beta!)^\kappa\langle\xi\rangle^{m-|\alpha|}\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\,.$$

REMARK. As in the remark of Lemma 1.3 the estimate (2.31) holds even if we replace $\Phi_{\nu+1}(x, \xi)$ by $x\cdot\xi$ in (2.2).

Proof. Let $q_{\nu+1,\delta,k}(x, \xi^k, x^k, \xi)$ denote a symbol defined by (2.9) with $p_{\nu+1}$ replaced by $p^k_{\nu+1,\delta}$ and set

$$(2.32)\qquad \begin{aligned} q^k_{\nu+1,\delta}(x, \xi) = O_s-\iint & e^{i(\Phi_{\nu+1,k}(x,\xi^k,x^k,\xi)-\Phi_{\nu+1}(x,\xi))} \\ & \times q_{\nu+1,\delta,k}(x, \xi^k, x^k, \xi)dx^k\,d\xi^k\,. \end{aligned}$$

Then we have $r_{\nu+1,\delta}(x, \xi)=\sum_{j=1}^{\nu+1} q^k_{\nu+1,\delta}(x, \xi)$. So, it suffices to estimate each $q^k_{\nu+1}(x, \xi)$. Since $(X^k_\nu, \Xi^k_\nu)\,(x, \xi)$ is the solution of the equation

$$\begin{cases} x^k = \nabla_\xi(\phi_1\#\cdots\#\phi_k)(x, \xi^k)\,, \\ \xi^k = \nabla_x(\phi_{k+1}\#\cdots\#\phi_{\nu+1})(x^k, \xi)\,, \end{cases}$$

it follows from (2.8), (2.30) and Remark of Lemma 2.2 that

$$(2.33)\qquad \begin{aligned} \langle\xi\rangle|\nabla_{\xi^k}\Phi_{\nu+1\,k}(x, \xi^k, x^k, \xi)| + |\nabla_{x^k}\Phi_{\nu+1,k}(x, \xi^k, x^k, \xi)| \\ \geqq c_1\langle\xi\rangle \qquad on \quad \mathrm{supp}\, q_{\nu+1,\delta,k} \end{aligned}$$

for a constant c_1 determined by δ and τ^o in (A–1). Set $L=-i(\langle\xi\rangle^2|\nabla_{\xi^k}\Phi_{\nu+1,k}|^2 + |\nabla_{x^k}\Phi_{\nu+1,k}|^2)^{-1}(\langle\xi\rangle^2\nabla_{\xi^k}\Phi_{\nu+1,k}\cdot\nabla_{\xi^k} + \nabla_{x^k}\Phi_{\nu+1\,k}\cdot\nabla_{x^k})$. Then, we get $L[\exp(i\times(\Phi_{\nu+1,k}(x, \xi^k, x^k, \xi)-\Phi_{\nu+1}(x, \xi)))] = \exp(i(\Phi_{\nu+1,k}(x, \xi^k, x^k, \xi)-\Phi_{\nu+1}(x, \xi)))$. Now, we integrate (2.32) by parts. Then, we have

$$q^k_{\nu+1,\delta}(\xi, x) = O_s-\iint e^{i(\Phi_{\nu+1,k}-\Phi_{\nu+1})}(L^t)^N q_{\nu+1,\delta,k}dx^k\,d\xi^k\,,$$

where L^t is the transposed operator of L. It follows from (2.10) and (2.33) that there exist constants A_2 and M_2 independent of ν such that we have for any N

$$\begin{aligned} |\partial^\alpha_\xi\partial^{\alpha'}_{\xi'}\partial^\beta_x\partial^{\beta'}_{x'}((L^t)^N q_{\nu+1,\delta,k})| \leqq (C_0 A_2^{\nu-1}) M_2^{-(|\alpha|+|\alpha'|+|\beta|+|\beta'|)-N} \\ \times(\alpha!\alpha'!\beta!\beta'!N!)^\kappa\langle\xi\rangle^{m-|\alpha|-|\alpha'|-N} \qquad (x'=x^k,\ \xi'=\xi^k)\,. \end{aligned}$$

Now, we apply Lemma 2.3 for the case $\zeta=\xi$ by setting $p_{\nu+1}=(L^t)^N q_{\nu+1,\delta,k}$ and C_0 and m in (2.6) as $C_0 A_2^{\nu-1} N!^\kappa$ and $m-N$, respectively. Then, for any N the estimate

$$|\partial_\xi^\alpha \partial_x^\beta q_{\nu+1,\delta}^k| \leqq C_0 A_3^\nu M_3^{-(|\alpha|+|\beta|+N)}(\alpha!\beta!N!)^\kappa \langle\xi\rangle^{m-|\alpha|-N} \tag{2.34}$$

holds with constants A_3 and M_3 independent of N and ν. Consequently, by means of the remark after Definition 1.1 we obtain (2.31) from (2.34). Q.E.D.

We end this section by the following remark: In the following sections we may assume that there exists a constant M_0 independent of ν such that for a multiple symbol $p_{\nu+1}(x, \xi^\nu, \tilde{x}^\nu, \xi)$ satisfying (2.5) and (2.6) we have $|x-x^\nu|\leqq M_0$ on supp $p_{\nu+1}$. Indeed, it follows from (1.5) in [11] that $|x-X_\nu^\nu(x, \xi)|\leqq c_0$ for a constant c_0 independent of ν and the symbol $q_{\nu+1}(x, \xi)$ defined by (2.2) satisfies (2.31) if $p_{\nu+1}(x, \xi^\nu, \tilde{x}^\nu, \xi)$ vanishes when $|x-x^\nu|\leqq M_0$. This result also follows from Lemma 2.4.

3. Integration by parts with respect to time variables

Let $\phi(t, s; x, \xi)$ be a solution of an eiconal equation

$$\begin{cases} \partial_t\phi = \lambda(t, x, \nabla_x\phi)\,, \\ \phi|_{t=s} = x\cdot\xi\,. \end{cases} \tag{3.1}$$

Proposition 3.1. *Assume that $\lambda(t, x, \xi)$ is a real symbol in $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$. Then there exists a $T_0>0$ such that the solution $\phi(t, s; x, \xi)$ of (3.1) exists uniquely in $\{(t, s);\ 0\leqq t, s\leqq T_0\}\equiv[0, T_0]^2$ and belongs to $\mathcal{P}_{G(\kappa)}(\tilde{c}_0|t-s|)$ for a constant $\tilde{c}_0$ independent of t and s. Furthermore, there exist constants C, M_1 and M_2 such that $\phi(t, s; x, \xi)$ satisfies for any α, β, γ and γ'*

$$|\partial_t^\gamma \partial_s^{\gamma'} \partial_x^\beta \partial_\xi^\alpha \phi| \leqq C M_1^{-(|\alpha|+|\beta|)} M_2^{-(\gamma+\gamma')}(\alpha!\beta!\gamma!\gamma'!)^\kappa \langle\xi\rangle^{1-|\alpha|} \tag{3.2}$$

for $t, s\in[0, T_0]$.

Proof. Assertions except the last one are the same as those of Proposition 3.1 in [18]. Since it follows that $\partial_t\phi=\lambda(t, x, \nabla_x\phi)$ and $\partial_s\phi=-\lambda(s, \nabla_\xi\phi, \xi)$ we obtain (3.2) by the inductive method with respect to γ and γ'. Q.E.D.

Let $\{\lambda_j(t, x, \xi)\}_{j=1}^\infty$ be a bounded set of real symbols in $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$ and let $\psi_j(t, s; x, \xi)$ denote the solution of the eiconal equation (3.1) with $\lambda=\lambda_j$. Let ν and μ be non-negative integers such that $\nu\geqq 3$ and $0\leqq\mu\leqq\nu-2$. For a fixed positive $t_0\leqq T_0$ we set

$$\Delta_\mu = \{\tilde{t}^\mu = (t_1, \cdots, t_\mu)\in R^\mu;\ 0\leqq t_\mu\leqq t_{\mu-1}\leqq\cdots\leqq t_1\leqq t_0\} \tag{3.3}$$

when $\mu\geqq 1$. Let Σ_μ be a subset of $\{2, \cdots, \nu\}$ and denote it as

$$\Sigma_\mu = \{j_1, \cdots, j_{\mu+1}\} \tag{3.4}$$

with $2\leqq j_1<j_2<\cdots<j_{\mu+1}\leqq\nu$. For convenience we use, in the following, the notation $(t_0, \tilde{t}^\mu)=t_0$ when $\mu=0$. For a $\tilde{t}^\mu\in\Delta_\mu$ (when $\mu\geqq 1$) and Σ_μ we define a set of phase functions $\{\phi_j(t_0, \tilde{t}^\mu; x, \xi)\}_{j=1}^{\nu+1}\equiv\{\phi_j(t_0, \tilde{t}^\mu)\}_{j+1}^{\nu-1}$ by

$$\begin{cases} \phi_j(t_0, \tilde{t}^\mu; x, \xi) = x\cdot\xi & \text{for } j\notin\Sigma_\mu \\ \phi_{j_k}(t_0, \tilde{t}^\mu; x, \xi) = \psi_{j_k}(t_{k-1}, t_k; x, \xi) & \text{for } j_k\in\Sigma_\mu \end{cases}$$
$$(k=1, \cdots, \mu+1, t_{\mu+1} = 0)\,.$$

Then it follows from Proposition 3.1 that the set $\{\phi_j(t_0, \tilde{t}^\mu)\}_{j=1}^{\nu+1}$ satisfies the assumption (A–1) in § 1 if T_0 is small enough. Define $\Psi^\mu_{\nu+1,\Sigma_\mu}(t_0, \tilde{t}^\mu; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ by the formula (1.9) for the set $\{\phi_j(t_0, \tilde{t}^\mu)\}_{j=1}^{\nu+1}$. In the following we shorten $\Psi^\mu_{\nu+1,\Sigma_\mu}(t_0, \tilde{t}^\mu; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ to $\Psi^\mu_{\nu+1}(t_0, \tilde{t}^\mu)$ or $\Psi^\mu_{\nu+1}$. Then, we have

$$\begin{aligned} \partial_{t_k}\Psi^\mu_{\nu+1} &= (\partial_s\psi_{j_k})(t_{k-1}, t_k; x^{j_k-1}, \xi^{j_k}) \\ &\quad +(\partial_t\psi_{j_{k+1}})(t_k, t_{k+1}; x^{j_{k+1}-1}, \xi^{j_{k+1}}) \\ &= -\lambda_{j_k}(t_k, \nabla_\xi\psi_{j_k}(t_{k-1}, t_k; x^{j_k-1}, \xi^{j_k}), \xi^{j_k}) \\ &\quad +\lambda_{j_{k+1}}(t_k, x^{j_{k+1}-1}, \nabla_x\psi_{j_{k+1}}(t_k, t_{k+1}; x^{j_{k+1}-1}, \xi^{j_{k+1}})) \\ &\qquad (k = 1, \cdots, \mu, t_{\mu+1} = 0)\,. \end{aligned} \tag{3.5}$$

We note that (3.5) depends only on $(3+4n)$ variables $(t_{k-1}, t_k, t_{k+1}, x^{j_k-1}, \xi^{j_k}, x^{j_{k+1}}, \xi^{j_{k+1}})$, which is the key point of our discussions. Let $\mu\geqq 1$. For k $k\in\{1, \cdots, \mu+1\}$ and Σ_μ in (3.4) we denote $\Sigma_{\mu,k}=\{j_1, \cdots, j_{k-1}, j_{k+1}, \cdots, j_{\mu+1}\}$. Noting $\psi_{j_k|t=s}=x\cdot\xi$, we have

$$\Psi^{\mu-1}_{\nu+1,\Sigma_{\mu,k}}(t_0, \tilde{s}^{\mu-1}; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi) = \Psi^\mu_{\nu+1,\Sigma_\mu}(t_0, \tilde{t}^\mu; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)_{|t_k=t_{k-1}}$$

by setting $\tilde{s}^{\mu-1}=(t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_\mu)$, where $\Psi^\mu_{\nu+1,\Sigma_\mu|t_k=t_{k-1}}$ for $k=\mu+1$ means $\Psi^\mu_{\nu+1,\Sigma_\mu|t_\mu=0}$. We denote this by $\Psi^{\mu-1,k}_{\nu+1}(t_0, s^{\mu-1}; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ or simply by $\Psi^{\mu-1,k}_{\nu+1}(t_0, \tilde{s}^{\mu-1})$ or $\Psi^{\mu-1,k}_{\nu+1}$ in what follows. From (3.5) it is easy to see

$$\begin{aligned} \partial_{s_{k-1}}\Psi^{\mu-1,k}_{\nu+1}(t_0, \tilde{s}^{\mu-1}) &= (\partial_s\psi_{j_{k-1}})(s_{k-2}, s_{k-1}; x^{j_{k-1}-1}, \xi^{j_{k-1}}) \\ &\quad +(\partial_t\psi_{j_{k+1}})(s_{k-1}, s_k; x^{j_{k+1}-1}, \xi^{j_{k+1}})\,, \\ &\qquad (k = 2, \cdots, \mu, s_0 = t_0, s_\mu = 0)\,, \end{aligned} \tag{3.6}$$

and

$$\partial_{s_j}\Psi^{\mu-1,k}_{\nu+1}(t_0, \tilde{s}^{\mu-1}) = \begin{cases} \partial_{t_j}\Psi_{\nu+1}(t_0, \tilde{t}^\mu) & \text{for } j<k-1\,, \\ \partial_{t_{j+1}}\Psi_{\nu+1}(t_0, \tilde{t}^\mu) & \text{for } j\geqq k\,, \end{cases} \qquad (k = 1, \cdots, \mu+1)\,, \tag{3.7}$$

with

$$(t_0, \tilde{t}^\mu) = \begin{cases} (s_0, s_1, \cdots, s_{k-2}, s_{k-1}, s_{k-1}, s_k, \cdots, s_{\mu-1}), (s_0=t_0)\,, \\ \qquad\qquad\qquad \text{for } k\in\{1, \cdots, \mu\}\,, \\ (t_0, s_1, s_2, \cdots, s_{\mu-1}, 0) \quad \text{for } k=\mu+1\,. \end{cases}$$

Let $p^{\mu}_{\nu+1}(t_0, \tilde{t}^{\nu}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)$ be a multiple symbol with parameters t_0 and $\tilde{t}^{\mu}\in\Delta_{\mu}$ satisfying (2.5). We say that $p^{\mu}_{\nu+1}$ satisfies the condition $\mathcal{G}(C, h, M, m)$ for an integer $h\geqq 0$, a real m and constants C and M if we have for any α, β, $\tilde{\alpha}^{\nu}$, $\tilde{\beta}^{\nu}$ and k

$$\begin{aligned}(3.8)\qquad &\sum_{\tilde{\gamma}^{\mu}=k}\frac{k!}{\tilde{\gamma}^{\mu}!}|\partial^{\tilde{\gamma}^{\mu}}_{\tilde{t}^{\mu}}\partial^{\alpha}_{\xi}\partial^{\beta}_{x}\partial^{\tilde{\alpha}^{\nu}}_{\tilde{\xi}^{\nu}}\partial^{\tilde{\beta}^{\nu}}_{\tilde{x}^{\nu}}p^{\mu}_{\nu+1}|\\ &\leqq CM^{-(|\tilde{\alpha}^{\nu}|+|\tilde{\beta}^{\nu}|+|\alpha|+|\beta|+k)}(\alpha!\beta!\tilde{\alpha}^{\nu}!\tilde{\beta}^{\nu}!(k+h)!)^{\kappa}\langle\xi\rangle^{m-|\alpha|-|\tilde{\alpha}^{\nu}|}\,.\end{aligned}$$

We consider the symbol $q^{\mu}_{\nu+1}(t_0; x, \xi)$ defined by

$$\begin{aligned}(3.9)\qquad q^{\mu}_{\nu+1}(t_0; x, \xi) &= \int_{\Delta_{\mu}}d\tilde{t}^{\mu}O_s\text{-}\iint e^{i(\Psi^{\mu}_{\nu+1}(t_0,\tilde{t}^{\mu})-x\cdot\xi)}\\ &\qquad\times p^{\mu}_{\nu+1}(t_0, \tilde{t}^{\mu}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)d\tilde{x}^{\nu}d\tilde{\xi}^{\nu}\\ &\equiv O_s\text{-}\iint\Big(\int_{\Delta_{\mu}}e^{i(\Psi^{\mu}_{\nu+1}(t_0,\tilde{t}^{\mu})-x\cdot\xi)}\\ &\qquad\times p^{\mu}_{\nu+1}(t_0, \tilde{t}^{\mu}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)d\tilde{t}^{\mu}\Big)d\tilde{x}^{\nu}d\tilde{\xi}^{\nu}\end{aligned}$$

when $\mu\geqq 1$ and

$$(3.9)'\qquad q^{0}_{\nu+1}(t_0; x, \xi) = O_s\text{-}\iint e^{i(\Phi^{0}_{\nu+1}(t_0)-x\cdot\xi)}p^{0}_{\nu+1}(t_0; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)d\tilde{x}^{\nu}d\tilde{\xi}^{\nu}$$

for $\mu=0$. Let $\{X^{j}_{\nu}, \Xi^{j}_{\nu}\}(t_0, \tilde{t}^{\mu}; x, \xi)$ be a solution of (1.7) with $\{\phi_j(t_0, \tilde{t}^{\mu})\}^{\nu+1}_{j=1}$. First, we assume that there exist positive constants ε and δ such that

$$\begin{aligned}(3.10)\qquad &\operatorname{supp}p^{\mu}_{\nu+1}\cap\Big(\bigcap_{k=1}^{\mu}\{|\partial_{t_k}\Psi^{\mu}_{\nu+1}|\leqq\varepsilon\langle\xi\rangle/2\}\Big)\\ &\cap\Big(\bigcap_{j=1}^{\nu}\{X^{j}_{\nu}-x^{j}|\leqq\delta,\ |\Xi^{j}_{\nu}-\xi^{j}|\leqq\delta\langle\xi\rangle\}\Big) = \phi\,.\end{aligned}$$

In (3.10) the second factor is the whole space if $\mu=0$. Let $\mu\geqq 1$ and let $p^{\mu-1,k}_{\nu+1}(t_0, \tilde{s}^{\mu-1}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)$ denote $p^{\mu}_{\nu+1}(t_0, \tilde{t}^{\mu}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)_{|t_k=t_{k-1}}$ for some $k\in\{1, 2, \cdots, \mu+1\}$ with $\tilde{s}^{\mu-1}=(t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_{\mu})$, where $p^{\mu}_{\nu+1|t_k=t_{k-1}}$ for $k=\mu+1$ means $p^{\mu}_{\nu+1|t_{\mu}=0}$. Then $p^{\mu-1,k}_{\nu+1}$ satisfies the condition $\mathcal{G}(C, h, M, m)$ if $p^{\mu}_{\nu+1}$ so does.

By setting $t_k=t_{k-1}$ for a k with $1\leqq k\leqq\mu$ or $t_{\mu}=0$ for $k=\mu+1$ we can define a "child" $\{p^{\mu-1}_{\nu+1}, \Psi^{\mu-1}_{\nu+1}\}\equiv\{p^{\mu-1,k}_{\nu+1}, \Psi^{\mu-1,k}_{\nu+1}\}$ $(k=1, \cdots, \mu+1)$ of $\{p^{\mu}_{\nu+1}, \Psi^{\mu}_{\nu+1}\}$, and moreover can define descendent sequences of $\{p^{\mu}_{\nu+1}, \Psi^{\mu}_{\nu+1}\}$ successively until $\mu=0$. We finally assume that (3.10) holds for $\{p^{\mu}_{\nu+1}, \Psi^{\mu}_{\nu+1}\}$ and all children of all its descendent sequences. In what follows we denote this assumption by $B(\varepsilon, \delta)$.

Lemma 3.2. *Let $p^{\mu}_{\nu+1}(t_0, \tilde{t}^{\mu}; x, \tilde{\xi}^{\nu}, \tilde{x}^{\nu}, \xi)$ be a multiple symbol satisfying (2.5) and the condition $\mathcal{G}(C_O, 0, M, m)$. Assume that $\{p^{\mu}_{\nu+1}, \Psi^{\mu}_{\nu+1}\}$ satisfies $B(\varepsilon, \delta)$. Then there exist constants C, $\varepsilon'>0$, A_1 and M_1 independent of ν, μ*

such that for any α *and* β

$$|\partial_\xi^\alpha \partial_x^\beta q_{\nu+1}^\mu(t_0;\, x,\, \xi)| \leqq CC_0(A_1^{\nu+\mu}/\mu!)M_1^{-(|\alpha|+|\beta|)}(\alpha!\beta!)^\kappa e^{-\varepsilon'\langle\xi\rangle^{1/\kappa}}, \tag{3.11}$$

where $q_{\nu+1}^\mu$ *is the symbol defined by* (3.9) *or* (3.9)'. *Furthermore, if we resolve the constant* C *in* (3.11) *into* $\tilde{C}C_1^m[m]!^\kappa$ *when* $m>0$, *we can take constants* $\tilde{C}$, C_1, A_1 *and* M_1 *independent also of* m *and have*

$$|\partial_\xi^\alpha \partial_x^\beta q_{\nu+1}^\mu(t_0;\, x,\, \xi)| \leqq \tilde{C}C_0 C_1^m[m]!^\kappa(A_1^{\nu+\mu}/\mu!)M_1^{-(|\alpha|+|\beta|)}(\alpha!\beta!)^\kappa e^{-\varepsilon'\langle\xi\rangle^{1/\kappa}}. \tag{3.11$'$}$$

For the proof of this lemma it suffices to show

$$|q_{\nu+1}^\mu(t_0;\, x,\, \xi)| \leqq CC_0(A_1^{\nu+\mu}/\mu!)\langle\xi\rangle^m e^{-\varepsilon'\langle\xi\rangle^{1/\kappa}} \tag{3.12}$$

for constants C and A_1 independent of ν, μ and m. In fact, differentiating with respect to x and ξ, we get in view of $\phi_1 = \phi_{\nu+1} = x\cdot\xi$

$$\partial_\xi^\alpha \partial_x^\beta q_{\nu+1}^\mu(t_0;\, x,\, \xi) = \int_{\Delta_\mu} d\tilde{t}^\mu O_s\!-\!\iint e^{i(\Psi_{\nu+1}-x\cdot\xi)}$$
$$\times \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} \frac{\alpha!}{\alpha'!\alpha''!}\frac{\beta!}{\beta'!\beta''!}(i(\xi^1-\xi))^{\beta'}\partial_x^{\beta''}((i(x^\nu-x))^{\alpha'}\partial_\xi^{\alpha''}p_{\nu+1}^\mu)\,\tilde{x}^\nu\, d\tilde{\xi}^\nu,$$

when $\mu \geqq 1$. It follows from the last remark in Section 2 that we may assume $|x^\nu - x| \leqq M_0$ on $\operatorname{supp} p_{\nu+1}^\mu$. Noting that $|\xi^j-\xi| \leqq \langle\xi\rangle/2$ on $\operatorname{supp} p_{\nu+1}^\mu$ and for $N=|\beta'|+[\max(m,\, 0)]$ an estimate $\langle\xi\rangle^N \exp(-\varepsilon'\langle\xi\rangle^{1/\kappa}) \leqq M^{-N}N!^\kappa \times \exp(-\varepsilon'\langle\xi\rangle^{1/\kappa}/2)$ holds for a sufficiently small $M>0$, we get the assertion by means of the Leibniz formula. Similarly, we get the assertion for the case of $\mu=0$.

For the proof of Lemma 3.2 we prepare the following lemma, which is the direct consequence of Lemma 2.4.

Lemma 3.3. *Let* $p_{\nu+1}^\mu$ *satisfy the same condition as in Lemma* 3.2. *Assume that we have* (3.10) *and*

$$\operatorname{supp} p_{\nu+1}^\mu \subset \bigcap_{k=1}^{\mu} \{|\partial_{t_k}\Psi_{\nu+1}^\mu| \leqq \varepsilon\langle\xi\rangle/2\} \tag{3.13}$$

when $\mu \geqq 1$. *Then there exist constants* $\varepsilon'>0$, A_1 *and* M_1 *independent of* ν *such that*

$$|q_{\nu+1}^\mu(t_0;\, x,\, \xi)| \leqq C_0(A_1^{\nu+\mu}/\mu!)\langle\xi\rangle^m e^{-\varepsilon'\langle\xi\rangle^{1/\kappa}}. \tag{3.14}$$

Proof. Set

$$\tilde{q}_{\nu+1}(t_0,\, \tilde{t}^\mu;\, x,\, \xi) = O_s\!-\!\iint e^{i(\Psi_{\nu+1}^\mu - x\cdot\xi)} p_{\nu+1}^\mu\, d\tilde{x}^\nu\, d\tilde{\xi}^\nu.$$

It follows from Lemma 2.4 and its remark that $\tilde{q}_{\nu+1}$ satisfies the estimate (2.31) with $\alpha=\beta=0$ uniformly with respect to $\tilde{t}^{\mu}\in\Delta_{\mu}$, since the term defined by (2.29) for $p^{\mu}_{\nu+1}$ vanishes from (3.10) and (3.13). Noting that the volume of Δ_{μ} is equal to $t_0^{\mu}/\mu!$ we get the desired estimate. Q.E.D.

Let $\tilde{\chi}(t)$ be a function in $\gamma^{(\kappa)}(R^1_t)$ satisfying

$$0\leqq\tilde{\chi}\leqq 1,\quad \tilde{\chi}=1\ \ (|t|\leqq 1/4),\quad \tilde{\chi}=0\ \ (|t|\geqq 1/2)\,.$$

We set

$$\begin{cases} \chi^0_k(t_0,\tilde{t}^{\mu};x,\tilde{\xi}^{\nu},\tilde{x}^{\nu},\xi)=\tilde{\chi}(\partial_{t_k}\Psi^{\mu}_{\nu+1}/(\varepsilon\langle\xi\rangle))\,, \\ \chi^1_k=1-\chi^0_k\,, \qquad\qquad k\in\{1,\cdots,\mu\}\,. \end{cases}$$

Setting

$$H_{\mu}=\{\tilde{h}^{\mu}=(h_1,\cdots,h_{\mu});\ h_j=0,1\}\ ,$$

we divide $p^{\mu}_{\nu+1}$ into 2^{μ} terms:

$$p^{\mu}_{\nu+1}=\sum_{\tilde{h}^{\mu}\in H_{\mu}}p^{\mu}_{\nu+1,\tilde{h}^{\mu}}\ ,$$

where $p^{\mu}_{\nu+1,\tilde{h}^{\mu}}=\prod_{k=1}^{\mu}\chi^{h_k}_k p^{\mu}_{\nu+1}$. In view of this division, for the proof of Lemma 3.2, that is, for the proof of (3.12), it suffices to show the following:

Lemma 3.4. *Let $p^{\mu}_{\nu+1}(t_0,\tilde{t}^{\mu};x,\tilde{\xi}^{\nu},\tilde{x}^{\nu},\xi)$ be a multiple symbol satisfying (2.5) and $\mathcal{G}(C_{\mu},h,M,m)$. Assume that $\{p^{\mu}_{\nu+1},\Psi^{\mu}_{\nu+1}\}$ satisfies the condition $B(\varepsilon,\delta)$. We assume furthermore that*

$$(*)\qquad \begin{cases} \textit{for each } k\in\{1,\cdots,\mu\} \textit{ it follows that either} \\ |\partial_{t_k}\Psi^{\mu}_{\nu+1}|\leqq\varepsilon\langle\xi\rangle/2 \qquad \textit{for all}\quad (t_0,\tilde{t}^{\mu},x,\tilde{\xi}^{\nu},\tilde{x}^{\nu},\xi)\ \textit{on}\ \mathrm{supp}\,p^{\mu}_{\nu+1} \\ \textit{or} \\ |\partial_{t_k}\Psi^{\mu}_{\nu+1}|\geqq\varepsilon\langle\xi\rangle/4 \qquad \textit{for all}\quad (t_0,\tilde{t}^{\mu},x,\tilde{\xi}^{\nu},\tilde{x}^{\nu},\xi)\ \textit{on}\ \mathrm{supp}\,p^{\mu}_{\nu+1} \end{cases}$$

when $\mu\geqq 1$. Then there exist constants A, A_1 and M_1 independent of ν and μ such that for any N we have

$$(3.15)\qquad |q^{\mu}_{\nu+1}(t_0;x,\xi)|\leqq 5C_{\mu}A^{\nu}M_1^{-N}\langle\xi\rangle^{m-N}\sum_{j=0}^{N\wedge\mu}\frac{(N-j+h)!^{\kappa}}{(\mu-j)!}A_1^{\mu-j}\,,$$

where $N\wedge\mu=\min(N,\mu)$.

Proof. We shall prove the lemma by induction on μ. When $\mu=0$ the estimate (3.15) for any integer $N\geqq 0$ follows from Lemma 3.3. Suppose $\mu\geqq 1$ and that the lemma for any h is valid until $\mu-1$. If $|\partial_{t_k}\Psi^{\mu}_{\nu+1}|\leqq\varepsilon\langle\xi\rangle/2$ on $\mathrm{supp}\,p^{\mu}_{\nu+1}$ for all k then the estimate (3.15) follows also from Lemma 3.3. Suppose that there exists a $k\in\{1,\cdots,\mu\}$ such that $|\partial_{t_k}\Psi^{\mu}_{\nu+1}|\geqq\varepsilon\langle\xi\rangle/4$ on supp $p^{\mu}_{\nu+1}$. Note that

$$\int_{\Delta_\mu} d\tilde{t}^\mu = \int_{\Delta'_{\mu-1}} dt_1 \cdots dt_{k-1} dt_{k+1} \cdots dt_\mu \int_{t_{k-1}}^{t_{k-1}} dt_k,$$

where $\Delta'_{\mu-1}$ is defined by (3.3) with respect to $t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_\mu$. Integrating by parts with respect to t_k we have

$$\begin{aligned}\int_{t_{k+1}}^{t_{k-1}} e^{i\Psi^\mu_{\nu+1}} p^\mu_{\nu+1} dt_k = &-\int_{t_{k+1}}^{t_{k-1}} e^{i\Phi^\mu_{\nu+1}} \partial_{t_k}((i\partial_{t_k}\Psi^\mu_{\nu+1})^{-1} p^\mu_{\nu+1}) dt_k \\ &+ e^{i\Psi^{\mu-1,k}_{\nu+1}}[(i\partial_{t_k}\Psi^\mu_{\nu+1})^{-1} p^\mu_{\nu+1}]|_{t_k=t_{k-1}} \\ &- e^{i\Psi^{\mu-1,k}_{\nu+1}}[(i\partial_{t_k}\Psi^\mu_{\nu+1})^{-1} p^\mu_{\nu+1}]|_{t_k=t_{k+1}}.\end{aligned} \tag{3.16}$$

Hence we get

$$q^\mu_{\nu+1}(t_0; x, \xi) = q^{\mu,1}_{\nu+1}(t_0; x, \xi) + q^{\mu-1,1}_{\nu+1,0}(t_0; x, \xi) + q^{\mu-1,1}_{\nu+1,1}(t_0; x, \xi),$$

where the multiple symbol of each term corresponds to the one of (3.16). In order to apply the hypothesis of the induction, in view of (3.6) and (3.7) we divide the multiple symbol of $q^{\mu-1,1}_{\nu+1,j}(t_0; x, \xi)$, $j=0, 1$, into two terms by multiplying the partition of unity $\{\tilde{\chi}(\partial_{s_{k'}} \Psi^{\mu-1,k+j}_{\nu+1}/(\varepsilon\langle\xi\rangle)), 1-\chi(\partial_{s_{k'}} \Psi^{\mu-1,k+j}_{\nu+1}/(\varepsilon\langle\xi\rangle))\}$ with $k'=k-1+j$. Then we get the division

$$q^\mu_{\nu+1}(t_0; x, \xi) = q^{\mu,1}_{\nu+1}(t_0; x, \xi) + \sum_{j=1}^{4} \tilde{q}^{\mu-1,1}_{\nu+1,j}(t_0; x, \xi),$$

where each term satisfies the assumptions of the lemma. Repeating the same procedure as above for $q^{\mu,1}_{\nu+1}$ again and moreover repeating N times, we finally obtain

$$q^\mu_{\nu+1} = q^{\mu,N}_{\nu+1} + \sum_{K=1}^{N}\sum_{j=1}^{4} \tilde{q}^{\mu-1,K}_{\nu+1,j}. \tag{3.17}$$

Each term satisfies the assumptions of the lemma. More precisely, multiple symbols of $q^{\mu,N}_{\nu+1}$ and $\tilde{q}^{\mu-1,K}_{\nu+1}$ satisfy conditions $\mathcal{I}(C_\mu M_2^{-N}, h+N, M, m-N)$ and $\mathcal{I}(C_\mu M_2^{-K}, h+K-1, M, m-K)$, respectively, if we take another constant M_2 independent of ν, h and K. To prove this, taking another small M if necessary, we may assume that $r(\theta_1, \theta_2, \theta_3, y^1, \eta^1, y^2, \eta^2) \equiv ((\partial_{t_k}\Psi^\mu_{\nu+1})(\theta_1, \theta_2, \theta_3, y^1, \eta^1, y^2, \eta^2))^{-1}$ satisfies

$$|\partial^{\tilde{\gamma}^3}_{\tilde{\theta}^3} \partial^{\tilde{\beta}^2}_{\tilde{y}^2} \partial^{\tilde{\alpha}^2}_{\tilde{\eta}^2} r| \leqq C(2M)^{-|\tilde{\alpha}^2+\tilde{\beta}^2+\tilde{\gamma}^3|}(\tilde{\alpha}^2!\tilde{\beta}^2!\tilde{\gamma}^3!)^\kappa \langle\xi\rangle^{-(1+|\tilde{\alpha}^2|)}$$

for a constant C. Then, noting that r depends only on $3+4n$ variables, we get the desired properties for multiple symbols of $q^{\mu,N}_{\nu+1}$ and $q^{\mu-1,K}_{\nu+1}$.

Now, we use Lemma 2.1 by setting $C_O = C_\mu M_2^{-N}(h+N)!^\kappa$ and replacing m by $m-N$. Then, from the fact that the multiple symbol of $q^{\mu,N}_{\nu+1}$ satisfies

$\mathcal{Q}(C_\mu M_2^{-N}, h+N, M, m-N)$ we have

$$|q_{\nu+1}^{\mu;N}| \leqq C_\mu M_2^{-N} A^\nu A_1^\mu (h+N)!^\kappa \langle\xi\rangle^{m-N}/\mu! \tag{3.18}$$

for a constant A_1. Similarly we have

$$\sum_{j=1}^{4} |\tilde{q}_{\nu+1;j}^{\mu-1;N}| \leqq 4C_\mu M_2^{-N} A^\nu A_1^{\mu-1}(h+N-1)!^\kappa \langle\xi\rangle^{m-N}/(\mu-1)!\,. \tag{3.19}$$

For $\tilde{q}_{\nu+1}^{\mu-1,K}$, $K=1, \cdots, N-1$ we use the hypothesis of the induction with N replaced by $N-K$. Then we have

$$\begin{aligned} \sum_{K=1}^{N-1}\sum_{j=1}^{4} |\tilde{q}_{\nu+1,j}^{\mu-1;K}| &\leqq 4\sum_{K=1}^{N} (5C_\mu M_2^{-K} A^\nu M_1^{-(N-K)} \langle\xi\rangle^{(m-K)-(N-K)} \\ &\quad\times \sum_{j=0}^{(N-K)\wedge(\mu-1)} \frac{(N-K-j+h+K-1)!^\kappa}{(\mu-1-j)!} A_1^{\mu-1-j} \\ &\leqq 20C_\mu A^\nu M_1^{-N} \sum_{K=1}^{N} (M_1/M_2)^K \langle\xi\rangle^{m-N} \sum_{j=0}^{(N\wedge\mu)-1} \frac{(N-(j+1)+h)!}{(\mu-(j+1))!} A_1^{\mu-(j+1)} \end{aligned} \tag{3.20}$$

Summing up (3.17)–(3.20) we get (3.15) if M_1 is sufficiently smaller than M_2. This concludes the proof. Q.E.D.

Finally, we give a simple proposition for the argument of the next section.

Proposition 3.5. *Assume that*

$$|\partial_{t_k} \Psi_{\nu+1}^\mu(t_0, \tilde{t}^\mu, x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)| \leqq \varepsilon\langle\xi\rangle/2 \tag{3.21}$$

for a $k \in \{1, \cdots, \mu\}$ *and that*

$$|X_\nu^j(t_0, \tilde{t}^\mu, x, \xi) - x^j| \leqq \delta, \quad |\Xi_\nu^j(t_0, \tilde{t}_\mu, x, \xi) - \xi^j| \leq \delta\langle\xi\rangle \quad \textit{for any} \quad j \in \{1, \cdots, \nu\}. \tag{3.22}$$

If δ *is sufficiently small, we have*

$$|\lambda_{j_k}(t_k, Y^k, H^k) - \lambda_{j_{k+1}}(t_k, Y^k, H^k)| \leqq \varepsilon\langle\xi\rangle\,, \tag{3.23}$$

where $\{Y_\mu^k, H_\mu^k\}$ *is the solution of* (1.7) *with* $\nu=\mu$ *and with* $\{\phi_j\}_{j=1}^{\nu+1}$ *replaced by* $\{\psi_j\}_{j=1}^{\mu+1}$.

Proof. The estimate (3.23) follows easily from (3.5) and (1.7) because we have $(X_\nu^j, \Xi_\nu^j) = (X_\nu^{j_k}, \Xi_\nu^{j_k}) = (Y_\mu^k, H_\mu^k)$ for $j_k \leqq j < j_{k+1}$, $k \in \{0, \cdots, \mu+1\}$, where $j_0=1$ and $j_{\mu+2}=\nu+1$. Q.E.D.

4. Proof of Theorem 1

Before the proof we state the definition of ε-admissible trajectories, following [4] and [5]. Let $\lambda_j(t, x, \xi)$, $j \in \{1, \cdots, l\}$ be characteristic roots of $\mathcal{L}$, given in Introduction. Namely, $\lambda_j(t, x, \xi)$ belong to $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$ and satisfy $\lambda_j(t, x, \theta\xi)=\theta\lambda_j(t, x, \xi)$ for $\theta\geqq 1$ and $|\xi|\geqq 1$. We say that a curve $\{(t, x(t), \xi(t))\}\subset[0, T]\times T^*(R^n_x)$ is the bicharacteristic curve with respect to λ_j through (s, y, η) if $\{x(t), \xi(t)\}$ satisfies the equation

$$\begin{cases} dx/dt=-\nabla_\xi\lambda_j(t, x, \xi)\,, & d\xi/dt=\nabla_x\lambda_j(t, x, \xi)\,, \\ (x, \xi)_{|t=s}=(y, \eta)\,. \end{cases}$$

We denote by $\chi_j(t, s)$ a transformation

$$T^*(R^n)\backslash 0\ni\rho\equiv(y, \eta)\to\chi_j(t, s)\rho=(x(t), \xi(t))\in T^*(R^n)\backslash 0\,.$$

For an integer $\nu\geqq 0$, let $\Pi_{\nu+1}$ denote a set of $(\nu+1)$-repeated permutations $(j_1, j_2, \cdots, j_{\nu+1})$ with $j_k\in\{1, \cdots, l\}$ and let $\Pi^o_{\nu+1}$ denote a subset of $\Pi_{\nu+1}$ whose elements $J_\nu\equiv(j_1\cdots, j_{\nu+1})$ satisfy $j_k\neq j_{k+1}$ for any k. Let t_0 be a fixed point in $(0, T]$ and let Δ^o_ν denote the interior of Δ_ν defined by (3.3). A continuous curve $\{(t, x(t), \xi(t)); t\in[0, t_0]\}$ is called a trajectory of step ν, issuing from ρ, if for some $J_\nu=(j_1, \cdots, j_{\nu+1})\in\Pi^o_{\nu+1}$ and some $\tilde{t}^\nu\in\Delta^o_\nu$ it is the bicharacteristic curve with respect to λ_{j_k} when $t\in[t_k, t_{k-1}]$ ($k=1, \cdots, \nu+1$, $t_{\nu+1}=0$) and $(x(0), \xi(0))=\rho$. We often denote the trajectory by $C(J_\nu, \tilde{t}^\nu, \rho)$. A point

$$\chi_{j_1}(t_0, t_1)\chi_{j_2}(t_1, t_2)\cdots\chi_{j_{\nu+1}}(t_\nu, 0)\rho$$

is called the end point (at $t=t_0$) of the trajectory. For an $\varepsilon\geqq 0$ we say that the trajectory is ε-admissible if

$$|\lambda_{j_k}(t_k, \rho_k)-\lambda_{j_{k+1}}(t_k, \rho_k)|\leqq\varepsilon\langle\eta\rangle\,, \qquad k=1, \cdots, \nu\,,$$

where $\rho_k\equiv(x^k, \xi^k)=\chi_{j_{k+1}}(t_k, t_{k+1})\cdots\chi_{j_{\nu+1}}(t_\nu, 0)\rho$. We remark that the bicharacteristic curve is also a trajectory of step 0 and it is always 0-admissible.

Since $\{\lambda_j\}$ is bounded in the symbol class S^1 and each $\lambda_j(t, x, \xi)$ is homogeneous for $|\xi|\geqq 1$, it is easy to see

Proposition 4.1. *Let $\{(t, \rho(t; \rho_o)); t\in[0, t_0]\}$ denote a trajectory $C(J_\nu, \tilde{t}^\nu, \rho_o)$ for $J_\nu\in\Pi^o_{\nu+1}$, $\tilde{t}^\nu\in\Delta^o_\nu$ and $\rho_o\in T^*(R^n)\backslash 0$. Then, there exists a positive constant $c>0$ independent of J_ν and $\tilde{t}^\nu$ such that for $\rho_o\equiv(x_o, \xi_o)$, $\rho'_o\equiv(x'_o, \xi'_o)$*

$$|\rho(t; \rho_o)-\rho(t; \rho'_o)|\leqq e^{ct}|\rho_o-\rho'_o|\,,$$

if $|\xi_o|$ and $|\xi'_o|$ are large enough. Here $|\rho-\rho'|=|x-x'|+|\xi/|\xi|-\xi'/|\xi'||$ for $\rho\equiv(x, \xi)$, $\rho'\equiv(x', \xi')\in T^*(R^n)\backslash 0$.

By means of Proposition 4.1 we may replace the definition of $\Gamma(t_0; V)$ in Introduction by the following

$$(4.1)\qquad \begin{cases} \Gamma_\varepsilon(t_0;\,V) = \textit{the closure of } \bigcup_{\nu=0}^{\infty}\Gamma_\varepsilon^\nu(t_0;\,V_\varepsilon) \\ \Gamma(t_0;\,V) = \bigcap_{\varepsilon>0}\Gamma_\varepsilon(t_0;\,V)\,. \end{cases}$$

In fact, for any $\varepsilon>0$ there exists an ε' $(0<\varepsilon'<\varepsilon)$ such that $\Gamma_{\varepsilon'}(t_0;\,V)\subset \bigcup_{\nu=0}^{\infty}\Gamma_\varepsilon^\nu(t_0;\,V_\varepsilon)$. It follows from (4.1) that $\Gamma(t_0;\,V)$ is closed in $T^*(R^n)\backslash 0$.

Let $\phi_j(t,\,s;\,x,\,\xi)$ be the solution of (3.1) with $\lambda=\lambda_j$ $(j\in\{1,\cdots,l\})$, where λ_j are characteristic roots of $\mathcal{L}$ of (1). From Proposition 3.1 we can find a small constant T_O such that the following property holds: Let t_0 be a positive constant smaller than T_O. Then, for any fixed $\tilde{t}^\nu\in\Delta_\nu$ and $J_\nu=(j_1,\cdots,j_{\nu+1})\in\Pi_{\nu+1}$ the set of phase functions $\{\phi_{j_k}(t_{k-1},\,t_k;\,x,\,\xi)\}_{k=1}^{\nu+1}$ $(t_{\nu+1}=0)$ satisfies the assumption (A–1) in Section 1. As in the C^∞-case ([10], pp. 185–186) the fundamental solution $E(t,\,s)$ of (1) is constructed in the form

$$(4.2)\qquad \begin{aligned} E(t,\,s) = \sum_{j=1}^{l} I_{j,\phi_j}(t,\,s) + \sum_{\nu=1}^{\infty}\sum_{\substack{j_k\in\{1,\cdots,l\}\\ k=1,\cdots,\nu+1}} \int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} I_{j_1,\phi_{j_1}}(t,\,t_1) \\ \times W_{j_2,\phi_{j_2}}(t_1,\,t_2)\cdots W_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu,\,s)dt_\nu\cdots dt_1 \\ (t_0=t)\qquad \textit{for}\quad 0\leqq t,\,s\leqq T_O\,, \end{aligned}$$

where $I_{j,\phi_j}(t,\,s)$ is a matrix of Fourier integral operators with phase function $\phi_j(t,\,s;\,x,\,\xi)$ and with symbol 1 ($(j,\,j)$ element) or 0 (others), and $W_{j,\phi_j}(t,\,s)$ is the one with symbol $w_j(t,\,s;\,x,\,\xi)=w_j^0(t,\,s;\,x,\,\xi)+\tilde{w}_j(t,\,s;\,x,\,\xi)$. Here $w_j^0(t,\,s;\,x,\,\xi)\in G^{(\kappa)}([0,\,T_O]\times[0,\,T_O];\,S^\sigma_{G(\kappa)})$ and $\tilde{w}_j(t,\,s;\,x,\,\xi)\in G^{(\kappa)}([0,\,T_O]\times[0,\,T_O];\,\mathcal{R}_{G(\kappa)})$, that is, w_j^0 satisfies

$$(4.3)\qquad |\partial_t^\gamma\partial_s^{\gamma'}\partial_\xi^\alpha\partial_x^\beta w_j^0(t,\,s;\,x,\,\xi)|\leqq CM^{-(\gamma+\gamma'+|\alpha|+|\beta|)}(\alpha!\beta!\gamma!\gamma'!)^\kappa\langle\xi\rangle^{\sigma-|\alpha|}$$

for constants C and M independent of α, β, γ and γ', and $\tilde{w}_j$ satisfies

$$(4.4)\qquad |\partial_t^\gamma\partial_s^{\gamma'}\partial_\xi^\alpha\partial_x^\beta \tilde{w}_j(t,\,s;\,x,\,\xi)|\leqq C_\alpha M^{-(\gamma+\gamma'+|\beta|)}(\beta!\gamma!\gamma'!)^\kappa e^{-\varepsilon\langle\xi\rangle^{1/\kappa}}$$

for constants M and $\varepsilon>0$ independent of α, β, γ and γ', and for a constant C_α depending only on α. It follows from Theorem 2.1 in [18] that the right hand side of (4.2) is transformed to that of (1.12) in [20] (cf. Theorem 3.2 in [18]). Therefore, from Proposition 1.1 in [20] we can find a solution $U(t)$ in $\mathcal{B}^\infty([0,\,T_O];\,\mathcal{D}_{L^2}^{\{\kappa\}'})$ for the problem (7) as $U(t)=E(t,\,0)G$. For the proof of the existence of the solution $U(t)$ for $t\in(T_O,\,T]$ it suffices to consider the product $E(t,\,0)\equiv E(t,\,kT_O)E(kT_O,\,(k-1)T_O)\cdots E(T_O,\,0)$ of the fundamental solutions if $t\in[kT_O,\,(k+1)T_O]$. Finally, we note that $E(t,\,s)$ of (1.12) in [20] maps $\mathcal{B}^\infty([0,\,T_O];\,\mathcal{B}_{L^2}^{\{\kappa\}})$ to itself. So, the uniqueness of the solution also follows from the usual duality method.

For the proof of the inclusion (8) we prepare

Proposition 4.2. *Let V be a closed conic set in $T^*(R^n_x)$ and let $\Gamma_\varepsilon(t_0; V)$ be a set defined by* (4.1) *for an $\varepsilon>0$ and $0<t_0\leqq T$. Let $a(x, \xi)$ and $b(x, \xi)$ be symbols in $S^0_{G(\kappa)}$ satisfying*

$$(4.5)\qquad \begin{cases} \text{supp}\, b\subset V_{\varepsilon/2}\,, \\ |x-y|\geqq\varepsilon/2 \quad or \quad |\xi/|\xi|-\eta/|\eta||\geqq\varepsilon/2 \\ \qquad\qquad if\ (x, \xi)\in\text{supp}\ a \quad and \quad (y, \eta)\in\Gamma_\varepsilon(t_0; V)\,. \end{cases}$$

Then, for the fundamental solution $E(t, s)$ of (4.2) *the operator $a(X, D_x)E(t_0, 0)\times b(X, D_x)$ is a pseudo-differential operator with symbol in $\mathcal{R}_{G(\kappa)}$.*

Admitting this proposition for a moment, we first give the proof of (8) by using this. Let $U(t_0)$ be a solution of (7) and set $V=\text{WF}_{G(\kappa_1)}(G)$ for $\kappa\leqq\kappa_1<1/\sigma$. Assume that (x^o, ξ^o) does not belong to $\Gamma(t_0; V)$. Then there exists an $\varepsilon>0$ such that $(x^o, \xi^o)\notin\Gamma_0(t_\varepsilon; V)$. Since $\Gamma_\varepsilon(t_0; V)$ and V are closed conic sets, taking another small $\varepsilon>0$ if necessary, we can find symbols $a(x, \xi)$ and $b(x, \xi)$ in $S^0_{G(\kappa)}$ satisfying (4.5), $b(x, \xi)=1$ in a conic neighborhood of V and $a(x^o, \theta\xi^o)\neq0$ for $\theta\geqq1$. Then we have

$$\begin{aligned} AU(t_0) &= AE(t_0, 0)G \\ &= AE(t_0, 0)BG+AE(t_0, 0)(I-B)G\in\gamma^{(\kappa_1)}\,. \end{aligned}$$

Here, we used Proposition 4.2 above and Lemma 2.1 in [20] for the proof of $AE(t_0, 0)BG\in\gamma^{(\kappa_1)}$, and for the proof of $AE(t_0, 0)(I-B)G\in\gamma^{(\kappa_1)}$ we used the similar discussions as in the proof of Proposition 1.1 and Theorem 4 in [20]. Then, in view of Definition in Introduction we have $(x^o, \xi^o)\notin\text{WF}_{G(\kappa_1)}(U(t_0))$. This proves (8).

Now, we return to the proof of Proposition 4.2. First, we consider the case $T=T_O$. Regard $AE(t_0, 0)B$ as a pseudo-differential operator. Then its symbol is a sum of $\sigma(a(X, D_x)I_{j,\phi_j}(t_0, 0)b(X, D_x))$ and the terms of the form

$$(4.6)\qquad \begin{aligned} &q_{\nu+1}(t_0; x, \xi) \\ &\equiv\int_{\Delta_{\nu-2}} d\tilde{t}^{\nu-2}O_s\!-\!\iint e^{i(\Psi_{\nu+1}(t_0, \tilde{t}^{\nu-2}\,;\, x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)-x\cdot\xi)} \\ &\qquad\qquad \times p_{\nu+1}(t_0, \tilde{t}^{\nu-2}; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)dx\, d\tilde{\xi}^\nu\,, \end{aligned}$$

with $(j_1, \cdots, j_{\nu-1})\in\Pi_{\nu-1}$, $\nu\geqq3$. Here $\Psi_{\nu+1}$ is defined by (1.9) with $\phi_1=\phi_{\nu+1}=x\cdot\xi$ and ϕ_{k+1} replaced by ϕ_{j_k} $(k=1, \cdots, \nu-1)$, and $p_{\nu+1}\equiv p_{\nu+1}(t_0, \tilde{t}^{\nu-1}; x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ is a multiple symbol defined by

$$p_{\nu+1} = a(x, \xi^1)\Big(\prod_{k=1}^{\nu-1} w_{j_k}(t_{k-1}, t_k; x^k, \xi^{k+1})\Big)\, b(x^\nu, \xi^\nu) \qquad (t_{\nu-1} = 0)\,.$$

So, for the proof of Proposition 4.2 with $T=T_O$ it suffices to show

Proposition 4.3. *Let $p_{\nu+1}$ be as above. Then the symbol $q_{\nu+1}(t_0; x, \xi)$ ($\nu\geqq 2$) defined by (4.6) belongs to $\mathcal{R}_{G(\kappa)}$ and it satisfies for positive constants ε', A and M independent of ν*

$$|\partial_\xi^\alpha \partial_x^\beta q_{\nu+1}(t_0; x, \xi)| \leqq C_\alpha A^\nu M^{-|\beta|}\beta!^\kappa \nu!^{\sigma\kappa-1} e^{-\varepsilon'\langle\xi\rangle^{1/\kappa}}$$

for α and β. Here C_α is a constant independent of β and ν.

Proof. In view of Lemma 1.3 and its remark we may assume that $p_{\nu+1}$ satisfies (2.5). Hence it suffices to check the conditions $\mathcal{G}(A_O^\nu, 0, M, (\nu-2)\sigma)$ (for a constant A_O independent of ν) and $B(\varepsilon, \delta)$ in Lemma 3.2 by means of (3.11)$'$ with $m=(\nu-2)\sigma$ and the fact that we have $[(\nu-2)\sigma]!^\kappa \leqq A^\nu \nu!^{\sigma\kappa}$ for a constant A independent of ν. The condition $\mathcal{G}(A_O^\nu, 0, M, (\nu-2)\sigma)$ follows from (4.3) clearly. If $|\partial_{t_k}\Psi_{\nu+1}(t_0, \tilde{t}^{\nu-2}; x, \xi)| \leqq \varepsilon\langle\xi\rangle/2$ for all $k\in\{1, \cdots, \nu-2\}$ and if $(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ satisfies (3.22) for a sufficiently small $\delta>0$ then it follows from Proposition 3.5 that there exists an ε-admissible trajectory of step $\mu+1$, from (X_ν^ν, ξ), whose end point is (x, Ξ_ν^1). Here μ is a number of elements in $\{k; j_k \neq j_{k+1}\}$ for a permutation $(j_1, \cdots, j_{\nu-1})$ which determines $\Psi_{\nu+1}$. Since (x^ν, ξ) and (x, ξ^1) are contained in δ-conic neighborhoods of (X_ν^ν, ξ) and (x, Ξ_1^ν), respectively, we see by means of the choice of $a(x, \xi)$ and $b(x, \xi)$ that $\{p_{\nu+1}, \Psi_{\nu+1}\}$ satisfies (3.10) if δ also satisfies $\delta\leqq\varepsilon/2$. Next, we consider a child $\{p_{\nu+1}^{\nu-3}, \Psi_{\nu+1}^{\nu-3}\}$ of $\{p_{\nu+1}, \Psi_{\nu+1}\}$. Then under $|\partial_{t_j}\Psi_{\nu+1}^{\nu-3}| \leqq \varepsilon\langle\xi\rangle/2$, $j=1, \cdots, \nu-3$, we find the existence of an ε-admissible trajectory of step, at most, $\mu+1$ from (X_ν^ν, ξ), whose end point is (x, Ξ_ν^1). Noting the choice of $a(x, \xi)$ and $b(x, \xi)$ we also see that this child satisfies (3.10). Repeating this procedure, we finally see that $\{p_{\nu+1}, \Psi_{\nu+1}\}$ satisfies the condition $B(\varepsilon, \delta)$. Q.E.D.

We proceed to the proof of Proposition 4.2 for the general case. For simplicity we assume $T_O\leqq T\leqq 3T_O/2$. Then, from the uniqueness of the problem (7) it follows that for any s with $T_O/2\leqq s\leqq T_O$

$$E(t_0, 0) = E(t_0, s)E(s, 0) \qquad (T_O\leqq t_0\leqq T)\,. \tag{4.7}$$

Let $\omega(s)$ be a function in $\gamma^{(\kappa)}(R_t^1)$ satisfying $\operatorname{supp}\omega\subset(T_O/2, T_O)$ and $\int_{T_O/2}^{T_O}\omega(s)ds=1$. Then, from (4.7) we have

$$E(t_0, 0) = \int_{T_O/2}^{T_O} E(t_0, 0)\omega(s)ds = \int_{T_O/2}^{T_O} E(t_0, s)\omega(s)E(s, 0)ds\,.$$

So, in order to show Proposition 4.2 it suffices to show

$$\int_{T_O/2}^{T_O} AE(t_0, s)\omega(s)E(s, 0)Bds \in \mathcal{R}_{G(\kappa)} \tag{4.8}$$

with $A=a(X, D_x)$ and $B=b(X, D_x)$ whose symbols $a(x, \xi)$ and $b(x, \xi)$ satisfy

(4.5). Since $t_0-s\leqq T_O$ and $s\leqq T_O$, we can apply the discussions of Section 1 to each term in $E(t_0, s)$ and $E(s, 0)$, and we obtain

$$\begin{aligned}(4.9)\quad &\sigma\Big(\int_{T_O/2}^{T_O} AE(t_0, s)\omega(s)E(s, 0)Bds\Big)(x, \xi)\\ &=\sum_{\nu=0}^{\infty}\sum_{\nu'=0}^{\infty}\sum\nolimits_{J_{\nu+1},J'_{\nu'+1}}\int_{T_O/2}^{T_O}\Big\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\\ &\cdot\int_0^s\int_0^{t'_1}\cdots\int_0^{t'_{\nu'-1}}\{O_s-\int\cdots\int\exp(i(\Psi_{J_{\nu+1},J'_{\nu'+1}}-x\cdot\xi))\\ &\times p_{J_{\nu+1},J'_{\nu'+1}}(t_0, \tilde{t}^\nu, s, \tilde{t}'^{\nu'}; x, \xi^0, x^0, \xi^{\nu+\nu'+1}, \tilde{x}^{\nu+\nu'+1}, \xi)\\ &\cdot dx^0 d\xi^0 d\tilde{x}^{\nu+\nu'+1} d\tilde{\xi}^{\nu+\nu'+1}\}d\tilde{t}^\nu d\tilde{t}'^{\nu'}\Big\}ds + r(t_0, s)\end{aligned}$$

with $r(t_0, s)\in\mathcal{R}_{G(\kappa)}$, where

$$\begin{aligned}\Psi_{J_{\nu+1},J'_{\nu'+1}} &= x\cdot\xi^0-x^0\cdot\xi^0\\ &+\sum_{k=1}^{\nu}(\phi_{j_k}(t_{k-1}, t_k; x^{k-1}, \xi^k)-x^k\cdot\xi^k)\\ &+\phi_{j_{\nu+1}}(t_\nu, s; x^\nu, \xi^{\nu+1})-x^{\nu+1}\cdot\xi^{\nu+1}\\ &+\sum_{k=1}^{\nu'}(\phi_{j'_k}(t'_{k-1}, t'_k; x^{\nu+k}, \xi^{\nu+k+1})-x^{\nu+k+1}\cdot\xi^{\nu+k+1})\\ &+\phi_{j'_{\nu'+1}}(t'_{\nu'}, 0; x^{\nu+\nu'}, \xi^{\nu+\nu'+1})\\ &-x^{\nu+\nu'+1}\cdot\xi^{\nu+\nu'+1}+x^{\nu+\nu'+1}\cdot\xi \qquad (t'_0=s)\,.\end{aligned}$$

Here, $\sum_{J_{\nu+1},J'_{\nu'+1}}$ means the summation which is taken over all $J_{\nu+1}=(j_1, \cdots, j_{\nu+1})$ and $J'_{\nu'+1}=(j'_1, \cdots, j'_{\nu'+1})$ with $j_k, j'_k=1, \cdots, l$; and in (4.9), $\int_s^{t_0}\cdots\int_s^{t_{\nu-1}}$(resp. $\int_0^s\cdots\int_0^{t'_{\nu'-1}}$) for the case $\nu=0$ (resp. $\nu'=0$) means that we do not integrate the integrand with respect to $\tilde{t}^\nu$-variables (resp. $\tilde{t}^{\nu'}$-variables). In (4.9) the symbol $p_{J_{\nu+1},J'_{\nu'+1}}$ satisfies

$$\begin{cases}|\xi^j-\xi^{\nu+1}|\leqq\langle\xi^{\nu+1}\rangle/8 & (j=0, \cdots, \nu)\,,\\ |\xi^j-\xi|\leqq\langle\xi\rangle/8 & (j=\nu+2, \cdots, \nu+\nu'+1)\end{cases}$$

$$on\quad \operatorname{supp} p_{J_{\nu+1},J'_{\nu'+1}}\,,$$

$$p_{J_{\nu+1},J'_{\nu'+1}}|_{s=T_O/2}=0\,,\quad p_{J_{\nu+1},J'_{\nu'+1}}|_{s=T_O}=0$$

and the pair $\{p_{J_{\nu+1},J'_{\nu'+1}}, \Psi_{J_{\nu+1},J'_{\nu'+1}}\}$ with $\Psi_{J_{\nu+1},J'_{\nu'+1}}$ satisfies condition similar to $B(\varepsilon, \delta)$ (we note that in this case we pose the condition (3.10) with X^j_ν and Ξ^j_ν replaced by the points $\mathcal{X}_{j'_k}(t'_{k-1}, t'_k)\cdots\mathcal{X}_{j'_{\nu'+1}}(t'_{\nu'}, 0)(x^{\nu+\nu'+1}, \xi)$ $(1\leqq k\leqq\nu'+1;$ $t'_0=s$, $t'_{\nu'+1}=0)$ or $\mathcal{X}_{j_k}(t_{k-1}, t_k)\cdots\mathcal{X}_{j_{\nu+1}}(t_\nu, s)\mathcal{X}_{j'_1}(s, t'_1)\cdots\mathcal{X}_{j'_{\nu'+1}}(t'_{\nu'}, 0)(x^{\nu+\nu'+1}, \xi)$ $(2\leqq$ $k\leqq\nu+1;$ $t_{\nu+1}=s)$ in the trajectories). Replacing $r(t_0, s)$ in (4.9) by another symbol in $\mathcal{R}_{G(\kappa)}$ we may assume moreover that

$$|\xi^{\nu+1}-\xi^{\nu+2}|\leqq\langle\xi^{\nu+2}\rangle/2 \qquad on \quad \operatorname{supp} p_{J_{\nu+1}, J'_{\nu'+1}}.$$

Then, we can prove (4.8) by the similar discussions as the case of $T\leqq T_O$ if we use Lemma 2.1′ below instead of Lemma 2.1 and the fact that with $\Psi=\Psi_{J_{\nu+1}, J'_{\nu'+1}}$ and $p=p_{J_{\nu+1}, J'_{\nu'+1}}$

$$(4.10) \qquad \int_{T_O/2}^{T_O}\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\int_0^s\int_0^{t'_1}\cdots\int_0^{t'_{\nu'-1}}\partial_s e^{i(\Psi-x\cdot\xi)} \times(i\partial_s\Psi)^{-1}p d\tilde{t}^\nu d\tilde{t}'^{\nu'}\}ds$$
$$=\int_{T_O/2}^{T_O}\{\frac{\partial}{\partial s}\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\int_0^s\int_0^{t'_1}\cdots\int_0^{t'_{\nu'-1}}e^{i(\Psi-x\cdot\xi)} \times(i\partial_s\Psi)^{-1}p d\tilde{t}^\nu d\tilde{t}'^{\nu'}\}\}ds$$
$$-[-\int_{T_O/2}^{T_O}\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-2}}\int_0^s\int_0^{t'_1}\cdots\int_0^{t'_{\nu'-1}}e^{i(\Psi-x\cdot\xi)} \times((i\partial_s\Psi)^{-1}p)_{|t_\nu=s}dt_1\cdots dt_{\nu-1}d\tilde{t}'^{\nu'}\}ds$$
$$+\int_{T_O/2}^{T_O}\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\int_0^s\int_0^{t'_2}\cdots\int_0^{t'_{\nu'-1}}e^{i(\Psi-x\cdot\xi)} \times((i\partial_s\Psi)^{-1}p)_{|t'_1=s}d\tilde{t}^\nu dt'_2\cdots dt'_\nu\}ds$$
$$+\int_{T_O/2}^{T_O}\{\int_s^{t_0}\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\int_0^s\int_0^{t'_1}\cdots\int_0^{t'_{\nu'-1}}e^{i(\Psi-x\cdot\xi)} \times\partial_s\{(i\partial_s\Psi)^{-1}p\}d\tilde{t}^\nu d\tilde{t}'^{\nu'}\}ds]$$

holds and that the first term in the right member of (4.10) is zero. Hence, the proof of Proposition 4.2 for the case of $T_O\leqq T\leqq 3T_O/2$ is reduced to the proof of Lemma 2.1′ below.

Lemma 2.1′. *Let* $p_{\nu+1}(x, \tilde{\xi}^\nu, \tilde{x}^\nu, \xi)$ *satisfy* (2.6) *and consider* $q_{\nu+1}(x, \xi)$ *defined by* (2.2) *with* $\Phi_{\nu+1}(x, \xi)$ *replaced by* $x\cdot\xi$. *Suppose that for a k the variables* $(\xi^1, \cdots, \xi^\nu)$ *are divided into two groups* $(\xi^1, \cdots, \xi^k)$ *and* $(\xi^{k+1}, \cdots, \xi^\nu)$ *and they satisfy*

$$\begin{cases} |\xi^j-\xi^k|\leqq\langle\xi^k\rangle/8 & (j=1, \cdots, k-1), \\ |\xi^j-\xi|\leqq\langle\xi\rangle/8 & (j=k+1, \cdots, \nu), \\ |\xi^k-\xi^{k+1}|\leqq\langle\xi^{k+1}\rangle/2 & on \quad \operatorname{supp} p_{\nu+1}. \end{cases}$$

Moreover, let $\phi_j\in\mathcal{P}_{G(\kappa)}(\tau_j)$ *and assume* $\sum_{j=1}^k\tau_j\leqq\tau^o$ *and* $\sum_{j=k+1}^{\nu+1}\tau_j\leqq\tau^o$ *with* τ^o *in* (A–1). *Then, there exist constants A and* C_α, *and for any* ε (>0) *there exists a constant* $M=M_\varepsilon$ *such that*

$$(4.11) \qquad |q_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)|\leqq C_0C_\alpha A^\nu M^{-|\beta|}\beta!^\kappa\langle\xi\rangle^m \exp(\varepsilon\langle\xi\rangle^{1/\kappa})$$

hold.

REMARK. In the above lemma we need not assume $\sum_{j=1}^{\nu+1} \tau_j \leqq \tau^o$.

Proof. Let $q_{\nu+1,k}(x, \xi^k, x^k, \xi)$ be the symbol defined by (2.9). Then, since $\sum_{j=1}^{k} \tau_j \leqq \tau^o$ and $\sum_{j=k+1}^{\nu+1} \tau_j \leqq \tau^o$, we can apply the discussions in Section 2 to the integrals with respect to $(x^1, \cdots, x^{k-1}, \xi^1, \cdots, \xi^{k-1})$ and $(x^{k+1}, \cdots, x^\nu, \xi^{k+1}, \cdots, \xi^\nu)$ individually and obtain (2.10) with

$$(4.12) \qquad q_{\nu+1}(x, \xi) = O_s\text{-}\iint e^{i(\Phi_{\nu+1,k}(x,\xi^k,x^k,\xi)-x\cdot\xi)} q_{\nu+1,k}(x, \xi^k, x^k, \xi) dx^k d\xi^k ,$$

where $\Phi_{\nu+1,k}(x, \xi^k, x^k, \xi)$ is defined by (2.8). Write (4.12) as

$$q_{\nu+1}(x, \xi) = O_s\text{-}\iint e^{i(x-x^k)\cdot(\xi-\xi^k)} \tilde{q}_{\nu+1,k}(x, \xi^k, x^k, \xi) dx^k d\xi^k ,$$

where

$$\tilde{q}_{\nu+1,k}(x, \xi^k, x^k, \xi) = e^{i\tilde{J}_{\nu+1}(x,\xi^k,x^k,\xi)} q_{\nu+1,k}(x, \xi^k, x^k, \xi)$$

and

$$\begin{aligned} \tilde{J}_{\nu+1}(x, \xi^k, x^k, \xi) = &\{(\phi_1 \# \cdots \# \phi_k)(x, \xi^k) - x\cdot\xi^k\} \\ &+ \{(\phi_{k+1} \# \cdots \# \phi_{\nu+1})(x^k, \xi) - x^k\cdot\xi\} . \end{aligned}$$

Then, together with the discussion (1.6)–(1.7) in [18] for $\tilde{J}_{\nu+1}(x, \xi^k, x^k, \xi)$, we can easily prove (4.11) by using the factor $\exp(\varepsilon\langle\xi\rangle^{1/\kappa})$ in the right member of (4.11). Q.E.D.

5. Hyperbolic differential operators

Let L be a single hyperbolic operator of order m which has a form

$$(5.1) \qquad L = L_1 L_2 \cdots L_r + A_1 L_2 \cdots L_r + A_2 L_3 \cdots L_r + \cdots + A_{r-1} L_r + A_r ,$$

where L_k $(k=1, \cdots, r)$ are regularly hyperbolic operators with coefficients in $\gamma^{(\kappa)}([0, T]\times R_x^n)$ and A_k $(k=1, \cdots, r)$ are differential operators with coefficients in $\gamma^{(\kappa)}([0, T]\times R_x^n)$ satisfying

$$(5.2) \qquad \mathrm{Ord}\, A_k \leqq \mathrm{Ord}\,(L_1 \cdots L_k) - k/\mu$$

for a constant $\mu \geqq 1$. We assume $\kappa < \mu/(\mu-1)$. The form (5.1) is a generalization of (12) in Introduction. In fact, (12) is derived from (5.1) by setting $A_1 = A_2 = \cdots = A_{r-1} = 0$. We remark that any hyperbolic operator with characteristic roots of constant multiplicity can be written in the form (5.1) if the constant μ is defined as the irregularity of the hyperbolic operator (cf. Theorem 3.1 of [7], see also Lemma 4.1 of [3]).

Let m_k denote the order of L_k and let $\lambda_{k,j}(t, x, \xi)$, $j=1, \cdots, m_k$, be characteristic roots of L_k. We may assume $\lambda_{k,j}\in G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$ by multiplying a cut function with respect to ξ if necessary. Since L_k is a regularly hyperbolic operator, there exist $\lambda'_{k,j}(t, x, \xi)\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$ such that

$$L_k = (D_t-\lambda_{k,1}(t, X, D_x)-\lambda'_{k,1}(t, X, D_x)) \cdots \times(D_t-\lambda_{k,m_k}(t, X, D_x)-\lambda'_{k,m_k}(t, X, D_x)) + \sum_{j=0}^{m_k-1} b_{k,j}(t, X, D_x)D_t^j \tag{5.3}$$

with $b_{k,j}(t, x, \xi)\in G^{(\kappa)}([0, T]; S^{-2m}_{G(\kappa)})$. Here the equality (5.3) means that it holds modulo regularizers of the form $\sum_{j=0}^{m_k-1} r_j(t, X, D_x)D_t^j$ with $r_j\in \mathcal{R}_{G(\kappa)}$ for any fixed t. Since we may disregard the contribution of such regularizers in our discussion, till the end of this section the equality means that it holds modulo regularizers. Set $\bar{m}_0=0$, $\bar{m}_k=m_1+\cdots+m_k$ $(\bar{m}_r=m)$ and

$$\partial_j = D_t-\lambda_{k,j-\bar{m}_{k-1}}(t, X, D_x)-\lambda'_{k,j-\bar{m}_{k-1}}(t, X, D_x) \quad \text{if} \quad \bar{m}_{k-1}<j\leqq\bar{m}_k\,. \tag{5.4}$$

Proposition 5.1. *Let L be a hyperbolic operator of order m which has the form* (5.1). *Assume* (5.2). *Then, L can be written as*

$$\begin{aligned} L = {} & \partial_1 \cdots \partial_m \\ & + \sum_{p=1}^{m_1-1} \sum_{1\leqq j_1<\cdots<j_p\leqq m_1} b^1_{j_1\cdots j_p}\partial_{j_1}\cdots\partial_{j_p}\partial_{m_1+1}\cdots\partial_m \\ & + \sum_{p=1}^{\bar{m}_2-2} \sum_{1\leqq j_1<\cdots<j_p\leqq \bar{m}_2} b^2_{j_1\cdots j_p}\partial_{j_1}\cdots\partial_{j_p}\partial_{\bar{m}_2+1}\cdots\partial_m \\ & + \cdots \\ & + \sum_{p=0}^{m-r} \sum_{1\leqq j_1<\cdots<j_p\leqq m} b^r_{j_1\cdots j_p}\partial_{j_1}\cdots\partial_{j_p}\,, \end{aligned} \tag{5.5}$$

where ∂_j are defined by (5.4) *and $b^k_{j_1\cdots j_p}$ is a pseudo-differential operator $b^k_{j_1\cdots j_p}(t, X, D_x)$ with symbol $b^k_{j_1\cdots j_p}(t, x, \xi)\in G^{(\kappa)}([0, T]; S^{k-k/\mu}_{G(\kappa)})$.*

For the proof we prepare

Lemma 5.2. *Let s be a positive integer and let ∂_j $(j=1, \cdots, s)$ denote $D_t-\lambda_j(t, X, D_x)$ for some $\lambda_j(t, x, \xi)\in G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$. Assume $|\lambda_j(t, x, \xi)-\lambda_k(t, x, \xi)|\geqq c_0\langle\xi\rangle$ for a constant $c_0>0$ if $j\neq k$ and $|\xi|$ is large. Let A be an operator of the form*

$$A = \sum_{k=1}^{s-1} b_k(t, X, D_x)D_t^k \quad \text{for} \quad b_k(t, x, \xi)\in G^{(\kappa)}([0, T]; S^{s-1-k}_{G(\kappa)})\,. \tag{5.6}$$

Then, we can write A as

$$A=\sum_{\substack{1\leq j_1<\cdots<j_k\leq s\\ k<s}} a_{j_1\cdots j_k}(t, X, D_x)\partial_{j_1}\cdots\partial_{j_k}+a(t, X, D_x)$$

for some $a(t, x, \xi)$ *and* $a_{j_1\cdots j_k}(t, x, \xi)$ *in* $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$.

The proof of this lemma easily follows from the induction on s.

Proof of Proposition 5.1. Let $b(t, x, \xi)$ be a symbol in $G^{(\kappa)}([0, T]; S^{-m}_{G(\kappa)})$. Then for any integer $0\leqq k\leqq m$ there exist $a_j(t, x, \xi)\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$ $(j=0, \cdots, k)$ such that $bD_t^k=a_0+\sum_{j=2}^{k} a_j\partial_{m-j+1}\cdots\partial_{m-1}\partial_m$. On the other hand, from (5.3) we can write $L_1L_2\cdots L_r$ as

$$L_1L_2\cdots L_r=(\partial_1\cdots\partial_m)(\partial_{m_1+1}\cdots\partial_{\bar{m}_2})\cdots(\partial_{\bar{m}_{r-1}+1}\cdots\partial_{\bar{m}_r})$$
$$+\sum_{k=1}^{r} L_1\cdots L_{k-1}(\sum_{j=0}^{m_k-1} b_{k,j}(t, X, D_x)D_t^j)\partial_{\bar{m}_k+1}\cdots\partial_{\bar{m}_r}$$

and the second term of the right hand side can be rewritten as

$$\sum_{j=0}^{m-1} b_j(t, X, D_x)D_t^j$$

for some $b_j(t, x, \xi)\in G^{(\kappa)}([0, T]; S^{-m}_{G(\kappa)})$. Hence we may assume $L_k=\partial_{\bar{m}_{k-1}+1}\cdots\partial_{\bar{m}_k}$, and so $L_k\cdots L_r=\partial_{\bar{m}_{k-1}+1}\cdots\partial_m$. Since $D_t^{m_k}$ can be written as $L_k-\sum_{j=1}^{m_k} b_k^jD_t^{m_k-j}$ for some $b_k^j\in G^{(\kappa)}([0, T]; S^j_{G(\kappa)})$, we may also assume that the order of A_k with respect to D_t is smaller than or equal to $\bar{m}_k-k$. Consequently, A_k can be written as the finite sum of operators of the form $\Lambda^{k-k/\mu}A_{k,1}A_{k,2}\cdots A_{k,k}$, where $\Lambda=\langle D_x\rangle$ and $A_{k,j}$ $(j=1, \cdots, k)$ is the operator of the form (5.6) with $s=m_j$. Applying Lemma 5.2 to each $A_{k,j}$ and $\{\partial_{\bar{m}_{j-1}+1}, \cdots, \partial_{\bar{m}_j}\}$, we have

$$A_k=\sum_{p=1}^{\bar{m}_k-k}\sum_{1\leq j_1<j_2<\cdots<j_p\leq m_k} b^k_{j_1\cdots j_p}\partial_{j_1}\cdots\partial_{j_p},$$
$$b^k_{j_1\cdots j_p}\in G^{(\kappa)}([0, T]; S^{k-k/\mu}_{G(\kappa)}),$$

and this gives (5.5). Q.E.D.

Theorem 2 (cf. Proposition 3.3 in [18]). *Let L be a hyperbolic operator of the form* (5.1). *Assume* (5.2). *Set* $\sigma=1-1/\mu$. *Then there exists a hyperbolic system* $\mathcal{L}$ *of the form* (1) *with* $b_{j,k}(t, x, \xi)$ *in* $G^{(\kappa)}([0, T]; S^{\sigma}_{G(\kappa)})$ *such that the Cauchy problem* (10) *for L can be reduced to the equivalent Cauchy problem* (7) *for* $\mathcal{L}$.

Consequently we have (13) for κ_1 satisfying $\kappa\leqq\kappa_1<1/\sigma=\mu/(\mu-1)$ concerning the propagation of wave front sets in Gevrey classes of solutions of (10).

Proof. For $1\leqq p<m$ we set

$$\Pi_p^o=\{J=(j_1,\cdots,j_p);\ 1\leqq j_1<\cdots<j_p\leqq m\}$$

and for $J\in\Pi_p^o$ we denote the length p of J by $|J|$. Let Π_p be a subset of Π_o^p whose element $J=(j_1,\cdots,j_p)$ satisfies the following: Set $S_J=\{j_1,\cdots,j_p\}$. If there exists a $k\in\{1,\cdots,r\}$ such that $\{\bar{m}_{k-1}+1,\cdots,\bar{m}_k\}\supset S_J$ then the set $\{\bar{m}_k+1,\cdots,m\}$ is also contained in S_J. Set $\Pi=\{0\}\cup(\bigcup_{p=1}^{m-1}\Pi_p)$ and denote the number of elements of Π by l. Let u be a solution of (10) and set

$$\begin{cases} u_0=\Lambda^{(m-1)\sigma}u \\ u_J=\Lambda^{(m-1-p)\sigma}\partial_J u\,, \qquad J=(j_1,\cdots,j_p)\in\Pi_q,\ 1\leqq p\leqq m\,, \end{cases} \tag{5.7}$$

where $\partial_J=\partial_{j_1}\cdots\partial_{j_p}$. Then, from (5.5) we can write $Lu=0$ as

$$Lu=\partial_1 u_{(2,\cdots,m)}+\sum_{J\in\Pi} b_J(t,X,D_x)u_J=0 \tag{5.8}$$

for some $b_J(t,x,\xi)\in G^{(\kappa)}([0,T];S^\sigma_{G(\kappa)})$. For $J\in\Pi$ we set $j_0=\max\{j;\ 1\leqq j\leqq m,\ j\notin S_J\}$, where we denote $S_J=\phi$ for $J=0$. We shall show

$$\partial_{j_0}\Lambda^{(m-1-|J|)\sigma}\partial_J=\begin{cases}\sum_{\tilde{J}\in\Pi} b^J_{\tilde{J}}\Lambda^{(m-1-|\tilde{J}|)\sigma}\partial_{\tilde{J}} & \text{if } |J|\leqq m-2\,,\\ \partial_1\cdots\partial_m+\sum_{\tilde{J}\in\Pi} b^J_{\tilde{J}}\Lambda^{(m-1-|I|)\sigma}\partial_{\tilde{J}} & \text{if } |J|=m-1,\ J\neq(2,\cdots,m)\end{cases} \tag{5.9}$$
$$\text{for}\quad b^J_{\tilde{J}}(t,x,\xi)\in G^{(\kappa)}([0,T];S^\sigma_{G(\kappa)})\,.$$

Then, together with (5.8) we have for any $J\in\Pi$

$$\partial_{j_0}u_J=\begin{cases}\sum_{\tilde{J}\in\Pi} b^J_{\tilde{J}}(t,X,D_x)u_{\tilde{J}} & \text{if } |J|\leqq m-2\,,\\ -\sum_{\tilde{J}\in\Pi} b_{\tilde{J}}(t,X,D_x)u_{\tilde{J}}+\sum_{\tilde{J}\in\Pi} b^J_{\tilde{J}}(t,X,D_x)u_{\tilde{J}} & \text{if } |J|=m-1\,.\end{cases}$$

This shows that the l-dimensional vector $U=(u_J)_{J\in\Pi}$ satisfies $\mathcal{L}U=0$ for a system $\mathcal{L}$ of the form (1). In this way we reduce the problem (10) for L to a problem (7) for $\mathcal{L}$. The fact that (10) and (7) are equivalent is verified by the same way as in [13] and [8].

So, it remains to prove (5.9). To prove this it suffices to show

$$\partial_{j_0}\partial_J=\begin{cases}\sum_{\tilde{J}\in\Pi} a_{\tilde{J}}(t,X,D_x)\partial_{\tilde{J}} & \text{if } |J|\leqq m-2\\ \partial_1\cdots\partial_m+\sum_{\tilde{J}\in\Pi} a_{\tilde{J}}(t,X,D_x)\partial_{\tilde{J}} & \text{if } |J|=m-1\end{cases} \tag{5.10}$$
$$\text{for}\quad a_{\tilde{J}}(t,x,\xi)\in G^{(\kappa)}([0,T];S^0_{G(\kappa)})\,.$$

Since (5.10) follows immediately for the case when $J=0$ or $j_0<j$ for any $j\in S_J$, we take $J\in\Pi$ such that $j_0>j$ holds for some $j\in S_J$. Note that J can be written as $(j_1, \cdots, j_{\tilde{p}}, \bar{m}_{k-1}+1, \cdots, m)$ for some $\tilde{p}$ and k. Let $p'\in\{1, \cdots, \tilde{p}\}$ be a maximal integer such that $j_{p'}<j_0$ and write $\partial_{j_0}\partial_J$ as

$$(5.11)\qquad \partial_{j_0}\partial_J = \partial_{j_1}\cdots\partial_{j_{p'}}\partial_{j_0}\partial_{j_{p'+1}}\cdots\partial_{j_p}\partial_{\bar{m}_{k-1}+1}\cdots\partial_m + \sum_{q=1}^{p'}\partial_{j_1}\cdots\partial_{j_{q-1}}[\partial_{j_0}, \partial_{j_q}]\partial_{j_{q+1}}\cdots\partial_{j_p}\partial_{\bar{m}_{k-1}+1}\cdots\partial_m .$$

The definition of Π_p implies that for any j_q there exists a $j'\notin S_J$ such that $\bar{m}_{k'-1}+1\leqq j'\leqq\bar{m}_{k'}$, holds with k' satisfying $\bar{m}_{k'-1}+1\leqq j_q\leqq\bar{m}_{k'}$. Hence, by means of the regular hyperbolicity of $L_{k'}$ we have $|\lambda_{j_q}-\lambda_{j'}|\geqq c_o\langle\xi\rangle$ for some $c_o>0$ if $|\xi|$ is large enough. Consequently we have

$$[\partial_{j_0}, \partial_{j_q}] = a_1\partial_{j_q}+a_2\partial_{j'}+a_3 \quad \textit{for}\quad a_j(t, x, \xi)\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)}) .$$

In view of this, each term with commutator $[\partial_{j_0}, \partial_{j_q}]$ in (5.11) can be written as the linear combination of ∂_J, $\partial_{J'}$, ∂_{J_q} and their minor operators $\partial_{J''}$ with coefficients in $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$, where $J_q\in\Pi_{p-1}$ is defined by $S_{J_q}=S_J\backslash\{j_q\}$, J' is the permutation $(j_1, \cdots, j_{q-1}, j', j_{q+1}, \cdots, j_p, \bar{m}_{k-1}+1, \cdots, m)$ and the permutation J'' of the minor operator $\partial_{J''}$ is defined by $S_{J''}=S_J\backslash\tilde{S}$ or $S_{J'}\backslash\tilde{S}$ or $S_{J_q}\backslash\tilde{S}$ for a subset $\tilde{S}$ of $\{j_1, \cdots, j_{q-1}\}$. For the operator $\partial_{J'}$ or its minor operators $\partial_{J''}$, if $J'\notin\Pi$ or $J''\notin\Pi$, we repeat the above discussions until $\partial_{J'}$ and $\partial_{J''}$ are represented as the linear combination of ∂_J with J in Π. Then, we get (5.10) and we can complete the proof of the theorem. Q.E.D.

As another application of Theorem 1 we consider an operator L of the form

$$(5.12)\qquad L = L_1L_2L_3+P_1L_1+P_2L_2+P_3L_3+P_4 .$$

Here, L_j, $j=1, 2, 3$, are regularly hyperbolic differential operators of order m_j $(m_1+m_2+m_3=m)$ and P_1, P_2, P_3 and P_4 are differential operators of order, at most, $m-m_1-1$, $m-m_2-1$, $m-m_3-1$ and $m-1$, respectively, with coefficients in $\gamma^{(\kappa)}([0, T]\times R^n_x)$. If we admit L_j and P_j in (5.12) to be pseudodifferential operators with respect to x, a hyperbolic operator with characteristic roots of the maximal multiplicity at most three always has the form (5.12), provided that its characteristic roots belong to $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$. The assumption of differential operators with respect to x is not necessary for the argument in what follows.

Theorem 3. *Let L be a hyperbolic operator of the form* (5.12). *Then, the problem* (10) *for L can be reduced to the equivalent problem* (7) *for an operator of the form* (1) *with $\mathcal{L}$ satisfying the following*:

i) $\sigma=0$ *if* $\operatorname{Ord} P_1 \leqq m-m_1-2$, $\operatorname{Ord} P_2 \leqq m-m_2-3$
$\operatorname{Ord} P_3 \leqq m-m_3-2$ *and* $\operatorname{Ord} P_4 \leqq m-3$.

i)′ $\sigma_j=0$ *if* $\operatorname{Ord} P_j \leqq m-m_j-2$ $(j=1, 2, 3)$,
$\operatorname{Ord} P_4 \leqq m-3$ *and* $[L_2, L_3]=B_2L_2+B_3L_3+B_4$ *with differential operators* B_2, B_3 *and* B_4 *of order* m_3-1, m_2-1 *and* m_2+m_3-2, *respectively.*

ii) $\sigma=1/3$ *if* $\operatorname{Ord} P_j \leqq m-m_j-2$ $(j=1, 2, 3)$
and $\operatorname{Ord} P_4 \leqq m-2$.

iii) $\sigma=1/2$ *if* $\operatorname{Ord} P_j \leqq m-m_j-1$ $(j=1, 2, 3)$
and $\operatorname{Ord} P_4 \leqq m-2$.

iv) $\sigma=2/3$ *otherwise.*

REMARK. When the operator L is a differential operator whose maximal multiplicity is at most three, it seems that the cases i)–iv) cover all the cases which we can consider as the conditions on lower order terms for any given constant $\sigma<1$. In the above we make a convention: the terms of the forms $A_1L_2L_3$, $A_2L_1L_3$ and $A_3L_1L_2$ are absorbed in $L_1L_2L_3$ by modifying the lower order terms of L_j.

As in the proof of Proposition 5.1 we may assume $L_1=\partial_1\cdots\partial_{m_1}$, $L_2=\partial_{m_1+1}\cdots\partial_{m_1+m_2}$, $L_3=\partial_{m_1+m_2+1}\cdots\partial_m$, where ∂_j $(j=1, \cdots, m)$ are defined by (5.4). Let Π_p^o and Π_p denote the same sets as in the proof of Theorem 2.

Proof of Case i) (cf. [16]). Since $\operatorname{Ord}(P_2L_2)$ and $\operatorname{Ord}[P_1, L_1]$ are smaller than or equal to $m-3$ we can write

$$L = L_1(L_2L_3+P_1)+P_3L_3+P_4\,. \tag{5.13}$$

Let u be a solution of (10) and set $J_0=(m_1+1, \cdots, m)$. For $J\in\Pi$ we set

$$\begin{cases} u_J = \partial_J u & \text{if } S_J \not\supset S_{J_0} \\ u_J = \partial_{j_1}\cdots\partial_{j_k}(L_2L_3+P_1)u & \\ \qquad\qquad \textit{if } J=(j_1, \cdots, j_k, m_1+1, \cdots, m)\in\Pi\,, \end{cases} \tag{5.14}$$

where $\partial_J u=u$ if $J=0$. Using Lemma 5.2 as in the proof of Proposition 5.1 we have from (5.13)

$$\begin{aligned} \partial_1 u_{(2,\cdots,m)} &= -P_3L_3u-P_4u \\ &= \sum_{J\not\supset J_0} a_J\partial_J u = \sum_{J\not\supset J_0} a_J u_J\,, \\ &\textit{with} \quad a_J\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})\,, \end{aligned} \tag{5.15}$$

where we denote $J\supset J'$ if $J, J'\in\Pi$ satisfy $S_J\supset S_{J'}$. Let J be an element of Π for which the set S_J contains just $\tilde{p}_0\equiv m_2+m_3-1$ elements of $\{m_1+1, \cdots, m\}$.

Then, denoting $J=(j_1, \cdots, j_{p-\tilde{p}_0}, j_{p-\tilde{p}_0+1}, \cdots, j_p)$ with $j_1, \cdots, j_{p-\tilde{p}_0}\in\{1, \cdots, m_1\}$ and $j_k\geqq m_1+1$ for $k>p-\tilde{p}_0$, we have from (5.10)

$$\partial_{j_0}\partial_J = \partial_{j_1}\cdots\partial_{j_{p-\tilde{p}_0}}L_2L_3+\sum_{\tilde{J}\in\Pi} a_{\tilde{J}}\partial_{\tilde{J}} \\ = \partial_{j_1}\cdots\partial_{j_{p-\tilde{p}_0}}(L_2L_3+P_1)+\sum_{\tilde{J}\in\Pi} a'_{\tilde{J}}\partial_{\tilde{J}} \tag{5.16}$$

with $a_{\tilde{J}}$ and $a'_{\tilde{J}}\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$. Here we used the fact that the order of $\partial_{j_1}\cdots\partial_{j_{p-\tilde{p}_0}}P_1$ is smaller than or equal to $m-3$. Hence, we can reduce the problem (10) to (7) by (5.15), (5.10) and (5.16). This concludes the proof of Case i).

Proof of Case i)′. We add to the set Π the set $\Pi'=\{J=(j_1, \cdots, j_{p'}, m_1+1, \cdots, \bar{m}_2); j_1<j_2<\cdots<j_{p'}, j_1, \cdots, j_{p'}\in\{1, \cdots, m_1, \bar{m}_2+1, \cdots, m\}, \{1, \cdots, m_1\}\not\subset S_J, \{\bar{m}_2+1, \cdots, m\}\not\subset S_J\}$ and we define u_J as in (5.14). Then we have

$$\partial_1 u_{(2,\cdots,m)} = -P_2L_2u-P_3L_3u-P_4u \\ = \sum_{J\not\ni J_0} a_J u_J \\ \textit{for}\quad a_J\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)}), \tag{5.15$'$}$$

where J belong to $\Pi\cup\Pi'$. Consequently, in view of the proof of Case i) it suffices to derive equations for u_J with $J\in\Pi'$. Let $J=(j_1, \cdots, j_k, m_1+1, \cdots, \bar{m}_2)\in\Pi'$ and $j_0=\max\{j; 1\leqq j\leqq m, j\notin S_J\}$. For the case where the number of elements in $\{j_1, \cdots, j_k\}\cap\{\bar{m}_2+1, \cdots, m\}$ is smaller than m_3-1, from the same discussion to prove (5.10) we have

$$\partial_{j_0}\partial_J = \sum_{\tilde{J}\in\Pi'} a_{\tilde{J}}\partial_{\tilde{J}}u+aL_2u \\ \textit{for}\quad a_{\tilde{J}}, a\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)}). \tag{5.17}$$

So, we assume the number of elements in $\{j_1, \cdots, j_k\}\cap\{\bar{m}_2+1, \cdots, m\}$ is equal to m_3-1. Then, we have from $[L_2, L_3]=B_2L_2+B_3L_3+B_4$

$$\partial_{j_0}\partial_J u = \partial_{J_1}L_3L_2u+\sum_{\tilde{J}\in\Pi'} a_{\tilde{J}}\partial_{\tilde{J}}u+aL_2u \\ = \partial_{J_1}(L_2L_3+P_1)u+\partial_{J_1}B_2L_2u+\partial_{J_1}B_3L_3u \\ +\partial_{J_1}(B_4-P_1)u+\sum_{\tilde{J}\in\Pi'} a_{\tilde{J}}\partial_{\tilde{J}}u+aL_2u \\ = \partial_{J_1}(L_2L_3+P_1)u+\sum_{\tilde{J}\in\Pi\cup\Pi'} a'_{\tilde{J}}\partial_{\tilde{J}}u, \tag{5.18}$$

where $J_1\in\Pi_{k+1-m_3}$ with $S_{J_1}=\{j_1, \cdots, j_k\}\setminus\{\bar{m}_2+1, \cdots, m\}$ and $a_{\tilde{J}}, a, a'_{\tilde{J}}\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$. Here we used Lemma 5.2 to represent $\partial_{J_1}B_2L_2+\partial_{J_1}B_3L_3+\partial_{J_1}(B_4-P_1)$ as a linear combination of $\partial_{\tilde{J}}$ ($\tilde{J}\in\Pi\cup\Pi'$) with coefficients in $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$. Combining (5.15)′, (5.17), (5.18) and the results in the proof of Case i), we obtain the systemization for Case i)' and we can concludes the proof of this case.

REMARK. The condition that for any $j\in\{m_1+1, \cdots, \bar{m}_2\}$ and $j'\in\{\bar{m}_2+1, \cdots, m\}$ the equation

$$[\partial_j, \partial_{j'}] = a_{jj'}\partial_j+b_{jj'}\partial_{j'}+c_{jj'}$$

holds with symbols $a_{jj'}$, $b_{jj'}$ and $c_{jj'}$ in $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$ implies that $[L_2, L_3]=B_2L_2+B_3L_3+B_4$ in the condition for the case i)' if we admit B_j to be pseudo-differential operators with respect to x.

Proofs of Case ii) and iv). Proofs are the direct consequence of Theorem 2. Indeed, in the case ii) (resp. iv)) the operator L can be written as $L_1L_2L_3+P_4$, $P_4=\sum_{j=0}^{m-q} a_jD^j_{t'}$, $a_j\in G^{(\kappa)}([0, T]; S^{m-q-j}_{G(\kappa)})$, with $q=2$ (resp. $q=1$).

Proof of Case iii). Let $\{j_1, \cdots, j_p\}$ be a subset of $\{1, \cdots, m\}$. By induction on p it is easy to see

$$(5.19)\qquad \begin{aligned}&\partial_{j_1}\cdots\partial_{j_{q-1}}\partial_{j_q}\partial_{j_{q+1}}\cdots\partial_{j_q}\\ &\quad=\partial_{j_1}\cdots\partial_{j_{q-1}}\partial_{j_{q+1}}\partial_{j_q}\partial_{j_{q+2}}\cdots\partial_{j_p}+\sum_{\substack{J\in\Pi_{p'}\\ p'\leqq p-2}}\Lambda^{[(p-p')/2]}a_J\partial_J\\ &\qquad\qquad \textit{with}\quad a_J\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})\end{aligned}$$

because $[\partial_j, \partial_k]$, $[\partial_{j'}, [\partial_j, \partial_k]]$, $\cdots$ belong to $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$. Set $\tilde{\Pi}=\bigcup_{p=0}^{m-1}\Pi^o_p$ ($\Pi^o_0=\{0\}$). We shall prove

$$(5.20)\qquad \begin{aligned}L&=\partial_1\partial_2\cdots\partial_m+\sum_{J\in\Pi_{m-1}}a_J\partial_J+\sum_{\substack{J\in\Pi_p\\ p\leqq m-2}}\Lambda^{[(m-p)/2]}a_J\partial_J\\ &\qquad \textit{with}\quad a_J\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})\,.\end{aligned}$$

As in the beginning of the proof of Proposition 5.1 we may assume that the order of P_4 (resp. P_j, $j=1, 2, 3$) with respect to D_t is smaller than or equal to $m-3$ (resp. $m-m_j-2$) by adding the second term of the right hand side of (5.20). Next, we apply Lemma 5.2 for $\Lambda^{-1}P_j$ ($j=1, 2, 3, 4$) and use (5.20) for the terms of the form $\partial_{j_1}\cdots\partial_{j_{\tilde{p}}}L_k$ with $k=1$, 2 and $\tilde{p}\leqq m-m_k-2$. Then, $P_1L_1+P_2L_2+P_3L_3+P_4$ can be written as the linear combination of $\Lambda^{(m-p)\sigma}\partial_J$ ($J\in\Pi_p$, $p\leqq m-2$) with coefficients in $G^{(\kappa)}([0, T]; S^0_{G(\kappa)})$. Let $J\in\Pi^o_p$ ($0\leqq p\leqq m-1$) and set $j_0=\max\{j; 1\leqq j\leqq m, j\notin S_J\}$. By means of (5.19) we have

$$(5.21)\qquad \begin{aligned}\partial_{j_0}\Lambda^{(m-1-p)\sigma}\partial_J&=\Lambda^\sigma(\Lambda^{(m-1-(p+1))\sigma}\partial_{J'})+\sum_{\tilde{J}\in\Pi_{p'}}\Lambda^\sigma a_{\tilde{J}}\Lambda^{(m-1-p')\sigma}\partial_{\tilde{J}}\\ &\qquad \textit{with}\quad a_{\tilde{J}}\in G^{(\kappa)}([0, T]; S^0_{G(\kappa)})\,.\end{aligned}$$

Here $J'\in\Pi_{p+1}$ satisfies $S_{J'}=S_J\cup\{j_0\}$. The conjunction of (5.20) and (5.21) shows that the vector $U=(u_J)_{J\in\tilde{\Pi}}$ defined by (5.7) with Π_p replaced by Π_p^0 satisfies $\mathcal{L}U=0$ for a system $\mathcal{L}$ of the form (1). This proves the reduction of (10) to (7) for Case iii). Q.E.D.

As shown in Theorem 3 it seems to be very difficult to find the conditions on lower order terms of a hyperbolic operator which the problem (10) is reduced to an equivalent problem (7) of a hyperbolic system (1) with a given σ (<1).

6. Equivalence of two estimates

In this section we assume that characteristic roots $\lambda_j(t, x, \xi)$ of $\mathcal{L}$ belong to $\mathcal{B}^\infty([0, T]; S^1)$ instead of $G^{(\kappa)}([0, T]; S^1_{G(\kappa)})$ and are homogeneous for $|\xi|\geqq 1$. Set

$$(6.1)\qquad \begin{cases} p(\tilde{x}, \tilde{\xi}) = \sum_{j=1}^{l} p_j(\tilde{x}, \tilde{\xi}), \\ p_j(\tilde{x}, \tilde{\xi}) = \tau-\lambda_j(t, \tilde{x}, \tilde{\xi}), \qquad j = 1, \cdots, l, \end{cases}$$

where $\tilde{x}=(t, x)$ and $\tilde{\xi}=(\tau, \xi)$. In what follows we write $z=(\tilde{x}, \tilde{\xi})\in T^*(R^{n+1})\simeq R^{2n+2}$ and $\delta z=(\delta\tilde{x}, \delta\tilde{\xi})\in T_z(T^*(R^{n+1}))\simeq R^{2n+2}$.

For the case where $p(z)$ has the form (6.1), we shall define the "flows" $K_z^+(z\in T^*(R^{n+1})\backslash 0$ following [22] and [21]: We first define the localization $p_z(\delta z)$ at $z\in T^*(R^{n+1})\backslash 0$ by

$$p(z+s\delta z) = s^\mu(p_z(\delta z)+o(1)) \qquad as \quad s\to 0,$$

where $p_z(\delta z)\not\equiv 0$ (in δz) is a homogeneous polynomial of $\delta z\in T_z(T^*(R^{n+1}))$. Since p has the form (6.1) the localization $p_z(\delta z)$ is simply given by

$$(6.2)\qquad p_z(\delta) = (\prod_{j\notin\Sigma_z} p_j(z)) \prod_{j\in\Sigma_z} (\nabla_{\tilde{x}} p_j(z)\cdot\delta\tilde{x}+\nabla_{\tilde{\xi}} p_j(z)\cdot\delta\tilde{\xi}),$$

where Σ_z is a maximal subset of $\{1, \cdots, l\}$ satisfying $z\in \bigcap_{j\in\Sigma_z} p_j^{-1}(0)$. Here $p_j^{-1}(0)=\{z\in T^*(R^{n+1})\backslash 0; p_j(z)=0\}$. Let Γ_z denote the connected component of $\{\delta z\in T_z(T^*(R^{n+1})); p_z(\delta z)\neq 0\}$ which contains $(0; 1, 0, \cdots, 0)$. Then it follows from (6.2) that

$$(6.3)\qquad \Gamma_z = \bigcap_{j\in\Sigma_z} \{\delta z; \sigma(H_{p_j}(z), \delta z)>0\},$$

where $\sigma(\delta z', \delta z)=\delta\tilde{x}'\cdot\delta\tilde{\xi}-\delta\tilde{\xi}'\cdot\delta\tilde{x}$ and $H_{p_j}(z)$ denotes $(\nabla_{\tilde{\xi}} p_j(z), -\nabla_{\tilde{x}} p_j(z))$. Set

$$\Gamma_z^\sigma = \{\delta z; \sigma(\delta z, \delta z')\geqq 0 \qquad for\ any \quad \delta z'\in\Gamma_z\}.$$

Then by means of (6.3) we have

$$(6.4)\qquad \Gamma_z^\sigma = \{\sum_{j\in\Sigma_z} \alpha_j H_{p_j}(z); \alpha_j\geqq 0\}.$$

Now, we define K_z^+ as

$$(6.5)\qquad K_z^+ = \{z(s)\in T^*(R^{n+1});\ \{z(t)\} \textit{ is Lipschitz continuous curve satisfying } (d/ds)z(s)\in\Gamma^\sigma_{z(s)} \text{ (a.e. } s) \textit{ and } z(0)=z,\ s\geqq 0\}$$

for $z\in T^*(R^{n+1})\backslash 0$.

Theorem 4. *Let V be a closed conic set in $T^*(R^n)\backslash 0$ and let $t_0\in(0, T]$. Then we have*

$$(6.6)\qquad \Gamma(t_0;\ V) = \{\pi(K_z^+\cap\{t=t_0\});\ z\in\pi^{-1}(V)\cap\{t=0\}\cap p^{-1}(0)\}\,,$$

where π is the natural projection from $T^*(R^{n+1}_{\tilde{x}})$ to $T^*(R^n_x)$ and $p^{-1}(0)=\bigcup_{j=1}^{l} p_j^{-1}(0)$.

An inclusion relation $\Gamma(t_0;\ V)\supset\{\cdot\}$ was proved in Theorem 4.4 of [22]. We remark that the assumption (L.2) of [21] is verified because $\lambda_j(t, x, \xi)\in\mathscr{B}^\infty([0, T];\ S^1)$. So, in what follows we shall show another inclusion relation. Suppose that $\delta_0\in\Gamma(t_0;\ V)$. Then for any $\varepsilon>0$ there exists an ε-admissible trajectory linking δ_0 and $\rho_\varepsilon\in V$. Taking a subsequence of $\{\rho_\varepsilon\}_{\varepsilon>0}$, if necessary, we may assume that ρ_ε converges to a point $\rho_0\in V$ because V is closed. It follows from Proposition 4.1 that for any $\varepsilon>0$ there exists an ε-admissible trajectory issuing from ρ_0 whose end point δ_ε converges to δ_0. From the ε-admissible trajectory $\{(t, x(t), \xi(t));\ t\in[0, t_0]\}\subset R_t\times T^*(R^n_x)$ we make a lift $\tilde{C}_\varepsilon\equiv\{(t, x(t), \tau(t), \xi(t));\ t\in[0, t_0]\}\subset T^*(R^{n+1}_x)$ by setting $\tau(t)=\lambda_j(t, x(t), \xi(t))$ if $\{(t, x(t), \xi(t))\}$ is the bicharacteristic curve with respect to λ_j. It is clear that $\tilde{C}_\varepsilon\subset p^{-1}(0)$. Taking a subsequence $\{\tilde{C}_\varepsilon\}_{\varepsilon>0}$, if necessary, we may assume that the initial point of $\tilde{C}_\varepsilon$ for any $\varepsilon>0$ equals a point $z^0\in T^*(R^{n+1})\backslash 0$ with $\pi(z^0)=\rho_0$. Similarly we may assume that the end point z_ε of $\tilde{C}_\varepsilon$ converges to a point $z_0\in T^*(R^{n+1})\backslash 0$ with $\pi(z_0)=\delta_0$. Summing up, for the proof of (6.6) it suffices to show

$$(6.7)\qquad z_0\in K^+_{z^0}.$$

In order to show this we need to define the set $K^+_{z^0,t_0}(h)$ for $h>0$ which approximates $K^+_{z^0}$, following [22] and [21]. Let K be a compact neighborhood of z^0 in $T^*(R^{n+1})$. We assume that K is large enough to contain all lifts of ε-admissible trajectories for $\varepsilon<1$ from z^0. For $h>0$ and $z\in K$, there exists a compact set $M(z, h)$ in Γ_z such that $(0;\ 1, 0, \cdots, 0)\in\mathring{M}(z, h)$ and

$$\Gamma^\sigma_z\subset M(z, h)^\sigma\subset(\Gamma^\sigma_z)_h\,,$$

where $\mathring{M}$ denotes the interior of M and $(\Gamma)_h$ is defined by

$$(\Gamma)_h\equiv\{\delta z;\ \delta z=0 \text{ or } ||\delta z|^{-1}\delta z-|\delta z'|^{-1}\delta z'|<h \textit{ for some } \delta z'\in\Gamma\}\,.$$

Here we take $M(z, h)=\{0\}$ if $\Gamma^\sigma_z=\{0\}$, that is, if $z\notin p^{-1}(0)$. By Theorem 2.3

of [22], for each $h>0$ and $z\in K$ there exists $r(z, h)>0$ such that $r(z, h)<h$ and

$$M(z, h)\subset\Gamma_{z^1} \quad \text{for} \quad z^1\in U(z, h)\equiv\{z^1;\ |z^1-z|<r(z, h)\}. \tag{6.8}$$

(In our special case, this fact follows easily from (6.3)). Since K is compact, there exists a finite number of $z^{h,j}\in K$ $(1\leqq j\leqq N(h))$ such that $K\subset\bigcup_{j=1}^{N(h)} U'(z^{h,j}, h)$, where $U'(z, h)=\{z';\ |z'-z|\leqq r(z, h)/2\}$. We remark that $z\notin p^{-1}(0)$ for any $z\in U'(z^{h,j}, h)$ if $z^{h,j}\notin p^{-1}(0)$. In fact, if $z^{h,j}\notin p^{-1}(0)$ then we have $M(z^{h,j}, h)^\sigma=\{0\}$ and hence, it follows from (6.8) that for $z\in U'(z^{h,j}, h)$ we have $\Gamma^\sigma_z=\{0\}$, that is, $z\notin p^{-1}(0)$. Now, we define $K^+_{z^0,t_0}(h)$ as follows: A point $z\in K\cap\{0\leqq t\leqq t_0\}$ belongs to $K^+_{z^0,t_0}(h)$ if there exist $j_0, \cdots, j_\nu$ and $\tilde{z}^1, \cdots, \tilde{z}^{\nu-1}$ such that $\tilde{z}^k\in U'(z^{h,j_k}, h)$ $(0\leqq k\leqq\nu)$ and

$$\tilde{z}^{k+1}-\tilde{z}^k\in M(z^{h,j_k}, h)^\sigma\cap\{\delta z;\ |\delta z|<\rho(h)\} \qquad (0\leqq k\leqq\nu-1), \tag{6.9}$$

where $\tilde{z}^0=z^0$, $\tilde{z}^\nu=z$ and $\rho(h)=\min_{1\leqq j\leqq N(h)} r(z^{h,j}, h)/2$. We remark that $K^+_{z^0,t_0}(h)$ is well-defined because the assumption (L–2) of [22] is valid (see pp. 1160 in [22]).

Proposition 6.1 (see Theorem 2.4 of [22] and Theorem 3.3 of [21]). *It follows that*

$$\bigcap_{h>0}\overline{K^+_{z^0,t_0}(h)} = K^+_{z^0}\cap\{0\leqq t\leqq t_0\}, \tag{6.10}$$

where $\bar{K}$ denotes the closure of K.

By means of this proposition, for the proof of (6.7) it suffices to show

$$\begin{cases} \text{for auy } h>0 \text{ there exists an } \varepsilon_0>0 \text{ such that the end} \\ \text{poiut } z_\varepsilon \text{ of the lift } \tilde{C}_\varepsilon \text{ belongs to } K^+_{z^0,t_0}(h) \quad \text{if} \quad \varepsilon\leqq\varepsilon_0. \end{cases} \tag{6.11}$$

Lemma 6.2. *Let $\{z(t)=(t, x(t), \tau(t), \xi(t));\ t\in[0, t_0]\}$ be a lift of an ε-admissible trajectory. Then for any two continuous points $z(s_1)$ and $z(s_2)$ on the lift we have*

$$|z(s_1)-z(s_2)|\leqq C|s_1-s_2|+(l-1)\varepsilon, \tag{6.12}$$

where C it a positive constant and l is the size of the system $\mathcal{L}$.

Proof. Let π_0 denote the natural projection from $T^*(R^{n+1})$ to $R_t\times T^*(R^n_x)$. It is clear that

$$|\pi_0(z(s_1))-\pi_0(z(s_2))|\leqq C|s_1-s_2|. \tag{6.13}$$

Assume $\tau(s_k)=\lambda_{j_k}(\pi_0(z(s_k)))$, $k=1, 2$. If $j_1=j_2$ then (6.12) follows from the continuity of λ_j. Assume that $j_1\neq j_2$. For simplicity we consider the case for $l=2$. By taking a discontinuous point $z(t')$ between $z(s_1)$ and $z(s_2)$ we estimate

$$\begin{aligned}&|\tau(s_2)-\tau(s_1)|\\&\leqq|\tilde{\lambda}_{j_1}(s_1)-\tilde{\lambda}_{j_1}(t')|+|\tilde{\lambda}_{j_1}(t')-\tilde{\lambda}_{j_2}(t')|+|\tilde{\lambda}_{j_2}(t')-\tilde{\lambda}_{j_2}(s_2)|\\&\leqq C(|s_1-t'|+|t'-s_2|)+\varepsilon\end{aligned}$$

by using (6.13), where $\tilde{\lambda}_j(s)=\lambda_j(\pi_0(z(s)))$. In the general case for $l\geqq 3$ we can also estimate the difference between $\tau(s_1)$ and $\tau(s_2)$ by taking $l-1$ discontinuous points, at most, between $z(s_1)$ and $z(s_2)$. Then, we get (6.12). Q.E.D.

Let π_1 denote the natural projection from $T^*(R^{n+1})$ to R_t. Set

$$\mathcal{H}_z=\{\sum_{j=1}^{l}\alpha_j H_{p_j}(z);\ \sum_{j}^{l}\alpha_j=1,\ \alpha_j\geqq 0\}\ \text{(cf. (6.4))}.$$

Since the Hamilton field $H_{p_j}(z)$ depends only on $\pi_0(z)$ it follows that if z^k $(k=1, 2)$ are two points of the lift $\tilde{C}_\varepsilon$ of an ε-admissible trajectory then for any $v_{z^1}\in\mathcal{H}_{z^1}$ there exists a $v_{z^2}\in\mathcal{H}_{z^2}$ such that

$$|v_{z^2}-v_{z^1}|\leqq C|\pi_1(z^2-z^1)|, \tag{6.14}$$

where C is a constant independent of the lifts C_ε of ε-admissible trajectories.

Lemma 6.3. *For any $h>0$ there is an $\varepsilon(h, l)>0$ satisfying the following property: Assume that z^k $(k=1, 2)$ are two continuous points on a lift $\tilde{C}_\varepsilon$ of an ε-admissible trajectory such that $\varepsilon\leqq\varepsilon(h, l)$ and*

$$h\leqq\pi_1(z^2-z^1)\leqq 2h.$$

Then there exists a $v_{z^1}\in\mathcal{H}_{z^1}$ such that

$$|(z^2-z^1)/\pi_1(z^2-z^1)-v_{z^1}|\leqq Ch, \tag{6.15}$$

where C is a constant independent of h and $\tilde{C}_\varepsilon$.

REMARK. Let Σ be a subset of $\{1, \cdots, l\}$ such that the part of $\tilde{C}_\varepsilon$ between z^1 and z^2 is composed of bicharacteristic curves with respect to λ_j, $j\in\Sigma$. Then we can replace $\mathcal{H}_{z^1}$ in the lemma by $\mathcal{H}'_{z^1}\equiv\{\sum_{j\in\Sigma}\alpha_j H_{p_j}(z^1);\ \alpha_j\geqq 0,\ \sum_{j\in\Sigma}\alpha_j=1\}$.

Proof. We shall prove the lemma by the inductive method on l. The case for $l=1$ is trivial. So, we assume $l\geqq 2$ and suppose that the conclusion holds until $l-1$. Take continuous points $\tilde{z}^1, \cdots, \tilde{z}^{\nu-1}$ on $\tilde{C}_\varepsilon$ between z^1 and z^2 such that

$$h^3\leqq\pi_1(\tilde{z}^{k+1}-\tilde{z}^k)\leqq 2h^3,\qquad k=1, \cdots, \nu-1$$
$$(\tilde{z}^1=z^1\ \textit{and}\ \tilde{z}^\nu=z^2).$$

Setting $h_o=\pi_1(z^2-z^1)$ and $h_k=\pi_1(\tilde{z}^{k+1}-\tilde{z}^k)$ $(k=1, \cdots, \nu-1)$ we write

$$(z^2-z^1)/h_o=\sum_{k=1}^{\nu}(h_k/h_o)(\tilde{z}^{k+1}-\tilde{z}^k)/h_k. \tag{6.16}$$

Suppose that, for any k, the part of $\tilde{C}_\varepsilon$ between $\tilde{z}^k$ and $\tilde{z}^{k+1}$ is composed of bicharacteristic curves with respect to, at most, $l-1$ elements of $\{\lambda_j\}$. Then it follows from the hypothesis of the induction that if $\varepsilon \leqq \varepsilon(h^3, l-1)$ we have

$$(\tilde{z}^{k+1}-\tilde{z}^k)/h_k - v_{\tilde{z}^k} = O(h^3) \qquad \text{for} \quad v_{\tilde{z}^k} \in \mathcal{H}_{\tilde{z}^k}\,.$$

Here $v=O(h^3)$ means $|v| \leqq Ch^3$ with a constant C independent of h and the choice of the lift $\tilde{C}_\varepsilon$ of ε-admissible trajectory. By using (6.14) we have for some $v^k_{z^1} \in \mathcal{H}_{z^1}$

$$(\tilde{z}^{k+1}-\tilde{z}^k)/h_k - v^k_{z^1} = O(h)\,.$$

Consequently, using $\sum h_k/h_o=1$ we obtain (6.15) in this case.

Consider the case that the part of $\tilde{C}_\varepsilon$ between $\tilde{z}^k$ and $\tilde{z}^{k+1}$ for some k is composed of bicharacteristic curves with respect to full elements of $\{\lambda_j\}_{j=1}^l$. We denote by k_1 the minimum of such k and by k_2 the integer k_2 such that k_2-1 is the maximum of the k's stated above. Now, we write (6.16) as

$$(6.16)' \qquad (z^2-z^1)/h_o = \sum_{k=1}^{k_1} (h_k/h_o)(\tilde{z}^{k+1}-\tilde{z}^k)/h_k + (\tilde{z}^{k_2}-\tilde{z}^{k_1})/h_o + \sum_{k=k_2}^{\nu} (h_k/h_o)(\tilde{z}^{k+1}-\tilde{z}^k)/h_k\,.$$

Suppose that $\pi_1(\tilde{z}^{k_2}-\tilde{z}^{k_1}) \leqq h^2$. Then it follows from Lemma 6.2 that we have $|\tilde{z}^{k_2}-\tilde{z}^{k_1}|=O(h^2)$ if $\varepsilon \leqq \min(\varepsilon(h^3, l-1), h^2)$. Hence, the second term of the right hand side of (6.16)$'$ is estimated by the constant times of h. So, we get (6.15) by using the discussions of the preceding paragraph and $\sum_{k=k_1+1}^{k_2-1} (h_k/h_o)v = O(h)$ for any $v \in \mathcal{H}_{z^1}$.

Assume that $\pi_1(\tilde{z}^{k_2}-\tilde{z}^{k_1}) \geqq h^2$ and let $\varepsilon \leqq \min(\varepsilon(h^3, l-1), h^3)$. To complete the proof it suffices to show

$$(6.17) \qquad (\tilde{z}^{k_2}-\tilde{z}^{k_1})/\tilde{h} - v_{\tilde{z}^{k_1}} = O(h) \qquad \text{for some} \quad v_{\tilde{z}^{k_1}} \in \mathcal{H}_{\tilde{z}^{k_1}}\,,$$

where $\tilde{h}=\pi_1(\tilde{z}^{k_2}-\tilde{z}^{k_1})$. Since the bicharacteristic curves with respect to full elements of $\{\lambda_j\}_{j=1}^l$ appear on the part of $\tilde{C}_\varepsilon$ between $\tilde{z}^{k_1}$ and $\tilde{z}^{k_1+1}$ (also between $\tilde{z}^{k_2-1}$ and $\tilde{z}^{k_2}$), it follows from the continuity of λ_j that we have

$$(6.18) \qquad \lambda_j(\pi_0(\tilde{z}^{k_i})) - \lambda_{j'}(\pi_0(\tilde{z}^{k_i})) = O(h^3)\,, \qquad i=1, 2\,, \quad \text{for any} \quad j, j' \in \{1, \cdots, l\}\,.$$

In order to simplify the notation below we denote $\tilde{z}^{k_1}$ and $\tilde{z}^{k_2}$ by $\tilde{z}^1$ and $\tilde{z}^2$, respectively, in what follows. Since $\pi_0\tilde{C}_\varepsilon$ is continuous in $R_t \times T^*(R^n_x)$ we have

$$(6.19) \qquad \pi_0(\tilde{z}^2-\tilde{z}^1)/\tilde{h} - \sum_{j=1}^{l} \alpha_j \pi_0(H_{p_j}(\tilde{z}^1)) = O(\tilde{h})$$

for some $\alpha_j \geqq 0$ with $\sum \alpha_j = 1$. Let π'_1 be the natural projection from $T^*(R^{n+1})$ to R_τ, where τ is the dual variable of t. We shall show

$$\pi'_1(\tilde{z}^2 - \tilde{z}^1)/\tilde{h} - \sum_{j=1}^{l} \alpha_j \pi'_1(H_{p_j}(\tilde{z}^1)) = O(h) . \tag{6.20}$$

In view of $\tilde{h} \geqq h^2$ it follows from (6.18) that

$$\pi'_1(\tilde{z}^2 - \tilde{z}^1)/\tilde{h} = \sum_{j=1}^{l} \alpha_j (\lambda_j(\pi_0(\tilde{z}^2)) - \lambda_j(\pi_0(\tilde{z}^1))/\tilde{h} + O(h) . \tag{6.21}$$

Set $s_k = \pi_1(\tilde{z}^k)$ $(k=1, 2)$ and denote the $\mathcal{E}$-admissible trajectory $\pi_0(\tilde{C}_\varepsilon)$ by $\{(t, x(t), \xi(t))\}$. Then we have

$$\begin{aligned}
&\sum_{j=1}^{l} \alpha_j (\lambda_j(\pi_0(\tilde{z}^2)) - \lambda_j(\pi_0(\tilde{z}^1)))/\tilde{h} \\
&= \sum_{j=1}^{l} (\alpha_j/\tilde{h}) \int_{s_1}^{s_2} \partial_t \lambda_j(t, x(t), \xi(t)) dt \\
&\quad + (1/\tilde{h}) \int_{s_1}^{s_2} \sum_{j=1}^{l} \alpha_j (\nabla_x \lambda_j(t, x(t), (t)) dx/dt \\
&\quad + \nabla_\xi \lambda_j(t, x(t), \xi(t)) d\xi/dt) dt .
\end{aligned} \tag{6.22}$$

The second term of the right hand side is equal to

$$(\sum_{j=1}^{l} \alpha_j \nabla_x \lambda_j(\tilde{z}^1), \sum_{j=1}^{l} \alpha_j \nabla_\xi \lambda_j(\tilde{z}^1)) \cdot \left(\frac{1}{\tilde{h}} \int_{s_1}^{s_2} (dx(t), d\xi(t)) \right) + O(h) .$$

It follows from (6.19) that

$$\begin{aligned}
\frac{1}{\tilde{h}} \int_{s_1}^{s_2} (dx(t), d\xi(t)) &= \frac{1}{\tilde{h}} (\pi(\tilde{z}^2) - \pi(\tilde{z}^1)) \\
&= \sum_{j=1}^{l} \alpha_j \pi(H_{p_j}(\tilde{z}^1)) + O(h) .
\end{aligned}$$

Hence, we can estimate the second term of (6.22) by

$$\{\sum_{j=1}^{l} \alpha_j \lambda_j(\tilde{z}^1) + O(h), \sum_{j=1}^{l} \alpha_j \lambda_j(\tilde{z}^1) + O(h)\} = O(h) ,$$

where $\{\ ,\ \}$ denotes the Poisson bracket in $T^*(R^n_x)$. Note that the first term of the right hand side of (6.22) equals $\sum \alpha_j \pi'_1(H_{p_j}(\tilde{z}^1)) + O(h)$. Hence we get (6.20) from (6.21). This concludes the proof of (6.17), and hence, the proof of (6.15). Q.E.D.

Proof of (6.11). Let h be a fixed positive number. As in the proof of Lemma 3.2 of [21], for any $j \in \{1, \cdots, N(h)\}$ we can find an $h(j) > 0$ such that

$$\begin{aligned}
&(\Gamma^\sigma_z)_{h(j)} \subset M(z^{h,j}, h)^\sigma \\
&\quad \textit{for} \quad z \in U''(z^{h,j}, h) \equiv \{z;\ |z - z^{h,j}| < 2r(z^{h,j}, h)/3\} ,
\end{aligned} \tag{6.23}$$

if $z^{h,j}\in p^{-1}(0)$ (see (3.4) of [21]). Set $h'=\min\{\rho(h), h(1), \cdots, h(N(h))\}$ with a convention $h(j)=\infty$ if $z^{h,j}\notin p^{-1}(0)$. Let h'' be another positive number sufficiently smaller than h', which is determined later on. For the moment we take h'' as the number for which we can find a constant $\varepsilon(h'')>0$ satisfying the following;

$$(6.24)\qquad \begin{cases} \textit{if two points } z^k\ (k=1, 2) \textit{ on the lift } \tilde{C}_\varepsilon \textit{ with } \varepsilon\leqq\varepsilon(h'') \\ \textit{satisfy } \pi_1(z^1-z^2)\leqq 2h'' \textit{ then } |z^1-z^2|<\rho(h)/3 \textit{ holds} \\ (\textit{and hence } z^1\in U'(z^{h,j}, h) \textit{ implies } z^2\in U''(z^{h,j}, h)) . \end{cases}$$

Here we used Lemma 6.2.

We shall apply Lemma 6.3 by setting $h=h''$. Assume that $\varepsilon\leqq\min(\varepsilon(h'', l), \varepsilon(h''))$. Take continuous points $\tilde{z}^1, \cdots, \tilde{z}^{\nu-1}$ on $\tilde{C}_\varepsilon$ linking z^0 and z_ε such that

$$h''\leqq\pi_1(\tilde{z}^{k+1}-\tilde{z}^k)\leqq 2h'' , \qquad k=0, \cdots, \nu-1$$
$$(\tilde{z}^0=z^0,\ \tilde{z}^\nu=z_\varepsilon) .$$

Then it follows from Lemma 6.3 and its remark that for any $k\in\{0, \cdots, \nu-1\}$ there exist $\{\alpha_j\}$ with $\alpha_j\geqq 0$ and $\sum\alpha_j=1$ such that

$$v\equiv(\tilde{z}^{k+1}-\tilde{z}^k)/\pi_1(\tilde{z}^{k+1}-\tilde{z}^k)-\sum_{j\in\Sigma_k}\alpha_j H_{p_j}(\tilde{z}^k) = O(h'') .$$

Here Σ_k is a subset of $\{1, \cdots, l\}$ satisfying the following: The part of the lift $\tilde{C}_\varepsilon$ between $\tilde{z}^k$ and $\tilde{z}^{k+1}$ is composed of bicharacteristic curves with respect to λ_j for $j\in\Sigma_k$. By means of (6.13), for any $j\in\Sigma_k$ there exists a point z^j on $\tilde{C}_\varepsilon$ between $\tilde{z}^k$ and $\tilde{z}^{k+1}$ such that $z^j\in p_j^{-1}(0)$ and

$$v_j\equiv H_{p_j}(\tilde{z}^k)-H_{p_j}(z^j) = O(h'') .$$

Note that $H_{p_j}(z^j)\in\Gamma^\sigma_{z^j}$. It follows from (6.24) that $z^j\in U''(z^{h,j_k}, h)$ holds if $z^k\in U'(z^{h,j_k}, h)$. Hence, using (6.24) and the convexity of $M(z^{h,j_k}, h)^\sigma$ we have

$$(6.25)\qquad |\tilde{z}^{k+1}-\tilde{z}^k|<\rho(h) ,$$

$$(6.26)\qquad (\tilde{z}^{k+1}-\tilde{z}^k)/|\tilde{z}^{k+1}-\tilde{z}^k|$$
$$=\sum_{j\in\Sigma_k}\alpha_j(\pi_1(\tilde{z}^{k+1}-\tilde{z}^k)/|\tilde{z}^{k+1}-\tilde{z}^k|)(H_{p_j}(z^j)+v_j+v)\in M(z^{h,j_k}, h)^\sigma$$

because $H_{p_j}(z^j)+v_j+v\in(\Gamma^\sigma_{z^j})_{h'}\subset M(z^{h,j_k}, h)^\sigma$ if h'' is sufficiently smaller than h'. Then, (6.25) and (6.26) show that the end point z_ε of $\tilde{C}_\varepsilon$ belongs to $K^+_{z^0,t_0}(h)$ if $\varepsilon\leqq\min(\varepsilon(h'', l), \varepsilon(h''))$, that is, we have proved the property (6.11). Q.E.D.

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