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ON QUASI-SUPPORTS OF SMOOTH MEASURES AND CLOSABILITY OF PRE-DIRICHLET FORMS

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1. Introduction

Consider a singular perturbation of the Laplacian Δ on the d-dimensional Euclidean space R^d:

$$\hat{L} = -\Delta + L_\Gamma .$$

Here L_Γ is a linear operator "living on" a closed subset $\Gamma \subset R^d$ which might be of zero Lebesgue measure and as irregular as a fractal set. The problem is how and when we can give $\hat{L}$ a proper sense. One way to formulate this is to introduce a perturbed bilinear form

$$\hat{\mathcal{E}}(f, g) = \boldsymbol{D}(f, g) + \mathcal{E}_\Gamma(f|_\Gamma, g|_\Gamma), \qquad f, g \in C_0^\infty(R^d),$$

where $\boldsymbol{D}$ is the Dirichlet integral and $\mathcal{E}_\Gamma$ is a closable pre-Dirichlet form on $L^2(\Gamma; \mu)$ for some positive Radon measure μ on Γ such that $C_0^\infty(R^d)|_\Gamma \subset \mathcal{D}[\mathcal{E}_\Gamma]$. If $\hat{\mathcal{E}}$ is proven to be closable on $L^2(R^d)$, the L^2-space based on the Lebesgue measure dx, then the associated self-adjoint operator on $L^2(R^d)$ may be thought of as a realization of $\hat{L}$.

Some sufficient conditions for the closability of the perturbed pre-Dirichlet form $\hat{\mathcal{E}}$ are known (M. Fukushima [2; §2.1], J.F. Brasche and W. Karwowski [1]). It is plausible that $\hat{\mathcal{E}}$ ought to be closable on $L^2(R^d)$ under the sole potential theoretic assumption that μ charges no set of zero (Newtonian) capacity. A purpose of the present paper is to affirm this in a more general context as will be stated in §2 and proven in §4.

The proof in §4 involves the notion of the quasi-support of a measure and its characterization crucially. The quasi-notions have appeared in potential theory in diverse contexts. Another aim of the present paper is to show in §3 the existence of the quasi-support along with its useful characterizations in terms of classes of quasi-continuous functions.

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Using the characterization, the quasi-support of a smooth measure is seen to be identical with the probabilistic notion of the support of the additive functional associated with the measure. The latter notion has been adopted recently by M. Fukushima, K. Sato and S. Taniguchi[3] to describe the closable part of a pre-Dirichlet form and to give a necessary and sufficient criterion for the closability. In §5, we restate some basic results of [3] in terms of the quasi-support together with alternative proof using a characterization of §3 instead of the usage of additive functionals. In particular, a characterization due to M. Röckner and N. Wielens[5] for the closability is recovered. The arguments in §4 can be regarded as a reduction of those in §5 to a simpler specific situation.

2. Statements on closability of perturbed pre-Dirichlet forms

In what follows, we fix a locally compact separable metric space X. $C_0(X)$ denotes the family of continuous functions on X with compact support. Suppose that a pair $(\mathcal{E}, \mathcal{C})$ satisfies the following conditions:

(1) $\mathcal{C}$ is a dense subalgebra of $C_0(X)$ such that, for any compact set K and relatively compact open set $G \supset K$, there exists $u \in \mathcal{C}$ with $u=1$ on K, $u=0$ on $X-G$ and $0 \leq u \leq 1$ on X. Furthermore, for any $\varepsilon>0$, there exists a real function $\varphi_\varepsilon(t)$ with $\varphi_\varepsilon(t)=1, t \in [0, 1]$, $-\varepsilon \leq \varphi_\varepsilon(t) \leq 1+\varepsilon$, $t \in R$ and $0 \leq \varphi_\varepsilon(t') - \varphi_\varepsilon(t) \leq t'-t$ for $t<t'$ such that $\varphi_\varepsilon(\mathcal{C}) \subset \mathcal{C}$.

(2) $\mathcal{E}$ is a non-negative definite symmetric bilinear form on $\mathcal{C}$ such that, for each $\varepsilon>0$ and for some function φ_ε satisfying the condition in (1), $\mathcal{E}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leq \mathcal{E}(u, u)$, $u \in \mathcal{C}$.

Then we call $\mathcal{E}$ or the pair $(\mathcal{E}, \mathcal{C})$ a *pre-Dirichlet form* over X.

Denote by $\mathcal{M}$ the family of positive Radon measure on X and let

$$\mathcal{M}' = \{m \in \mathcal{M}: \text{supp } m = X\}.$$

where supp m denotes the topological support of m. A pre-Dirichlet form $(\mathcal{E}, \mathcal{C})$ is called *closable* on $L^2(X; m)$ for $m \in \mathcal{M}'$ if $\mathcal{E}(u_n, u_n) \to 0$ whenever $u_n \in \mathcal{C}$, $\{u_n\}$ is $\mathcal{E}$-Cauchy and $u_n \to 0$ in $L^2(X; m)$. If this is the case, the closure $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; m)$ is a regular Dirichlet form on $L^2(X; m)$. Conversely, given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, the restriction of $\mathcal{E}$ to any subfamily $\mathcal{C} \subset \mathcal{F}$ satisfying condition (1) is a pre-Dirichlet form closable on $L^2(X; m)$. In this case, $\mathcal{E}$ satisfies a stronger condition than (2) in the sense that the statement "for some function φ_ε" in (2) can be strengthened to "for any function φ_ε".

If $(\mathcal{E}, \mathcal{C})$ is a pre-Dirichlet form closable on $L^2(X; m)$ for some $m \in \mathcal{M}'$, we have the associated notion of capacity which can be evaluated for compact set K as

$$\text{Cap}(K) = \inf\{\mathcal{E}_1(u, u): u \in \mathcal{C}, u \geq 1 \text{ on } K\},$$

where $\mathcal{E}_1(u, v)$ is the sum of $\mathcal{E}(u, v)$ and the inner product $(u, v)_m$ in $L^2(X; m)$. The set of zero capacity is called $\mathcal{E}_1$*-polar* to indicate its relevence to $\mathcal{E}$ and m.

Theorem 2.1. *Let* $(\mathcal{E}, \mathcal{C})$ *and* $(\hat{\mathcal{E}}, \mathcal{C})$ *be pre-Dirichlet forms closable on* $L^2(X; m)$ *and* $L^2(X; \hat{m})$ *respectively for some* $m, \hat{m} \in \mathcal{M}'$. *We assume that*

$$\hat{\mathcal{E}}(u, u) \geq \mathcal{E}(u, u)\,, \qquad u \in \mathcal{C}\,,$$

and that $\hat{m}$ *charges no* $\mathcal{E}_1$*-polar set. Then* $(\hat{\mathcal{E}}, \mathcal{C})$ *is closagle on* $L^2(X; m)$.

Theorem 2.1 will be proven in §4 by using quasi-notions studied in the next section. We now show an immediate consequence of Theorem 2.1.

Theorem 2.2. (superposition of closable forms). *Let* $(\mathcal{E}, \mathcal{C})$ *be a pre-Dirichlet form closable on* $L^2(X; m)$ *for some* $m \in \mathcal{M}'$. *Consider a collection* $\{\Gamma_\theta; \theta \in \Theta\}$ *of closed subsets of* X *and suppose that, for each* $\theta \in \Theta$, *there exist a positive Radon measure* μ_θ *on* X *charging no* $\mathcal{E}_1$*-polar set with supp* $\mu_\theta = \Gamma_\theta$, *and a pre-Dirichlet form* $\mathcal{E}_\theta$ *over* Γ_θ *closable on* $L^2(\Gamma_\theta, \mu_\theta)$ *with* $\mathcal{C}|_{\Gamma_\theta} \subset \mathcal{D}[\mathcal{E}_\theta]$. *Further let* $(\Theta, \mathcal{A}, \nu)$ *be an auxiliary* σ*-finite measure space such that* $\mathcal{E}_\theta(f|_{\Gamma_\theta}, f|_{\Gamma_\theta})$ *and* $\mu_\theta(K)$ *are, as functions of* $\theta \in \Theta$, ν*-integrable for every* $f \in \mathcal{C}$ *and every compact* $K \subset X$. *Then the form* $\hat{\mathcal{E}}$ *defined by*

$$\hat{\mathcal{E}}(f, g) = \mathcal{E}(f, g) + \int_\theta \mathcal{E}_\theta(f|_{\Gamma_\theta}, g|_{\Gamma_\theta})\, \nu(d\theta)\,, \qquad f, g \in \mathcal{C}\,,$$

is a pre-Dirichlet form closable on $L^2(X; m)$.

Proof. In view of the remark made after the definition of the pre-Dirichlet form, we can see that $(\hat{\mathcal{E}}, \mathcal{C})$ is a pre-Dirichlet form. If we let

$$\hat{m} = m + \int_\theta \mu_\theta(\cdot)\nu(d\theta)\,,$$

the m is a positive Radon measure on X charging no $\mathcal{E}_1$-polar set. Furthermore in the same manner as in the proof of [2; Theorem 2.1.3], we can show that $\hat{\mathcal{E}}$ is also closable on $L^2(X; \hat{m})$. Hence $\hat{\mathcal{E}}$ is closable on $L^2(X; m)$ by Theorem 2.1.

3. Quasi-supports of smooth measures and their characterizations

Let m be in $\mathcal{M}'$ and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$. Then we have the associated capacity Cap, and we use the terms "quasi-continuous", "polar" and "q.e." in relation to Cap. We add "$\mathcal{E}_1$-" in front of these terms when it is necessary to emphasize their relevence to $\mathcal{E}$ and m. Without loss of generality, we assume that each element of $\mathcal{F}$ is quasi-continuous. Two elements of $\mathcal{F}$ represent an equivalence class of $\mathcal{F} \subset L^2(X; m)$ iff they coincide q.e. An increasing sequence $\{F_n\}$ of closed sets with $\lim_{n\to\infty}$ Cap $(X - F_n) = 0$ is said to be a *nest*.

A set $E \subset X$ is called *quasi-open* if there exists a nest $\{F_n\}$ such that $E \cap F_n$ is open in F_n in the relative topology for each n. The complement of a quasi-open set is called *quasi-closed*.

Lemma 3.1. *(i) A set $F \in X$ is quasi-closed if and only if there exists a quasi-continuous function u with $F = u^{-1}(\{0\})$ q.e.*

(ii) If F is quasi-closed, then, for any relatively compact open set G, there exists a non-negative quasi-continuous function u in $\mathcal{F}$ such that $u=0$ q.e. on F and $u>0$ on $G-F$.

Proof. The "if" part of (i) is evident. We give probabilistic constructions for the rest of the proof. Consider a Hunt process $\boldsymbol{M}=(X_t, P_x)$ on X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$. For any Borel set B, denote by e_B the 1-order hitting probability of B:

$$e_B(x) = E_x(e^{-\sigma_B}), \qquad x \in X,$$

where $\sigma_B = \inf \{t>0: X_t \in B\}$. When Cap $(B) < \infty$, e_B is a quasi-coninuous version of the (1-)equilibrium potential of B ([2; Th.4.3.5]). But we can see that e_B is quasi-continuous for any Borel B. In fact, for any relatively compact $E \subset X$, $w = e_B \wedge e_E$ is a 1-excessive function dominated by $e_E \in \mathcal{F}$. Hence $w \in \mathcal{F}$ (by [2; Lemma 3.3.2]) and w (and consequently e_B) is quasi-continuous (by [2; Th.4.3.2]).

Now, for any quasi-closed set F, the function

$$u(x) = 1 - e_F(x), \qquad x \in X,$$

has the required property in (i) because $F - F^r$ is polar (by [2; Th.4.2.3]) and $e_F(x) < 1$ *q.e.* $x \in X - F$ (by [2; (4.3.5)]). The properties stated in (ii) is satisfied by

$$u(x) = (e_G(x) - e_{F \cap \bar{G}}(x)) \cdot v(x)$$

where v is a (quasi-continuous) bounded function in $\mathcal{F}$ such that $v>0$ on G and $v=0$ *q.e.* on $X-G$ (eg. $v(x) = E_x(\int_0^{\sigma_{X-G}} e^{-t} f(X_t) dt)$ for bounded $f>0$, $f \in L^2(X; m)$).

Corollary 3.2. *Any m-negligible quasi-open set is polar.*

Proof. Let E be an m-negligible quasi-open set. By Lemma 3.1(i), $E = \{u \neq 0\}$ for some quasi-continuous u. Then $u=0$ $m-a.e.$ and consequently q.e., namely, E is polar.

A measure $\mu \in \mathcal{M}$ is said to be *of finite energy integral* if $\mathcal{F} \subset L^1(X; \mu)$ and

$$\int |v(x)| \mu(dx) \leq C\sqrt{\mathcal{E}_1(v, v)}, \qquad v \in \mathcal{F}, \tag{3.1}$$

for some positive constant C. The family of all measures of finite energy integrals is denoted by S_0. A positive Borel measure μ is said to be *smooth* if μ charges no polar set and there exists an incresaing sequence $\{F_n\}$ of closed sets such that

$$(3.2) \qquad \mu\left(X-\bigcup_{n=1}^{\infty} F_n\right)=0,\ \lim_{n\to\infty} \mathrm{Cap}\,(K-F_n)=0 \quad \text{for any compaat set } K$$

and $\mu(F_n)<\infty$ for each n. S will denote the totality of smooth measures. S contains the class $\mathcal{M}_0$ defined by

$$\mathcal{M}_0=\{\mu\in\mathcal{M}:\quad \mu \text{ charges no polar set}\}.$$

It is known ([2; Th.3.2.3]) that $\mu\in S$ iff there exists an increasing sequence $\{F_n\}$ of closed sets satisfying (3.2) and $I_{F_n}\cdot\mu\in S_0$ for each n.

For set $A, B\subset X$, we write

$$A\subset B \quad q.e. \qquad (resp.\ A=B\ q.e.)$$

if the set $A-B$ (resp. the symmetric differnece $A\ominus B$) is polar. For $\mu\in S$, a set $\tilde{F}\subset X$ is said to be a *quasi-support* of μ if

(a) $\tilde{F}$ is quasi-closed and $\mu(X-\tilde{F})=0$

(b) if $\check{F}$ is another set with property (a), then $\tilde{F}\subset\check{F}$ *q.e.*

The quasi-support $\tilde{F}$ of $\mu\in S$ is unique up to a polar set. Let $F=\mathrm{supp}\ \mu$ be the toplogical support of μ. Since any closed set is quasi-closed, we have $\tilde{F}\subset F$ *q.e.*, and by deleting a polar set from $\tilde{F}$ if necessary, we can always assume that $\tilde{F}\subset F$.

Theorem 3.3.

(i) Any $\mu\in S$ admits a quasi-support.

(ii) For $\mu\in S$ and quasi-closed $F\subset X$, the following conditions are equivalent:

(1) F is a quasi-support of μ.

(2) $u=0$ μ—a.e. on X if and only if $u=0$ q.e. on F for any $u\in\mathcal{F}$.

(3) Condition (2) holds for any quasi-continuous function u.

Proof. We first prove (ii).

(2)⇒(1): For $\mu\in S$ and quasi-closed F, we set

$$(3.3) \qquad \mathcal{N}_\mu=\left\{u\in\mathcal{F}:\int|u|\,d\mu=0\right\}$$

$$(3.4) \qquad \mathcal{F}_{F^c}=\{u\in\mathcal{F}:\ u=0 \quad q.e. \quad \text{on } F\}$$

and assume that $\mathcal{N}_\mu=\mathcal{F}_{F^c}$. For any relatively compact set G, take a function u of Lemma 3.1 (ii). Then $u\in\mathcal{F}_{F^c}$ and hence $u\in\mathcal{N}_\mu$, which means $\mu(G-F)=0$. Consequently we get $\mu(X-F)=0$. Consider other quasi-closed set F_1 with

$\mu(X-F_1)=0$. Take again a function u_1 of Lemma 3.1 (ii) for G and F_1. Then $u_1\in\mathcal{N}_\mu$ and hence $u_1\in\mathcal{F}_{F^c}$, which means $F\cap G\subset F_1\cap G$ *q.e.* Accordingly $F\subset F_1$ *q.e.* proving that F is a quasi-supo support of μ.

(1)$\Rightarrow$(3): Suppose (1) is satisfied. Then the "if" part of condition (2) is celarly satisfied for any Borel function u. If u is quasi-continuous and $u=0$ $\mu-a.e.$, then the set $\check{F}=\{u=0\}$ has the property (a) and hence $F\subset\check{F}$ *q.e.* and $u=0$ *q.e.* on F.

The implication (3)$\Rightarrow$(2) is trivial.

(i) can be proved as follows. For any $\mu\in S$, the space $\mathcal{N}_\mu$ defined by (3.3) is a closed subspace of the separable Hilbert space $(\mathcal{F},\mathcal{E}_1)$ because $I_{F_n}\cdot\mu\in S_0$ for some increasing closed sets F_n satisfying (3.2) and $\mathcal{F}$ is continuously embedded into $L^1(X;I_{F_n}\cdot\mu)$ for each n by (3.1). Choose a countable dense subcollection $\{u_k\}$ of $\mathcal{N}_\mu$ and let

$$F=v^{-1}(\{0\})\quad\text{for}\quad v(x)=\sum_{k=1}^{\infty}2^{-k}\frac{|u_k(x)|}{||u_k||_{e_1}} \tag{3.5}$$

Since $v\in\mathcal{N}_\mu$, F is quasi-closed by Lemma 3.1(i) and further $\mu(X-F)=0$. Hence we arrive at the equality $\mathcal{N}_\mu=\mathcal{F}_{F^c}$ for $\mathcal{F}_{F^c}$ defined by (3.4) for this F. We can then conclude that F is a quasi-support of μ from (ii).

Corollary 3.4. *The underlying measure m has the full quasi-support X.*

Finally we state an important probabilistic consequence of Theorem 3.3 although we shall not use it in this paper.

Corollary 3.5. *For $\mu\in S$, the support of the associated positive continuous additive functional (PCAF) of $\boldsymbol{M}$ is a quasi-support of μ.*

Proof. Denote by A a PCAF of $\boldsymbol{M}$ associated with the smooth measure $\mu\in S$ (cf. [2; Chap. 5]). The support F_A of A is defined by

$$F_A=\{x\in X-N\colon P_x(A_t>0\text{ for any }t>0)=1\},$$

where N is an ecxeptional (polar) set for A. Then

$$F_A=\{x\in X-N\colon e_{F_A}(x)=1\}$$

and consequently F_A is quasi-closed since e_{F_A} is quasi-continuous as was seen in the proof of Lemma 3.1. Furthermore we can check the property (3) of Theorem 3.3 (ii) for μ and F_A in the same way as in the last paragraph of the proof of [2; Th.5.5.1].

4. Proof of Theorem 2.1.

We prove Theorem 2.1 by a series of lemmas. Suppose that $\mathcal{E}$, $\hat{\mathcal{E}}$, m and

$\hat{m}$ satisfy the conditions of Theorem 2.1. Let $(\mathcal{E}, \mathcal{F})$ and $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ be the closures of $(\mathcal{E}, C)$ and $(\hat{\mathcal{E}}, C)$ on $L^2(X; m)$ and $L^2(X, \hat{m})$ respectively. $(u, v)_m$ (resp. $(u, v)_{\hat{m}}$) denotes the inner product in $L^2(X; m)$ (resp. $L_2(X; \hat{m})$). Recall that $\mathcal{E}_1(u, v)$ (resp. $\hat{\mathcal{E}}_1(u, v)$) stands for $\mathcal{E}_1(u, v)+(u, v)_m$ (resp. $\hat{\mathcal{E}}_1(u, v)+(u, v)_{\hat{m}}$). Note further that the conditions of Theorem 2.1 are never destroyed if we replace $\hat{m}$ by $m+\hat{m}$. Hence we can assume without loss of generality that

$$\hat{m} \geq m . \tag{4.1}$$

Lemma 4.1. *We let*

$$\hat{\mathcal{E}}^m(u, v) = \hat{\mathcal{E}}(u, v)+(u, v)_m , \qquad u, v \in \hat{\mathcal{F}} . \tag{4.2}$$

Then $(\hat{\mathcal{E}}^m, \hat{\mathcal{F}})$ is a Dirichlet form on $L^2(X; \hat{m})$ possessing C as a core. Moreover this is transient, namely, there exists a strictly positive $\hat{m}$-integrable function $\hat{g}$ such that

$$\int_X |v(x)|\,\hat{g}(x)\hat{m}(dx) \leq \sqrt{\hat{\mathcal{E}}^m(v, v)} , \qquad v \in \hat{\mathcal{F}} . \tag{4.3}$$

Proof. The first assertion is evident because $\hat{\mathcal{E}}_1^m(u, u)=\hat{\mathcal{E}}^m(u, u)+(u, u)_{\hat{m}}$ is equivalent to $\hat{\mathcal{E}}_1(u, u)$ for $u \in \hat{\mathcal{F}}$. Since $\hat{m} \in \mathcal{M}_0$ with respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ by assumption, there exist increasing closed sets F_n such that (3.2) holds for $\hat{m}$ and (3.1) holds for $I_{F_n} \cdot \hat{m}$ and some positive constant C_n for each n. Hence we can find a function $\hat{g}$ with the required properties.

Lemma 4.2. *The measure m has the full quasi-support X with respect to the Dirichlet form of Lemma* 4.1.

Proof. Let $\check{S}$ be a quasi-support of m with respect to the Dirichlet form $(\hat{\mathcal{E}}^m, \hat{\mathcal{F}})$ on $L^2(X; \hat{m})$ of Lemma 4.1. $E=X-\check{S}$ is then $\hat{\mathcal{E}}_1^m$-quasi-open and $m(E)=0$. Due to the domination of $\hat{\mathcal{E}}_1^m$ over $\mathcal{E}_1$, E is also $\mathcal{E}_1$-quasi-open and consequently $\mathcal{E}_1$-polar by Corollary 3.2. Then E is $\hat{m}$-negligible by the assumption and $\hat{\mathcal{E}}_1^m$-polar by Corollary 3.2 again.

The next lemma particularly implies Theorem 2.1. Denote by $(\hat{\mathcal{E}}^m, \mathcal{G})$ the extended Dirichlet space of the transient Dirichlet form of Lemma 4.1. $\mathcal{G}$ is the completion of $\hat{\mathcal{F}}$ with respect to the metric $\hat{\mathcal{E}}^m$. We may assume that each element of $\mathcal{G}$ is $\hat{\mathcal{E}}_1^m$-quasi-continuous.

Lemma 4.3. $(\hat{\mathcal{E}}, \mathcal{G})$ *is a Dirichlet form on $L^2(X; m)$ possessing C as its core.*

Proof. If $u \in \mathcal{G}$ and $u=0$ $m-a.e.$ on X, then $u=0$ $\hat{\mathcal{E}}_1^m-q,e.$ on X by Lemma 4.2 and Theorem 3.3, and hence $\hat{\mathcal{E}}(u, u)=0$. Therefore $(\hat{\mathcal{E}}, \mathcal{G})$ can be regarded as a symmetric form on $L^2(X; m)$. The rest of the proof is clear from Lemma 4.1.

5. Closable part and closability criterion in terms of quasi support

In this section, we restate some basic results of [3] in terms of the quasi-support and we see how an analytical characterization of §3 simplifies the arguments. For $m\in\mathcal{M}'$, let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ possessing as its core a set C satisfying condition (1) in §2. $(\mathcal{E}, \mathcal{F})$ is assumed to be either transient or irreducible. For simplicity of presentation, we only describe the case that $(\mathcal{E}, \mathcal{F})$ is transient. The irreducible case can be treated in the same way however by considering the transient Dirichlet form $\mathcal{E}^{\mu}(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot)+(\cdot, \cdot)_{L^2(X;\mu)}$ ([3], [4]).

Take any non-trivial $\mu\in\mathcal{M}_0$. Denote by F and $\tilde{F}$ the (topological) support and the quasi-support of μ respectively. As was noticed in §3, we may asume that $\tilde{F}\subset F$. For a moment, we do not assume that $F=X$. Consider the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of $(\mathcal{F}, \mathcal{E})$. $\mathcal{F}_e$ is a Hilbert space with inner product $\mathcal{E}$ and each element of $\mathcal{F}_e$ can be assumed to be $\mathcal{E}_1$-quasi-continuous. Let

$$\mathcal{F}_{e,X-\tilde{F}} = \{u\in\mathcal{F}_e : u = 0 \ q.e. \text{ on } \tilde{F}\}.$$

This is a closed subspace of $(\mathcal{F}_e, \mathcal{E})$. Denote by $P_{\tilde{F}}$ the orthogonal projection on the orthogonal complement of $\mathcal{F}_{e,X-\tilde{F}}$.

Note that, if $v_1, v_2\in\mathcal{F}_e$ and $v_1=v_2$ $\mu-a.e.$ on F, then $v_1=v_2$ $q.e.$ on $\tilde{F}$ by virtue of Theorem 3.3, and consequently $P_{\tilde{F}}v_1=P_{\tilde{F}}v_2$. Therefore the following definition makes sense:

$$\mathcal{F}^{\mu} = \{u\in L^2(F; \mu): u = v \quad \mu-a.e. \text{ on } F \text{ for some } v\in\mathcal{F}_e\}$$
$$\mathcal{E}^{\mu}(u, u) = \mathcal{E}(P_{\tilde{F}}v, P_{\tilde{F}}v) \text{ for } v \text{ as in the above braces.}$$

Lemma 5.1. *$(\mathcal{F}^{\mu}, \mathcal{E}^{\mu})$ is a Dirichlet form on $L^2(F; \mu)$ possessing $C|_F$ as a core. Here $C|_F$ denotes the restrictions to F of elements of C.*

Proof. It can be readily seen that, for $u\in\mathcal{F}^{\mu}$,

$$\mathcal{E}^{\mu}(u, u) = \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e \quad v = u \quad \mu-a.e. \text{ on } F\}.$$

Denote by Tu the unit contraation $0\vee u\wedge 1$ of $u\in\mathcal{F}^{\mu}$. Then $Tu\in\mathcal{F}^{\mu}$ and

$$\begin{aligned}\mathcal{E}^{\mu}(Tu, Tu) &= \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e, v=Tu \ \mu-a.e. \text{ on } F\}\\ &\leq \inf\{\mathcal{E}(Tv, Tv): v\in\mathcal{F}_e, v=u \ \mu-a.e. \text{ on } F\}\\ &\leq \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e, v=u \ \mu-a.e. \text{ on } F\}\\ &= \mathcal{E}(u, u),\end{aligned}$$

proving that the unit contraction operates. The closedness of $(\mathcal{F}^{\mu}, \mathcal{E}^{\mu})$ on $L^2(F; \mu)$ is easily verified. We refer to [3] for the last statement about the core.

A pre-Dirichlet form $(\mathcal{A}, C)$ is called the *closable part* of a pre-Dirichlet form $(\mathcal{E}, C)$ with respect to $\mu\in\mathcal{M}'$ if $(\mathcal{A}, C)$ is closable on $L^2(X; \mu)$, $\mathcal{A}(u, u)\leq\mathcal{E}(u, u)$, $u\in C$, and $(\mathcal{A}, C)$ is the maximum among those. Lemma 5.1

leads us to the next assertion (cf. [3; Lemma 4.3]).

Theorem 5.2. *For $\mu\in\mathcal{M}'\cap\mathcal{M}_0$, $(\mathcal{E}^{\mu},\mathcal{C})$ is the closable part of $(\mathcal{E},\mathcal{C})$ with respect to μ.*

Theorem 5.3. *For $\mu\in\mathcal{M}'\cap\mathcal{M}_0$, $(\mathcal{E},\mathcal{C})$ is closable on $L^2(X;\mu)$ if and only if μ has the full quasi-support X.*

Proof. By Theorem 5.2, we have the following series of equivalent conditions:

$$\begin{aligned}&(\mathcal{E},\mathcal{C}) \text{ is closable on } L^2(X;\mu)\\ &\qquad\Leftrightarrow P_{\tilde{F}}f=f \quad \text{for any } f\in\mathcal{F}_e\\ &\qquad\Leftrightarrow f=0 \ q.e. \text{ on } \tilde{F} \text{ iff } f=0 \ q.e. \text{ on } X \text{ for any } f\in\mathcal{F}_e\end{aligned}$$

Since $\tilde{F}$ is a quasi-support of μ, we see by Theorem 3.3 that the last condition is equivalent to

$$\text{“}f=0 \ \mu-a.e. \text{ on } X \quad \text{iff } f=0 \ q.e. \text{ on } X \text{ for any } f\in\mathcal{F}_e\text{”},$$

which is in turn equivalent to "X is a quasi-support of μ" by the same theorem.

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