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PROJECTIVE MODULES OVER VON NEUMANN REGULAR RINGS HAVE THE FINITE EXCHANGE PROPERTY

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It is an open problem to determine which ring satisfies the condition that every projective module over the ring has the (finite) exchange property.

For this problem, Harada and Ishii ([1]) and Yamagata ([5]) have shown that projective modules over perfect rings have the exchange property, and recently Kutami and Oshiro ([3]) have shown that projective modules over a certain Boolean ring have the exchange property.

The purpose of the present note is to show that projective modules over (von Neumann) regular rings have the finite exchange property.

Throughout this paper we assume a ring R has identity and all R-modules are unitary. A right R-module M has the *exchange property* if for any right R-module X and any two decompositions:

$$X = M' \oplus N = \sum_I \oplus A_\alpha$$

with $M' \simeq M$, there exist submodules $A'_\alpha \subseteq A_\alpha$ such that

$$X = M' \oplus (\sum_I \oplus A'_\alpha) .$$

M has the *finite exchange property* if this holds whenever the index set I is finite.

For a given projective right module P, the following conditions due to Nicholson [4] are useful in the study of the exchange property:

(C_1): If $P = P_1 + P_2$ where P_i are submodules there exists a submodule $P'_i \subseteq P_i$ for $i = 1, 2$ such that

$$P = P'_1 \oplus P'_2 .$$

(C_2): If $P = \sum_I P_\alpha$ where P_α are submodules there exists a decomposition

$$P = \sum_I \oplus P'_\alpha$$

with $P'_\alpha \subseteq P_\alpha$ for each $\alpha \in I$.

Nicholson ([4]) has shown that a projective right R-module P has the

finite exchange property if and only if it satisfies the condition (C_1). It has been shown by Kutami and Oshiro ([3]) that a projective right R-module over a right hereditary ring has the exchange property if it satisfies the condition (C_2).

Let M be a right R-module with the two expressions:

$$M = \sum_I \oplus n_\alpha R$$
$$= \sum_J a_\beta R$$

with the cardinal $|I| > \chi_0$. For a subset I' (resp. J') of I (resp. J) we put

$$N(I') = \sum_{I'} \oplus n_\alpha R ,$$
$$A(J') = \sum_{J} a_\beta R .$$

Under this situation we show the following result which is a key lemma in this note:

Lemma 1. *Let $I = I' \cup I''$ be a partition with $|I'| > \chi_0$. Then there exist subsets $I_1, I_2, \cdots \subseteq I'$ and $J_1, J_2, \cdots \subseteq J$ such that*

$$\begin{aligned}
&|I_i| = \chi_0 \qquad \text{for each } i ,\\
&I_j \cap I_k = \phi \qquad \text{if } j \neq k ,\\
&|J_i| \leq \chi_0 \qquad \text{for each } i ,\\
&N(I'') \oplus N(I_1)\\
\subseteq\ &N(I'') + A(J_1)\\
\subseteq\ &N(I'') \oplus N(I_1) \oplus N(I_2)\\
\subseteq\ &N(I'') + A(J_1) + A(J_2)\\
&\ \ \vdots\\
\subseteq\ &N(I'') \oplus N(I_1) \oplus \cdots\cdots\cdots \oplus N(I_n)\\
\subseteq\ &N(I'') + A(J_1) + \cdots\cdots\cdots + A(J_n)\\
\subseteq\ &N(I'') \oplus N(I_1) \oplus \cdots\cdots\cdots \oplus N(I_n) \oplus N(I_{n+1})\\
&\ \ \vdots
\end{aligned}$$

Proof. First we take a countably infinite subset $I_1 \subseteq I'$. (Note $|I| > \chi_0$). Since $M = \sum_J a_\beta R$ there exists a subset $J_1 \subseteq J$ such that $|J_1| \leq \chi_0$ and

$$N(I_1) \subseteq A(J_1) .$$

We proceed the proof by the induction. So, assume that there exist subsets $I_1, I_2, \cdots, I_n \subseteq I'$ and $J_1, J_1, \cdots, J_n \subseteq J$ such that

$$|I_i| = \chi_0 \qquad \text{for } i = 1, \cdots, n ,$$

$$
\begin{aligned}
& I_i \cap I_k = \phi \qquad \text{if } j \neq k\,, \\
& |J_j| \leq \aleph_0 \qquad \text{for } i = 1, \cdots, n\,, \\
& N(I'') \oplus N(I_1) \\
\subseteq & N(I'') + A(J_1) \\
\subseteq & N(I'|) \oplus N(I_1) \oplus N(I_2) \\
\subseteq & N(I'') + A(J_1) + A(J_2) \\
& \vdots \\
\subseteq & N(I'') \oplus N(I_1) \oplus \cdots\cdots\cdots \oplus N(I_n) \\
\subseteq & N(I'') + A(J_1) + \cdots\cdots\cdots + A(J_n)
\end{aligned}
$$

We set $I^* = I - \bigcup_{i=1}^{n} I_i$, and by π we denote the projection: $M = N(I'') \oplus \sum_{i=1}^{n} \oplus N(I_i) \oplus N(I^*) \to N(I^*)$. Then $|I^*| > \aleph_0$ and $\pi_n(A(J_n))$ is countably generated; so we can choose a countably infinite subset $I_{n+1} \subseteq I'$ such that

$$
\begin{aligned}
& N(I'') + A(J_1) + \cdots\cdots\cdots + A(J_n) \\
\subseteq & N(I'') \oplus N(I_1) \oplus \cdots\cdots\cdots \oplus N(I_n) \oplus N(I_{n+1})\,.
\end{aligned}
$$

Thus the proof is now complete.

REMARK. In the above lemma we note that

$$
\begin{aligned}
& N(I'') \oplus N(I_\beta) \oplus N(I_2) \oplus \cdots \\
= & N(I'') + A(J_1) + A(J_2) + \cdots
\end{aligned}
$$

and $\bigcup_{i=1}^{\infty} J_i$ is a countable subset of J.

Lemma 2. *Let R be a regular ring and P a projective right R-module with the two expressions*:

$$
\begin{aligned}
P &= P^* \oplus \sum_I \oplus x_\alpha R \\
&= \sum_J a_\beta R\,.
\end{aligned}
$$

Assume the cardinal $|I| \leq \aleph_0$. Then there exist a countable subset $J' \subseteq J$ and $b_\beta \in a_\beta R$ for each $\beta \in J'$ such that

$$
P = P^* \oplus \sum_{J'} \oplus b_\beta R\,.
$$

Proof. Since R is a regular ring, every finitely generated submodule of P is a direct summand of P (see [2]). Now, since $|I| \leq \aleph_0$ and $P = \sum_J a_\beta R$, there exists a countable subset $J' \subseteq J$ such that

$$
P = P^* + \sum_{J'} a_\beta R\,.
$$

We may assume $|J'|=\aleph_0$. Put $J'=\{\beta_1, \beta_2, \cdots\}$. Since $P^*<\oplus P$ and $a_{\beta_1}R\subseteq P$ we see $P^*+a_{\beta_1}R<\oplus P$ and there exists $b_{\beta_1}\in a_{\beta_1}R$ such that

$$\begin{aligned} &P^*+a_{\beta_1}R \\ &=P^*\oplus b_{\beta_1}R\,. \end{aligned}$$

By doing the same work on $P^*\oplus b_{\beta_1}R<\oplus P$ and $a_{\beta_2}R\subseteq P$ we can take $b_{\beta_2}\in a_{\beta_2}R$ such that

$$\begin{aligned} &P^*+a_{\beta_1}R+a_{\beta_2}R \\ &=P^*\oplus b_{\beta_1}R\oplus b_{\beta_2}R\,. \end{aligned}$$

Continuing this manner the proof is established.

Theorem 3. *If R is a regular ring then every projective right R-module satisfies the condition* (C_2).

Proof. It is enough to show that every free right R-module satisfies the condition (C_2). So, let F be a free right R-module and consider an expression $F=\sum_J a_\beta R$. We put $F=\sum_I\oplus x_\alpha R$. with each $x_\alpha R\cong R$. Consider a triple $(I'\subseteq I;\ J'\in J;\ \{b_\beta\}_{J'})$ such that $b_\beta\in a_\beta R$ for all $\beta\in J'$ and

$$\sum_{I'}\oplus x_\alpha R=\sum_{J'}\oplus b_\beta R\,.$$

Such a triple exists by Lemma 1, Remark and Lemma 2. We denote the set of all such triples by $\sum$. Then $\sum$ becomes a partially ordered set as follows: For $\delta_i=(I_i; J_i; \{b_\beta^i\})\in\sum$, $i=1, 2$, we define an order '$\delta_1\leq\delta_2$' if $I_1\subseteq I_2$, $J_1\subseteq J_2$ and $b_\beta^1=b_\beta^2$ for all $\beta\in J_1$. By the Zorn's lemma, we see that $\sum$ has a maxmal member; say $\delta_0=(I_0; J_0; \{b_\beta\}_{J_0})$. We put

$$F'=\sum_{I_0}\oplus x_\alpha R=\sum_{J_0}\oplus b_\beta R\,.$$

In view of Lemma 1, Remark and Lemma 2, we first see $|I-I_0|\leq\aleph_0$, and next it follows from Lemma 2 that $|I-I_0|$ is empty. As a result we get

$$F=\sum_{J_0}\oplus b_\beta R\,.$$

Corollary 4. *Projective modules over regular rings have the finite exchange property.*

Proof. This follows from Theorem 3 and [4, Proposition 2.9].

Corollary 5. *Projective right modules over right hereditary regular rings have the exchange property.*

Proof. Combine Theorem 3 to [3, Lemma 1].

Corollary 6 (Kaplansky). *Projective modules over regular rings are written as direct sums of cyclic modules.*

Proof. This is a direct consequence of Theorem 3.

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