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<th>Projective modules over von Neumann regular rings have the finite exchange property</th>
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It is an open problem to determine which ring satisfies the condition that every projective module over the ring has the (finite) exchange property.

For this problem, Harada and Ishii ([1]) and Yamagata ([5]) have shown that projective modules over perfect rings have the exchange property, and recently Kutami and Oshiro ([3]) have shown that projective modules over a certain Boolean ring have the exchange property.

The purpose of the present note is to show that projective modules over (von Neumann) regular rings have the finite exchange property.

Throughout this paper we assume a ring \( R \) has identity and all \( R \)-modules are unitary. A right \( R \)-module \( M \) has the exchange property if for any right \( R \)-module \( X \) and any two decompositions:

\[
X = M' \oplus N = \sum_{\alpha} A_{\alpha}
\]

with \( M' \cong M \), there exist submodules \( A'_{\alpha} \subseteq A_{\alpha} \) such that

\[
X = M' \oplus (\sum_{\alpha} A'_{\alpha}) .
\]

\( M \) has the finite exchange property if this holds whenever the index set \( I \) is finite.

For a given projective right module \( P \), the following conditions due to Nicholson [4] are useful in the study of the exchange property:

\( (C_1) \): If \( P = P_1 + P_2 \) where \( P_i \) are submodules there exists a submodule \( P_i' \subseteq P_i \) for \( i = 1, 2 \) such that

\[
P = P_1' \oplus P_2' .
\]

\( (C_2) \): If \( P = \sum_i P_i \) where \( P_i \) are submodules there exists a decomposition

\[
P = \sum_i P_i' ,
\]

with \( P_i' \subseteq P_i \) for each \( i \in I \).

Nicholson ([4]) has shown that a projective right \( R \)-module \( P \) has the
finite exchange property if and only if it satisfies the condition \((C_1)\). It has been shown by Kutami and Oshiro ([3]) that a projective right \(R\)-module over a right hereditary ring has the exchange property if it satisfies the condition \((C_2)\).

Let \(M\) be a right \(R\)-module with the two expressions:

\[
M = \sum_I \oplus n_{\alpha} R = \sum_I a_{\beta} R
\]

with the cardinal \(|I| > \aleph_0\). For a subset \(I'\) (resp. \(J'\)) of \(I\) (resp. \(J\)) we put

\[
N(I') = \sum_I \oplus n_{\alpha} R, \\
A(J') = \sum_I a_{\beta} R.
\]

Under this situation we show the following result which is a key lemma in this note:

**Lemma 1.** Let \(I=I' \cup I''\) be a partition with \(|I'| > \aleph_0\). Then there exist subsets \(I_1, I_2, \cdots \subseteq I'\) and \(J_1, J_2, \cdots \subseteq J\) such that

\[
|I_i| = \aleph_0 \quad \text{for each } i, \\
I_j \cap I_k = \emptyset \quad \text{if } j \neq k, \\
|J_i| \leq \aleph_0 \quad \text{for each } i, \\
N(I'') \oplus N(I_i) \subseteq N(I'') + A(J_i) \\
\subseteq N(I'') \oplus N(I_1) \oplus N(I_2) \\
\subseteq N(I'') + A(J_1) + A(J_2) \\
\vdots \\
\subseteq N(I'') \oplus N(I_1) \oplus \cdots \oplus N(I_n) \\
\subseteq N(I'') + A(J_1) + \cdots + A(J_n) \\
\subseteq N(I'') \oplus N(I_1) \oplus \cdots \oplus N(I_n) \oplus N(I_{n+1}) \\
\vdots
\]

**Proof.** First we take a countably infinite subset \(I_i \subseteq I'\). (Note \(|I| > \aleph_0\)). Since \(M=\sum_I a_{\beta} R\) there exists a subset \(J_1 \subseteq J\) such that \(|J_1| \leq \aleph_0\) and

\[
N(I_1) \subseteq A(J_1).
\]

We proceed the proof by the induction. So, assume that there exist subsets \(I_1, I_2, \cdots, I_n \subseteq I'\) and \(J_1, J_2, \cdots, J_n \subseteq J\) such that

\[
|I_i| = \aleph_0 \quad \text{for } i = 1, \cdots, n,
\]
We set \( I^* = I - \bigcup_{i=1}^{n} I_i \), and by \( \pi \) we denote the projection: \( M = N(I^*) \oplus \bigoplus_{i=1}^{n} N(I_i) \oplus N(I^*) \rightarrow N(I^*) \). Then \( |I^*| > \aleph_0 \) and \( \pi_n(A(J_n)) \) is countably generated; so we can choose a countably infinite subset \( I_{n+1} \subseteq J' \) such that

\[
N(I^*) + A(J_1) + \cdots + A(J_n) \\
\subseteq N(I^*) + A(J_1) + \cdots + N(I_n) + N(I_{n+1})
\]

Thus the proof is now complete.

**Remark.** In the above lemma we note that

\[
N(I^*) \oplus N(I_n) \oplus N(I_3) \oplus \cdots = N(I^*) + A(J_1) + A(J_2) + \cdots
\]

and \( \bigcup_{i=1}^{n} J_i \) is a countable subset of \( J \).

**Lemma 2.** Let \( R \) be a regular ring and \( P \) a projective right \( R \)-module with the two expressions:

\[
P = P^* \oplus \bigoplus_{J} \alpha_J \cdot R
\]

\[
= \sum_{J} a_J \cdot R
\]

Assume the cardinal \( |I| \leq \aleph_0 \). Then there exist a countable subset \( J' \subseteq J \) and \( b_\beta \in a_\beta \cdot R \) for each \( \beta \in J' \) such that

\[
P = P^* \oplus \bigoplus_{J'} b_\beta \cdot R
\]

**Proof.** Since \( R \) is a regular ring, every finitely generated submodule of \( P \) is a direct summand of \( P \) (see [2]). Now, since \( |I| \leq \aleph_0 \) and \( P = \sum_{J} a_J \cdot R \), there exists a countable subset \( J' \subseteq J \) such that

\[
P = P^* + \sum_{J'} a_J \cdot R
\]
We may assume \(|J'|=\mathcal{C}_0\). Put \(J' = \{\beta_1, \beta_2, \ldots\}\). Since \(P* \otimes P\) and \(a_{\beta_1}R \subseteq P\) we see \(P* + a_{\beta_1}R \otimes P\) and there exists \(b_{\beta_1} \in a_{\beta_1}R\) such that

\[
P* + a_{\beta_1}R \subseteq P* \otimes b_{\beta_1}R.
\]

By doing the same work on \(P* \otimes b_{\beta_1}R \otimes P\) and \(a_{\beta_2}R \subseteq P\) we can take \(b_{\beta_2} \in a_{\beta_2}R\) such that

\[
P* + a_{\beta_1}R + a_{\beta_2}R \subseteq P* \otimes b_{\beta_1}R \otimes b_{\beta_2}R.
\]

Continuing this manner the proof is established.

**Theorem 3.** If \(R\) is a regular ring then every projective right \(R\)-module satisfies the condition \((C_2)\).

**Proof.** It is enough to show that every free right \(R\)-module satisfies the condition \((C_2)\). So, let \(F\) be a free right \(R\)-module and consider an expression \(F = \sum a_{\gamma}R\). We put \(F = \sum \otimes x_{\gamma}R\), with each \(x_{\gamma}R = R\). Consider a triple \((I', J; \{b_{\beta}\})\) such that \(b_{\beta} \in a_{\beta}R\) for all \(\beta \in J'\) and

\[
\sum_{J'} \otimes x_{\gamma}R = \sum_{J'} \otimes b_{\beta}R.
\]

Such a triple exists by Lemma 1, Remark and Lemma 2. We denote the set of all such triples by \(\Sigma\). Then \(\Sigma\) becomes a partially ordered set as follows: For \(\delta_i = (I_i; J_i; \{b_{\beta_i} \}) \in \Sigma\), \(i = 1, 2\), we define an order \(\delta_i \leq \delta_2\) if \(I_1 \subseteq I_2\), \(J_1 \subseteq J_2\) and \(b_{\beta_1} = b_{\beta_2}\) for all \(\beta \in J_1\). By the Zorn's lemma, we see that \(\Sigma\) has a maximal member; say \(\delta_0 = (I_0; J_0; \{b_{\beta} \})\). We put

\[
F' = \sum_{I_0} \otimes x_{\gamma}R = \sum_{J_0} \otimes b_{\beta}R.
\]

In view of Lemma 1, Remark and Lemma 2, we first see \(|I-I_0| \leq \mathcal{C}_0\), and next it follows from Lemma 2 that \(|I-I_0|\) is empty. As a result we get

\[
F = \sum_{J_0} \otimes b_{\beta}R.
\]

**Corollary 4.** Projective modules over regular rings have the finite exchange property.

**Proof.** This follows from Theorem 3 and [4, Proposition 2.9].

**Corollary 5.** Projective right modules over right hereditary regular rings have the exchange property.

**Proof.** Combine Theorem 3 to [3, Lemma 1].
Corollary 6 (Kaplansky). Projective modules over regular rings are written as direct sums of cyclic modules.

Proof. This is a direct consequence of Theorem 3.

References


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