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<th><strong>Title</strong></th>
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Let $I$ be a compact interval of the real line. For a continuous map $f : I \to I$ by Misiurewicz et al. ([1, 12, 13]) the following relation between the topological entropy $h(f)$ and the growth rate of the number of periodic points is known:

\[
(*) \quad h(f) \leq \limsup_{n \to \infty} \frac{1}{n} \log \# \text{Per}(f, n)
\]

where $\text{Per}(f, n)$ denotes the set of all fixed points of $f^n$ for $n \geq 1$, and $\#A$ the number of elements of a set $A$. (The equality of the expression $(*)$ does not hold in general. For instance, the topological entropy of the identity map is zero, nevertheless all of points of the interval are fixed by this map.)

For a periodic point $p$ of $f$ with period $n$ we put

\[
\mathcal{O}_f^+(p) = \{p, f(p), \ldots, f^{n-1}(p)\}.
\]

Then we say that $q$ is a homoclinic point of $p$ if $q \notin \mathcal{O}_f^+(p)$ and there are a positive integer $m$ with $f^m(q) = p$ and a sequence $q_0, q_1, \ldots, q_k, \ldots \in I$ with $q_0 = q$ such that

\[
f(q_k) = q_{k-1} \quad (k \geq 1), \quad \lim_{k \to \infty} |q_k - \mathcal{O}_f^+(p)| = 0
\]

where $|x - A| = \inf\{|x - y| : y \in A\}$ for $x \in I, A \subseteq I$. It is known by Block ([2, 3]) that $h(f)$ is positive if and only if $f$ has a homoclinic point of a periodic point.

In this paper we shall establish more results (Theorems 1 and 2) for differentiable maps of intervals. To describe them we need some notations.

Let $f : I \to I$ be a $C^{1+\alpha}$ map $(\alpha > 0)$. A periodic point $p$ of $f$ with period $n$ is a source if

\[
v(p) = |(f^n)'(p)|^{1/n} > 1.
\]

For $n \geq 1$, $v > 1$ and $\delta > 0$ we define an $f$-invariant set by
Per\((f, n, v, \delta) = \{ p \in \text{Per}(f, n) : v(p) \geq v, \ |f^i(p)| \geq \delta \text{ for all } 0 \leq i \leq n - 1 \}.\)

Then we have

\[
\text{Per}(f, n, v_1, \delta_1) \subset \text{Per}(f, n, v_2, \delta_2) \text{ if } v_1 \geq v_2, \delta_1 \geq \delta_2.
\]

and

\[
\{ p : \text{source of } f \} = \bigcup_{v > 1} \bigcup_{\delta > 0} \bigcup_{n = 1}^{\infty} \text{Per}(f, n, v, \delta).
\]

One of our results is the following:

**Theorem 1.**

\[
h(f) = \max \left\{ 0, \lim_{v \to 1} \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \# \text{Per}(f, n, v, \delta) \right\}.
\]

By Theorem 1 it is clear that for a \(C^{1+\varepsilon}\) map of a compact interval if the topological entropy is positive then the map has infinitely many sources. However, the converse is not true in general. In fact, for any \(r \geq 1\) it is easy to construct a \(C^r\) diffeomorphism of a compact interval having infinitely many source fixed points. But every diffeomorphism of an interval has zero entropy.

**Remark.** It is known that if \(f\) is a \(C^2\) map with non-flat critical points, then any periodic point of \(f\) with sufficiently large period is a source ([10]). In Theorem 1 we do not assume any condition concerned with critical points. Then the map \(f\) may have flat critical points.

For a source \(p\) of \(f\) with period \(n\) we denote by \(W^u_{\text{loc}}(p)\) the maximal interval \(J\) of \(I\) containing \(p\) such that

\[
|(f^n)'(x)| \geq [(1 + v(p))/2]^n \text{ for all } x \in J.
\]

We say that a homoclinic point \(q\) of \(p\) is *transversal* if there are non-negative integers \(m_1, m_2\) and a point \(q' \in W^u_{\text{loc}}(p)\) such that

\[
f^{m_1}(q') = q, \quad f^{m_1+m_2}(q') = f^{m_2}(q) = p \quad \text{and} \quad (f^{m_1+m_2})'(q') \neq 0.
\]

If \(f\) has a transversal homoclinic point of a source, then there is a \(C^1\) neighborhood \(\mathcal{U}\) of \(f\) such that every map \(g\) belonging to \(\mathcal{U}\) has a transversal homoclinic point of a source. We denote the set of transversal homoclinic points of a source \(p\) of \(f\) by \(\text{TH}(p)\), and its closure by \(\text{TH}(p)\). We call \(\text{TH}(p)\) the *transversal homoclinic closure of
p. It is easy to see that \( p \in \overline{\text{TH}(p)} \), and \( \overline{\text{TH}(p)} \) is \( f \)-invariant. For \( m \geq 1 \) and \( \delta > 0 \) define

\[
H(p, m, \delta) = \{ q \in W^u_{\text{loc}}(p) : f^m(q) = p, |f^i(f(q))| \geq \delta \text{ for all } 0 \leq i \leq m - 1 \}.
\]

Then we have

\[
H(p, m, \delta_1) \subset H(p, m, \delta_2) \quad \text{if} \quad \delta_1 \geq \delta_2.
\]

and

\[
\text{TH}(p) = \bigcup_{\delta > 0} \bigcup_{m=1}^{\infty} \bigcup_{i=0}^{m-1} f^i H(p, m, \delta) \setminus \mathcal{O}_f^+(p).
\]

The second result of this paper is the following:

**Theorem 2.** If \( h(f) > 0 \) then

\[
h(f) = \sup \{ h(f_{\mid \text{TH}(p)}) : p \text{ is a source of } f \},
\]

and for a source \( p \) of \( f \) we have

\[
h(f_{\mid \text{TH}(p)}) = \max \left\{ 0, \lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \# H(p, m, \delta) \right\}.
\]

A result corresponding to Theorem 2 is known for surface diffeomorphisms by Mendoza ([11]). As an easy corollary of Theorem 2 we have:

**Corollary 3.** The following statements are equivalent:

(i) \( h(f) > 0 \);
(ii) \( f \) has a transversal homoclinic point of a source;
(iii) \( f \) has a homoclinic point of a periodic point.

1. **Proofs of Theorems**

Let \( f : I \to I \) be a continuous map. For integers \( k, l \geq 1 \) we say that a closed \( f \)-invariant set \( \Gamma \) is a \((k, l)\)-horseshoe of \( f \) if there are subsets \( \Gamma^0, \ldots, \Gamma^{k-1} \) of \( I \) such that

\[
\Gamma = \Gamma^0 \cup \cdots \cup \Gamma^{k-1}, \quad f(\Gamma^j) = \Gamma^{j+1} \pmod{k}
\]

and \( f^k \mid_{\Gamma^0} : \Gamma^0 \to \Gamma^0 \) is topologically conjugate to a one-sided full shift in \( l \)-symbols. If \( \Gamma \) is a \((k, l)\)-horseshoe, then it is clear that

\[
h(f_{\mid \Gamma^n}) = \frac{1}{k} \log l \quad \text{and} \quad l^n \leq \# \{ \text{Per}(f, kn) \cap \Gamma \} \leq kl^n.
\]
for all $n \geq 1$. It was proved by Misiurewicz et al. ([1, 12, 13]) that if the topological entropy of $f$ is positive then there are sequences $k_j, l_j$ of positive integers with a $(k_j, l_j)$-horseshoe $\Gamma_j$ of $f$ ($j \geq 1$) such that

$$h(f) = \lim_{j \to \infty} h(f |_{\Gamma_j}) = \lim_{j \to \infty} \frac{1}{k_j} \log l_j.$$ 

Then the formula (*) follows from this fact.

In order to prove our results, we need the notion of hyperbolic horseshoe and ideas of the theory of hyperbolic measures ([14, 15]). Katok ([9]) has proved that if a $C^{1+\alpha}$ diffeomorphism of a manifold has a hyperbolic measure then its metric entropy is approximated by the entropy of a hyperbolic horseshoe. The author has shown in [5] that the result of Katok is also valid for $C^{1+\alpha}$ (non-invertible) maps.

Let $f : I \to I$ be a differentiable map. For integers $k, l \geq 1$, numbers $\nu > 1$ and $\delta > 0$ we say that $\Gamma$ is a $(k, l, \nu, \delta)$-hyperbolic horseshoe of $f$ if $\Gamma$ is a $(k, l)$-horseshoe and

$$|f^k(y(x))| \geq \nu^k, \quad |f^k(x)| \geq \delta \quad (x \in \Gamma).$$

The following lemma plays an important role for the proofs of Theorems 1 and 2.

**Lemma 4.** Let $f : I \to I$ be a $C^{1+\alpha}$ map. If $h(f) > 0$, then for a number $v_0$ with $1 < v_0 < \exp[h(f)]$ there exist sequences $k_j, l_j$ of positive integers and $\delta_j > 0$ ($j \geq 1$) such that for $j \geq 1$ there is a $(k_j, l_j, v_0, \delta_j)$-hyperbolic horseshoe $\Gamma_j$ of $f$ so that

$$h(f) = \lim_{j \to \infty} h(f |_{\Gamma_j}) = \lim_{j \to \infty} \frac{1}{k_j} \log l_j.$$ 

This is corresponding to the result obtained by Katok for surface diffeomorphisms ([9]). For the proof we use the result stated in [5].

Proof of Lemma 4. For a number $v_0$ with $1 < v_0 < \exp[h(f)]$ we take a sequence $\eta_j$ of positive numbers ($j \geq 1$) such that $\exp[h(f) - 3\eta_j] > v_0$ and $\eta_j \to 0$ as $j \to \infty$. By the variational principle for the topological entropy ([6, 7, 8]), we have an $f$-invariant ergodic Borel probability measure $\mu_j$ on $I$ such that

$$h_j \geq h(f) - \eta_j > 0$$

where $h_j$ denotes the metric entropy of $\mu_j$ with respect to $f$. If $\lambda_j$ denotes the Lyapunov exponent of $\mu_j$, that is,

$$\lambda_j = \int \log |f'(x)| d\mu_j(x),$$

for all $n \geq 1$. It was proved by Misiurewicz et al. ([1, 12, 13]) that if the topological entropy of $f$ is positive then there are sequences $k_j, l_j$ of positive integers with a $(k_j, l_j)$-horseshoe $\Gamma_j$ of $f$ ($j \geq 1$) such that

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Then the formula (*) follows from this fact.
then by the Ruelle inequality ([17]) we have
\[ \lambda_j \geq h_j > 0, \]
and so \( \mu_j \) is a hyperbolic measure of \( f \). Then by Theorem C (and its proof) of [5], we can construct sequences of integers \( k_j, l_j \geq 1 \) with \((1/k_j) \cdot \log l_j \geq h_j - \eta_j\), numbers \( c_j \geq 1 \) and closed sets \( \Lambda_j \subset I \ (j \geq 1) \) such that:
(1) \( f^{k_j} \cdot (\Lambda_j) = \Lambda_j \);
(2) \( |(f^{k_j}) \cdot (x)| \geq c_j^{-1} \cdot \exp[k_j \cdot i(\lambda_j - \eta_j)] \) for all \( x \in \Lambda_j \) and \( i \geq 1 \);
(3) \( f^{k_j} \mid_{\Lambda_j} : \Lambda_j \rightarrow \Lambda_j \) is topologically conjugate to a one-sided full shift in \( l_j \)-symbols.

For \( j \geq 1 \) we set
\[ \Gamma_j = \Lambda_j \cup f \Lambda_j \cdots \cup f^{k_j-1} \Lambda_j. \]

Then \( \Gamma_j \) is \( f \)-invariant. Moreover we put
\[ \delta_j = \min[|f^i (x)| : x \in \Gamma_j] > 0, \]
\[ e_j = \max \left\{ \frac{|f^i (x)|}{|f^i (y)|} : x, y \in \Gamma_j \right\} \in [1, \infty) \]
and take an integer \( n_j \geq 1 \) large enough so that
\[ \exp[k_j n_j \eta_j] \geq c_j e_j^{k_j}. \]

Then we have
\[ |(f^{k_j n_j}) \cdot (x)| \geq v_0^{k_j n_j} \quad (x \in \Gamma_j). \]

This follows from the fact that for \( 0 \leq i \leq k_j - 1 \) and \( x \in f^i \Lambda_j \)
\[ |(f^{k_j n_j}) \cdot (x)| = |(f^{k_j n_j}) \cdot (f^{k_j-1} \cdot (x))| \cdot |(f^{k_j-1}) \cdot (f^{k_j n_j} (x))|^{-1} \]
\[ \geq c_j^{-1} \cdot \exp[k_j n_j (\lambda_j - \eta_j)] \cdot e_j^{k_j-1} \]
\[ \geq \exp[k_j n_j (\lambda_j - 2\eta_j)] \]
\[ \geq \exp[k_j n_j (h(f) - 3\eta_j)] \]
\[ \geq v_0^{k_j n_j}. \]

It is easy to see that \( f^{k_j n_j} \mid_{\Lambda_j} : \Lambda_j \rightarrow \Lambda_j \) is topologically conjugate to a one-sided full shift in \( l_j \)-symbols. Thus \( \Gamma_j \) is a \((k_j n_j, l_{j}^{n_j}, v_0, \delta_j)\)-hyperbolic horseshoe, and from which
\[ h(f \mid_{\Gamma_j}) = \frac{1}{k_j n_j} \log l_{j}^{n_j}. \]
\[
= \frac{1}{k_j} \log l_j \\
\geq h_j - \eta_j \\
\geq h(f) - 2\eta_j.
\]

Since \(\eta_j \to 0\) as \(j \to \infty\), we have
\[
h(f) = \lim_{j \to \infty} h(f \mid \tau_j).
\]

Lemma 4 was proved. \(\square\)

Proof of Theorem 1. For \(v > 1\) and \(\delta > 0\) we want to find \(\gamma_0 = \gamma_0(v, \delta) > 0\) such that \(\text{Per}(f, n, v, \delta)\) is an \((n, \gamma_0)\)-separated set of \(f\) for all \(n \geq 1\). Take \(\gamma_1 = \gamma_1(\delta) > 0\) so small that if \(x, y \in I\) satisfy \(|x - y| \leq \gamma_1\) then
\[
|f'(x) - f'(y)| \leq \frac{\delta}{2}.
\]

We put
\[
I_\delta = \left\{ x \in I : |f'(x)| \geq \frac{\delta}{2} \right\}.
\]

Obviously, \(I_\delta\) is closed. For \(n \geq 1\) and \(x \in I\),
\[
|x - \text{Per}(f, n, v, \delta)| \leq \gamma_1 \quad \text{implies that} \quad x \in I_\delta.
\]

Since a function \(x \mapsto \log |f'(x)|\) is bounded and varies continuously on \(I_\delta\), there is \(\gamma_2 = \gamma_2(v, \delta) > 0\) such that if \(x, y \in I_\delta\) satisfy \(|x - y| \leq \gamma_2\) then
\[
\left| \log |f'(x)| - \log |f'(y)| \right| \leq \frac{1}{2} \log v.
\]

We put \(\gamma_0 = \min\{\gamma_1, \gamma_2\}\). Then it is checked that \(\text{Per}(f, n, v, \delta)\) is an \((n, \gamma_0)\)-separated set of \(f\) for \(n \geq 1\). Indeed, if a pair \(p, p' \in \text{Per}(f, n, v, \delta)\) with \(p \leq p'\) satisfies
\[
\max\{|f^i(p) - f^i(p')| : 0 \leq i \leq n - 1\} \leq \gamma_0,
\]
then we see that for \(x \in [p, p']\) and \(0 \leq i \leq n - 1\),
\[
|f^i(x) - f^i(p)| \leq \gamma_0, \quad f^i(x) \in I_\delta.
\]

On the other hand, by the mean value theorem there is a point \(\xi \in [p, p']\) such that
\[
|f^n(p) - f^n(p')| = |(f^n)'(\xi)| \cdot |p - p'|.
\]
Since \( f^i(\xi) \), \( f^i(p) \in I_\delta \) and \( |f^i(\xi) - f^i(p)| \leq \gamma_0 \) for \( 0 \leq i \leq n - 1 \), we have
\[
| \log |(f^n)'(\xi)| - \log |(f^n)'(p)| | \leq \sum_{i=0}^{n-1} | \log |f^i'(f^i(\xi))| - \log |f^i'(f^i(p))| |
\leq \frac{n}{2} \log v,
\]
and so
\[
\frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \geq \exp \left( -\frac{n}{2} \log v \right) = v^{-n/2}.
\]
Since \( p, p' \in \text{Per}(f, n, v, \delta) \), we have
\[
|p - p'| = |f^n(p) - f^n(p')| \\
= |(f^n)'(\xi)| \cdot |p - p'| \\
= \frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \cdot |(f^n)'(p)| \cdot |p - p'| \\
\geq v^{-n/2} \cdot v^n \cdot |p - p'| \\
= v^{n/2} \cdot |p - p'|,
\]
and so \( p = p' \) because of \( v > 1 \). Thus \( \text{Per}(f, n, v, \delta) \) is an \((n, \gamma_0)\)-separated set of \( f \), and then
\[
(1.1) \quad \limsup_{n \to \infty} \frac{1}{n} \log \# \text{Per}(f, n, v, \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log \# \text{Per}(f, n, \gamma) \\
\leq \lim_{\gamma \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \# \text{Per}(f, n, \gamma) \\
= h(f)
\]
for \( v > 1 \) and \( \delta > 0 \), where \( \# \text{Per}(f, n, \gamma) \) denotes the maximal cardinality of \((n, \gamma)\)-separated sets for \( f \). Therefore we have the conclusion of Theorem 1 when \( h(f) = 0 \). Thus it remains to give the proof for the case when \( h(f) > 0 \). Fix \( 1 < v_0 < \exp[h(f)] \). Take sequences \( k_j, l_j, \delta_j \) and \( \Gamma_j \) \((j \geq 1)\) as in Lemma 4. Since
\[
I_{j^n} \leq \# \{ \text{Per}(f, nk_j, v_0, \delta_j) \cap \Gamma_j \} \leq k_j l_{j^n}
\]
for all \( n \geq 1 \), we have
\[
\lim \limsup_{\delta \to 0} \frac{1}{n} \log \# \text{Per}(f, n, v_0, \delta) \geq \lim_{n \to \infty} \frac{1}{nk_j} \log \# \{ \text{Per}(f, nk_j, v_0, \delta_j) \cap \Gamma_j \} \\
= \frac{1}{k_j} \log l_j.
\]
If \( j \to \infty \), then

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{n} \text{Per}(f, n, v_0, \delta) \geq h(f).
\]

Combining (1.1) and (1.2) we have

\[
h(f) \leq \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{n} \text{Per}(f, n, v_0, \delta)
\]
\[
\leq \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{n} \text{Per}(f, n, v, \delta)
\]
\[
\leq h(f).
\]

Theorem 1 was proved. \( \square \)

**Remark.** In fact, from the proof of Theorem 1 it follows that

\[
h(f) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{n} \text{Per}(f, n, v_0, \delta)
\]

if \( 1 < v_0 < \exp[h(f)] \).

**Proof of Theorem 2.** Under the assumption of Theorem 2 we fix a number \( v_0 \) with \( 1 < v_0 < \exp[h(f)] \). By Lemma 4, for \( j \geq 1 \) there are \( k_j, l_j \geq 1 \) and \( \delta_j > 0 \) with a \((k_j, l_j, v_0, \delta_j)\)-hyperbolic horseshoe \( \Gamma_j = \Gamma_j^0 \cup \cdots \cup \Gamma_j^{k_j-1} \) such that

\[
h(f \mid \Gamma_j) = \frac{1}{k_j} \log l_j \to h(f)
\]
as \( j \to \infty \). For \( j \geq 1 \) define a product space

\[
\Sigma_j = \prod_{m=1}^{\infty} \{1, \ldots, l_j\}
\]
with the product topology and a shift \( \sigma_j : \Sigma_j \to \Sigma_j \) by

\[
\sigma_j((a_m)_{m \geq 1}) = (a_{m+1})_{m \geq 1}, \quad ((a_m)_{m \geq 1} \in \Sigma_j).
\]

From the definition of hyperbolic horseshoe, there is a homeomorphism \( \varphi_j : \Sigma_j \to \Gamma_j^0 \) such that \( \varphi_j \circ \sigma_j = (f^{k_j} \mid_{\Gamma_j^0}) \circ \varphi_j \). Then \( p_j = \varphi_j(1, 1, \ldots) \) is a source of \( f \). For \( m \geq 1 \) and \( a_1, \ldots, a_m \in \{1, \ldots, l_j\} \) with \( a_i \neq 1 \) for some \( 1 \leq i \leq m \), \( \varphi_j(a_1, \ldots, a_m, 1, 1, \ldots) \) is a transversal homoclinic point of \( p_j \). Thus, \( \text{TH}(p_j) \supset \Gamma_j \), from which
\[ h(f) = \lim_{j \to \infty} h(f|_r) \]
\[ \leq \lim_{j \to \infty} h(f|_{TH(p)}) \]
\[ \leq \sup \{ h(f|_{TH(p)}): p \text{ is a source of } f \} \]
\[ \leq h(f). \]

The first statement was proved.

**Proof of the second statement.** Let \( p \) be a source of \( f \). Without loss of generality we may assume that \( p \) is a fixed point, i.e., \( f(p) = p \). To show that for \( \delta > 0 \)
\[ h(f|_{TH(p)}) \geq \lim_{m \to \infty} \frac{1}{m} \log \frac{\varepsilon}{H(p, m, \delta)}, \]
take \( \gamma_0 = \gamma_0(\delta) > 0 \) so small that if \( x, y \in I \) satisfy \( |x - y| \leq \gamma_0 \) then
\[ |f^i(x) - f^i(y)| \leq \frac{\delta}{2}. \]
Then, for \( m \geq 1 \) and a pair \( q, q' \in H(p, m, \delta) \) satisfying
\[ \max \{|f^i(q) - f^i(q')|: 0 \leq i \leq m - 1\} \leq \gamma_0, \]
we can find a sequence \( \xi_0, \ldots, \xi_{m-1} \in I \) such that
\[ |\xi_i - f^i(q)| \leq \gamma_0 \]
and
\[ |f^{i+1}(q) - f^{i+1}(q')| = |f^i(\xi_i)| : |f^i(q) - f^i(q')| \quad (0 \leq i \leq m - 1). \]
Since \( f^m(q) = f^m(q') = p \), we have
\[ 0 = |f^m(q) - f^m(q')| = |f^i(\xi_{m-1})| \cdot |f^{m-1}(q) - f^{m-1}(q')| \]
\[ = \cdots = \prod_{i=0}^{m-1} |f^i(\xi_i)| \cdot |q - q'| \]
\[ \geq \prod_{i=0}^{m-1} \left( \frac{|f^i(f^i(q))|}{\delta} - \frac{\delta}{2} \right) \cdot |q - q'| \]
\[ \geq \left( \frac{\delta}{2} \right)^m \cdot |q - q'|, \]
and so \( q = q' \). Thus \( H(p, m, \delta) \) is an \((m, \gamma_0)\)-separated set of \( f|_{TH(p)} \), from which it follows that
\[ \limsup_{m \to \infty} \frac{1}{m} \log \mathcal{H}(p, m, \delta) \leq \limsup_{m \to \infty} \frac{1}{m} \log s(f |_{\mathcal{H}(p)}, m, \gamma_0) \leq h(f |_{\mathcal{H}(p)}). \]

If \( h(f |_{\mathcal{H}(p)}) = 0 \), then nothing to prove for the second statement. Thus we must check the conclusion for the case when \( h(f |_{\mathcal{H}(p)}) > 0 \). To do so fix a number \( v_0 \) with \( 1 < v_0 < \min \{ v(p), \exp h(f |_{\mathcal{H}(p)}) \} \). By the same way as in the proof of Lemma 4, we can take sequences of integers \( k_j, l_j \geq 1 \), numbers \( \delta_j > 0 \) with \( (k_j, l_j, v_0, \delta_j) \)-hyperbolic horseshoes \( \Gamma_j = \Gamma_0^0 \cup \cdots \cup \Gamma_{k_j-1}^0 \) containing \( p \ (j \geq 1) \) such that

\[
\lim_{j \to \infty} \frac{1}{k_j} \log l_j \to h(f |_{\mathcal{H}(p)}) \quad \text{as} \quad j \to \infty.
\]

Then there is a homeomorphism \( \varphi_j : \Sigma_j \to \Gamma^0_j \) such that \( \varphi_j \circ \sigma_j = (f^{k_j} |_{\Gamma_j^0}) \circ \varphi_j \), where \( \sigma_j : \Sigma_j \to \Sigma_j \) is the shift defined as in the proof of the first statement. Without loss of generality we may assume that \( \varphi_j(1, 1, \ldots) = p \). By taking an integer \( n_j \geq 1 \) large enough we have

\[
\varphi_j([1, \ldots, 1]_{n_j}) \subset W^u_{\text{loc}}(p)
\]

where

\[
[1, \ldots, 1]_{n_j} = \{ (b_m)_{m \geq 1} \in \Sigma_j : b_m = 1 \text{ for all } 1 \leq m \leq n_j \}.
\]

Since

\[
\underbrace{\varphi_j(1, \ldots, 1, a_1, \ldots, a_{m-n_j}, 1, 1, \ldots)}_{n_j \text{ times}} \in H(p, mk_j, \delta_j)
\]

holds for all \( m \geq n_j + 1 \) and \( a_1, \ldots, a_{m-n_j} \in [1, \ldots, l_j] \), we have

\[
\mathcal{H}(p, mk_j, \delta_j) \geq l_j^{m-n_j}.
\]

Thus,

\[
\lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \mathcal{H}(p, m, \delta) \geq \limsup_{m \to \infty} \frac{1}{mk_j} \log \mathcal{H}(p, mk_j, \delta_j) \geq \lim_{m \to \infty} \frac{m-n_j}{mk_j} \log l_j = \frac{1}{k_j} \log l_j
\]

for \( j \geq 1 \). If \( j \to \infty \), then we have

\[
\lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \mathcal{H}(p, m, \delta) \geq h(f |_{\mathcal{H}(p)}).
\]
Combining (1.3) and (1.4),
\[
h(f |_{\mathcal{H}(p)}) \leq \lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log m H(p, m, \delta) \leq h(f |_{\mathcal{H}(p)}).
\]

The second statement was proved. This completes the proof of Theorem 2.

2. Circle Maps

In the same way as above, it can be checked that our results (Theorems 1 and 2) are also valid for \(C^{1+\alpha}\) maps \((\alpha > 0)\) of the circle \(S^1\). However, the existence of a homoclinic point does not imply that the topological entropy is positive. In fact, we know an example of a \(C^\infty\) map \(g : S^1 \to S^1\) such that \(g\) has a homoclinic point of a source fixed point, nevertheless \(h(g) = 0\) ([16]). It is known that the topological entropy of a continuous circle map is positive if and only if the map has a nonwandering homoclinic point of a periodic point ([4]). Since any transversal homoclinic point of a source is nonwandering, we have:

**Corollary 5.** For a \(C^{1+\alpha}\) map \(f : S^1 \to S^1\) \((\alpha > 0)\) the following statements are equivalent:

(i) \(h(f) > 0\);
(ii) \(f\) has a transversal homoclinic point of a source;
(iii) \(f\) has a nonwandering homoclinic point of a periodic point.

**Added in proof.** After this manuscript was completed the author learned from A. Katok that he and A. Mezhirov had obtained a result that overlaps with Theorem 1 for \(C^1\) maps with finitely many critical points ([18]).

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**References**


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