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TOPOLOGICAL ENTROPY FOR DIFFERENTIABLE MAPS OF INTERVALS

YONG MOO CHUNG

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Let I be a compact interval of the real line. For a continuous map $f : I \to I$ by Misiurewicz et al. ([1, 12, 13]) the following relation between the topological entropy $h(f)$ and the growth rate of the number of periodic points is known:

$$(*) \qquad h(f) \le \limsup_{n\to\infty} \frac{1}{n} \log \sharp \operatorname{Per}(f, n)$$

where $\operatorname{Per}(f, n)$ denotes the set of all fixed points of f^n for $n \ge 1$, and $\sharp A$ the number of elements of a set A. (The equality of the expression $(*)$ does not hold in general. For instance, the topological entropy of the identity map is zero, nevertheless all of points of the interval are fixed by this map.)

For a periodic point p of f with period n we put

$$\mathcal{O}_f^+(p) = \{p, f(p), \cdots, f^{n-1}(p)\}.$$

Then we say that q is a *homoclinic point* of p if $q \notin \mathcal{O}_f^+(p)$ and there are a positive integer m with $f^m(q) = p$ and a sequence $q_0, q_1, \ldots, q_k, \ldots \in I$ with $q_0 = q$ such that

$$f(q_k) = q_{k-1} \ (k \ge 1), \qquad \lim_{k\to\infty} |q_k - \mathcal{O}_f^+(p)| = 0$$

where $|x - A| = \inf\{|x - y| : y \in A\}$ for $x \in I$, $A \subset I$. It is known by Block ([2, 3]) that $h(f)$ is positive if and only if f has a homoclinic point of a periodic point.

In this paper we shall establish more results (Theorems 1 and 2) for differentiable maps of intervals. To describe them we need some notations.

Let $f : I \to I$ be a $C^{1+\alpha}$ map ($\alpha > 0$). A periodic point p of f with period n is a *source* if

$$\nu(p) = |(f^n)'(p)|^{1/n} > 1.$$

For $n \ge 1$, $\nu > 1$ and $\delta > 0$ we define an f-invariant set by

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$$\mathrm{Per}(f, n, \nu, \delta) = \{p \in \mathrm{Per}(f, n) : \nu(p) \geq \nu, \ |f'(f^i(p))| \geq \delta \text{ for all } 0 \leq i \leq n-1\}.$$

Then we have

$$\mathrm{Per}(f, n, \nu_1, \delta_1) \subset \mathrm{Per}(f, n, \nu_2, \delta_2) \quad \text{if} \quad \nu_1 \geq \nu_2, \delta_1 \geq \delta_2,$$

and

$$\{p : \text{source of } f\} = \bigcup_{\nu > 1} \bigcup_{\delta > 0} \bigcup_{n=1}^{\infty} \mathrm{Per}(f, n, \nu, \delta).$$

One of our results is the following:

Theorem 1.

$$h(f) = \max \left\{ 0, \lim_{\nu \to 1} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sharp \, \mathrm{Per}(f, n, \nu, \delta) \right\}.$$

By Theorem 1 it is clear that for a $C^{1+\alpha}$ map of a compact interval if the topological entropy is positive then the map has infinitely many sources. However, the converse is not true in general. In fact, for any $r \geq 1$ it is easy to constract a C^r diffeomorphism of a compact interval having infinitely many source fixed points. But every diffeomorphism of an interval has zero entropy.

REMARK. It is known that if f is a C^2 map with non-flat critical points, then any periodic point of f with sufficiently large period is a source ([10]). In Theorem 1 we do not assume any conditition concerned with critical points. Then the map f may have flat critical points.

For a source p of f with period n we denote by $W^u_{\mathrm{loc}}(p)$ the maximal interval J of I containing p such that

$$|(f^n)'(x)| \geq \{(1 + \nu(p))/2\}^n \quad \text{for all} \quad x \in J.$$

We say that a homoclinic point q of p is *transversal* if there are non-negative integers m_1, m_2 and a point $q' \in W^u_{\mathrm{loc}}(p)$ such that

$$f^{m_1}(q') = q, \quad f^{m_1+m_2}(q') = f^{m_2}(q) = p \quad \text{and} \quad (f^{m_1+m_2})'(q') \neq 0.$$

If f has a transversal homoclinic point of a source, then there is a C^1 neighborhood $\mathcal{U}$ of f such that every map g belonging to $\mathcal{U}$ has a transversal homoclinic point of a source. We denote the set of transversal homoclinic points of a source p of f by $\mathrm{TH}(p)$, and its closure by $\overline{\mathrm{TH}(p)}$. We call $\overline{\mathrm{TH}(p)}$ the *transversal homolinic closure* of

p. It is easy to see that $p \in \overline{\mathrm{TH}(p)}$, and $\overline{\mathrm{TH}(p)}$ is f-invariant. For $m \geq 1$ and $\delta > 0$ define

$$H(p, m, \delta) = \{q \in W^u_{\mathrm{loc}}(p) : f^m(q) = p, |f'(f^i(q))| \geq \delta \text{ for all } 0 \leq i \leq m-1\}.$$

Then we have

$$H(p, m, \delta_1) \subset H(p, m, \delta_2) \quad \text{if} \quad \delta_1 \geq \delta_2$$

and

$$\mathrm{TH}(p) = \bigcup_{\delta>0} \bigcup_{m=1}^{\infty} \bigcup_{i=0}^{m-1} f^i H(p, m, \delta) \setminus \mathcal{O}^+_f(p).$$

The second result of this paper is the following:

Theorem 2. *If $h(f) > 0$ then*

$$h(f) = \sup\{h(f\,|_{\overline{\mathrm{TH}(p)}}) : p \textit{ is a source of } f\},$$

and for a source p of f we have

$$h(f\,|_{\overline{\mathrm{TH}(p)}}) = \max\left\{0, \lim_{\delta\to 0} \limsup_{m\to\infty} \frac{1}{m} \log \sharp H(p, m, \delta)\right\}.$$

A result corresponding to Theorem 2 is known for surface diffeomorphisms by Mendoza ([11]). As an easy corollary of Theorem 2 we have:

Corollary 3. *The following statements are equivalent*:
(i) $h(f) > 0$;
(ii) *f has a transversal homoclinic point of a source*;
(iii) *f has a homoclinic point of a periodic point.*

1. Proofs of Theorems

Let $f : I \to I$ be a continuous map. For integers $k, l \geq 1$ we say that a closed f-invariant set Γ is a (k, l)-*horseshoe* of f if there are subsets $\Gamma^0, \ldots, \Gamma^{k-1}$ of I such that

$$\Gamma = \Gamma^0 \cup \cdots \cup \Gamma^{k-1}, \qquad f(\Gamma^j) = \Gamma^{j+1} \pmod{k}$$

and $f^k\,|_{\Gamma^0} : \Gamma^0 \to \Gamma^0$ is topologically conjugate to a one-sided full shift in l-symbols. If Γ is a (k, l)-horseshoe, then it is clear that

$$h(f\,|_\Gamma) = \frac{1}{k} \log l \quad \text{and} \quad l^n \leq \sharp[\mathrm{Per}(f, kn) \cap \Gamma] \leq k l^n$$

for all $n \geq 1$. It was proved by Misiurewicz et al. ([1, 12, 13]) that if the topological entropy of f is positive then there are sequences k_j, l_j of positive integers with a (k_j, l_j)-horseshoe Γ_j of f ($j \geq 1$) such that

$$h(f) = \lim_{j\to\infty} h(f|_{\Gamma_j}) = \lim_{j\to\infty} \frac{1}{k_j} \log l_j.$$

Then the formula ($*$) follows from this fact.

In order to prove our results, we need the notion of hyperbolic horseshoe and ideas of the theory of hyperbolic measures ([14, 15]). Katok ([9]) has proved that if a $C^{1+\alpha}$ diffeomorphism of a manifold has a hyperbolic measure then its metric entropy is approximated by the entropy of a hyperbolic horseshoe. The author has shown in [5] that the result of Katok is also valid for $C^{1+\alpha}$ (non-invertible) maps.

Let $f : I \to I$ be a differentiable map. For integers $k, l \geq 1$, numbers $\nu > 1$ and $\delta > 0$ we say that Γ is a (k, l, ν, δ)-*hyperbolic horseshoe* of f if Γ is a (k, l)-horseshoe and

$$|(f^k)'(x)| \geq \nu^k, \quad |f'(x)| \geq \delta \quad (x \in \Gamma).$$

The following lemma plays an important role for the proofs of Theorems 1 and 2.

Lemma 4. *Let $f : I \to I$ be a $C^{1+\alpha}$ map. If $h(f) > 0$, then for a number ν_0 with $1 < \nu_0 < \exp\{h(f)\}$ there exist sequences k_j, l_j of positive integers and $\delta_j > 0$ ($j \geq 1$) such that for $j \geq 1$ there is a $(k_j, l_j, \nu_0, \delta_j)$-hyperbolic horseshoe Γ_j of f so that*

$$h(f) = \lim_{j\to\infty} h(f|_{\Gamma_j}) = \lim_{j\to\infty} \frac{1}{k_j} \log l_j.$$

This is corresponding to the result obtained by Katok for surface diffeomorphisms ([9]). For the proof we use the result stated in [5].

Proof of Lemma 4. For a number ν_0 with $1 < \nu_0 < \exp\{h(f)\}$ we take a sequence η_j of positive numbers ($j \geq 1$) such that $\exp\{h(f) - 3\eta_j\} > \nu_0$ and $\eta_j \to 0$ as $j \to \infty$. By the variational principle for the topological entropy ([6, 7, 8]), we have an f-invariant ergodic Borel probability measure μ_j on I such that

$$h_j \geq h(f) - \eta_j > 0$$

where h_j denotes the metric entropy of μ_j with respect to f. If λ_j denotes the Lyapunov exponent of μ_j, that is,

$$\lambda_j = \int \log |f'(x)| d\mu_j(x),$$

then by the Ruelle inequality ([17]) we have

$$\lambda_j \geq h_j > 0,$$

and so μ_j is a hyperbolic measure of f. Then by Theorem C (and its proof) of [5], we can construct sequences of integers k_j, $l_j \geq 1$ with $(1/k_j) \cdot \log l_j \geq h_j - \eta_j$, numbers $c_j \geq 1$ and closed sets $\Lambda_j \subset I$ $(j \geq 1)$ such that:
(1) $f^{k_j}(\Lambda_j) = \Lambda_j$;
(2) $|(f^{k_j i})'(x)| \geq c_j{}^{-1} \cdot \exp\{k_j i(\lambda_j - \eta_j)\}$ for all $x \in \Lambda_j$ and $i \geq 1$;
(3) $f^{k_j} |_{\Lambda_j} \colon \Lambda_j \to \Lambda_j$ is topologically conjugate to a one-sided full shift in l_j-symbols.
For $j \geq 1$ we set

$$\Gamma_j = \Lambda_j \cup f\Lambda_j \cdots \cup f^{k_j - 1}\Lambda_j.$$

Then Γ_j is f-invariant. Moreover we put

$$\begin{aligned} \delta_j &= \min\{|f'(x)| : x \in \Gamma_j\} > 0, \\ e_j &= \max\left\{\frac{|f'(x)|}{|f'(y)|} : x, y \in \Gamma_j\right\} \in [1, \infty) \end{aligned}$$

and take an integer $n_j \geq 1$ large enough so that

$$\exp\{k_j n_j \eta_j\} \geq c_j e_j{}^{k_j}.$$

Then we have

$$|(f^{k_j n_j})'(x)| \geq \nu_0{}^{k_j n_j} \quad (x \in \Gamma_j).$$

This follows from the fact that for $0 \leq i \leq k_j - 1$ and $x \in f^i \Lambda_j$

$$\begin{aligned} |(f^{k_j n_j})'(x)| &= |(f^{k_j n_j})'(f^{k_j - i}(x))| \cdot |(f^{k_j - i})'(x)| \cdot |(f^{k_j - i})'(f^{k_j n_j}(x))|^{-1} \\ &\geq c_j{}^{-1} \cdot \exp\{(k_j n_j)(\lambda_j - \eta_j)\} \cdot e_j{}^{-k_j + i} \\ &\geq \exp\{k_j n_j(\lambda_j - 2\eta_j)\} \\ &\geq \exp\{k_j n_j(h(f) - 3\eta_j)\} \\ &\geq \nu_0{}^{k_j n_j}. \end{aligned}$$

It is easy to see that $f^{k_j n_j} |_{\Lambda_j} \colon \Lambda_j \to \Lambda_j$ is topologically conjugate to a one-sided full shift in $l_j{}^{n_j}$-symbols. Thus Γ_j is a $(k_j n_j, l_j{}^{n_j}, \nu_0, \delta_j)$-hyperbolic horseshoe, and from which

$$h(f |_{\Gamma_j}) = \frac{1}{k_j n_j} \log l_j{}^{n_j}$$

$$
\begin{aligned}
&= \frac{1}{k_j}\log l_j \\
&\geq h_j - \eta_j \\
&\geq h(f) - 2\eta_j .
\end{aligned}
$$

Since $\eta_j \to 0$ as $j \to \infty$, we have

$$h(f) = \lim_{j\to\infty} h(f|_{\Gamma_j}).$$

Lemma 4 was proved. $\square$

Proof of Theorem 1. For $\nu > 1$ and $\delta > 0$ we want to find $\gamma_0 = \gamma_0(\nu, \delta) > 0$ such that $\mathrm{Per}(f, n, \nu, \delta)$ is an (n, γ_0)-separated set of f for all $n \geq 1$. Take $\gamma_1 = \gamma_1(\delta) > 0$ so small that if $x, y \in I$ satisfy $|x - y| \leq \gamma_1$ then

$$|f'(x) - f'(y)| \leq \frac{\delta}{2}.$$

We put

$$I_\delta = \left\{ x \in I : |f'(x)| \geq \frac{\delta}{2} \right\}.$$

Obviously, I_δ is closed. For $n \geq 1$ and $x \in I$,

$$|x - \mathrm{Per}(f, n, \nu, \delta)| \leq \gamma_1 \quad \text{implies that} \quad x \in I_\delta.$$

Since a function $x \mapsto \log|f'(x)|$ is bounded and varies continuously on I_δ, there is $\gamma_2 = \gamma_2(\nu, \delta) > 0$ such that if $x, y \in I_\delta$ satisfy $|x - y| \leq \gamma_2$ then

$$\big| \log|f'(x)| - \log|f'(y)| \big| \leq \frac{1}{2}\log\nu.$$

We put $\gamma_0 = \min\{\gamma_1, \gamma_2\}$. Then it is checked that $\mathrm{Per}(f, n, \nu, \delta)$ is an (n, γ_0)-separated set of f for $n \geq 1$. Indeed, if a pair $p, p' \in \mathrm{Per}(f, n, \nu, \delta)$ with $p \leq p'$ satisfies

$$\max\{|f^i(p) - f^i(p')| : 0 \leq i \leq n-1\} \leq \gamma_0,$$

then we see that for $x \in [p, p']$ and $0 \leq i \leq n-1$,

$$|f^i(x) - f^i(p)| \leq \gamma_0, \qquad f^i(x) \in I_\delta.$$

On the other hand, by the mean value theorem there is a point $\xi \in [p, p']$ such that

$$|f^n(p) - f^n(p')| = |(f^n)'(\xi)| \cdot |p - p'|.$$

Since $f^i(\xi),\ f^i(p) \in I_\delta$ and $|f^i(\xi) - f^i(p)| \le \gamma_0$ for $0 \le i \le n-1$, we have

$$\Big|\log|(f^n)'(\xi)| - \log|(f^n)'(p)|\Big| \le \sum_{i=0}^{n-1} \Big|\log|f'(f^i(\xi))| - \log|f'(f^i(p))|\Big| \le \frac{n}{2}\log\nu,$$

and so

$$\frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \ge \exp\left(-\frac{n}{2}\log\nu\right) = \nu^{-n/2}.$$

Since $p,\ p' \in \mathrm{Per}(f, n, \nu, \delta)$, we have

$$\begin{aligned}|p-p'| &= |f^n(p) - f^n(p')| \\ &= |(f^n)'(\xi)| \cdot |p-p'| \\ &= \frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \cdot |(f^n)'(p)| \cdot |p-p'| \\ &\ge \nu^{-n/2} \cdot \nu^n \cdot |p-p'| \\ &= \nu^{n/2} \cdot |p-p'|,\end{aligned}$$

and so $p = p'$ because of $\nu > 1$. Thus $\mathrm{Per}(f, n, \nu, \delta)$ is an (n, γ_0)-separated set of f, and then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\log \sharp\,\mathrm{Per}(f, n, \nu, \delta) &\le \limsup_{n\to\infty}\frac{1}{n}\log s(f, n, \gamma_0) \\ &\le \lim_{\gamma\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s(f, n, \gamma) \\ &= h(f)\end{aligned} \tag{1.1}$$

for $\nu > 1$ and $\delta > 0$, where $s(f, n, \gamma)$ denotes the maximal cardinality of (n, γ)-separated sets for f. Therefore we have the conclusion of Theorem 1 when $h(f) = 0$. Thus it remains to give the proof for the case when $h(f) > 0$. Fix $1 < \nu_0 < \exp\{h(f)\}$. Take sequences k_j, l_j, δ_j and Γ_j $(j \ge 1)$ as in Lemma 4. Since

$${l_j}^n \le \sharp[\mathrm{Per}(f, nk_j, \nu_0, \delta_j) \cap \Gamma_j] \le k_j {l_j}^n$$

for all $n \ge 1$, we have

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sharp\,\mathrm{Per}(f, n, \nu_0, \delta) &\ge \lim_{n\to\infty}\frac{1}{nk_j}\log\sharp[\mathrm{Per}(f, nk_j, \nu_0, \delta_j)\cap\Gamma_j] \\ &= \frac{1}{k_j}\log l_j.\end{aligned}$$

If $j \to \infty$, then

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sharp\,\mathrm{Per}(f,n,\nu_0,\delta)\geq h(f). \tag{1.2}$$

Combining (1.1) and (1.2) we have

$$\begin{aligned} h(f) &\leq \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sharp\,\mathrm{Per}(f,n,\nu_0,\delta) \\ &\leq \lim_{\nu\to 1}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sharp\,\mathrm{Per}(f,n,\nu,\delta) \\ &\leq h(f). \end{aligned}$$

Theorem 1 was proved. □

REMARK. In fact, from the proof of Theorem 1 it follows that

$$h(f) = \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sharp\,\mathrm{Per}(f,n,\nu_0,\delta)$$

if $1 < \nu_0 < \exp\{h(f)\}$.

Proof of Theorem 2.

Proof of the first statement. Under the assumption of Theorem 2 we fix a number ν_0 with $1 < \nu_0 < \exp\{h(f)\}$. By Lemma 4, for $j \geq 1$ there are k_j, $l_j \geq 1$ and $\delta_j > 0$ with a $(k_j, l_j, \nu_0, \delta_j)$-hyperbolic horseshoe $\Gamma_j = \Gamma_j^0 \cup \cdots \cup \Gamma_j^{k_j-1}$ such that

$$h(f\,|_{\Gamma_j}) = \frac{1}{k_j}\log l_j \to h(f)$$

as $j \to \infty$. For $j \geq 1$ define a product space

$$\Sigma_j = \prod_{m=1}^{\infty}\{1,\ldots,l_j\}$$

with the product topology and a shift $\sigma_j : \Sigma_j \to \Sigma_j$ by

$$\sigma_j((a_m)_{m\geq 1}) = (a_{m+1})_{m\geq 1} \quad ((a_m)_{m\geq 1}\in\Sigma_j).$$

From the definition of hyperbolic horseshoe, there is a homeomorphism $\varphi_j : \Sigma_j \to \Gamma_j^0$ such that $\varphi_j \circ \sigma_j = (f^{k_j}\,|_{\Gamma_j^0}) \circ \varphi_j$. Then $p_j = \varphi_j(1,1,\ldots)$ is a source of f. For $m \geq 1$ and $a_1,\ldots,a_m \in \{1,\ldots,l_j\}$ with $a_i \neq 1$ for some $1 \leq i \leq m$, $\varphi_j(a_1,\ldots,a_m,1,1,\ldots)$ is a transversal homoclinic point of p_j. Thus, $\overline{\mathrm{TH}(p_j)} \supset \Gamma_j$, from which

$$\begin{aligned} h(f) &= \lim_{j\to\infty} h(f|_{\Gamma_j}) \\ &\le \lim_{j\to\infty} h(f|_{\overline{TH(p_j)}}) \\ &\le \sup\{h(f|_{\overline{TH(p)}}) : p \text{ is a source of } f\} \\ &\le h(f). \end{aligned}$$

The first statement was proved.

Proof of the second statement. Let p be a source of f. Without loss of generality we may assume that p is a fixed point, i.e., $f(p) = p$. To show that for $\delta > 0$

$$h(f|_{\overline{\mathrm{TH}(p)}}) \ge \limsup_{m\to\infty} \frac{1}{m} \log \sharp H(p, m, \delta),$$

take $\gamma_0 = \gamma_0(\delta) > 0$ so small that if $x, y \in I$ satisfy $|x - y| \le \gamma_0$ then

$$|f'(x) - f'(y)| \le \frac{\delta}{2}.$$

Then, for $m \ge 1$ and a pair $q, q' \in H(p, m, \delta)$ satisfying

$$\max\{|f^i(q) - f^i(q')| : 0 \le i \le m-1\} \le \gamma_0,$$

we can find a sequence $\xi_0, \ldots, \xi_{m-1} \in I$ such that

$$|\xi_i - f^i(q)| \le \gamma_0$$

and

$$|f^{i+1}(q) - f^{i+1}(q')| = |f'(\xi_i)| \cdot |f^i(q) - f^i(q')| \quad (0 \le i \le m-1).$$

Since $f^m(q) = f^m(q') = p$, we have

$$\begin{aligned} 0 = |f^m(q) - f^m(q')| &= |f'(\xi_{m-1})| \cdot |f^{m-1}(q) - f^{m-1}(q')| \\ = \cdots &= \prod_{i=0}^{m-1} |f'(\xi_i)| \cdot |q - q'| \\ &\ge \prod_{i=0}^{m-1} \left(|f'(f^i(q))| - \frac{\delta}{2}\right) \cdot |q - q'| \\ &\ge \left(\frac{\delta}{2}\right)^m \cdot |q - q'|, \end{aligned}$$

and so $q = q'$. Thus $H(p, m, \delta)$ is an (m, γ_0)-separated set of $f|_{\overline{\mathrm{TH}(p)}}$, from which it follows that

$$\limsup_{m\to\infty}\frac{1}{m}\log\sharp H(p,m,\delta)\le\limsup_{m\to\infty}\frac{1}{m}\log s(f\,|_{\overline{\mathrm{TH}(p)}},m,\gamma_0)\le h(f\,|_{\overline{\mathrm{TH}(p)}}).\tag{1.3}$$

If $h(f\,|_{\overline{\mathrm{TH}(p)}})=0$, then nothing to prove for the second statement. Thus we must check the conclusion for the case when $h(f\,|_{\overline{\mathrm{TH}(p)}})>0$. To do so fix a number ν_0 with $1<\nu_0<\min\{\nu(p),\exp h(f\,|_{\overline{\mathrm{TH}(p)}})\}$. By the same way as in the proof of Lemma 4, we can take sequences of integers k_j, $l_j\ge 1$, numbers $\delta_j>0$ with (k_j,l_j,ν_0,δ_j)-hyperbolic horseshoes $\Gamma_j=\Gamma_j^0\cup\cdots\cup\Gamma_j^{k_j-1}$ containing p $(j\ge 1)$ such that

$$h(f\,|_{\Gamma_j})=(1/k_j)\cdot\log l_j\to h(f\,|_{\overline{\mathrm{TH}(p)}})\quad\text{as}\quad j\to\infty.$$

Then there is a homeomorphism $\varphi_j:\Sigma_j\to\Gamma_j^0$ such that $\varphi_j\circ\sigma_j=(f^{k_j}\,|_{\Gamma_j^0})\circ\varphi_j$, where $\sigma_j:\Sigma_j\to\Sigma_j$ is the shift defined as in the proof of the first statement. Without loss of generality we may assume that $\varphi_j(1,1,\dots)=p$. By taking an integer $n_j\ge 1$ large enough we have

$$\varphi_j([1,\dots,1]_{n_j})\subset W^u_{\mathrm{loc}}(p)$$

where

$$[1,\dots,1]_{n_j}=\{(b_m)_{m\ge 1}\in\Sigma_j : b_m=1 \text{ for all } 1\le m\le n_j\}.$$

Since

$$\varphi_j(\overbrace{1,\dots,1}^{n_j\ \text{times}},a_1,\dots,a_{m-n_j},1,1,\dots)\in H(p,mk_j,\delta_j)$$

holds for all $m\ge n_j+1$ and $a_1,\dots,a_{m-n_j}\in\{1,\dots,l_j\}$, we have

$$\sharp H(p,mk_j,\delta_j)\ge l_j^{m-n_j}.$$

Thus,

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{m\to\infty}\frac{1}{m}\log\sharp H(p,m,\delta)&\ge\limsup_{m\to\infty}\frac{1}{mk_j}\log\sharp H(p,mk_j,\delta_j)\\&\ge\lim_{m\to\infty}\frac{m-n_j}{mk_j}\log l_j\\&=\frac{1}{k_j}\log l_j\end{aligned}$$

for $j\ge 1$. If $j\to\infty$, then we have

$$\lim_{\delta\to 0}\limsup_{m\to\infty}\frac{1}{m}\log\sharp H(p,m,\delta)\ge h(f\,|_{\overline{\mathrm{TH}(p)}}).\tag{1.4}$$

Combining (1.3) and (1.4),

$$
\begin{aligned}
h(f\,|_{\overline{\mathrm{TH}(p)}}) &\leq \lim_{\delta\to 0}\limsup_{m\to\infty}\frac{1}{m}\log\sharp H(p,m,\delta)\\
&\leq h(f\,|_{\overline{\mathrm{TH}(p)}}).
\end{aligned}
$$

The second statement was proved. This completes the proof of Theorem 2. □

2. Circle Maps

In the same way as above, it can be checked that our results (Theorems 1 and 2) are also valid for $C^{1+\alpha}$ maps ($\alpha > 0$) of the circle S^1. However, the existence of a homoclinic point does not imply that the topological entropy is positive. In fact, we know an example of a C^∞ map $g : S^1 \to S^1$ such that g has a homoclinic point of a source fixed point, nevertheless $h(g) = 0$ ([16]). It is known that the topological entropy of a continuous circle map is positive if and only if the map has a nonwandering homocinic point of a periodic point ([4]). Since any transversal homoclinic point of a source is nonwandering, we have:

Corollary 5. *For a* $C^{1+\alpha}$ *map* $f : S^1 \to S^1$ ($\alpha > 0$) *the following statements are equivalent*:
(i) $h(f) > 0$;
(ii) f *has a transversal homoclinic point of a source*;
(iii) f *has a nonwandering homoclinic point of a periodic point.*

Added in proof. After this manuscript was completed the author learned from A. Katok that he and A. Mezhirov had obtained a result that overlaps with Theorem 1 for C^1 maps with finitely many critical points ([18]). □

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Department of Applied Mathematics
Faculty of Engineering
Hiroshima University
Higashi-Hiroshima 739-8527, Japan
e-mail: chung@amath.hiroshima-u.ac.jp