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## TOPOLOGICAL ENTROPY FOR DIFFERENTIABLE MAPS OF INTERVALS

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Let  $I$  be a compact interval of the real line. For a continuous map  $f : I \rightarrow I$  by Misiurewicz et al. ([1, 12, 13]) the following relation between the topological entropy  $h(f)$  and the growth rate of the number of periodic points is known:

$$(*) \quad h(f) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n)$$

where  $\text{Per}(f, n)$  denotes the set of all fixed points of  $f^n$  for  $n \geq 1$ , and  $\#A$  the number of elements of a set  $A$ . (The equality of the expression  $(*)$  does not hold in general. For instance, the topological entropy of the identity map is zero, nevertheless all of points of the interval are fixed by this map.)

For a periodic point  $p$  of  $f$  with period  $n$  we put

$$\mathcal{O}_f^+(p) = \{p, f(p), \dots, f^{n-1}(p)\}.$$

Then we say that  $q$  is a *homoclinic point* of  $p$  if  $q \notin \mathcal{O}_f^+(p)$  and there are a positive integer  $m$  with  $f^m(q) = p$  and a sequence  $q_0, q_1, \dots, q_k, \dots \in I$  with  $q_0 = q$  such that

$$f(q_k) = q_{k-1} \ (k \geq 1), \quad \lim_{k \rightarrow \infty} |q_k - \mathcal{O}_f^+(p)| = 0$$

where  $|x - A| = \inf\{|x - y| : y \in A\}$  for  $x \in I$ ,  $A \subset I$ . It is known by Block ([2, 3]) that  $h(f)$  is positive if and only if  $f$  has a homoclinic point of a periodic point.

In this paper we shall establish more results (Theorems 1 and 2) for differentiable maps of intervals. To describe them we need some notations.

Let  $f : I \rightarrow I$  be a  $C^{1+\alpha}$  map ( $\alpha > 0$ ). A periodic point  $p$  of  $f$  with period  $n$  is a *source* if

$$v(p) = |(f^n)'(p)|^{1/n} > 1.$$

For  $n \geq 1$ ,  $v > 1$  and  $\delta > 0$  we define an  $f$ -invariant set by

$$\text{Per}(f, n, \nu, \delta) = \{p \in \text{Per}(f, n) : \nu(p) \geq \nu, |f'(f^i(p))| \geq \delta \text{ for all } 0 \leq i \leq n-1\}.$$

Then we have

$$\text{Per}(f, n, \nu_1, \delta_1) \subset \text{Per}(f, n, \nu_2, \delta_2) \quad \text{if} \quad \nu_1 \geq \nu_2, \delta_1 \geq \delta_2,$$

and

$$\{p : \text{source of } f\} = \bigcup_{\nu > 1} \bigcup_{\delta > 0} \bigcup_{n=1}^{\infty} \text{Per}(f, n, \nu, \delta).$$

One of our results is the following:

**Theorem 1.**

$$h(f) = \max \left\{ 0, \lim_{\nu \rightarrow 1} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n, \nu, \delta) \right\}.$$

By Theorem 1 it is clear that for a  $C^{1+\alpha}$  map of a compact interval if the topological entropy is positive then the map has infinitely many sources. However, the converse is not true in general. In fact, for any  $r \geq 1$  it is easy to construct a  $C^r$  diffeomorphism of a compact interval having infinitely many source fixed points. But every diffeomorphism of an interval has zero entropy.

REMARK. It is known that if  $f$  is a  $C^2$  map with non-flat critical points, then any periodic point of  $f$  with sufficiently large period is a source ([10]). In Theorem 1 we do not assume any condition concerned with critical points. Then the map  $f$  may have flat critical points.

For a source  $p$  of  $f$  with period  $n$  we denote by  $W_{\text{loc}}^u(p)$  the maximal interval  $J$  of  $I$  containing  $p$  such that

$$|(f^n)'(x)| \geq \{(1 + \nu(p))/2\}^n \quad \text{for all } x \in J.$$

We say that a homoclinic point  $q$  of  $p$  is *transversal* if there are non-negative integers  $m_1, m_2$  and a point  $q' \in W_{\text{loc}}^u(p)$  such that

$$f^{m_1}(q') = q, \quad f^{m_1+m_2}(q') = f^{m_2}(q) = p \quad \text{and} \quad (f^{m_1+m_2})'(q') \neq 0.$$

If  $f$  has a transversal homoclinic point of a source, then there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$  such that every map  $g$  belonging to  $\mathcal{U}$  has a transversal homoclinic point of a source. We denote the set of transversal homoclinic points of a source  $p$  of  $f$  by  $\text{TH}(p)$ , and its closure by  $\overline{\text{TH}(p)}$ . We call  $\overline{\text{TH}(p)}$  the *transversal homoclinic closure* of

$p$ . It is easy to see that  $p \in \overline{\text{TH}(p)}$ , and  $\overline{\text{TH}(p)}$  is  $f$ -invariant. For  $m \geq 1$  and  $\delta > 0$  define

$$H(p, m, \delta) = \{q \in W_{\text{loc}}^u(p) : f^m(q) = p, |f'(f^i(q))| \geq \delta \text{ for all } 0 \leq i \leq m-1\}.$$

Then we have

$$H(p, m, \delta_1) \subset H(p, m, \delta_2) \quad \text{if } \delta_1 \geq \delta_2$$

and

$$\text{TH}(p) = \bigcup_{\delta > 0} \bigcup_{m=1}^{\infty} \bigcup_{i=0}^{m-1} f^i H(p, m, \delta) \setminus \mathcal{O}_f^+(p).$$

The second result of this paper is the following:

**Theorem 2.** *If  $h(f) > 0$  then*

$$h(f) = \sup\{h(f|_{\overline{\text{TH}(p)}}) : p \text{ is a source of } f\},$$

and for a source  $p$  of  $f$  we have

$$h(f|_{\overline{\text{TH}(p)}}) = \max \left\{ 0, \lim_{\delta \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta) \right\}.$$

A result corresponding to Theorem 2 is known for surface diffeomorphisms by Mendoza ([11]). As an easy corollary of Theorem 2 we have:

**Corollary 3.** *The following statements are equivalent:*

- (i)  $h(f) > 0$ ;
- (ii)  $f$  has a transversal homoclinic point of a source;
- (iii)  $f$  has a homoclinic point of a periodic point.

## 1. Proofs of Theorems

Let  $f : I \rightarrow I$  be a continuous map. For integers  $k, l \geq 1$  we say that a closed  $f$ -invariant set  $\Gamma$  is a  $(k, l)$ -horseshoe of  $f$  if there are subsets  $\Gamma^0, \dots, \Gamma^{k-1}$  of  $I$  such that

$$\Gamma = \Gamma^0 \cup \dots \cup \Gamma^{k-1}, \quad f(\Gamma^j) = \Gamma^{j+1} \pmod{k}$$

and  $f^k|_{\Gamma^0} : \Gamma^0 \rightarrow \Gamma^0$  is topologically conjugate to a one-sided full shift in  $l$ -symbols. If  $\Gamma$  is a  $(k, l)$ -horseshoe, then it is clear that

$$h(f|_{\Gamma}) = \frac{1}{k} \log l \quad \text{and} \quad l^n \leq \sharp[\text{Per}(f, kn) \cap \Gamma] \leq k l^n$$

for all  $n \geq 1$ . It was proved by Misiurewicz et al. ([1, 12, 13]) that if the topological entropy of  $f$  is positive then there are sequences  $k_j, l_j$  of positive integers with a  $(k_j, l_j)$ -horseshoe  $\Gamma_j$  of  $f$  ( $j \geq 1$ ) such that

$$h(f) = \lim_{j \rightarrow \infty} h(f|_{\Gamma_j}) = \lim_{j \rightarrow \infty} \frac{1}{k_j} \log l_j.$$

Then the formula (\*) follows from this fact.

In order to prove our results, we need the notion of hyperbolic horseshoe and ideas of the theory of hyperbolic measures ([14, 15]). Katok ([9]) has proved that if a  $C^{1+\alpha}$  diffeomorphism of a manifold has a hyperbolic measure then its metric entropy is approximated by the entropy of a hyperbolic horseshoe. The author has shown in [5] that the result of Katok is also valid for  $C^{1+\alpha}$  (non-invertible) maps.

Let  $f : I \rightarrow I$  be a differentiable map. For integers  $k, l \geq 1$ , numbers  $v > 1$  and  $\delta > 0$  we say that  $\Gamma$  is a  $(k, l, v, \delta)$ -hyperbolic horseshoe of  $f$  if  $\Gamma$  is a  $(k, l)$ -horseshoe and

$$|(f^k)'(x)| \geq v^k, \quad |f'(x)| \geq \delta \quad (x \in \Gamma).$$

The following lemma plays an important role for the proofs of Theorems 1 and 2.

**Lemma 4.** *Let  $f : I \rightarrow I$  be a  $C^{1+\alpha}$  map. If  $h(f) > 0$ , then for a number  $v_0$  with  $1 < v_0 < \exp\{h(f)\}$  there exist sequences  $k_j, l_j$  of positive integers and  $\delta_j > 0$  ( $j \geq 1$ ) such that for  $j \geq 1$  there is a  $(k_j, l_j, v_0, \delta_j)$ -hyperbolic horseshoe  $\Gamma_j$  of  $f$  so that*

$$h(f) = \lim_{j \rightarrow \infty} h(f|_{\Gamma_j}) = \lim_{j \rightarrow \infty} \frac{1}{k_j} \log l_j.$$

This is corresponding to the result obtained by Katok for surface diffeomorphisms ([9]). For the proof we use the result stated in [5].

**Proof of Lemma 4.** For a number  $v_0$  with  $1 < v_0 < \exp\{h(f)\}$  we take a sequence  $\eta_j$  of positive numbers ( $j \geq 1$ ) such that  $\exp\{h(f) - 3\eta_j\} > v_0$  and  $\eta_j \rightarrow 0$  as  $j \rightarrow \infty$ . By the variational principle for the topological entropy ([6, 7, 8]), we have an  $f$ -invariant ergodic Borel probability measure  $\mu_j$  on  $I$  such that

$$h_j \geq h(f) - \eta_j > 0$$

where  $h_j$  denotes the metric entropy of  $\mu_j$  with respect to  $f$ . If  $\lambda_j$  denotes the Lyapunov exponent of  $\mu_j$ , that is,

$$\lambda_j = \int \log |f'(x)| d\mu_j(x),$$

then by the Ruelle inequality ([17]) we have

$$\lambda_j \geq h_j > 0,$$

and so  $\mu_j$  is a hyperbolic measure of  $f$ . Then by Theorem C (and its proof) of [5], we can construct sequences of integers  $k_j, l_j \geq 1$  with  $(1/k_j) \cdot \log l_j \geq h_j - \eta_j$ , numbers  $c_j \geq 1$  and closed sets  $\Lambda_j \subset I$  ( $j \geq 1$ ) such that:

- (1)  $f^{k_j}(\Lambda_j) = \Lambda_j$ ;
- (2)  $|(f^{k_j i})'(x)| \geq c_j^{-1} \cdot \exp\{k_j i(\lambda_j - \eta_j)\}$  for all  $x \in \Lambda_j$  and  $i \geq 1$ ;
- (3)  $f^{k_j}|_{\Lambda_j}: \Lambda_j \rightarrow \Lambda_j$  is topologically conjugate to a one-sided full shift in  $l_j$ -symbols.

For  $j \geq 1$  we set

$$\Gamma_j = \Lambda_j \cup f\Lambda_j \cdots \cup f^{k_j-1}\Lambda_j.$$

Then  $\Gamma_j$  is  $f$ -invariant. Moreover we put

$$\begin{aligned} \delta_j &= \min\{|f'(x)| : x \in \Gamma_j\} > 0, \\ e_j &= \max \left\{ \frac{|f'(x)|}{|f'(y)|} : x, y \in \Gamma_j \right\} \in [1, \infty) \end{aligned}$$

and take an integer  $n_j \geq 1$  large enough so that

$$\exp\{k_j n_j \eta_j\} \geq c_j e_j^{k_j}.$$

Then we have

$$|(f^{k_j n_j})'(x)| \geq v_0^{k_j n_j} \quad (x \in \Gamma_j).$$

This follows from the fact that for  $0 \leq i \leq k_j - 1$  and  $x \in f^i \Lambda_j$

$$\begin{aligned} |(f^{k_j n_j})'(x)| &= |(f^{k_j n_j})'(f^{k_j-i}(x))| \cdot |(f^{k_j-i})'(x)| \cdot |(f^{k_j-i})'(f^{k_j n_j}(x))|^{-1} \\ &\geq c_j^{-1} \cdot \exp\{(k_j n_j)(\lambda_j - \eta_j)\} \cdot e_j^{-k_j+i} \\ &\geq \exp\{k_j n_j(\lambda_j - 2\eta_j)\} \\ &\geq \exp\{k_j n_j(h(f) - 3\eta_j)\} \\ &\geq v_0^{k_j n_j}. \end{aligned}$$

It is easy to see that  $f^{k_j n_j}|_{\Lambda_j}: \Lambda_j \rightarrow \Lambda_j$  is topologically conjugate to a one-sided full shift in  $l_j^{n_j}$ -symbols. Thus  $\Gamma_j$  is a  $(k_j n_j, l_j^{n_j}, v_0, \delta_j)$ -hyperbolic horseshoe, and from which

$$h(f|_{\Gamma_j}) = \frac{1}{k_j n_j} \log l_j^{n_j}$$

$$\begin{aligned}
&= \frac{1}{k_j} \log l_j \\
&\geq h_j - \eta_j \\
&\geq h(f) - 2\eta_j.
\end{aligned}$$

Since  $\eta_j \rightarrow 0$  as  $j \rightarrow \infty$ , we have

$$h(f) = \lim_{j \rightarrow \infty} h(f|_{\Gamma_j}).$$

Lemma 4 was proved. □

**Proof of Theorem 1.** For  $\nu > 1$  and  $\delta > 0$  we want to find  $\gamma_0 = \gamma_0(\nu, \delta) > 0$  such that  $\text{Per}(f, n, \nu, \delta)$  is an  $(n, \gamma_0)$ -separated set of  $f$  for all  $n \geq 1$ . Take  $\gamma_1 = \gamma_1(\delta) > 0$  so small that if  $x, y \in I$  satisfy  $|x - y| \leq \gamma_1$  then

$$|f'(x) - f'(y)| \leq \frac{\delta}{2}.$$

We put

$$I_\delta = \left\{ x \in I : |f'(x)| \geq \frac{\delta}{2} \right\}.$$

Obviously,  $I_\delta$  is closed. For  $n \geq 1$  and  $x \in I$ ,

$$|x - \text{Per}(f, n, \nu, \delta)| \leq \gamma_1 \quad \text{implies that} \quad x \in I_\delta.$$

Since a function  $x \mapsto \log |f'(x)|$  is bounded and varies continuously on  $I_\delta$ , there is  $\gamma_2 = \gamma_2(\nu, \delta) > 0$  such that if  $x, y \in I_\delta$  satisfy  $|x - y| \leq \gamma_2$  then

$$|\log |f'(x)| - \log |f'(y)|| \leq \frac{1}{2} \log \nu.$$

We put  $\gamma_0 = \min\{\gamma_1, \gamma_2\}$ . Then it is checked that  $\text{Per}(f, n, \nu, \delta)$  is an  $(n, \gamma_0)$ -separated set of  $f$  for  $n \geq 1$ . Indeed, if a pair  $p, p' \in \text{Per}(f, n, \nu, \delta)$  with  $p \leq p'$  satisfies

$$\max\{|f^i(p) - f^i(p')| : 0 \leq i \leq n-1\} \leq \gamma_0,$$

then we see that for  $x \in [p, p']$  and  $0 \leq i \leq n-1$ ,

$$|f^i(x) - f^i(p)| \leq \gamma_0, \quad f^i(x) \in I_\delta.$$

On the other hand, by the mean value theorem there is a point  $\xi \in [p, p']$  such that

$$|f^n(p) - f^n(p')| = |(f^n)'(\xi)| \cdot |p - p'|.$$

Since  $f^i(\xi), f^i(p) \in I_\delta$  and  $|f^i(\xi) - f^i(p)| \leq \gamma_0$  for  $0 \leq i \leq n-1$ , we have

$$\begin{aligned} \left| \log |(f^n)'(\xi)| - \log |(f^n)'(p)| \right| &\leq \sum_{i=0}^{n-1} \left| \log |f'(f^i(\xi))| - \log |f'(f^i(p))| \right| \\ &\leq \frac{n}{2} \log v, \end{aligned}$$

and so

$$\frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \geq \exp\left(-\frac{n}{2} \log v\right) = v^{-n/2}.$$

Since  $p, p' \in \text{Per}(f, n, v, \delta)$ , we have

$$\begin{aligned} |p - p'| &= |f^n(p) - f^n(p')| \\ &= |(f^n)'(\xi)| \cdot |p - p'| \\ &= \frac{|(f^n)'(\xi)|}{|(f^n)'(p)|} \cdot |(f^n)'(p)| \cdot |p - p'| \\ &\geq v^{-n/2} \cdot v^n \cdot |p - p'| \\ &= v^{n/2} \cdot |p - p'|, \end{aligned}$$

and so  $p = p'$  because of  $v > 1$ . Thus  $\text{Per}(f, n, v, \delta)$  is an  $(n, \gamma_0)$ -separated set of  $f$ , and then

$$\begin{aligned} (1.1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sharp \text{Per}(f, n, v, \delta) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(f, n, \gamma_0) \\ &\leq \lim_{\gamma \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(f, n, \gamma) \\ &= h(f) \end{aligned}$$

for  $v > 1$  and  $\delta > 0$ , where  $s(f, n, \gamma)$  denotes the maximal cardinality of  $(n, \gamma)$ -separated sets for  $f$ . Therefore we have the conclusion of Theorem 1 when  $h(f) = 0$ . Thus it remains to give the proof for the case when  $h(f) > 0$ . Fix  $1 < v_0 < \exp\{h(f)\}$ . Take sequences  $k_j, l_j, \delta_j$  and  $\Gamma_j$  ( $j \geq 1$ ) as in Lemma 4. Since

$$l_j^n \leq \sharp[\text{Per}(f, nk_j, v_0, \delta_j) \cap \Gamma_j] \leq k_j l_j^n$$

for all  $n \geq 1$ , we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sharp \text{Per}(f, n, v_0, \delta) &\geq \lim_{n \rightarrow \infty} \frac{1}{nk_j} \log \sharp[\text{Per}(f, nk_j, v_0, \delta_j) \cap \Gamma_j] \\ &= \frac{1}{k_j} \log l_j. \end{aligned}$$



If  $j \rightarrow \infty$ , then

$$(1.2) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n, v_0, \delta) \geq h(f).$$

Combining (1.1) and (1.2) we have

$$\begin{aligned} h(f) &\leq \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n, v_0, \delta) \\ &\leq \lim_{v \rightarrow 1} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n, v, \delta) \\ &\leq h(f). \end{aligned}$$

Theorem 1 was proved. □

REMARK. In fact, from the proof of Theorem 1 it follows that

$$h(f) = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}(f, n, v_0, \delta)$$

if  $1 < v_0 < \exp\{h(f)\}$ .

Proof of Theorem 2.

*Proof of the first statement.* Under the assumption of Theorem 2 we fix a number  $v_0$  with  $1 < v_0 < \exp\{h(f)\}$ . By Lemma 4, for  $j \geq 1$  there are  $k_j, l_j \geq 1$  and  $\delta_j > 0$  with a  $(k_j, l_j, v_0, \delta_j)$ -hyperbolic horseshoe  $\Gamma_j = \Gamma_j^0 \cup \dots \cup \Gamma_j^{k_j-1}$  such that

$$h(f|_{\Gamma_j}) = \frac{1}{k_j} \log l_j \rightarrow h(f)$$

as  $j \rightarrow \infty$ . For  $j \geq 1$  define a product space

$$\Sigma_j = \prod_{m=1}^{\infty} \{1, \dots, l_j\}$$

with the product topology and a shift  $\sigma_j : \Sigma_j \rightarrow \Sigma_j$  by

$$\sigma_j((a_m)_{m \geq 1}) = (a_{m+1})_{m \geq 1} \quad ((a_m)_{m \geq 1} \in \Sigma_j).$$

From the definition of hyperbolic horseshoe, there is a homeomorphism  $\varphi_j : \Sigma_j \rightarrow \Gamma_j^0$  such that  $\varphi_j \circ \sigma_j = (f^{k_j}|_{\Gamma_j^0}) \circ \varphi_j$ . Then  $p_j = \varphi_j(1, 1, \dots)$  is a source of  $f$ . For  $m \geq 1$  and  $a_1, \dots, a_m \in \{1, \dots, l_j\}$  with  $a_i \neq 1$  for some  $1 \leq i \leq m$ ,  $\varphi_j(a_1, \dots, a_m, 1, 1, \dots)$  is a transversal homoclinic point of  $p_j$ . Thus,  $\overline{\text{TH}(p_j)} \supset \Gamma_j$ , from which

$$\begin{aligned}
 h(f) &= \lim_{j \rightarrow \infty} h(f|_{\Gamma_j}) \\
 &\leq \lim_{j \rightarrow \infty} h(f|_{\overline{TH(p_j)}}) \\
 &\leq \sup\{h(f|_{\overline{TH(p)}}) : p \text{ is a source of } f\} \\
 &\leq h(f).
 \end{aligned}$$

The first statement was proved.

*Proof of the second statement.* Let  $p$  be a source of  $f$ . Without loss of generality we may assume that  $p$  is a fixed point, i.e.,  $f(p) = p$ . To show that for  $\delta > 0$

$$h(f|_{\overline{TH(p)}}) \geq \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta),$$

take  $\gamma_0 = \gamma_0(\delta) > 0$  so small that if  $x, y \in I$  satisfy  $|x - y| \leq \gamma_0$  then

$$|f'(x) - f'(y)| \leq \frac{\delta}{2}.$$

Then, for  $m \geq 1$  and a pair  $q, q' \in H(p, m, \delta)$  satisfying

$$\max\{|f^i(q) - f^i(q')| : 0 \leq i \leq m - 1\} \leq \gamma_0,$$

we can find a sequence  $\xi_0, \dots, \xi_{m-1} \in I$  such that

$$|\xi_i - f^i(q)| \leq \gamma_0$$

and

$$|f^{i+1}(q) - f^{i+1}(q')| = |f'(\xi_i)| \cdot |f^i(q) - f^i(q')| \quad (0 \leq i \leq m - 1).$$

Since  $f^m(q) = f^m(q') = p$ , we have

$$\begin{aligned}
 0 &= |f^m(q) - f^m(q')| = |f'(\xi_{m-1})| \cdot |f^{m-1}(q) - f^{m-1}(q')| \\
 &= \dots = \prod_{i=0}^{m-1} |f'(\xi_i)| \cdot |q - q'| \\
 &\geq \prod_{i=0}^{m-1} \left( |f'(f^i(q))| - \frac{\delta}{2} \right) \cdot |q - q'| \\
 &\geq \left( \frac{\delta}{2} \right)^m \cdot |q - q'|,
 \end{aligned}$$

and so  $q = q'$ . Thus  $H(p, m, \delta)$  is an  $(m, \gamma_0)$ -separated set of  $f|_{\overline{TH(p)}}$ , from which it follows that

$$(1.3) \quad \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta) \leq \limsup_{m \rightarrow \infty} \frac{1}{m} \log s(f|_{\overline{\text{TH}(p)}}, m, \gamma_0) \\ \leq h(f|_{\overline{\text{TH}(p)}}).$$

If  $h(f|_{\overline{\text{TH}(p)}}) = 0$ , then nothing to prove for the second statement. Thus we must check the conclusion for the case when  $h(f|_{\overline{\text{TH}(p)}}) > 0$ . To do so fix a number  $\nu_0$  with  $1 < \nu_0 < \min\{\nu(p), \exp h(f|_{\overline{\text{TH}(p)}})\}$ . By the same way as in the proof of Lemma 4, we can take sequences of integers  $k_j, l_j \geq 1$ , numbers  $\delta_j > 0$  with  $(k_j, l_j, \nu_0, \delta_j)$ -hyperbolic horseshoes  $\Gamma_j = \Gamma_j^0 \cup \dots \cup \Gamma_j^{k_j-1}$  containing  $p$  ( $j \geq 1$ ) such that

$$h(f|_{\Gamma_j}) = (1/k_j) \cdot \log l_j \rightarrow h(f|_{\overline{\text{TH}(p)}}) \quad \text{as } j \rightarrow \infty.$$

Then there is a homeomorphism  $\varphi_j : \Sigma_j \rightarrow \Gamma_j^0$  such that  $\varphi_j \circ \sigma_j = (f^{k_j}|_{\Gamma_j^0}) \circ \varphi_j$ , where  $\sigma_j : \Sigma_j \rightarrow \Sigma_j$  is the shift defined as in the proof of the first statement. Without loss of generality we may assume that  $\varphi_j(1, 1, \dots) = p$ . By taking an integer  $n_j \geq 1$  large enough we have

$$\varphi_j([1, \dots, 1]_{n_j}) \subset W_{\text{loc}}^u(p)$$

where

$$[1, \dots, 1]_{n_j} = \{(b_m)_{m \geq 1} \in \Sigma_j : b_m = 1 \text{ for all } 1 \leq m \leq n_j\}.$$

Since

$$\varphi_j(\overbrace{[1, \dots, 1]_{n_j}}^{n_j \text{ times}}) \in H(p, mk_j, \delta_j)$$

holds for all  $m \geq n_j + 1$  and  $a_1, \dots, a_{m-n_j} \in \{1, \dots, l_j\}$ , we have

$$\sharp H(p, mk_j, \delta_j) \geq l_j^{m-n_j}.$$

Thus,

$$\lim_{\delta \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta) \geq \limsup_{m \rightarrow \infty} \frac{1}{mk_j} \log \sharp H(p, mk_j, \delta_j) \\ \geq \lim_{m \rightarrow \infty} \frac{m - n_j}{mk_j} \log l_j \\ = \frac{1}{k_j} \log l_j$$

for  $j \geq 1$ . If  $j \rightarrow \infty$ , then we have

$$(1.4) \quad \lim_{\delta \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta) \geq h(f|_{\overline{\text{TH}(p)}}).$$

Combining (1.3) and (1.4),

$$\begin{aligned} h(f|_{\overline{\text{TH}(p)}}) &\leq \lim_{\delta \rightarrow 0} \limsup_{m \rightarrow \infty} \frac{1}{m} \log \sharp H(p, m, \delta) \\ &\leq h(f|_{\overline{\text{TH}(p)}}). \end{aligned}$$

The second statement was proved. This completes the proof of Theorem 2.  $\square$

## 2. Circle Maps

In the same way as above, it can be checked that our results (Theorems 1 and 2) are also valid for  $C^{1+\alpha}$  maps ( $\alpha > 0$ ) of the circle  $S^1$ . However, the existence of a homoclinic point does not imply that the topological entropy is positive. In fact, we know an example of a  $C^\infty$  map  $g : S^1 \rightarrow S^1$  such that  $g$  has a homoclinic point of a source fixed point, nevertheless  $h(g) = 0$  ([16]). It is known that the topological entropy of a continuous circle map is positive if and only if the map has a nonwandering homoclinic point of a periodic point ([4]). Since any transversal homoclinic point of a source is nonwandering, we have:

**Corollary 5.** *For a  $C^{1+\alpha}$  map  $f : S^1 \rightarrow S^1$  ( $\alpha > 0$ ) the following statements are equivalent:*

- (i)  $h(f) > 0$ ;
- (ii)  $f$  has a transversal homoclinic point of a source;
- (iii)  $f$  has a nonwandering homoclinic point of a periodic point.

**Added in proof.** After this manuscript was completed the author learned from A. Katok that he and A. Mezhirova had obtained a result that overlaps with Theorem 1 for  $C^1$  maps with finitely many critical points ([18]).  $\square$

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