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UNIQUENESS OF THE STATIONARY SOLUTIONS FOR A FLUID DYNAMICAL MODEL OF SEMICONDUCTORS

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Abstract
We study a one-dimensional fluid dynamical model of semiconductors. Our goal in this paper is to prove the uniqueness of stationary solutions.

1. Introduction

The present paper is concerned with the uniqueness of stationary solutions to the boundary value problem for a one-dimensional fluid dynamical model of semiconductors. The motion of electrons in semiconductors is governed by the system of equations

$$\begin{cases} \rho_t + j_x = 0, \\ j_t + \left(\dfrac{j^2}{\rho} + p(\rho) \right)_x = \rho\phi_x - \dfrac{1}{\tau} j, \quad (x, t) \in (0, 1) \times [0, \infty), \\ \phi_{xx} = \rho - D, \end{cases} \tag{1.1}$$

where ρ, j and ϕ are the electron density, the current density and the electron potential respectively. The electron velocity is defined as $u = j/\rho$. The pressure $p(\rho)$ is a function of the electron density ρ with the form $p(\rho) = \rho^\gamma/\gamma$, where γ is a constant satisfying $\gamma \geq 1$. A constant τ is the relaxation time. For simplicity, we assume $\tau = 1$. The doping profile D is a given function of the spatial variable $x \in \Omega := [0, 1]$ and satisfies

$$D \in C(\Omega), \quad \min_{x \in \Omega} D(x) > 0. \tag{1.2}$$

In the present paper, for the time-dependent system (1.1), we shall investigate stationary solutions $(\rho(x), j(x), \phi(x))$ satisfy the system of equations

$$\begin{cases} j_x = 0, \\ \left(\dfrac{j^2}{\rho} + p(\rho) \right)_x = \rho\phi_x - j, \quad x \in (0, 1), \\ \phi_{xx} = \rho - D \end{cases} \tag{1.3}$$

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and the boundary condition

$$\rho(0) = \rho_l > 0, \quad \rho(1) = \rho_r > 0, \tag{1.4}$$

$$\phi(0) = 0, \quad \phi(1) = \phi_r > 0. \tag{1.5}$$

We consider the classical solutions in the region where the subsonic condition (i.e. the elliptic condition)

$$\inf_{x\in(0,1)} (p'(\rho) - u^2) > 0, \tag{1.6}$$

and the positivity of the density

$$\inf_{x\in(0,1)} \rho(x) > 0 \tag{1.7}$$

hold.

Multiply the equation $(1.3)_2$ by $1/\rho$, and then differentiate the resultant equation with respect to x. Since the solutions satisfy the elliptic condition (1.6), applying the maximum principle, we obtain

$$C_m \leq \rho \leq C_M, \tag{1.8}$$

where

$$C_m := \min\left\{\rho_l,\, \rho_r,\, \inf_{x\in\Omega} D(x)\right\}, \quad C_M := \max\left\{\rho_l,\, \rho_r,\, \sup_{x\in\Omega} D(x)\right\}.$$

On the other hand, we deduce from $(1.3)_2$

$$\left(\frac{j^2}{2\rho^2} + h(\rho)\right)_x = \phi_x - \frac{j}{\rho}, \tag{1.9}$$

where $h(\rho) := \rho^{\gamma-1}/(\gamma-1)$. Then, from (1.4)–(1.5), we obtain

$$\left(\frac{1}{\rho_r^2} - \frac{1}{\rho_l^2}\right) j^2 + 2\int_0^1 \frac{dx}{\rho} \cdot j - 2C_b = 0, \tag{1.10}$$

where $C_b := \phi_r + h(\rho_l) - h(\rho_r)$. This equation yields

$$j = 2C_b\left\{\int_0^1 \frac{dx}{\rho} \pm \sqrt{\left(\int_0^1 \frac{dx}{\rho}\right)^2 + 2C_b\left(\frac{1}{\rho_r^2} - \frac{1}{\rho_l^2}\right)}\right\}^{-1}. \tag{1.11}$$

Now, we survey the related results for (1.1). This model was introduced by Bløtekjær [1]. It is important for engineering to study the bounded domain with the Dirichlet boundary condition (1.4)–(1.5) (see [4] and [5]). Moreover, considering the application of this model to engineering, it suffices to consider the case where $\rho_r = \rho_l$ and $\gamma = 1$.

For the boundary value problem (1.3)–(1.5), Degond and Markowich [2] discussed the uniqueness of stationary solutions for sufficiently large τ. Subsequently, Nishibata and Suzuki [3] showed the following:

Theorem 1.1 (Nishibata-Suzuki). *We assume that*

$$(C_m)^{\gamma+1} > 4C_b^2\left\{C_M^{-1} + \sqrt{C_M^{-2} + 2C_b(\rho_r^{-2} - \rho_l^{-2})}\right\}^{-2}, \tag{1.12}$$
$$C_M^{-2} + 2C_b(\rho_r^{-2} - \rho_l^{-2}) \geq 0 \quad \textit{if} \quad \rho_l < \rho_r.$$

Then the boundary value problem (1.3)–(1.5) *has a solution.*

Moreover we assume that

$$(C_m)^{\gamma+1} > (J_M)^2 + 2C_M(C_M + \phi_r)J_M, \tag{1.13}$$

where $J_M := C_M(C_M^{\gamma+1}|\rho_r^{-2} - \rho_l^{-2}|/2 + |C_b|)$.

Then there exists at most one classical solution to the boundary value problem (1.3)–(1.5) *satisfying* (1.6) *and* (1.7).

Comparing (1.13) with (1.12), (1.13) is the stronger condition than (1.12) in the case where $\rho_l \geq \rho_r$. The purpose of the present paper is to prove the uniqueness under the weaker condition in the case where $\rho_l \geq \rho_r$. Our main theorem is as follows.

Theorem 1.2. *We assume that* $\rho_l \geq \rho_r$. *Then there exists at most one classical solution to the boundary value problem* (1.3)–(1.5) *satisfying* (1.6), (1.7),

$$(C_m)^{\gamma+1} \leq 4(C_b)^2\left\{-\int_0^1 \frac{dx}{\rho} + \sqrt{\left(\int_0^1 \frac{dx}{\rho}\right)^2 + 2C_b\left(\frac{1}{\rho_r^2} - \frac{1}{\rho_l^2}\right)}\right\}^{-2} \tag{1.14}$$

and

$$(C_m)^{\gamma+1} > j^2. \tag{1.15}$$

REMARK 1. We mention the conditions (1.14) and (1.15) in the above theorem.

The quadratic equation (1.10) of j has two solutions. Consequently the uniqueness does not hold. To overcome this problem, we assume (1.14). From (1.14) and (1.15), the quadratic equation (1.10) has at most one solution

$$j = 2C_b\left\{\int_0^1 \frac{dx}{\rho} + \sqrt{\left(\int_0^1 \frac{dx}{\rho}\right)^2 + 2C_b\left(\frac{1}{\rho_r^2} - \frac{1}{\rho_l^2}\right)}\right\}^{-1} \tag{1.16}$$

in this case. If $|\rho_l - \rho_r|$ is small enough, (1.14) holds. By the way, the other solution of (1.10) tends to $-\infty$, as $|\rho_l - \rho_r| \to 0$.

On the other hand, (1.15) is weaker than (1.13). In addition, in view of (1.8) and (1.16), (1.15) is weaker than (1.12), which is necessary to prove the existence of solution.

2. Proof of Theorem 1.2

Proof. Before proving Theorem 1.2, we consider the current density j. From $(1.3)_1$, j is a constant. Moreover, since $\rho_l \geq \rho_r$, in view of (1.16), we find $j > 0$.

Now let (ρ_1, j_1, ϕ_1) and (ρ_2, j_2, ϕ_2) be classical solutions to the boundary value problem (1.3)–(1.5) satisfying (1.6), (1.7) and (1.15). This proof consists of four steps. In the first three steps, we prove $j_1 = j_2$ by contradiction. To do this, we assume that $j_2 > j_1$ without loss of generality.

STEP 1. We first prove the following inequality

$$\frac{(C_m)^{\gamma-1}}{\gamma-1}\left\{\left(\frac{j_2}{j_1}\right)^{\gamma-1} - 1\right\} > \frac{1}{2(C_m)^2}\{(j_2)^2 - (j_1)^2\}. \tag{2.1}$$

We set $r = j_2/j_1$ and consider

$$f(r) = \frac{(C_m)^{\gamma-1}}{\gamma-1}(r^{\gamma-1} - 1) + \frac{(j_2)^2}{2(C_m)^2}\left(\frac{1}{r^2} - 1\right).$$

Then we find $f(1) = 0$ and deduce from (1.8) and (1.15) $f'(r) > 0$ $(r > 1)$. Since our assumption means that $r > 1$, we conclude (2.1).

STEP 2. From (1.9) and the boundary conditions, we have

$$\int_0^1 \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1}\right) dx = \frac{(j_2)^2 - (j_1)^2}{2}\left(\frac{1}{\rho_l^2} - \frac{1}{\rho_r^2}\right) := \kappa\{(j_2)^2 - (j_1)^2\}. \tag{2.2}$$

Then there exists an interval $I = [x_-, x_+] \subset [0, 1]$ satisfying the following conditions. The proof is discussed in Appendix A.

(C1)

$$\int_I \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1}\right) dx \leq 0; \tag{2.3}$$

(C2) On the interval I, $\rho_2 \geq \rho_1$ holds;

(C3) At x_- and x_+, $\rho_2 = \rho_1$ holds.

We denote the value ρ_1 $(= \rho_2)$ at x_- and x_+ by ρ_- and ρ_+ respectively. On the other hand, from (C3), $j_2/\rho_2 - j_1/\rho_1 > 0$ holds at x_- and x_+. Therefore, from (C1), there exists a set of points on I such that $j_2/\rho_2 = j_1/\rho_1$ at each point in the set. Let $\tilde{x}$ be the first point on the left (i.e. the smallest point) in the set.

Finally, we observe the following.

(P1) From $(1.3)_3$ and (C2), $(\phi_2 - \phi_1)$ is convex on I.

(P2) From the choice of the point $\tilde{x}$, we have

$$\int_{x_-}^{\tilde{x}} \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1} \right) dx \geq 0. \tag{2.4}$$

STEP 3. We integrate (1.9) from x_- to x. Then, from (P1),

$$\left(\frac{(j_2)^2}{2(\rho_2)^2} + h(\rho_2) \right)(x) - \left(\frac{(j_1)^2}{2(\rho_1)^2} + h(\rho_1) \right)(x) + \int_{x_-}^{x} \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1} \right) dy$$

is a convex function of x. Therefor we obtain

$$\begin{aligned} &(1-\tau)\frac{(j_2)^2 - (j_1)^2}{2(\rho_-)^2} + \tau \frac{(j_2)^2 - (j_1)^2}{2(\rho_+)^2} + \tau \int_{x_-}^{x_+} \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1} \right) dx \\ &\geq \left\{ \left(\frac{j_2}{j_1} \right)^{\gamma-1} - 1 \right\} h(\tilde{\rho}) + \int_{x_-}^{\tilde{x}} \left(\frac{j_2}{\rho_2} - \frac{j_1}{\rho_1} \right) dx, \end{aligned}$$

where τ $(0 < \tau < 1)$ is a constant satisfying $\tilde{x} = (1-\tau)x_- + \tau x_+$ and $\tilde{\rho}$ is the value ρ_1 at $\tilde{x}$.

Then, from (2.3) and (2.4), we have

$$(1-\tau)\frac{(j_2)^2 - (j_1)^2}{2(\rho_-)^2} + \tau \frac{(j_2)^2 - (j_1)^2}{2(\rho_+)^2} \geq \left\{ \left(\frac{j_2}{j_1} \right)^{\gamma-1} - 1 \right\} h(\tilde{\rho}). \tag{2.5}$$

However, from (1.8), this inequality contradicts (2.1). Therefore we conclude $j_1 = j_2$.

STEP 4. We consider the case where $j := j_1 = j_2$. The following argument is the almost same as Lemma 2.3 in [3].

We show $(\phi_1 - \phi_2)_x \leq 0$ by contradiction. We assume that $(\phi_1 - \phi_2)_x$ attains the positive maximum at a point x_M on I.

If $0 < x_M < 1$, it holds that $(\phi_1 - \phi_2)_x(x_M) > 0$ and $(\rho_1 - \rho_2)(x_M) = (\phi_1 - \phi_2)_{xx}(x_M) = 0$. Then, from $(1.3)_2$, the following inequality holds at x_M.

$$\left(p'(\rho_1) - \frac{j^2}{(\rho_1)^2} \right)(\rho_1 - \rho_2)_x = \rho_1(\phi_1 - \phi_2)_x > 0. \tag{2.6}$$

However, since $(\rho_1 - \rho_2)_x(x_M) = (\phi_1 - \phi_2)_{xxx}(x_M) \leq 0$, this is a contradiction.

If $x_M = 0$, since $(\rho_1 - \rho_2)(0) = 0$, the similar observation yields (2.6). It follows from (2.6) that $(\phi_1 - \phi_2)_{xxx}(0) = (\rho_1 - \rho_2)_x(0) > 0$. From the continuity of solutions, there exists $\delta > 0$ such that $(\phi_1 - \phi_2)_{xx}(x) = (\rho_1 - \rho_2)(x) > 0$ for $0 < x < \delta$. Then $(\phi_1 - \phi_2)_x(x) > (\phi_1 - \phi_2)_x(0)$ for $0 < x < \delta$, which also contradicts the assumption that $(\phi_1 - \phi_2)_x(x)$ attains the positive maximum at $x_M = 0$. We can handle the case where $x_M = 1$ in the similar manner.

Consequently, we obtain $(\phi_1 - \phi_2)_x \leq 0$. Since $(\phi_1 - \phi_2)(0) = (\phi_1 - \phi_2)(1) = 0$, we have $\phi_1 \equiv \phi_2$. Moreover it follows from $(1.3)_3$ that $\rho_1 \equiv \rho_2$. This completes the proof. □

Appendix A. Existence of the interval I

In this section, we prove the existence of the interval $I \subset [0,1]$ satisfying (C1)–(C3).

Proof. At 0 and 1, since $\rho_2 = \rho_1$, we first find $j_2/\rho_2 - j_1/\rho_1 > 0$. Then, from $\kappa \leq 0$ and (2.2), there exists a set of points such that $j_2/\rho_2 = j_1/\rho_1$ holds at each point of the set. Let this set be $\{x_\lambda\}_{\lambda\in\Lambda}$. At x_λ, $\rho_2 = (j_2/j_1)\rho_1 > \rho_1$ holds.

Next, for each point x_λ, we set $x_{\lambda-} = \inf\{a;\ \rho_2 > \rho_1,\ x \in [a, x_\lambda]\}$, $x_{\lambda+} = \sup\{a;\ \rho_2 > \rho_1,\ x \in [x_\lambda, a]\}$. Then, in view of the boundary condition, we find $0 \leq x_{\lambda-}, x_{\lambda+} \leq 1$. Moreover, from the continuity of ρ_2 and ρ_1, $\rho_2 = \rho_1$ holds at $x_{\lambda-}$ and $x_{\lambda+}$. Then, for $x_{\lambda-}$ and $x_{\lambda+}$, we set $I_\lambda := (x_{\lambda-}, x_{\lambda+})$. We notice that I_λ satisfies the following. If $x_{\lambda'} \in I_\lambda$, $I_\lambda = I_{\lambda'}$; If $x_{\lambda'} \notin I_\lambda$, $I_\lambda \cap I_{\lambda'} = \emptyset$. We then define an equivalence relation $\lambda \sim \lambda'$ by $I_\lambda = I_{\lambda'}$. Then $\Lambda/\sim$ is a countable set. We denote the set of open intervals with the index set $\Lambda/\sim$ by I_k, $k = 1, 2, \ldots$.

Now, if there exists a k such that $\int_{I_k}(j_2/\rho_2 - j_1/\rho_1)\,dx \leq 0$, $\overline{I_k}$ is the desired interval. Therefore, for any k, we assume that $\int_{I_k}(j_2/\rho_2 - j_1/\rho_1)\,dx > 0$ holds and shall deduce a contradiction.

Set $\sum_{k=1}^{\infty}\int_{I_k}(j_2/\rho_2 - j_1/\rho_1)\,dx = \delta$. From our assumption, we find $\delta > 0$. Then there exists a n_0 such that $\sum_{k=1}^{n_0}\int_{I_k}(j_2/\rho_2 - j_1/\rho_1)\,dx > \delta/2$.

Set $J = [0, 1] - \bigcup_{k=1}^{n_0} I_k$. We then have $\int_J (j_2/\rho_2 - j_1/\rho_1)\,dx < -\delta/2 + \kappa\{(j_2)^2 - (j_1)^2\} \leq -\delta/2$.

Moreover we set $I = \bigcup_{k=1}^{\infty} I_k$. Since $\sum_{k=n_0+1}^{\infty}\int_{I_k}(j_2/\rho_2 - j_1/\rho_1)\,dx < \delta/2$, there exists a point x_* on $[0,1] - I$ such that $j_2/\rho_2 < j_1/\rho_1$ holds at x_*. Notice that $\rho_2 > \rho_1$ holds at x_*.

From the construction, J is a finite set which consists of points and closed intervals. Moreover, $\rho_2 = \rho_1$ holds at the points and the extremal points of the closed intervals. Therefore x_* is the interior point of a closed interval J_*.

On the other hand, we set $x_{*-} = \inf\{a;\ \rho_2 > \rho_1,\ x \in [a, x_*]\}$, $x_{*+} = \sup\{a;\ \rho_2 > \rho_1,\ x \in [x_*, a]\}$. The points x_{*-} and x_{*+} satisfy the following:

(Q1) $x_{*-}, x_{*+} \in J_*$;

(Q2) At x_{*-} and x_{*+}, $\rho_2 = \rho_1$ holds. Therefore, from $j_2 > j_1$, $j_2/\rho_2 > j_1/\rho_1$ holds at x_{*-} and x_{*+}.

Since $j_2/\rho_2 < j_1/\rho_1$ at x_*, from (Q2), there exists a point on $[x_{*-}, x_{*+}]$ such that $j_2/\rho_2 = j_1/\rho_1$ at the point. This means that $x_* \in (x_{*-}, x_{*+}) \subset I$. However this contradicts the fact that $x_* \in [0, 1] - I$. □

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