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## A NOTE ON PIVOTAL MEASURES IN MAJORIZED EXPERIMENTS

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### 1. Introduction

An *experiment*  $\mathbf{E}=(X, \mathbf{A}, P)$ , i.e. a triplet with a family  $P$  of probability measures on a measurable space  $(X, \mathbf{A})$ , is said to be majorized by a measure  $\mu$  equivalent to  $P$  (called a “*majorizing*” measure) if each  $p \in P$  has a density  $dp/d\mu$  with respect to  $\mu$ . Let  $\mathbf{S}_0$  be the  $\sigma$ -ring generated by all the  $\mathbf{E}$ -supports  $S(p)=\{x \in X; (dp/d\mu)(x) > 0\}$ ,  $p \in P$ . Then there exists a “*maximal decomposition*”  $\mathbf{F}$  such that  $\mathbf{F} \subset \mathbf{S}_0[P]$  (see [4], Lemma 2). A maximal decomposition  $\mathbf{F}$  is defined as a covering of  $X$  of almost disjoint elements each of which is included by an  $\mathbf{E}$ -support  $S(p^{(F)})$  of some  $p^{(F)}$  in  $P$ . For each  $F \in \mathbf{F}$ , we define a dominated sub-experiment  $\mathbf{E}(F)=(F, \mathbf{A} \cap F, P_F)$  by setting  $\mathbf{A} \cap F = \{A \cap F; A \in \mathbf{A}\}$  and  $P_F = \{p_F; p \in P, p(F) > 0\}$ , where  $p_F(A \cap F) = p(A \cap F)/p(F)$ . The  $\sigma$ -ring generated by  $\{dp/d(p+q) \cdot I_{S(p)}; p, q \in P\}$  is called the  $\sigma$ -ring of *pairwise likelihood ratios* and denoted by  $\mathbf{S}$ . The  $\sigma$ -field  $\mathbf{D}$  generated by  $\mathbf{S}$  is known to be the smallest PSS (*pairwise sufficient with supports*) subfield, and is equal to  $\mathbf{S}$  and minimal sufficient when  $\mathbf{E}$  is dominated (see [3]). A majorizing measure  $m$  on  $\mathbf{A}$  is said to be *pivotal* for  $\mathbf{E}$  if it holds that for each subfield  $\mathbf{B}$ ,  $\mathbf{B}$  is PSS if and only if each  $p$  has a  $\mathbf{B}$ -measurable version of the density  $dp/dm$ . A real valued function  $f: X \rightarrow \mathbf{R}$  is said to be  $\mathbf{S}$ -measurable if for each Borel subset  $B$  of  $\mathbf{R}$ ,  $f^{-1}(B) \cap \{x \in X; f(x) \neq 0\} \in \mathbf{S}$ .

The notion of pivotal measures was devised to obtain a minimal sufficient subfield by Halmos & Savage [6] and Bahadur [1] for the dominated experiment. Ramamoorthi & Yamada [9] generalized it to the majorized experiment. Recently Luschgy, Mussmann & Yamada [8] proved the following characterization theorem of pivotal measures by the method and in the terminology of vector lattices: Every pivotal measure is represented as the sum of a maximal orthogonal system in the minimal  $\mathbf{L}$ -space.

The *minimal  $\mathbf{L}$ -space* is the closed vector sublattice generated by  $P$ , and a *maximal orthogonal system* is a family of non-zero measures on  $\mathbf{A}$  such that any two distinct elements in the family are singular with each other and a measure which is singular with all the elements of the family is zero.

In the present note, we show that every pivotal measure in a majorized experiment is expressed as the sum of pivotal measures in the dominated sub-experiments  $\mathbf{E}(F)$  for some maximal decomposition  $\mathbf{F}(\subset \mathbf{S}_0[P])$  and that each pivotal measure on  $\mathbf{E}(F)$  has a positive  $\mathbf{S}$ -measurable density with respect to a pivotal measure. Further, pointing out that the minimal  $\mathbf{L}$ -space coincides with the totality of the signed measures which have  $\mathbf{S}$ -measurable and integrable densities with respect to a pivotal measure, we show that the present result is a measure theoretical version of the above characterization theorem.

## 2. Pivotal measures

We consider the relation between a pivotal measure for  $\mathbf{E}$  and a family of pivotal measures for the dominated sub-experiments  $\mathbf{E}(F)$ .

**Theorem 1.** *Let  $\mathbf{E}=(X, \mathbf{A}, P)$  be a majorized experiment. If  $\mathbf{F}(\subset \mathbf{S}_0[P])$  is a maximal decomposition and a measure  $m_F$  on  $\mathbf{A} \cap F$  is pivotal for each  $F \in \mathbf{F}$ , then  $m(\mathbf{A}) = \sum_{F \in \mathbf{F}} m_F(\mathbf{A} \cap F)$  is pivotal for  $\mathbf{E}$ .*

*Proof.* It is enough to show that each  $p \in P$  has an  $\mathbf{S}$ -measurable version of  $dp/dm$ . Fix  $p \in P$ . There exists a countable subfamily  $\{F_n; n \geq 1\}$  of  $\mathbf{F}$  such that  $S(p) \subset \bigcup_{n \geq 1} F_n[P]$  and  $S(p) \cap F_n \neq \emptyset[P]$  for all  $n$ . We may assume that  $\{F_n; n \geq 1\}$  is a disjoint family. As each  $m_{F_n}$  is pivotal for  $\mathbf{E}(F_n)$ , there exists an  $\mathbf{S} \cap F_n$ -measurable version  $g_n$  of  $dp_{F_n}/dm_{F_n}$ . For each  $n \geq 1$ , put  $f_n(x) = p(F_n) \cdot g_n(x)$  if  $x \in F_n$ , and  $= 0$  if  $x \in F_n^c$ . As each  $F_n$  is in  $\mathbf{S}_0$ ,  $f_n$  is  $\mathbf{S}$ -measurable. It is immediate that  $\sum_{n \geq 1} f_n$  is an  $\mathbf{S}$ -measurable version of  $dp/dm$ .

**Theorem 2.** *Let  $\mathbf{E}=(X, \mathbf{A}, P)$  be a majorized experiment. A measure  $m$  on  $\mathbf{A}$  is pivotal for  $\mathbf{E}$  if and only if for each maximal decomposition  $\mathbf{F}(\subset \mathbf{S}_0[P])$ , there exists a family of pivotal measures  $\{m_F; F \in \mathbf{F}\}$  for the sub-experiments  $\mathbf{E}(F)$  such that  $m(\mathbf{A}) = \sum_{F \in \mathbf{F}} m_F(\mathbf{A} \cap F)$ .*

*Proof.* "If" part is Theorem 1 itself, and so we prove "only if" part. Let  $m$  be a pivotal measure for  $\mathbf{E}$  and take a maximal decomposition  $\mathbf{F}(\subset \mathbf{S}_0[P])$ . Fix  $F \in \mathbf{F}$ . We define a measure  $m_F$  on  $\mathbf{E}(F)$  by  $m_F(\mathbf{A} \cap F) = m(\mathbf{A} \cap F)$ . It is clear that  $m_F \equiv P_F$  and  $m(\mathbf{A}) = \sum_{F \in \mathbf{F}} m_F(\mathbf{A} \cap F)$ . Take  $p \in P$  with  $p(F) > 0$ . Let  $f_p$  be an  $\mathbf{S}$ -measurable version of  $dp/dm$ . Then it is easily checked that  $f_p \cdot I_F/p(F)$  is an  $\mathbf{S}$ -measurable version of  $dp_F/dm_F$ . This implies that  $m_F$  is a pivotal measure for  $\mathbf{E}(F)$  for each  $F \in \mathbf{F}$ .

Next we consider pivotal measures on each dominated sub-experiment.

Let  $\mathbf{F}(\subset \mathbf{S}_0[P])$  be a maximal decomposition and  $n$  a pivotal measure for  $\mathbf{E}$ . Each  $F$  is included by a support  $S(p^{(F)})$ , and so  $p^{(F)} \equiv P$ . Fix  $F \in \mathbf{F}$ . Then the restriction  $n_F$  of  $n$  to  $\mathbf{A} \cap F$  is pivotal for  $\mathbf{E}(F)$  as in the proof of Theorem 2. Let  $u$  be a pivotal measure for  $\mathbf{E}(F)$ . Notice that  $du/dn_F = (dp^{(F)}/du)^{-1}$ .

$(dp_F^{(F)}/dn_F)^{-1}[P_F]$  as  $p_F^{(F)} \equiv n_F \equiv u$ . Two densities in the right side have  $\mathcal{S}$ -measurable versions as  $u$  and  $n_F$  are pivotal. That is,  $u(A \cap F) = \int_{F \cap F} f \, dn$  for some  $\mathcal{S} \cap F$ -measurable function  $f: F \rightarrow \mathbf{R}$ , with  $f > 0[P_F]$ .

Conversely a measure  $u$  of this form is pivotal for  $\mathbf{E}(F)$ . Because,  $n_F$  is pivotal and  $dp_F/du = (dp_F/dn_F)/f[P_F]$  for all  $p_F \in P_F$ .

Thus we have proved the following

**Theorem 3.** *Let  $\mathbf{E} = (X, \mathcal{A}, P)$  be a majorized experiment and  $n$  a pivotal measure for  $\mathbf{E}$ . Then, a measure  $m$  on  $\mathcal{A}$  is pivotal for  $\mathbf{E}$  if and only if it is expressed as  $m(A) = \sum_{F \in \mathbf{F}} \int_{A \cap F} f_F \, dn$  for some maximal decomposition  $\mathbf{F} (\subset \mathcal{S}_0[P])$  and some family  $\{f_F: F \in \mathbf{F}\}$  of  $\mathcal{S}$ -measurable functions such that  $f_F > 0[P]$  on  $F$  and  $f_F = 0[P]$  on  $F^c$ .*

In case  $\mathbf{E}$  is weakly dominated, we have the following simpler expression.

**Corollary.** *Let  $\mathbf{E} = (X, \mathcal{A}, P)$  be a weakly dominated experiment and  $n$  a pivotal measure for  $\mathbf{E}$ . A measure  $m$  on  $\mathcal{A}$  is pivotal for  $\mathbf{E}$  if and only if  $m(A) = \int_A f \, dn$  for some  $\mathcal{B}$ -measurable function  $f$  with  $f > 0[P]$ , where  $\mathcal{B}$  is a minimal sufficient subfield.*

**Proof.** Let  $\mathcal{B}$  be a minimal sufficient subfield. It follows from Theorem 1.1 in [7] that  $(X, \mathcal{B}, P | \mathcal{B})$  is weakly dominated. Hence for the family of  $\mathcal{S}$ -measurable functions  $\{f_F; F \in \mathbf{F}\}$  in Theorem 3, there exists a  $\mathcal{B}$ -measurable function  $f$  such that  $f \cdot I_F = f_F[P]$  for all  $F \in \mathbf{F}$ .

**REMARK.** In Theorem 3, “if” part remains true with “any” maximal decomposition and “any” family  $\{f_F; F \in \mathbf{F}\}$ . “Only if” part also holds true with “any” maximal decomposition and some family  $\{f_F; F \in \mathbf{F}\}$ . Gooßen [5] attained to a similar characterization by making use of a pivotal measure  $\sum_{F \in \mathbf{F}} p^{(F)}(A \cap F)$ , which was obtained by Diepenbrock.

In the following Examples 1 to 4, we observe that the totality of pivotal measures is fairly large.

**EXAMPLE 1** ( $N$ -th product normal family with a location parameter).  $X = \mathbf{R}^N$ ,  $\mathcal{A}$  = the Borel  $\sigma$ -field on  $X$ .  $P = \{p_\xi; \xi \in \mathbf{R}\}$ .  $(dp_\xi/d\mu)(x) = (2\pi)^{-N/2} \exp \{(-1/2) \sum_{i=1}^N (x_i - \xi)^2\}$ , where  $\mu$  is the Lebesgue measure on  $X$ . The statistic  $t(x) = \sum_{i=1}^N x_i$  is minimal sufficient and the  $\sigma$ -ring  $\mathcal{S}$  is a minimal sufficient subfield, which is induced by  $t$ . Each  $p_\xi$  is pivotal for  $\mathbf{E}$  as  $p_\xi \equiv P$  and  $\mathbf{E}$  is dominated. According to Corollary, a measure  $m$  on  $\mathcal{A}$  is pivotal for  $\mathbf{E}$  if and only if  $m(A) = \int_A f \, dp_0$  for some  $\mathcal{S}$ -measurable function  $f$  with  $f > 0[P]$ . Thus the totality of pivotal measures coincides with all the measures whose density with respect to  $\mu$  are of the form  $\exp \{(-1/2) \cdot \sum_{i=1}^N x_i^2 \cdot f(\sum_{i=1}^N x_i)\}$  for some

$f > 0[\mu]$ . In case  $N=1$ , all the measures which are equivalent to  $\mu$  are pivotal, and in particular so is  $\mu$ . In case  $N \geq 2$ ,  $\mu$  is not pivotal.

EXAMPLE 2.  $X = \mathbf{R}^3$ .  $\mathbf{F}$  = all the planes parallel to a coordinate plane.  $\mathbf{A} = \{A \subset X; A \cap F \in \mathbf{B}_{\mathbf{R}^2} \text{ for all } F \in \mathbf{F}\}$ , where  $\mathbf{B}_{\mathbf{R}^2}$  denotes the Borel  $\sigma$ -field on  $\mathbf{R}^2$ . On each plane  $F \in \mathbf{F}$ , consider the same normal family as in Example 1 and extend each element of the family to  $\mathbf{A}$  in an obvious way that it vanishes outside  $F$ .  $P$  = the union of such families. Define a measure  $\mu$  by  $\mu(A) = \sum_{F \in \mathbf{F}} \nu(A \cap F)$ , where  $\nu$  is the Lebesgue measure on  $\mathbf{R}^2$ .

The experiment  $\mathbf{E} = (X, \mathbf{A}, P)$  is majorized by  $\mu$ , and  $\mathbf{F}$  is a maximal decomposition such that  $\mathbf{F} \subset \mathbf{S}_0[P]$ . In each sub-experiment, the totality of pivotal measures is described in Example 1. By Theorem 2, a pivotal measure on  $\mathbf{E}$  is expressed as the sum of pivotal measures on  $\mathbf{E}(F)$ . The measure  $\mu$  is not pivotal for  $\mathbf{E}$  as it is pointed out in Example 1 that  $\nu$  is not pivotal on  $\mathbf{E}(F)$ .

EXAMPLE 3.  $X = \mathbf{R}^2$ .  $\mathbf{F}$  = all horizontal and vertical lines in  $X$ .  $\mathbf{A} = \{A \subset X; A \cap F \in \mathbf{B}_{\mathbf{R}} \text{ for all } F \in \mathbf{F}\}$ , where  $\mathbf{B}_{\mathbf{R}}$  denotes the Borel  $\sigma$ -field on  $\mathbf{R}$ . On each  $F \in \mathbf{F}$ , consider the same normal family as in Example 1 and extend each element of the family to  $\mathbf{A}$  as in Example 2.  $P$  = the union of such families. Define a measure  $\mu$  by  $\mu(A) = \sum_{F \in \mathbf{F}} \nu(A \cap F)$ , where  $\nu$  is the Lebesgue measure on  $\mathbf{R}$ .

The experiment  $\mathbf{E} = (X, \mathbf{A}, P)$  is majorized by  $\mu$ , and  $\mathbf{F}$  is a maximal decomposition such that  $\mathbf{F} \subset \mathbf{S}_0[P]$ . As in Example 1, every measure on  $F$  which is equivalent to  $\nu$  is pivotal for  $\mathbf{E}(F)$ . By Theorem 2, all the measures which are the sums of those measures are pivotal. In particular,  $\mu$  is pivotal unlike in Example 2.

EXAMPLE 4.  $X = \mathbf{R}_+^N$ .  $\mathbf{A}$  = the Borel  $\sigma$ -field of  $X$ . For each  $\theta > 0$ ,  $p_\theta$  denotes the  $N$ -th product probability measure of the uniform distribution on  $(0, \theta]$ , i.e.  $(dp_\theta/d\mu)(x) = \theta^{-N} \cdot \prod_{k=1}^N I_{(0, \theta]}(x_k)$ , where  $\mu$  is the Lebesgue measure on  $X$ . The statistic  $t(x) = \max_{1 \leq i \leq N} x_i$  is minimal sufficient, and the  $\sigma$ -ring  $\mathbf{S}$  is induced by  $t$ . We construct a pivotal measure. Take  $p_k (k \geq 1)$  from  $P$  and put  $F_1 = S(p_1) = (0, 1]^N$  and  $F_k = S(p_k) - \cup_{j=1}^{k-1} F_j = (0, k]^N - (0, k-1]^N$  for  $k \geq 2$ . Then it is easily seen that  $\mathbf{F} = \{F_k; k \geq 1\}$  is a maximal decomposition with  $\mathbf{F} \subset \mathbf{S}_0[P]$ . For each  $k \geq 1$ ,  $p_k(A \cap F_k)$  is pivotal for  $\mathbf{E}(F_k)$  as it is equivalent to  $P_{F_k}$ . By Theorem 1, the measure  $n$  defined by  $n(A) = \sum_{k \geq 1} p_k(A \cap F_k)$  is pivotal for  $\mathbf{E}$ . This is the same pivotal measure as that obtained by Diepenbrock (see Remark). A pivotal measure  $m$  is of the form  $m(A) = \int_A f dn$  for some  $\mathbf{S}$ -measurable function  $f$  with  $f > 0[P]$ . Hence  $m(A) = \int_A f \sum_{k \geq 1} dp_k | F_k = \sum_{k \geq 1} \int_{A \cap F_k} f \cdot (dp_k/d\mu) \cdot d\mu$ , and so  $dm/d\mu = k^{-N} \cdot f(\max_{1 \leq i \leq N} x_i)$  on each  $F_k, k \geq 1$ . Therefore the totality of pivotal measures coincides with all the measures whose densities with respect to  $\mu$  are functions through  $t$ .

### 3. Relationship between Theorem 3 and the Luschgy, Mussmann & Yamada-characterization theorem

In this section, we show that the characterization theorem in [8] can be derived from Theorem 3.

We first describe the minimal  $L$ -space through the  $\sigma$ -ring  $S$ . Let  $E=(X, A, P)$  be a majorized experiment. The minimal  $L$ -space is the closed vector sublattice (of the vector lattice of all bounded signed measures on  $A$ ) generated by  $P$ , where the topology of the vector lattice is induced by the total variation norm. Let  $n$  be a pivotal measure and  $L^1(S, n)$  be the set of all  $S$ -measurable functions which are integrable with respect to  $n$ . For each  $f \in L^1(S, n)$ ,  $u_f(A) = \int_A f \, dn$  is a signed measure. Notice that the total variation norm of  $u_f$  coincides with the usual  $L^1$ -norm of  $f$  with respect to  $n$ . As  $n$  is a pivotal measure, for each  $p \in P$ , there is an  $S$ -measurable density  $f_p$ , i.e.  $p(A) = \int_A f_p \, dn$ . This correspondence between  $p$  and  $f_p$  implies that the minimal  $L$ -space is isomorphic to  $L^1(S, n)$  as an  $L$ -space. Hence the minimal  $L$ -space coincides with the totality of all the signed measures whose densities with respect to  $n$  belong to  $L^1(S, n)$ .

Next we show that Theorem 3 implies the characterization theorem in [8]. Let  $m$  be a pivotal measure. Then  $m$  is expressed as in Theorem 3. The  $S$ -measurable function  $f_F$  in Theorem 3 is not necessarily integrable with respect to  $n$  for every  $F \in \mathbf{F} (\subset S_0[P])$ . However for each  $F \in \mathbf{F}$ , there exists a countable partition of  $F$  consisting of  $S$ -measurable sets such that the restriction of  $f_F$  to each element of the partition is integrable with respect to  $n$  (see [2], Lemma 3.1), and hence  $f_F$  is the sum of such integrable functions. As the union of all such partitions forms a maximal decomposition consisting of  $S$ -measurable sets, we can replace  $\mathbf{F}$  by this maximal decomposition and denote the latter by  $\mathbf{F}'$  again. Consequently,  $m$  is expressed as  $m(A) = \sum_{F \in \mathbf{F}'} \int_{A \cap F} f_F \, dn$  for a family  $\{f_F; F \in \mathbf{F}'\}$  of  $S$ -measurable functions such that each  $f_F$  is integrable with respect to  $n$ . Put  $m_F(A) = \int_{A \cap F} f_F \, dn$ . Then it follows that  $f_F \geq 0[P]$  and  $f_F \neq 0[P]$  for all  $F \in \mathbf{F}'$ ,  $\min(f_F, f_G) = 0[P]$  for all  $F \neq G \in \mathbf{F}'$  and for each  $f \in L^1(S, n)$ ,  $\min(f, f_F) = 0[P]$  for all  $F \in \mathbf{F}'$  implies  $f = 0[P]$ . This implies that  $\{m_F; F \in \mathbf{F}'\}$  is a maximal orthogonal system, and hence  $m$  is the sum of the maximal orthogonal system  $\{m_F; F \in \mathbf{F}'\}$ .

Conversely, suppose that a measure  $m$  is the sum of a maximal orthogonal system. Take a maximal decomposition  $\mathbf{F}' (\subset S_0[P])$  and fix  $F \in \mathbf{F}'$ . Adding up the densities of the maximal orthogonal system which are positive  $[P]$  on  $F$ , we have the function  $f_F$  required in Theorem 3. Hence Theorem 3 implies that  $m$  is a pivotal measure.

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