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A NOTE ON PIVOTAL MEASURES IN MAJORIZED EXPERIMENTS

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1. Introduction

An *experiment* $E=(X, A, P)$, i.e. a triplet with a family P of probability measures on a measurable space (X, A), is said to be majorized by a measure μ equivalent to P (called a *"majorizing"* measure) if each $p\in P$ has a density $dp/d\mu$ with respect to μ. Let S_0 be the σ-ring generated by all the E-supports $S(p)=\{x\in X; (dp/d\mu)(x)>0\}, p\in P$. Then there exists a *"maximal decomposition"* F such that $F\subset S_0[P]$ (see [4], Lemma 2). A maximal decomposition F is defined as a covering of X of almost disjoint elements each of which is included by an E-support $S(p^{(F)})$ of some $p^{(F)}$ in P. For each $F\in F$, we define a dominated sub-experiment $E(F)=(F, A \cap F, P_F)$ by setting $A \cap F=\{A\cap F; A\in A\}$ and $P_F=\{p_F; p\in P, p(F)>0\}$, where $p_F(A\cap F)=p(A\cap F)/p(F)$. The σ-ring generated by $\{dp/d(p+q)\cdot I_{S(p)}; p, q\in P\}$ is called the σ-ring of *pairwise likelihood ratios* and denoted by S. The σ-field D generated by S is known to be the smallest PSS (*pairwise sufficient with supports*) subfield, and is equal to S and minimal sufficient when E is dominated (see [3]). A majorizing measure m on A is said to be *pivotal* for E if it holds that for each subfield B, B is PSS if and only if each p has a B-measurable version of the density dp/dm. A real valued function $f: X\rightarrow R$ is said to be S-measurable if for each Borel subset B of R, $f^{-1}(B) \cap \{x\in X; f(x)\neq 0\} \in S$.

The notion of pivotal measures was devised to obtain a minimal sufficient subfield by Halmos & Savage [6] and Bahadur [1] for the dominated experiment. Ramamoorthi & Yamada [9] generalized it to the majorized experiment. Recently Luschgy, Mussmann & Yamada [8] proved the following characterization theorem of pivotal measures by the method and in the terminology of vector lattices: Every pivotal measure is represented as the sum of a maximal orthogonal system in the minimal L-space.

The *minimal L-space* is the closed vector sublattice generated by P, and a *maximal orthogonal system* is a family of non-zero measures on A such that any two distinct elements in the family are singular with each other and a measure which is singular with all the elements of the family is zero.

In the present note, we show that every pivotal measure in a majorized experiment is expressed as the sum of pivotal measures in the dominated sub-experiments $E(F)$ for some maximal decomposition $F(\subset S_0[P])$ and that each pivotal measure on $E(F)$ has a positive S-measurable density with respect to a pivotal measure. Further, pointing out that the minimal L-space coincides with the totality of the signed measures which have S-measurable and integrable densities with respect to a pivotal measure, we show that the present result is a measure theoretical version of the above chatacterization theorem.

2. Pivotal measures

We consider the relation between a pivotal measure for E and a family of pivotal measures for the dominated sub-experiments $E(F)$.

Theorem 1. *Let $E=(X, A, P)$ be a majorized experiment. If $F(\subset S_0[P])$ is a maximal decomposition and a measure m_F on $A \cap F$ is pivotal for each $F \in F$, then $m(A)=\Sigma_{F \in F}\, m_F(A \cap F)$ is pivotal for E.*

Proof. It is enough to show that each $p \in P$ has an S-measurable version of dp/dm. Fix $p \in P$. There exists a countable subfamily $\{F_n;\, n \geq 1\}$ of F such that $S(p) \subset \cup_{n \geq 1} F_n[P]$ and $S(p) \cap F_n \neq \phi[P]$ for all n. We may assume that $\{F_n;\, n \geq 1\}$ is a disjoint family. As each m_{F_n} is pivotal for $E(F_n)$, there exists an $S \cap F_n$-measurable version g_n of dp_{F_n}/dm_{F_n}. For each $n \geq 1$, put $f_n(x)=p(F_n) \cdot g_n(x)$ if $x \in F_n$, and $=0$ if $x \in F_n^c$. As each F_n is in S_0, f_n is S-measurable. It is immediate that $\Sigma_{n \geq 1} f_n$ is an S-measurable version of dp/dm.

Theorem 2. *Let $E=(X, A, P)$ be a majorized experiment. A measure m on A is pivotal for E if and only if for each maximal decomposition $F(\subset S_0[P])$, there exists a family of pivotal measures $\{m_F;\, F \in F\}$ for the sub-experiments $E(F)$ such that $m(A)=\Sigma_{F \in F}\, m_F(A \cap F)$.*

Proof. "If" part is Theorem 1 itself, and so we prove "only if" part. Let m be a pivotal measure for E and take a maximal decomposition $F(\subset S_0[P])$. Fix $F \in F$. We define a measure m_F on $E(F)$ by $m_F(A \cap F)=m(A \subset F)$. It is clear that $m_F \equiv P_F$ and $m(A)=\Sigma_{F \in F} m_F(A \cap F)$. Take $p \in P$ with $p(F)>0$. Let f_p be an S-measurable version of dp/dm. Then it is easily checked that $f_p \cdot I_F/p(F)$ is an S-measurable version of dp_F/dm_F. This implies that m_F is a pivotal measure for $E(F)$ for each $F \in F$.

Next we consider pivotal measures on each dominated sub-experiment.

Let $F(\subset S_0[P])$ be a maximal decomposition and n a pivotal measure for E. Each F is included by a support $S(p^{(F)})$, and so $p_F^{(F)} \equiv P$. Fix $F \in F$. Then the restriction n_F of n to $A \cap F$ is pivotal for $E(F)$ as in the proof of Theorem 2. Let u be a pivotal measure for $E(F)$. Notice that $du/dn_F=(dp_F^{(F)}/du)^{-1} \cdot$

$(dp_F^{(F)}/dn_F)^{-1}[P_F]$ as $p_F^{(F)} \equiv n_F \equiv u$. Two densities in the right side have S-measurable versions as u and n_F are pivotal. That is, $u(A \cap F) = \int_{F \cap F} f \, dn$ for some $S \cap F$-measurable function $f \colon F \to R$, with $f > 0 [P_F]$.

Conversely a measure u of this form is pivotal for $E(F)$. Because, n_F is pivotal and $dp_F/du = (dp_F/dn_F)/f[P_F]$ for all $p_F \in P_F$.

Thus we have proved the following

Theorem 3. *Let $E = (X, A, P)$ be a majorized experiment and n a pivotal measure for E. Then, a measure m on A is pivotal for E if and only if it is expressed as $m(A) = \Sigma_{F \in F} \int_{A \cap F} f_F \, dn$ for some maximal decomposition $F(\subset S_0[P])$ and some family $\{f_F \colon F \in F\}$ of S-measurable functions such that $f_F > 0 [P]$ on F and $f_F = 0 [P]$ on F^c.*

In case E is weakly dominated, we have the following simpler expression.

Corollary. *Let $E = (X, A, P)$ be a weakly dominated experiment and n a pivotal measure for E. A measure m on A is pivotal for E if and only if $m(A) = \int_A f \, dn$ for some B-measurable function f with $f > 0 [P]$, where B is a minimal sufficient subfield.*

Proof. Let B be a minimal sufficient subfield. It follows from Theorem 1.1 in [7] that $(X, B, P | B)$ is weakly dominated. Hence for the family of S-measurable functions $\{f_F; F \in F\}$ in Theorem 3, there exists a B-measurable function f such that $f \cdot I_F = f_F [P]$ for all $F \in F$.

REMARK. In Theorem 3, "if" part remains true with "any" maximal decomposition and "any" family $\{f_F; F \in F\}$. "Only if" part also holds true with "any" maximal decomposition and some family $\{f_F; F \in F\}$. Gooßen [5] attained to a similar characterization by making use of a pivotal measure $\Sigma_{F \in F} p^{(F)} (A \cap F)$, which was obtained by Diepenbrock.

In the following Examples 1 to 4, we observe that the totality of pivotal measures is fairly large.

EXAMPLE 1 (*N*-th product normal family with a location parameter). $X = R^N$, $A = $ the Borel σ-field on X. $P = \{p_\xi; \xi \in R\}$. $(dp_\xi/d\mu)(x) = (2\pi)^{-N/2} \exp \{(-1/2) \Sigma_{i=1}^N (x_i - \xi)^2\}$, where μ is the Lebesgue measure on X. The statistic $t(x) = \Sigma_{i=1}^N x_i$ is minimal sufficient and the σ-ring S is a minimal sufficient subfield, which is induced by t. Each p_ξ is pivotal for E as $p_\xi \equiv P$ and E is dominated. According to Corollary, a measure m on A is pivotal for E if and only if $m(A) = \int_A f \, dp_0$ for some S-measurable function f with $f > 0 [P]$. Thus the totality of pivotal measures coincides with all the measures whose density with respect to μ are of the form $\exp \{(-1/2) \cdot \Sigma_{i=1}^N x_i^2 \cdot f(\Sigma_{i=1}^N x_i)\}$ for some

$f>0\,[\mu]$.   In case $N=1$, all the measures which are equivalent to μ are pivotal, and in particular so is μ.   In case $N\geq 2$, μ is not pivotal.

EXAMPLE 2. $X=\boldsymbol{R}^3$. $\boldsymbol{F}=$all the planes parallel to a coordinate plane. $\boldsymbol{A}=\{A\subset X;\ A\cap F\in\boldsymbol{B}_{R^2}$ for all $F\in\boldsymbol{F}\}$, where $\boldsymbol{B}_{R^2}$ denotes the Borel σ-field on $\boldsymbol{R}^2$. On each plane $F\in\boldsymbol{F}$, consider the same normal family as in Example 1 and extend each element of the family to $\boldsymbol{A}$ in an obvious way that it vanishes outside F. $P=$the union of such families.   Define a measure μ by $\mu(A)=\Sigma_{F\in F}\,\nu(A\cap F)$, where ν is the Lebesgue measure on $\boldsymbol{R}^2$.

The experiment $\boldsymbol{E}=(X,\boldsymbol{A},P)$ is majorized by μ, and $\boldsymbol{F}$ is a maximal decomposition such that $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$.   In each sub-experiment, the totality of pivotal measures is described in Example 1.   By Theorem 2, a pivotal measure on $\boldsymbol{E}$ is expressed as the sum of pivotal measures on $\boldsymbol{E}(F)$.   The measure μ is not pivotal for $\boldsymbol{E}$ as it is pointed out in Example 1 that ν is not pivotal on $\boldsymbol{E}(F)$.

EXAMPLE 3. $X=\boldsymbol{R}^2$. $\boldsymbol{F}=$all horizontal and vertical lines in X. $\boldsymbol{A}=\{A\subset X;\ A\cap F\in\boldsymbol{B}_R$ for all $F\in\boldsymbol{F}\}$, where $\boldsymbol{B}_R$ denotes the Borel σ-field on $\boldsymbol{R}$.   On each $F\in\boldsymbol{F}$, consider the same normal family as in Example 1 and extend each element of the family to $\boldsymbol{A}$ as in Example 2. $P=$the union of such families.   Define a measure μ by $\mu(A)=\Sigma_{F\in F}\,\nu(A\cap F)$, where ν is the Lebesgue measure on $\boldsymbol{R}$.

The experiment $\boldsymbol{E}=(X,\boldsymbol{A},P)$ is majorized by μ, and $\boldsymbol{F}$ is a maxiaml decomposition such that $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$.   As in Example 1, every measure on F which is equivalent to ν is pivotal for $\boldsymbol{E}(F)$.   By Theorem 2, all the measures which are the sums of those measures are pivotal.   In particular, μ is pivotal unlike in Example 2.

EXAMPLE 4. $X=\boldsymbol{R}_+^N\cdot\boldsymbol{A}=$the Borel σ-field of X.   For each $\theta>0$, p_θ denotes the N-th product probability measure of the uniform distribution on $(0,\theta]$, i.e. $(dp_\theta/d\mu)(x)=\theta^{-N}\cdot\Pi_{k=1}^N\,I_{(0,\theta]}(x_k)$, where μ is the Lebesgue measure on X. The statistic $t(x)=\max_{1\leq i\leq N}\,x_i$ is minimal sufficient, and the σ-ring $\boldsymbol{S}$ is induced by t.   We construct a pivotal measure.   Take $p_k\,(k\geq 1)$ from P and put $F_1=S(p_1)=(0,1]^N$ and $F_k=S(p_k)-\cup_{j=1}^{k-1}F_j=(0,k]^N-(0,k-1]^N$ for $k\geq 2$.   Then it is easily seen that $\boldsymbol{F}=\{F_k;\ k\geq 1\}$ is a maximal decomposition with $\boldsymbol{F}\subset\boldsymbol{S}_0[P]$. For each $k\geq 1$, $p_k(A\cap F_k)$ is pivotal for $\boldsymbol{E}(F_k)$ as it is equivalent to P_F.   By Theorem 1, the measure n defined by $n(A)=\Sigma_{k\geq 1}\,p_k(A\cap F_k)$ is pivotal for $\boldsymbol{E}$. This is the same pivotal measure as that obtained by Diepenbrock (see Remark). A pivotal measure m is of the form $m(A)=\int_A f\,dn$ for some $\boldsymbol{S}$-measurable function f with $f>0\,[P]$.   Hence $m(A)=\int_A f\Sigma_{k\geq 1}\,dp_k\,|\,F_k=\Sigma_{k\geq 1}\int_{A\cap F_k} f\cdot(dp_k/d\mu)\cdot d\mu$, and so $dm/d\mu=k^{-N}\cdot f(\max_{1\leq i\leq N}\,x_i)$ on each F_k, $k\geq 1$.   Therefore the totality of pivotal measures coicides with all the measures whose densities with respect to μ are functions through t.

3. Relationship between Theorem 3 and the Luschgy, Mussmann & Yamada-characterization theorem

In this section, we show that the characterization theorem in [8] can be derived from Theorem 3.

We first describe the minimal L-space through the σ-ring S. Let $E = (X, A, P)$ be a majorized experiment. The minimal L-space is the closed vector sublacttice (of the vector lattice of all bounded signed measures on A) generated by P, where the topology of the vector lattice is induced by the total variation norm. Let n be a pivotal measure and $L^1(S, n)$ be the set of all S-measurable functions which are integrable with respect to n. For each $f \in L^1(S, n)$, $u_f(A) = \int_A f\, dn$ is a signed measure. Notice that the total variation norm of u_f coincides with the usual L^1-norm of f with respect to n. As n is a pivotal measure, for each $p \in P$, there is an S-measurable density f_p, i.e. $p(A) = \int_A f_p\, dn$. This correspondence between p and f_p implies that the minimal L-space is isomorphic to $L^1(S, n)$ as an L-space. Hence the minimal L-space coincides with the totality of all the signed measures whose densities with respect to n belong to $L^1(S, n)$.

Next we show that Theorem 3 implies the characterization theorem in [8]. Let m be a pivotal measure. Then m is expressed as in Theorem 3. The S-measurable function f_F in Theorem 3 is not necessarily integrable with respect to n for every $F \in F(\subset S_0[P])$. However for each $F \in F$, there exists a countable partition of F consisting of S-measurable sets such that the restriction of f_F to each element of the partition is integrable with respect to n (see [2], Lemma 3.1), and hence f_F is the sum of such integrable functions. As the union of all such partitions forms a maximal decomposition consisting of S-measurable sets, we can replace F by this maximal decomposition and denote the latter by F again. Consequently, m is expressed as $m(A) = \Sigma_{F \in F} \int_{A \cap F} f_F\, dn$ for a family $\{f_F; F \in F\}$ of S-measurable functions such that each f_F is integrable with respect to n. Put $m_F(A) = \int_{A \cap F} f_F\, dn$.. Then it follows that $f_F \geq 0\,[P]$ and $f_F \neq 0\,[P]$ for all $F \in F$, $\min (f_F, f_G) = 0\,[P]$ for all $F \neq G \in F$ and for each $f \in L^1(S, n)$, $\min (f, f_F) = 0\,[P]$ for all $F \in F$ implies $f = 0\,[P]$. This implies that $\{m_F; F \in F\}$ is a maximal orthogonal system, and hence m is the sum of the maximal orthogonal system $\{m_F; F \in F\}$.

Conversely, suppose that a measure m is the sum of a maximal orthogonal system. Take a maximal decomposition $F(\subset S_0[P])$ and fix $F \in F$. Adding up the densities of the maximal orthogonal system which are positive $[P]$ on F, we have the function f_F required in Theorem 3. Hence Theorem 3 implies that m is a pivotal measure.

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