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ON C-PROJECTIVE 8 AND 10 STEMS

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In [9] the author computed the stable C-projective homotopy of spheres through the 13-stem but left the 8-stem $\pi_{2n+7}^{SC}(S^{2n-1})$ for $n \equiv 0, 6 \pmod{8}$ and the 10-stem $\pi_{2n+9}^{SC}(S^{2n-1})$ for $n \equiv 2 \pmod{3}$ unsolved. The purpose of this note is to determine these stems. These stems are also dealt with by Snow [11] and Gilbert and Zvengrowski [3] which is a part of a master's thesis by Gilbert [7]. Our method is rather more elementary than the methods of [3] and [11] in the sense that their main tools are the spectral sequences but we do not use any spectral sequences.

Our results are the followings.

Theorem.

$$\begin{aligned} \text{(I)} \quad \pi_{2n+7}^{SC}(S^{2n-1}) &= \begin{cases} G_8 & \text{if } n \equiv 6 \pmod{8} \\ Z_2\{\eta\sigma\} & \text{if } n \equiv 0 \pmod{8} \end{cases}, \\ \text{(II)} \quad \pi_{2n+9}^{SC}(S^{2n-1}) &= \begin{cases} Z_2\{\eta\mu\} & \text{if } n \equiv 5 \pmod{6} \\ 0 & \text{if } n \equiv 2 \pmod{6} \end{cases}. \end{aligned}$$

We use the notations in [8], [9] and [12] freely.

By Atiyah's S-duality theorem we have

Lemma 1.

$$\pi_{2n+2l-1}^{SC}(S^{2n-k}) = \text{Ker} [i_{l*}: \pi_{2m+2l+k-1}^s(S^{2m}) \rightarrow \pi_{2m+2l+k-1}^s(P_{m+l+1, l+1})]$$

where $m = jM_{l+1}(C) - n - l - 1$ for large j , and $k=0$ or 1 . In particular,

$$\pi_{2n+7}^{SC}(S^{2n-1}) = \text{Ker} [i_{4*}: \pi_{2m+8}^s(S^{2m}) \rightarrow \pi_{2m+8}^s(P_{m+5,5})], \quad m = jM_5(C) - n - 5$$

and

$$\pi_{2n+9}^{SC}(S^{2n-1}) = \text{Ker} [i_{5*}: \pi_{2m+10}^s(S^{2m}) \rightarrow \pi_{2m+10}^s(P_{m+6,6})], \quad m = jM_6(C) - n - 6$$

for large j .

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Proof.

$$\begin{aligned}
 & \pi_{2n+2l-1}^{SC}(S^{2n-k}) \\
 &= \text{Im}[(Ep_{n+l,l})^*: \pi_s^{2n-k+1}(EP_{n+l,l}) \rightarrow \pi_s^{2n-k+1}(S^{2(n+l)})] \\
 &= \text{Ker}[q_{l*}: \pi_s^{2n-k+1}(S^{2(n+l)}) \rightarrow \pi_s^{2n-k+1}(P_{n+l+1,l+1})] \\
 &= \text{Ker}[i_{l*}: \pi_{2m+2l+k-1}^s(S^{2m}) \rightarrow \pi_{2m+2l+k-1}^s(P_{m+l+1,l+1})]
 \end{aligned}$$

for large j and $m=jM_{l+1}(C)-n-l-1$. Here the first equality follows from (1.3) of [9], the second one from the stable cohomotopy exact sequence associated with the cofibration $S^{2(n+l)-1} \xrightarrow{p_{n+l,l}} P_{n+l,l} \xrightarrow{i_1} P_{n+l+1,l+1}$, and the last one does from Atiyah's S -duality theorem [1] (or see (4.4) and (4.5) of [8]).

This lemma says that for our purpose it is enough to calculate some stable homotopy groups of the stunted complex projective spaces. Those we need are easily computed by the methods of [10], but they have been done in [2], [4], [5] and [6] so that we do not give proofs of (i), (ii), ..., (xiii) except (iv) below.

We explain some notations used below. We denote an element of $\pi_{2m+i}^s(P_{m+l,l})$ by $[\alpha]_l$ which is mapped to $\alpha \in G_{i-2l+2}$ by $q_{l-1*}: \pi_{2m+i}^s(P_{m+l,l}) \rightarrow \pi_{2m+i}^s(S^{2(m+l-1)}) = G_{i-2l+2}$. For a prime number p , ${}^p\pi_i^s(\)$ denotes the p -primary component of $\pi_i^s(\)$.

Lemma 2. *For m odd we have*

- (i) $\pi_{2m+5}^s(P_{m+2,2}) = Z_8\{[\nu]_2\} \oplus Z_3\{[\alpha_1]_2\},$
- (ii) $\pi_{2m+8}^s(P_{m+2,2}) = Z_4\{[\nu]_2\nu\},$
- (iii) $i_{1*}\bar{\nu} = i_{1*}\mathcal{E} = 2[\nu]_2\nu,$
- (iv) $\pi_{2m+8}^s(P_{m+5,5}) = Z\{[Q^s\{m+5, 5\}]_5\} \oplus \begin{cases} Z_4\{i_{3*}[\nu]_2\nu\} & \text{if } m \equiv 3 \pmod{8} \\ Z_2\{i_{3*}[\nu]_2\nu\} & \text{if } m \equiv 7 \pmod{8} \\ 0 & \text{if } m \equiv 1 \pmod{4}. \end{cases}$

Proof of (iv). Put

$$e_m = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4} \\ 2 & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

Then we have

- (v) ${}^2\pi_{2m+7}^s(P_{m+3,3}) = Z_{16}\{i_{2*}\sigma\} \oplus Z_{8/e_m}\{[e_m\nu]_3\},$
- (vi) ${}^2\pi_{2m+7}^s(P_{m+4,4}) = Z_{16}\{i_{3*}\sigma\} \oplus Z_{4/e_m}\{i_{1*}[e_m\nu]_3\},$
- (vii) $[e_m\nu]_3\eta = (m+3)e_m/2 i_{1*}[\nu]_2\nu,$
- (viii) $\pi_{2m+8}^s(P_{m+4,4}) = \begin{cases} Z_4\{i_{2*}[\nu]_2\nu\} & \text{if } m \equiv 3 \pmod{8} \\ Z_2\{i_{2*}[\nu]_2\nu\} & \text{if } m \equiv 7 \pmod{8} \\ 0 & \text{if } m \equiv 1 \pmod{4}. \end{cases}$

By (T_4) and (T'_4) of [10] we can choose an element $p' \in \pi_{2m+7}^s(P_{m+3,3})$ with

$i_1^*p' = p_{m+4,4}$ and $q_2^*p' = ((m+5)/2)(\nu + \alpha_1)$. We can put $p' = a_m i_2^* \sigma + b_m [e_m \nu]_3 + \text{odd torsions}$ for some integers a_m and b_m . Then

$$(1) \quad p_{m+4,4} = a_m i_3^* \sigma + b_m i_1^* [e_m \nu]_3 + \text{odd torsions}$$

and

$$\begin{aligned} ((m+5)/2)\nu + \text{odd torsions} &= q_2^* p' \\ &= b_m e_m \nu + \text{odd torsions} \end{aligned}$$

so that

$$(2) \quad b_m e_m \equiv (m+5)/2 \pmod{8}.$$

Consider the exact sequence (S_5) of [10]:

$$\cdots \rightarrow G_1 = Z_2\{\eta\} \xrightarrow{p_{m+4,4}^*} \pi_{2m+8}^s(P_{m+4,4}) \xrightarrow{i_1^*} \pi_{2m+8}^s(P_{m+5,5}) \xrightarrow{q_4^*} G_0 = Z\{\iota\} \rightarrow \cdots.$$

We have

$$\begin{aligned} p_{m+4,4}^*(\eta) &= a_m i_3^* \sigma \eta + b_m i_1^* [e_m \nu]_3 \eta, \quad \text{by (1)} \\ &= a_m i_3^* (\bar{\nu} + \varepsilon) + b_m i_1^* [e_m \nu]_3 \eta \\ &= b_m i_1^* [e_m \nu]_3 \eta, \quad \text{by (iii) and (viii)} \\ &= b_m ((m+3)/2) e_m i_2^* [\nu]_2 \nu, \quad \text{by (vii)} \\ &= ((m+3)(m+5)/4) i_2^* [\nu]_2 \nu, \quad \text{by (2) and (viii)} \\ &= 0, \quad \text{by (viii)}. \end{aligned}$$

Then we have a short exact sequence:

$$0 \rightarrow \pi_{2m+8}^s(P_{m+4,4}) \xrightarrow{i_1^*} \pi_{2m+8}^s(P_{m+5,5}) \xrightarrow{q_4^*} Z\{Q^s\{m+5, 5\}\iota\} \rightarrow 0,$$

since $\#p_{m+4,4} = Q^s\{m+5, 5\}$. Of course this splits. Thus the proof of (iv) is completed.

Now we prove (I). Since $i_4^* = i_3^* i_1^*$, by (iii) and (iv) we have

$$\begin{aligned} \text{Ker}[i_4^*: \pi_{2m+8}^s(S^{2m}) \rightarrow \pi_{2m+8}^s(P_{m+5,5})] \\ = \begin{cases} G_8 & \text{if } m \equiv 1 \pmod{4} \text{ or } 7 \pmod{8} \\ Z_2\{\bar{\nu} + \varepsilon\} & \text{if } m \equiv 3 \pmod{8}. \end{cases} \end{aligned}$$

Then (I) follows from (iii) of (2.8) of [9], since $M_5(C) = 2^6 \cdot 3^2 \cdot 5 \equiv 0 \pmod{8}$, $m \equiv -n - 5 \pmod{8}$, and $\eta\sigma = \bar{\nu} + \varepsilon$.

For (II) we only consider the 3-primary component by (2.11) of [9].

Lemma 3.

$$\begin{aligned}
(\text{ix}) \quad {}^3\pi_{2m+7}^s(P_{m+3,3}) &= \begin{cases} Z_9\{[\alpha_1]_3\} & \text{if } m \not\equiv 0 \pmod{3} \\ Z_3\{i_2^*\alpha_2\} \oplus Z_3\{[\alpha_1]_3\} & \text{if } m \equiv 0 \pmod{3}, \end{cases} \\
(\text{x}) \quad {}^3\pi_{2m+7}^s(P_{m+4,4}) &= \begin{cases} Z_9\{i_1^*[\alpha_1]_3\} & \text{if } m \not\equiv 0 \pmod{3} \\ Z_3\{i_3^*\alpha_2\} \oplus Z_3\{i_1^*[\alpha_1]_3\} & \text{if } m \equiv 0 \pmod{3}, \end{cases} \\
(\text{xi}) \quad {}^3\pi_{2m+10}^s(P_{m+3,3}) &= Z_3\{i_2^*\beta_1\} \\
(\text{xii}) \quad [\alpha_1]_3\alpha_1 &= \begin{cases} -((m+3)/2)i_2^*\beta_1 & \text{if } m \equiv 1 \pmod{2} \\ -(m/2)i_2^*\beta_1 & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\
(\text{xiii}) \quad {}^3\pi_{2m+10}^s(P_{m+4,4}) &= Z_3\{i_3^*\beta_1\}.
\end{aligned}$$

Note that $p_{m+4,4}$ is an element of $\pi_{2m+7}^s(P_{m+4,4})$. If $m \equiv 0 \pmod{3}$, then $[\alpha_1]_3\alpha_1 = 0$ by (xii) so that by (x)

$$(3) \quad p_{m+4,4}^*(\alpha_1) = 0 \quad \text{if } m \equiv 0 \pmod{3}$$

since $\alpha_2\alpha_1 = 0$.

As seen in [10]

$$\#p_{m+4,4} = Q^s\{m+5, 5\} = C\{jM_5(C) - m - 5, 5\} \quad \text{for a large } j,$$

so that by (3.1) of [8] we have

Lemma 4.

| $v_3(\#p_{m+4,4})$ | $m \pmod{\quad}$ |
|--------------------|------------------|
| 2 | 2(3) |
| 1 | 0(3), 1, 7(9) |
| 0 | 4(9) |

If $m \equiv 2 \pmod{3}$, then we may put $p_{m+4,4} = yi_1^*[\alpha_1]_3 + \text{other terms}$ for some integer y with $y \not\equiv 0 \pmod{3}$ by (x) and Lemma 4, and hence

$$(4) \quad p_{m+4,4}^*(\alpha_1) \not\equiv 0 \quad \text{if } m \equiv 2 \pmod{3}$$

by (xii) and (xiii). If $m \equiv 1 \pmod{3}$, then (x) and Lemma 4 say that $p_{m+4,4}$ is divisible by 3 so that

$$(5) \quad p_{m+4,4}^*(\alpha_1) = 0 \quad \text{if } m \equiv 1 \pmod{3}.$$

Now we prove (II). Consider the 3-primary part of the exact sequence $(S)_6$ of [10]:

$$\cdots \rightarrow 0 = {}^3\pi_{2m+10}^s(S^{2m+9}) \rightarrow {}^3\pi_{2m+10}^s(P_{m+5,5}) \xrightarrow{i_1^*} {}^3\pi_{2m+10}^s(P_{m+6,6}) \rightarrow \cdots$$

and of $(S)_5$:

$$\begin{aligned} \cdots \rightarrow {}^3\pi_{2m+10}^s(S^{2m+7}) = Z_3\{\alpha_1\} &\xrightarrow{p_{m+4,4}^*} {}^3\pi_{2m+10}^s(P_{m+4,4}) = Z_3\{i_3^*\beta_1\} \\ &\xrightarrow{i_1^*} {}^3\pi_{2m+10}^s(P_{m+5,5}) \rightarrow {}^3\pi_{2m+10}^s(S^{2m+8}) = 0 \rightarrow \cdots \end{aligned}$$

The first sequence implies

$$\begin{aligned} \text{Ker}[i_{5*}: {}^3\pi_{2m+10}^s(S^{2m}) \rightarrow {}^3\pi_{2m+10}^s(P_{m+6,6})] \\ = \text{Ker}[i_{4*}: {}^3\pi_{2m+10}^s(S^{2m}) \rightarrow {}^3\pi_{2m+10}^s(P_{m+5,5})], \end{aligned}$$

and the second one, (xiii), (3), (4) and (5) imply that this group is

$$\begin{cases} Z_3\{\beta_1\} & \text{if } m \equiv 2 \pmod{3} \\ 0 & \text{if } m \not\equiv 2 \pmod{3}. \end{cases}$$

Then (II) follows from (v) and (vi) of (2.11) of [9], since $M_6(C) = M_5(C) \equiv 0 \pmod{3}$ and $m \equiv -n \pmod{3}$.

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