

Title	On the spectrum representing algebraic K-theory for a finite field
Author(s)	Watanabe, Takashi
Citation	Osaka Journal of Mathematics. 1985, 22(3), p. 447-462
Version Type	VoR
URL	https://doi.org/10.18910/12328
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Watanabe, T. Osaka J. Math. 22 (1985), 447-462

# ON THE SPECTRUM REPRESENTING ALGEBRAIC K-THEORY FOR A FINITE FIELD

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

## TAKASHI WATANABE

(Received February 20, 1984)

Let r be an odd prime power. Let  $F_r$  denote the field with r elements. According to [11] and others, there exists a (-1)-connected  $\Omega$ -spectrum  $KF_r$  whose 0-th space is  $\mathbb{Z} \times BGLF_r^+$ , where  $BGLF_r^+$  is the plus construction of the classifying space of  $GLF_r$ .  $KF_r$  is a ring spectrum with a unit.

Let p be an odd prime. The object of this paper is the localization of  $KF_r$  at p,  $KF_{r(p)}$ , for the case that r gives a generator of the group of units  $(\mathbb{Z}/p^2)^{\times}$ . Then the associated generalized cohomology theory  $KF_r^*(; \mathbb{Z}_{(p)})$  appears as a secondary cohomology theory determined by a certain stable operation in connected complex K-theory localized at p. From this interpretation we deduce some results about the multiplicative structure on  $KF_{r(p)}$ , which are basic to the study of the ring structure of  $KF_{r^*}(CP^{\infty}; \mathbb{Z}_{(p)})$  etc. In particular we can characterize the product on  $KF_{r(p)}$  by a certain property.

For simplicity we write A for  $KF_{r(p)}$  (see [8]). We shall work in the homotopy category of CW-spectra (see [3, III]).

The paper is organized as follows. In §0 we collect several results on A. In §1 we compute  $H^*(A; \mathbb{Z}/p)$ . In §2 we compute  $H_*(A; \mathbb{Z}/p)$ . In §3 we consider the left coaction of  $\mathcal{A}_*$  on  $H_*(A; \mathbb{Z}/p)$  and discuss the  $\mathcal{B}$ -module structure of  $H^*(A; \mathbb{Z}/p)$ , where  $\mathcal{B}=\Lambda(Q_0, Q_1)\subset \mathcal{A}$ . In §4 we prove our main results, which are Theorems 4.3 and 4.5.

#### 0. The spectrum A

Let p be a fixed odd prime. Let  $bu_{(p)}$  be the  $\Omega$ -spectrum representing connected complex K-theory localized at p. This is a ring spectrum with a unit and  $\pi_*(bu_{(p)}) = \mathbb{Z}_{(p)}[u]$  where |u| = 2. It is known that

$$bu_{(p)} = \bigvee_{j=1}^{p-1} \Sigma^{2(j-1)} G$$

for a spectrum G [6]. This is a ring spectrum with a unit and  $\pi_*(G) = Z_{(p)}[v]$ where |v| = 2(p-1). According to [4], if  $\kappa: G \to bu_{(p)}$  is the injection, then the diagram

(0.1)  
$$\begin{array}{c} \Sigma^{2(p-1)}G \xrightarrow{\upsilon} G\\ \Sigma^{2(p-1)}\kappa \downarrow \\ \Sigma^{2(p-1)}bu_{(p)} \\ u^{p-2} \downarrow \\ \Sigma^{2}bu_{(p)} \xrightarrow{u} bu_{(p)} \end{array}$$

commutes, where (by abuse of notation) u, v denote the composites  $S^2 \wedge bu_{(p)}$  $\stackrel{u \wedge 1}{\longrightarrow} bu_{(p)} \wedge bu_{(p)} \rightarrow bu_{(p)}$  and  $S^{2(p-1)} \wedge G \xrightarrow{v \wedge 1} G \wedge G \rightarrow G$  respectively. Furthermore, for each r prime to p, there exists a map of ring spectra  $\psi^r \colon G \rightarrow G$  which makes the diagram

$$(0.2) \qquad \begin{array}{c} G \xrightarrow{\psi^{r}} G \\ & & \downarrow^{\kappa} \\ & & \downarrow^{\kappa} \\ & & bu_{(p)} \xrightarrow{\psi^{r}} bu_{(p)} \end{array}$$

commute, where the lower  $\psi'$  is derived from the Adams operation in complex K-theory.

Consider the fibre sequence

$$\Sigma^2 bu_{(p)} \xrightarrow{u} bu_{(p)} \xrightarrow{\rho} HZ_{(p)}$$

(where  $HZ_{(p)}$  denotes the Eilenberg-MacLane spectrum for  $Z_{(p)}$ ). This leads to an exact sequence

$$0 \rightarrow [\boldsymbol{b}\boldsymbol{u}_{(p)}, \Sigma^2 \boldsymbol{b}\boldsymbol{u}_{(p)}] \xrightarrow{\boldsymbol{u}_*} [\boldsymbol{b}\boldsymbol{u}_{(p)}, \boldsymbol{b}\boldsymbol{u}_{(p)}] \xrightarrow{\boldsymbol{\rho}_*} [\boldsymbol{b}\boldsymbol{u}_{(p)}, \boldsymbol{H}\boldsymbol{Z}_{(p)}]$$

where we have used the fact that  $H^{-1}(\boldsymbol{bu}_{(p)}; \boldsymbol{Z}_{(p)}) = 0$ . Consider the element  $\psi^r - 1 \in [\boldsymbol{bu}_{(p)}, \boldsymbol{bu}_{(p)}]$ . Since  $\rho_*(\psi^r - 1) = 0$ , there is a unique  $\theta \in [\boldsymbol{bu}_{(p)}, \Sigma^2 \boldsymbol{bu}_{(p)}]$  such that  $u_*(\theta) = \psi^r - 1$ . Denote by  $\boldsymbol{A}$  the fibre spectrum of  $\theta$ ; that is,

$$(0.3) A \xrightarrow{\eta} bu_{(p)} \xrightarrow{\theta} \Sigma^2 bu_{(p)}$$

is a fibre sequence.

From now on we deal with a case such that r is a generator of  $(\mathbb{Z}/p^2)^{\times}$ . Then A does not depend on the choice of r. In fact, since  $(\psi^r - 1)_*(u^s) = (r^s - 1)u^s$  in  $\pi_*(\boldsymbol{bu}_{(p)})$ , [1, Lemma (2.12)] yields

(0.4) 
$$\pi_i(A) = \begin{cases} Z_{(p)} & \text{if } i = 0 \\ Z/p^{1+\nu_p(t)} & \text{if } i = 2t(p-1)-1 \ (t>0) \\ 0 & \text{otherwise} \end{cases}$$

where  $\nu_{p}(t)$  is the power of p in t.

Consider the fibre sequence

$$\Sigma^{2(p-1)}G \xrightarrow{v} G \longrightarrow HZ_{(p)}$$
.

By a similar argument we have a unique lift  $\theta' \in [G, \Sigma^{2(p-1)}G]$  of  $\psi' - 1 \in [G, G]$ . Let A' denote the fibre of  $\theta'$ . Then from (0.1) and (0.2) it follows that there is a commutative diagram of fibre sequences



It is easily verified that the induced map  $\kappa': A' \rightarrow A$  is an equivalence. So we may identify them.

Choose r to be an odd prime power so that it satisfies our hypothesis. In view of [12, VIII] it seems that there exists a map of ring spectra  $Br: KF_{r(p)} \rightarrow bu_{(p)}$  and its lift  $KF_{r(p)} \rightarrow A$  in (0.3) becomes an equivalence. We identify them and then  $\eta$  can be regarded as a map of ring spectra (cf. [15, p. 252]). Since  $\kappa$  is a (split injective) map of ring spectra, so is  $\eta'$ . In §4 we give a different approach to this fact.

It is not an accident that  $\pi_*(\mathbf{A})$  is isomorphic to  $\text{Im } J_{(p)}$  which is a direct summand of  $\pi_*(\mathbf{S}^0)_{(p)}$ . In fact, Tornehave [19] showed that

(0.5) The unit  $\hat{\iota}: \mathbf{S}^0 \to \mathbf{A}$  realizes the projection of  $\pi_*(\mathbf{S}^0)_{(p)}$  onto  $\operatorname{Im} J_{(p)}$ .

Hereafter for brevity we write

(0.6) 
$$\Sigma^{2p-3} G \xrightarrow{\Delta} A \xrightarrow{\eta} G \xrightarrow{\theta} \Sigma^{2(p-1)} G$$

We will use only this fibre sequence in later sections.

## 1. The mod p cohomology of A

Let  $\mathcal{A}$  be the mod p Steenrod algebra. As an  $\mathcal{A}$ -module,

(1.1) 
$$H^*(G; \mathbb{Z}|p) \simeq \mathcal{A}|\mathcal{A}(Q_0, Q_1)$$

where  $Q_0 = \delta$ ,  $Q_1 = \mathcal{P}^1 \delta - \delta \mathcal{P}^1$  and  $\mathcal{A}()$  denotes the left ideal in  $\mathcal{A}$  generated by the set in parentheses. Apply the functor  $H^*(; \mathbb{Z}/p)$  to (0.6). Then we have

**Lemma 1.1.** If f is the generator of  $H^0(G; \mathbb{Z}|p)$ , then  $\theta^*(\sigma^{2(p-1)}f) = c \cdot \mathcal{L}^1 f$  for some non-zero  $c \in \mathbb{Z}|p$  (where  $\sigma^i$  denotes the increase of degrees by i).

Proof. By (1.1),  $H^{2(p-1)}(G; \mathbb{Z}/p) = \mathbb{Z}/p\{\mathcal{P}^1f\}$ . Hence we may set  $\theta^*(\sigma^{2(p-1)}f) = c \cdot \mathcal{P}^1f$  for some  $c \in \mathbb{Z}/p$ . It is sufficient to show that c is non-zero. Suppose c=0. Then it follows that  $\tilde{H}^*(\mathbf{A}; \mathbb{Z}/p) = \mathbb{Z}/p\{\eta^*(\mathcal{P}^1f)\}$  in degrees less than 2p(p-1)-1. On the other hand, by (0.5) or [17],  $\hat{\iota}_*: \pi_i(\mathbf{S}^0)_{(p)} \to \pi_i(\mathbf{A})$  is an isomorphism for  $i < |\beta_1| = 2p(p-1)-2$  (where  $\beta_1 \in \pi_*(\mathbf{S}^0)_{(p)}$  is the first element which does not belong to  $\mathrm{Im} J_{(p)}$ ). By the Whitehead theorem,  $\tilde{H}_*(\mathbf{A}; \mathbb{Z}/p) = 0$  in degrees less than 2p(p-1)-2. This is a contradiction.

REMARK. As in [5] one can prove this lemma by calculating the Adams spectral sequence for  $\pi_*(A)$  and using (0.4). See also [10, p. 421].

For  $a \in \mathcal{A}$  let  $L(a): \Sigma^{|a|} \mathcal{A} \to \mathcal{A}$  and  $R(a): \Sigma^{|a|} \mathcal{A} \to \mathcal{A}$  be defined by  $L(a)(\sigma^{|a|}b) = ab$  and  $R(a)(\sigma^{|a|}b) = ba$  respectively.

**Corollary 1.2.** The following square commutes :

From this corollary we see that

$$\operatorname{Coker}\left(\theta^*\colon \Sigma^{2(p-1)}\mathcal{A}/\mathcal{A}(Q_0, Q_1) \to \mathcal{A}/\mathcal{A}(Q_0, Q_1)\right) \simeq \mathcal{A}/\mathcal{A}(Q_0, \mathcal{Q}^1).$$

We also have an isomorphism

(1.2) Ker 
$$(\theta^*: \Sigma^{2(p-1)}\mathcal{A}/\mathcal{A}(Q_0, Q_1) \to \mathcal{A}/\mathcal{A}(Q_0, Q_1)) \simeq \Sigma^{2p(p-1)}\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)$$

the inverse of which is induced by  $R(\mathcal{P}^{p-1})$ . (Although it is easy for a specialist to prove this fact directly, we do it by a different method in § 2.) Combining these, we get a short exact sequence of  $\mathcal{A}$ -modules

$$0 \to \mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1) \xrightarrow{\hat{\gamma}^*} H^*(\mathbf{A}; \mathbf{Z}/p) \xrightarrow{\hat{\Delta}^*} \Sigma^{\mathfrak{q}} \mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1) \to 0$$

where q = 2p(p-1)-1. Put  $g = \hbar^*(1) \in H^0(\mathbf{A}; \mathbf{Z}/p)$  and let  $\sigma^q h \in H^q(\mathbf{A}; \mathbf{Z}/p)$ be the element such that  $\hat{\Delta}^*(\sigma^q h) = \sigma^q 1$ . Since  $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)^{q+1} = \mathbf{Z}/p\{\mathcal{P}^p\}$  and  $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)^{q+2(p-1)} = 0$ , we may set

$$H^*(\boldsymbol{A};\boldsymbol{Z}|p) = \mathcal{A}\{g\} \oplus \Sigma^q \mathcal{A}\{h\} / \mathcal{A}(Q_0g \oplus 0, \mathcal{P}^1g \oplus 0, d \cdot \mathcal{P}^pg \oplus \sigma^q Q_0h, 0 \oplus \sigma^q \mathcal{P}^1h)$$

for some  $d \in \mathbb{Z}/p$ . Here  $d \neq 0$ . For if d=0, then by looking at the cell structure of A, we find that there is a CW-spectrum  $(S^0 \cup e^{2p(p-1)})_{(p)}$  in which  $\mathcal{P}^p$  is non-zero. This contradicts the triviality of the mod p Hopf invariant [16].

**Theorem 1.3.** As a left A-module  $H^*(\mathbf{A}; \mathbf{Z}|p)$  is generated by g and  $\sigma^{q}h$  subject to the relations

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$$Q_0(g) = 0$$
,  $\mathcal{P}^1(g) = 0$ ,  $\mathcal{P}^p(g) = Q_0(\sigma^q h)$  and  $\mathcal{P}^1(\sigma^q h) = 0$ .

Proof. Change  $\sigma^{q}h$  for  $d \cdot \sigma^{q}h$  if necessary.

## 2. The mod p homology of A

Most of this section is an odd prime version of [14].

Let  $\mathcal{A}_*$  be the dual of  $\mathcal{A}$ . It is the tensor product of an exterior algebra and a polynomial algebra:

$$\mathcal{A}_* = \Lambda(\tau_0, \tau_1, \cdots) \otimes \mathbf{Z}/p[\xi_1, \xi_2, \cdots]$$

where  $|\tau_n| = 2p^n - 1$  and  $|\xi_n| = 2p^n - 2$ .  $\mathcal{A}_*$  is a left and right  $\mathcal{A}$ -module; respective actions are given by

$$\langle a(\alpha), b \rangle = \langle \alpha, ba \rangle$$
 and  $\langle (\alpha)a, b \rangle = \langle \alpha, ab \rangle$ 

for all  $a, b \in \mathcal{A}$  and  $\alpha \in \mathcal{A}_*$ . By abuse of notation, for  $a \in \mathcal{A}$  let  $L(a): \mathcal{A}_* \to \Sigma^{|a|} \mathcal{A}_*$  and  $R(a): \mathcal{A}_* \to \Sigma^{|a|} \mathcal{A}_*$  be defined by  $L(a)(\alpha) = \sigma^{|a|}a(\alpha)$  and  $R(a)(\alpha) = \sigma^{|a|}(\alpha)a$  respectively; note that  $R(a): \Sigma^{|a|} \mathcal{A} \to \mathcal{A}$  and  $L(a): \mathcal{A}_* \to \Sigma^{|a|} \mathcal{A}_*$  are dual. Define  $\mathcal{P}(), () \mathcal{P}: \mathcal{A}_* \to \mathcal{A}_*$  by  $\mathcal{P}(\alpha) = \sum_{i \geq 0} \mathcal{P}^i(\alpha)$  and  $(\alpha) \mathcal{P} = \sum_{i \geq 0} (\alpha) \mathcal{P}^i$  respectively. They are ring homomorphisms, since Cartan formulas  $\mathcal{P}^n(\alpha\beta) = \sum_{i \geq 0} \mathcal{P}^i(\alpha) \mathcal{P}^i(\alpha) \mathcal{P}^i(\alpha) \mathcal{P}^i(\alpha) \mathcal{P}^i(\alpha)$ 

 $\sum_{i+j=n} \mathcal{P}^i(\alpha) \mathcal{P}^j(\beta) \text{ and } (\alpha\beta) \mathcal{P}^n = \sum_{i+j=n} (\alpha) \mathcal{P}^i(\beta) \mathcal{P}^j \text{ hold.}$ 

**Proposition 2.1.** The following formulas hold :

(i) 
$$\mathcal{P}(\tau_n) = \tau_n$$
  
 $\mathcal{P}(\xi_n) = \xi_n + \xi_{n-1}^p$  (i.e.,  $\mathcal{P}^1(\xi_n) = \xi_{n-1}^p$ )  
 $\delta(\tau_n) = \xi_n$   
 $\delta(\xi_n) = 0$ .  
(ii)  $(\tau_n)\mathcal{P} = \tau_n + \tau_{n-1}$  (i.e.,  $(\tau_n)\mathcal{P}^{p^{n-1}} = \tau_{n-1}$ )  
 $(\xi_n)\mathcal{P} = \xi_n + \xi_{n-1}$  (i.e.,  $(\xi_n)\mathcal{P}^{p^{n-1}} = \xi_{n-1}$ )  
 $(\tau_n)\delta = \begin{cases} 0 & \text{if } n > 0\\ 1 & \text{if } n = 0 \end{cases}$   
 $(\xi_n)\delta = 0$ .

Proof. Recall the definitions of  $\tau_n$  and  $\xi_n$ .

By abuse of notation, let  $\chi$  denote the conjugation in  $\mathcal{A}$  or  $\mathcal{A}_*$ ; note that  $\chi: \mathcal{A} \to \mathcal{A}$  and  $\chi: \mathcal{A}_* \to \mathcal{A}_*$  are dual.

**Proposition 2.2.** For each  $a \in \mathcal{A}$  with  $\chi a = -a$ , the following squares commute:

The proof is immediate.

REMARK. This proposition can be applied to the cases  $a=Q_0$ ,  $\mathcal{P}^1$  and  $\mathcal{P}^p$  (see [13, §7]).

By Theorem 1.3 there is an exact sequence of  $\mathcal{A}$ -modules

$$\Sigma \mathcal{A} \oplus \Sigma^{2(p-1)} \mathcal{A} \oplus \Sigma^{q+1} \mathcal{A} \oplus \Sigma^{q+2(p-1)} \mathcal{A} \xrightarrow{R(Q_0 \oplus 0) \oplus R(\mathcal{P}^1 \oplus 0) \oplus \mathbb{P}} \frac{R(Q_0 \oplus 0) \oplus R(\mathcal{P}^1 \oplus 0) \oplus \mathbb{P}}{\mathbb{P}} \xrightarrow{R(Q_0 \oplus 0) \oplus R(\mathcal{P}^1 \oplus 0) \oplus \mathbb{P}} \mathcal{A} \oplus \Sigma^q \mathcal{A} \xrightarrow{\mathcal{E}} H^*(A \colon \mathbb{Z}/p) \to 0$$

Dualizing this gives

$$\Sigma \mathcal{A}_{*} \oplus \Sigma^{2(p-1)} \mathcal{A}_{*} \oplus \Sigma^{q+1} \mathcal{A}_{*} \oplus \Sigma^{q+2(p-1)} \mathcal{A}_{*} \xleftarrow{L(Q_{0}) \oplus L(\mathcal{P}^{1}) \oplus} \underbrace{(-L(\mathcal{P}^{p}) + L(\sigma^{q}Q_{0})) \oplus L(\sigma^{q}\mathcal{P}^{1})}_{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{\mathcal{E}_{*}} H_{*}(A; \mathbb{Z}/p) \leftarrow 0.$$

Using Proposition 2.2 (i) we get an exact sequence

$$\Sigma \mathcal{A}_{*} \oplus \Sigma^{2(p-1)} \mathcal{A}_{*} \oplus \Sigma^{q+1} \mathcal{A}_{*} \oplus \Sigma^{q+2(p-1)} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \oplus R(\sigma^{q}\mathcal{P}^{1}) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \oplus R(\sigma^{q}\mathcal{P}^{1}) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \oplus R(\sigma^{q}\mathcal{P}^{1}) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \oplus R(\sigma^{q}\mathcal{P}^{1}) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus R(\mathcal{P}^{1}) \oplus} (-R(\mathcal{P}^{p}) + R(\sigma^{q}Q_{0})) \xrightarrow{\mathcal{A}_{*} \oplus \Sigma^{q}} \mathcal{A}_{*} \xleftarrow{R(Q_{0}) \oplus} (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus) (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus)) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal{P}^{p}) \oplus)) \oplus (-R(\mathcal{P}^{p}) \oplus (-R(\mathcal$$

(where  $A_*($ ) denotes the generalized homology theory associated with A). In order to describe  $H_*(A; \mathbb{Z}/p)$ , we calculate the kernel of  $R(Q_0) \oplus R(\mathcal{L}^1) \oplus (-R(\mathcal{L}^p) + R(\sigma^q Q_0)) \oplus R(\sigma^q \mathcal{L}^1)$  and apply  $\chi \oplus \chi$  to it.

Using Proposition 2.1, we easily see that

$$\operatorname{Ker}\left(R(Q_{0}):\mathcal{A}_{*}\to\Sigma\mathcal{A}_{*}\right)=\Lambda(\tau_{1},\,\tau_{2},\,\cdots)\otimes \mathbb{Z}/p[\xi_{1},\,\xi_{2},\,\cdots]$$

and

Therefore

$$\operatorname{Ker} \left( R(Q_0) \oplus R(\mathcal{P}^1) \colon \mathcal{A}_* \to \Sigma \mathcal{A}_* \oplus \Sigma^{2(p-1)} \mathcal{A}_* \right) = \\ \Lambda(\tau_2, \tau_3, \cdots) \otimes \mathbb{Z}/p[\xi_1^p, \xi_2, \xi_3, \cdots]$$

We write B for this kernel.

**Lemma 2.3.** For any non-zero  $\alpha \in B$  there exists a unique  $\alpha' \in \text{Ker } R(\mathcal{P}^1)$ such that  $(\alpha')Q_0 = (\alpha)\mathcal{P}^p$  (where if  $\alpha \in \text{Ker } R(\mathcal{P}^p)$ ), we take  $\alpha' = 0$ ).

Proof. Direct calculations using Proposition 2.1.

Henceforth for each non-zero  $\alpha \in B$  we use  $\alpha'$  to denote such an element. Define two subsets of  $\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$  as

$$\tilde{B} = \{ \alpha \oplus \sigma^{q} \alpha' | \alpha \in B \}$$
 and  $\sigma^{q} B = \{ 0 \oplus \sigma^{q} \alpha | \alpha \in B \}.$ 

Then it is evident that  $A_*(HZ/p) \cong \tilde{B} + \sigma^q B$ . Thus we obtain

**Theorem 2.4.** As a Z/p-module,

$$H_*(\mathbf{A}; \mathbf{Z}|p) \simeq (\chi \oplus \chi)(\tilde{B}) + (\chi \oplus \chi)(\sigma^q B).$$

Proof of (1.2). Starting from (1.1), we go a similar way to the above and get

$$H_*(G; \mathbb{Z}/p) \cong \Lambda(\alpha_2, \alpha_3, \cdots) \otimes \mathbb{Z}/p[\beta_1, \beta_2, \cdots]$$

where  $\alpha_n = \chi \tau_n$  and  $\beta_n = \chi \xi_n$ . By the dual of Corollary 1.2,  $\theta_*$  can be identified with  $c \cdot L(\mathcal{Q}^1)$ . Using Propositions 2.1 and 2.2 (ii), we see that

$$\theta_*(\alpha_2^{\mathbf{e}_2}\alpha_3^{\mathbf{e}_3}\cdots\beta_1^{\mathbf{r}_1}\beta_2^{\mathbf{r}_2}\beta_3^{\mathbf{r}_3}\cdots) = \begin{cases} -cr_1 \cdot \sigma^{2(p-1)}\alpha_2^{\mathbf{e}_2}\alpha_3^{\mathbf{e}_3}\cdots\beta_1^{\mathbf{r}_1-1}\beta_2^{\mathbf{r}_2}\beta_3^{\mathbf{r}_3}\cdots & \text{if } r_1 > 0\\ 0 & \text{if } r_1 = 0 \end{cases}$$

where  $\varepsilon_i = 0, 1$  and  $r_i \ge 0$ . This shows that

Coker 
$$(\theta_* \colon H_*(G; \mathbb{Z}/p) \to H_*(\Sigma^{2(p-1)}G; \mathbb{Z}/p)) \cong$$
  
 $\Sigma^{2(p-1)}(\Lambda(\alpha_2, \alpha_3, \cdots) \otimes \mathbb{Z}/p[\beta_1^p, \beta_2, \beta_3, \cdots]) \{\beta_1^{p-1}\}.$ 

Since the dual of  $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)$  is just

$$\chi B = \Lambda(lpha_2, \, lpha_3, \, \cdots) \otimes {oldsymbol Z}/p[eta_1^p, \, eta_2, \, eta_3, \, \cdots] \, ,$$

the result follows by dualization.

## 3. The $\mathcal{A}_*$ -coaction on $H_*(A; \mathbb{Z}/p)$

Let  $\phi: H_*(\mathbf{A}; \mathbb{Z}/p) \to \mathcal{A}_* \otimes H_*(\mathbf{A}; \mathbb{Z}/p)$  be the dual of the usual  $\mathcal{A}$ -action map  $\mathcal{A} \otimes H^*(\mathbf{A}; \mathbb{Z}/p) \to H^*(\mathbf{A}; \mathbb{Z}/p)$ . It gives  $H_*(\mathbf{A}; \mathbb{Z}/p)$  the structure of an  $\mathcal{A}_*$ -comodule. We study this coaction.

Since  $\mathcal{E}_*$ :  $H_*(A; \mathbb{Z}/p) \to \mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$  is an injective homomorphism of  $\mathcal{A}_*$ comodules, it suffices to determine the  $\mathcal{A}_*$ -comodule structure of  $\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$ . Let  $\phi_*: \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$  be the coproduct on  $\mathcal{A}_*$ . It also gives an  $\mathcal{A}_*$ -comodule structure on  $\mathcal{A}_*$ . Recall the following properties of  $\phi_*$ : for  $\alpha, \beta \in \mathcal{A}_*$ ,

$$\begin{split} \phi_*(\alpha\beta) &= \phi_*(\alpha)\phi_*(\beta);\\ \phi_*\chi &= (\chi\otimes\chi)T\phi_* \quad \text{where} \quad T(\alpha\otimes\beta) = (-1)^{|\sigma||\beta|}\beta\otimes\alpha;\\ \phi_*(\xi_n) &= \sum_{i=0}^n \xi_{n-i}^{p^i}\otimes\xi_i \quad \text{and} \quad \phi_*(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^n \xi_{n-i}^{p^i}\otimes\tau_i \,. \end{split}$$

The composite

$$\Sigma^q \mathcal{A}_* \xrightarrow{\Sigma^q \phi_*} \Sigma^q (\mathcal{A}_* \otimes \mathcal{A}_*) \xrightarrow{\simeq} \mathcal{A}_* \otimes \Sigma^q \mathcal{A}_* ,$$

which we denote by  $\sigma^q \phi_*$ , gives an  $\mathcal{A}_*$ -comodule structure on  $\Sigma^q \mathcal{A}_*$ . Moreover the composite

$$\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_* \xrightarrow{\phi_* \oplus \Sigma^q \phi_*} (\mathcal{A}_* \otimes \mathcal{A}_*) \oplus (\mathcal{A}_* \otimes \Sigma^q \mathcal{A}_*) \xrightarrow{\simeq} \mathcal{A}_* \otimes (\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*),$$

which may be written as  $\phi_* + \sigma^q \phi_*$ , gives an  $\mathcal{A}_*$ -comodule structure on  $\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$ . Combining these and Theorem 2.4, one can evaluate  $\phi(x)$  for every  $x \in H_*(\mathbf{A}; \mathbf{Z}|p)$ .

It is convenient to introduce the following (artificial) multiplication on  $H_*(\mathbf{A}; \mathbf{Z}/p)$ . For non-zero  $\alpha, \beta \in B$  define

- (1)  $(\chi \alpha \oplus \sigma^q \chi \alpha') \circ (\chi \beta \oplus \sigma^q \chi \beta') = \chi(\alpha \beta) \oplus \sigma^q \chi(\alpha' \beta + \alpha \beta')$
- (2)  $(\chi \alpha \oplus \sigma^q \chi \alpha') \circ (0 \oplus \sigma^q \chi \beta) = 0 \oplus \sigma^q \chi(\alpha \beta)$
- (3)  $(0 \oplus \sigma^q \chi \alpha) \circ (\chi \beta \oplus \sigma^q \chi \beta') = 0 \oplus \sigma^q \chi (\alpha \beta)$
- (4)  $(0 \oplus \sigma^q \chi \alpha) \circ (0 \oplus \sigma^q \chi \beta) = 0$ .

This is well defined. To check this assertion we first observe that if  $\alpha \in B$  then  $(\alpha)Q_0=0$  and  $(\alpha)\mathcal{P}^i=0$  for 0 < i < p. Therefore, if  $\alpha, \beta \in B$  we have  $\alpha\beta \in B$ ,  $(\alpha'\beta+\alpha\beta')\mathcal{P}^1=0$  and

$$egin{aligned} & (lpha'eta+lphaeta')Q_{0}=(lpha')Q_{0}etaeta+lphaeta(eta')Q_{0}\ &=(lpha)\mathscr{D}^{p}etaeta+lphaelda(eta)\mathscr{D}^{p}\ &=(lphaeta)\mathscr{D}^{p}\,. \end{aligned}$$

This implies that (1) is well defined. The other cases are obvious.

We now show that the formula

$$\phi(x \circ y) = \phi(x) \circ \phi(y)$$

holds for all  $x, y \in H_*(\mathbf{A}; \mathbb{Z}/p)$ . For example, if  $x = \chi \alpha \oplus \sigma^q \chi \alpha'$  and  $y = \chi \beta \oplus \sigma^q \chi \beta'$ , then we have

$$\begin{split} \phi(x \circ y) &= \phi(\chi(\alpha\beta) \oplus \sigma^q \chi(\alpha'\beta + \alpha\beta')) \\ &= \phi_*(\chi \alpha \cdot \chi \beta) + \sigma^q \phi_*(\chi \alpha' \cdot \chi \beta + \chi \alpha \cdot \chi \beta') \\ &= \phi_*(\chi \alpha) \cdot \phi_*(\chi \beta) + \sigma^q (\phi_*(\chi \alpha') \phi_*(\chi \beta) + \phi_*(\chi \alpha) \phi_*(\chi \beta')) \end{split}$$

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$$= (\phi_*(\chi\alpha) + \sigma^q \phi_*(\chi\alpha')) \circ (\phi_*(\chi\beta) + \sigma^q \phi_*(\chi\beta'))$$
  
=  $\phi(x) \circ \phi(y)$ .

The other cases are obvious.

REMARK. As seen in [9],  $KF_r$ , has a natural product. So it induces a multiplication on  $H_*(A; \mathbb{Z}/p)$ . We cannot confirm whether  $\circ$  coincides with this one; however, we believe so (cf. Theorem 4.3).

By virtue of Lemma 2.3 we may put

$$\chi lpha = \chi lpha \oplus \sigma^q \chi lpha'$$
 and  $\sigma^q \chi lpha = 0 \oplus \sigma^q \chi lpha$ 

for each non-zero  $\alpha \in B$ . With this notation the multiplication  $\circ$  is given by

(1)  $\tilde{x} \circ \tilde{y} = \tilde{x} \tilde{y}$  (2)  $\tilde{x} \circ \sigma^q y = \sigma^q x y$  (3)  $\sigma^q x \circ \tilde{y} = \sigma^q x y$ 

$$(4) \quad \sigma^{q} x \circ \sigma^{q} y = 0$$

for all  $x, y \in \mathcal{XB}$ . Notice that as an algebra  $H_*(\mathbf{A}; \mathbb{Z}/p)$  is generated by the elements  $\sigma^{q_1}$ ,  $\tilde{\beta}_1^{p}$ ,  $\tilde{\beta}_n$ ,  $\tilde{\alpha}_n$  with  $n \ge 2$ .

**Theorem 3.1.** The  $\mathcal{A}_*$ -coaction on  $H_*(\mathbf{A}; \mathbf{Z}|p)$  is given by

$$\begin{split} \phi(\sigma^{q}1) &= 1 \otimes \sigma^{q}1 \\ \phi(\tilde{\beta}_{1}^{p}) &= \chi \xi_{1}^{p} \otimes \tilde{1} + \chi \tau_{0} \otimes \sigma^{q}1 + 1 \otimes \tilde{\beta}_{1}^{p} \\ \phi(\tilde{\beta}_{2}) &= \chi \xi_{2} \otimes \tilde{1} + \chi (\tau_{0}\xi_{1} - \tau_{1}) \otimes \sigma^{q}1 + \chi \xi_{1} \otimes \tilde{\beta}_{1}^{p} + 1 \otimes \tilde{\beta}_{2} \\ \phi(\tilde{\alpha}_{2}) &= \chi \tau_{2} \otimes \tilde{1} + \chi (\tau_{0}\tau_{1}) \otimes \sigma^{q}1 + \chi \tau_{1} \otimes \tilde{\beta}_{1}^{p} + \chi \tau_{0} \otimes \tilde{\beta}_{2} + 1 \otimes \tilde{\alpha}_{2} \\ \phi(\tilde{\beta}_{n}) &= \sum_{i=0}^{n} \chi \xi_{n-i} \otimes \tilde{\beta}_{i}^{p^{n-i}} \qquad for \quad n \geq 3 \\ \phi(\tilde{\alpha}_{n}) &= \sum_{i=0}^{n} \chi \tau_{n-i} \otimes \tilde{\beta}_{i}^{p^{n-i}} + 1 \otimes \tilde{\alpha}_{n} \qquad for \quad n \geq 3 . \end{split}$$

Let  $\mathcal{B}$  be the exterior subalgebra of  $\mathcal{A}$  generated by  $Q_0$  and  $Q_1$ . In the next section we need to know the  $\mathcal{B}$ -module structure of  $H^*(\mathbf{A}; \mathbf{Z}/p)$ . But it can be read off from Theorem 3.1. We give its details.

Define a left action of  $\mathcal{A}$  on  $H_*(A; \mathbb{Z}|p)$  by

$$\langle f, a(x) \rangle = (-1)^{|a||x|} \langle (\chi a)(f), x \rangle$$

for all  $a \in \mathcal{A}$ ,  $x \in H_*(\mathbf{A}; \mathbf{Z}/p)$  and  $f \in H^*(\mathbf{A}; \mathbf{Z}/p)$  (cf. [2, p. 76]).

**Corollary 3.2.** For i=0 or 1,  $Q_i$  acts on  $H_*(A; \mathbb{Z}|p)$  as a derivation (with respect to  $\circ$ ). So the  $\mathcal{B}$ -action on  $H_*(A; \mathbb{Z}|p)$  is given by

$$\begin{aligned} Q_0(\tilde{\beta}_1^p) &= \sigma^q 1, \qquad Q_0(\tilde{\alpha}_n) = \tilde{\beta}_n \qquad \text{for} \quad n \ge 2, \\ Q_1(\tilde{\beta}_2) &= -\sigma^q 1, \qquad Q_1(\tilde{\alpha}_n) = \tilde{\beta}_{n-1}^p \qquad \text{for} \quad n \ge 2. \end{aligned}$$

We define a weight function  $w: H_*(A; \mathbb{Z}/p) \rightarrow \mathbb{Z}$  by

$$\begin{aligned} &w(1) = 0, \quad w(\tilde{\beta}_1^p) = w(\sigma^q 1) = p, \\ &w(\tilde{\alpha}_n) = w(\tilde{\beta}_n) = p^{n-1} \quad \text{for} \quad n \ge 2 \end{aligned}$$

together with the rules

$$w(x+y) = \max{w(x), w(y)}$$
 and  
 $w(x \circ y) = w(x) + w(y)$ 

for all  $x, y \in H_*(A; \mathbb{Z}/p)$ . By Corollary 3.2 the  $\mathcal{B}$ -action preserves weight. For  $j \ge 0$  let  $N_j$  denote the submodule of  $H_*(A; \mathbb{Z}/p)$  spanned by elements of weight jp. Then  $H_*(A; \mathbb{Z}/p) \cong \bigoplus_{j \ge 0} N_j$  as  $\mathcal{B}$ -modules. It suffices to examine the  $\mathcal{B}$ -module structure of  $N_j$ . For this purpose the  $Q_i$ -homology

$$H_*(; Q_i) = \operatorname{Ker} Q_i / \operatorname{Im} Q_i$$

is useful.

**Lemma 3.3.** For  $j \ge 0$  we have

(i)  

$$H_{*}(N_{j}; Q_{0}) = \begin{cases} Z/p \{\tilde{1}\} & \text{if } j = 0 \\ Z/p \{\sigma^{a}(\beta_{1}^{p})^{np-1}, \ (\tilde{\beta}_{1}^{p})^{np}\} & \text{if } j = np \ (n \ge 1) \\ 0 & \text{otherwise} \end{cases}$$
(ii)  

$$H_{*}(N_{j}; Q_{1}) = \begin{cases} Z/p \{\sigma^{q}\beta_{2}^{p-1} \cdots \beta_{k+2}^{p-1}\beta_{k+3}^{nk-1}\beta_{k+4}^{nk-1} \cdots \beta_{l+3}^{n}, \\ \tilde{\beta}_{k+3}^{nk}\tilde{\beta}_{k+4}^{nk+1} \cdots \tilde{\beta}_{l+3}^{n}\} & \text{if } j = np \ (n \ge 0) \\ 0 & \text{otherwise} \end{cases}$$

where  $k = \nu_p(n)$  and  $n = n_k p^k + n_{k+1} p^{k+1} + \cdots + n_l p^l$  is the p-adic expansion of n.

Proof. From § 1 we have a short exact sequence of  $\mathcal{B}$ -modules

$$0 \to \Sigma^q \chi B \to H_*(\boldsymbol{A}; \boldsymbol{Z}/p) \to \chi B \to 0$$

This yields a long exact sequence

$$\cdots \to H_m(H_*(\mathbf{A}; \mathbf{Z}/p); Q_i) \to H_m(\chi B; Q_i) \xrightarrow{O} H_{m-1}(\Sigma^q \chi B; Q_i) \to \cdots$$

~

Since the  $\mathcal{B}$ -action on  $\chi B$  is given by

$$Q_0(\alpha_n) = \beta_n$$
 and  $Q_1(\alpha_n) = \beta_{n-1}^p$  for  $n \ge 2$ ,

it follows that

$$H_*(\chi B; Q_0) = \mathbf{Z}/p[\beta_1^{\flat}] \quad \text{and} \\ H_*(\chi B; Q_1) = \bigotimes_{n \ge 2} \mathbf{Z}/p[\beta_n]/(\beta_n^{\flat}).$$

An inspection of weight shows that to calculate  $H_*(N_j; Q_i)$  it suffices to determine the behavior of

$$egin{aligned} &H_{2j(p-1)}(\chi B;\,Q_0)=oldsymbol{Z}/p\,\{(eta_1^p)^j\}\ &igcup_{2j(p-1)-1}(\Sigma^q\chi B;\,Q_0)=oldsymbol{Z}/p\,\{\sigma^q(eta_1^p)^{j-1}\} \end{aligned}$$

and

$$\begin{array}{c} H_{2\nu_{p}((j_{p}^{2})_{1})(p-1)}(\chi B;\,Q_{1}) = \mathbf{Z}/p \{\beta_{2^{0}}^{j_{0}}\beta_{3^{1}}^{j_{1}}\cdots\beta_{s^{+}2}^{j_{s}}\} \\ \downarrow \partial \\ H_{2\nu_{p}((j_{p}^{2})_{1})(p-1)-(2p-1)}(\Sigma^{q}\chi B;\,Q_{1}) = \mathbf{Z}/p \{\sigma^{q}\beta_{2^{0}}^{j_{0}-1}\beta_{3^{1}}^{j_{1}}\cdots\beta_{s^{+}2}^{j_{s}}\} \end{array}$$

where  $j=j_0+j_1p+\cdots+j_sp^s$  is the *p*-adic expansion of *j*. By the definition of  $\partial$  and Corollary 3.2, we find that

$$\begin{array}{l} \partial((\beta_{1}^{p})^{j}) = j \cdot \sigma^{q}(\beta_{1}^{p})^{j-1} \quad \text{and} \\ \partial(\beta_{2}^{j_{0}}\beta_{3}^{j_{1}} \cdots \beta_{s+2}^{j_{s}}) = \begin{cases} -j_{0} \cdot \sigma^{q}\beta_{2}^{j_{0}-1}\beta_{3}^{j_{1}} \cdots \beta_{s+2}^{j_{s}} & \text{if} \quad j_{0} > 0 \\ 0 & \text{if} \quad j_{0} = 0 \end{cases}$$

This gives the result.

It is easy to carry these results to those for the usual  $\mathcal{B}$ -action (cf. [7, II]). Hereafter we talk about the usual action.

According to [3,III], there is a classification of finite dimensional  $\mathcal{B}$ -modules, which we use implicitly. We fix some notation. Let I be defined by the exact sequence of  $\mathcal{B}$ -modules

$$0 \to I \to \mathcal{B} \to \mathbf{Z}/p \to 0.$$

Put  $I^n = I \otimes \cdots \otimes I$  (*n*-factors). Note that  $H_*(I^n; Q_0) = \mathbb{Z}/p\{ \bigotimes_{\beta}^{n} Q_0 \}$  and  $H_*(I^n; Q_1) = \mathbb{Z}/p\{ \bigotimes_{\beta}^{n} Q_1 \}$  where  $|\bigotimes_{\beta}^{n} Q_0| = n$  and  $|\bigotimes_{\beta}^{n} Q_1| = n + 2n(p-1)$ .

The above discussion can be summarized as follows.

**Theorem 3.4.** As a  $\mathcal{B}$ -module, ignoring free summands,

$$H^*(\mathbf{A}; \mathbf{Z}|p) \cong \mathbf{Z}|p \bigoplus_{n \ge 1} (\Sigma^{a(n)} I^{b(n)} \bigoplus \Sigma^{c(n)} I^{d(n)})$$

where

$$\begin{aligned} a(n)+b(n) &= 2np^2(p-1)-1, \\ b(n) &= \nu_p((np^3)!)-np^2-\nu_p(n)-2, \\ c(n)+d(n) &= 2np^2(p-1), \\ d(n) &= \nu_p((np^3)!)-np^2. \end{aligned}$$

## 4. The multiplicative structure on A

The first half of this section is heavily influenced by [18]. Let  $\mu: G \wedge G \rightarrow G$  be the product on G. Consider the external product

$$\times : G^*(\mathbf{G}) \otimes G^*(\mathbf{G}) \xrightarrow{\wedge} (G \wedge G)^*(\mathbf{G} \wedge \mathbf{G}) \xrightarrow{'\mu_*} G^*(\mathbf{G} \wedge \mathbf{G}) .$$

**Lemma 4.1.** The element  $\theta \in G^{2(p-1)}(G)$  satisfies

$$heta \mu = (\Sigma^{2(p-1)}\mu)( heta \wedge 1_{G} + 1_{G} \wedge heta + v heta \wedge heta) \,.$$

Proof. Put  $1=1_{\mathbf{G}} \in G^{0}(\mathbf{G})$ . By the definition of  $\theta$ , we have

$$\psi'_{\ast}(1\times 1) = 1\times 1 + v_{\ast}\theta_{\ast}(1\times 1) = 1\times 1 + v_{\ast}(\theta\mu).$$

On the other hand, since  $\psi^r$  is multiplicative and  $\times$  is bilinear, we have

$$\begin{split} \psi^{*}*(1\times 1) &= \psi^{*}*\mu*(1\wedge 1) = \mu_{*}(\psi^{*}\wedge\psi^{*})*(1\wedge 1) \\ &= \psi^{*}*(1)\times\psi^{*}*(1) \\ &= (1+v_{*}\theta_{*}(1))\times(1+v_{*}\theta_{*}(1)) \\ &= 1\times 1+v_{*}(\theta_{*}(1)\times 1+1\times\theta_{*}(1)+v\theta_{*}(1)\times\theta_{*}(1)) \\ &= 1\times 1+v_{*}((\Sigma^{2(p-1)}\mu)(\theta\wedge 1+1\wedge\theta+v\theta\wedge\theta)) \,. \end{split}$$

Since  $v_*: G^{2(p-1)}(G \wedge G) \rightarrow G^0(G \wedge G)$  is injective, the result follows.

## Lemma 4.2. We have

(i) 
$$[A, \Sigma^{2p-3}G] = 0.$$

(i)  $[\mathbf{A} \wedge \mathbf{A}, \Sigma^{2p-3}\mathbf{G}] = 0.$ 

Proof. Consider the Adams spectral sequence  $\{E_r^{s,t}, d_r\}$  converging to  $G^*(X)$ , where X = A or  $A \wedge A$ . It has the form

$$E_2^{s,t} \simeq \operatorname{Ext}_{\mathcal{B}}^{s,t}(\mathbb{Z}|p, H^*(X; \mathbb{Z}|p)) \Rightarrow G^{t-s}(X).$$

(For this details see [3, III].) In view of Theorem 3.4 (where a similar result for  $A \wedge A$  follows from this and the Künneth theorem), all we need to do is the calculation of  $\operatorname{Ext}_{\mathcal{B}}^{*,*}(\mathbb{Z}/p, M)$  for  $M = \Sigma^m \mathcal{B}, \Sigma^m \mathbb{Z}/p, \Sigma^m \mathbb{I}^n$  and their direct sums. As is well known, for all  $\mathcal{B}$ -modules M and N,

$$\operatorname{Ext}_{\beta}^{s,t}(\boldsymbol{Z}|p, \ M \oplus N) \simeq \operatorname{Ext}_{\beta}^{s,t}(\boldsymbol{Z}|p, \ M) \oplus \operatorname{Ext}_{\beta}^{s,t}(\boldsymbol{Z}|p, \ N)$$
$$\operatorname{Ext}_{\beta}^{s,t}(\boldsymbol{Z}|p, \ \Sigma^{m}N) \simeq \operatorname{Ext}_{\beta}^{s,m+t}(\boldsymbol{Z}|p, \ N)$$

and

$$\begin{aligned} &\operatorname{Ext}_{\mathcal{B}}^{*,*}(\boldsymbol{Z}|p, \mathcal{B}) \cong \boldsymbol{Z}|p\{z\} \quad \text{where} \quad |z| = (0, -2p) \\ &\operatorname{Ext}_{\mathcal{B}}^{*,*}(\boldsymbol{Z}|p, \boldsymbol{Z}|p) \cong \boldsymbol{Z}|p[q_0, q_1] \quad \text{where} \quad |q_i| = (1, 2p^i - 1) \\ &\operatorname{Ext}_{\mathcal{B}}^{s,i}(\boldsymbol{Z}|p, I^n) \cong \operatorname{Ext}_{\mathcal{B}}^{s-n,i}(\boldsymbol{Z}|p, \boldsymbol{Z}|p) . \end{aligned}$$

Using these data, one can describe the figure of  $E_2^{*\prime*}$ ; in particular, we have  $E_2^{*,t}=0$  if t-s=2p-3. This implies the result.

**Theorem 4.3.** A is a ring spectrum and  $\eta: A \rightarrow G$  is a map of ring spectra. The product on A satisfying such property is unique.

Proof. Consider the exact sequence

$$0 \rightarrow [\mathbf{A} \land \mathbf{A}, \mathbf{A}] \xrightarrow{\eta_{*}} [\mathbf{A} \land \mathbf{A}, \mathbf{G}] \xrightarrow{\theta_{*}} [\mathbf{A} \land \mathbf{A}, \Sigma^{2(p-1)}\mathbf{G}]$$

where we have used Lemma 4.2 (ii). By Lemma 4.1 we have

$$egin{aligned} & heta_{m{*}}(\mu(\eta\wedge\eta))= heta\mu(\eta\wedge\eta)\ &=(\Sigma^{2(p-1)}\mu)( heta\wedge 1_{m{G}}{+}1_{m{G}}{\wedge} heta+v heta\wedge heta)(\eta\wedge\eta) \end{aligned}$$

which is clearly equal to zero, since  $\theta\eta=0$ . Hence there exists a unique  $\hat{\mu} \in [A \wedge A, A]$  such that  $\eta \hat{\mu} = \mu(\eta \wedge \eta)$ .

Let  $\iota: S^0 \to G$  be the unit on G. Then there is a unique  $\hat{\iota} \in [S^0, A]$  such that  $\eta \hat{\iota} = \iota$  (see (0.5)). Consider the exact sequence

$$0 \to [\mathbf{S}^{0} \land \mathbf{A}, \mathbf{A}] \xrightarrow{\eta_{*}} [\mathbf{S}^{0} \land \mathbf{A}, \mathbf{G}]$$

where we have used Lemma 4.2 (i). Then we have

$$egin{aligned} &\eta_{st}(\hat{\mu}(\hat{\iota}\wedge 1_{A})) = \eta\hat{\mu}(\hat{\iota}\wedge 1_{A}) = \mu(\eta\wedge\eta)(\hat{\iota}\wedge 1_{A}) \ &= \mu(\iota\wedge\eta) = \mu(\iota\wedge 1_{G})(1_{S^{0}}\wedge\eta) \ &= \eta = \eta_{st}(1_{A}) \,. \end{aligned}$$

This proves that  $\hat{\mu}(\hat{\imath} \wedge 1_A) = 1_A$ . Another equation  $\hat{\mu}(1_A \wedge \hat{\imath}) = 1_A$  is obtained similarly.

Lemma 4.4. Under the above notation we have (i)  $\hat{\mu}(1_A \wedge \Delta) = \Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}\eta \wedge 1_G).$ (ii)  $\hat{\mu}(\Delta \wedge 1_A) = \Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}1_G \wedge \eta).$ 

Proof. Because an argument is quite parallel, we show (ii) only. By smashing (0.6) to the right with A, we have a diagram

in which rows are fibre sequences. Part 2 commutes by Theorem 4.3. To prove the commutativity of part 1, it suffices to show that of part 3. But by Lemma 4.1 we have

$$egin{aligned} & heta\mu(1_G\wedge\eta)=(\Sigma^{2p-2}\mu)( heta\wedge1_G+1_G\wedge heta+v heta\wedge heta)(1_G\wedge\eta)\ &=(\Sigma^{2p-2}\mu)( heta\wedge\eta)\ &=(\Sigma^{2p-2}\mu)(\Sigma^{2p-2}1_G\wedge\eta)( heta\wedge1_A)\,. \end{aligned}$$

Let us consider  $\tilde{A}_*(CP^{\infty})$ . Since G-theory is complex oriented,  $\tilde{G}_n(CP^{\infty})=0$  if *n* is odd. From (0.6) we have an exact sequence

$$0 \to \tilde{A}_{2n}(CP^{\infty}) \xrightarrow{\eta} \tilde{G}_{2n}(CP^{\infty}) \xrightarrow{\theta} \tilde{G}_{2n-2(p-1)}(CP^{\infty}) \xrightarrow{\Delta} \tilde{A}_{2n-1}(CP^{\infty}) \to 0$$

for all  $n \ge 0$  (where of course  $\eta = (\eta \land 1_{CP^{\infty}})_*$  etc.). Thus we may use the following notation:

$$\eta(\tilde{x}) = x$$
 for  $x \in \text{Ker } \theta$ ;  
 $\Delta(x) = \bar{x}$  for  $x \in \tilde{G}_*(CP^{\infty})$ .

The multiplication  $m: CP^{\infty} \times CP^{\infty} \to CP^{\infty}$  induces a product  $\cdot$  on  $\tilde{G}_*(CP^{\infty})$  and a product \* on  $\tilde{A}_*(CP^{\infty})$ .

**Theorem 4.5.** The following formulas hold.

(i)  $\tilde{x} * \tilde{y} = x \cdot y$ . (ii)  $\tilde{x} * \tilde{y} = \overline{x \cdot y}$ . (iii)  $\bar{x} * \tilde{y} = \overline{x \cdot y}$ . (iv)  $\bar{x} * \tilde{y} = 0$ .

Proof. Since  $\eta$  is multiplicative by Theorem 4.3, (i) follows. Similarly, using  $\eta \Delta = 0$ , we have

$$\eta(\bar{x}*\bar{y}) = \eta(\bar{x})*\eta(\bar{y}) = \eta\Delta(x)*\eta\Delta(y) = 0$$

which proves (iv).

For (ii), by definition and Lemma 4.4 (i), we have

$$\begin{split} \tilde{x} * \bar{y} &= m_* (\tilde{x} \times \bar{y}) = m_* \hat{\mu} (\tilde{x} \wedge \bar{y}) \\ &= m_* \hat{\mu} (\tilde{x} \wedge \Delta (\Sigma^{2p-3} y)) \\ &= m_* \hat{\mu} (1_A \wedge \Delta) (\Sigma^{2p-3} \tilde{x} \wedge y) \\ &= m_* \Delta (\Sigma^{2p-3} \mu) (\Sigma^{2p-3} \eta \wedge 1_G) (\Sigma^{2p-3} \tilde{x} \wedge y) \\ &= m_* \Delta (\Sigma^{2p-3} \mu) (\Sigma^{2p-3} x \wedge y) \\ &= m_* \Delta (\Sigma^{2p-3} x \times y) \\ &= \Delta (\Sigma^{2p-3} m_* (x \times y)) \end{split}$$

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$$= \frac{\Delta(\Sigma^{2p-3}x \cdot y)}{= \overline{x \cdot y}}.$$

Similarly (iii) follows from Lemma 4.4 (ii).

REMARK. The argument of this section assures us that the same formulas as above hold with respect to the fibre sequences (0.3) and

$$AZ|p \rightarrow buZ|p \rightarrow \Sigma^2 buZ|p$$

where XZ/p represents the mod p X-theory (cf. [15, p. 254]).

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Department of Mathematics Osaka City University Sugimoto, Sumiyoshi-ku Osaka 558, Japan