Ishida, H.
Osaka J. Math.
49 (2012), 515–522

FILTERED COHOMOLOGICAL RIGIDITY OF BOTT TOWERS

HIROAKI ISHIDA

(Received July 5, 2010, revised November 19, 2010)

Abstract

A Bott tower is an iterated $\mathbb{C}\mathrm{P}^1$-bundle over a point, where each $\mathbb{C}\mathrm{P}^1$-bundle is the projectivization of a rank 2 decomposable complex vector bundle. For a Bott tower, the *filtered cohomology* is naturally defined. We show that isomorphism classes of Bott towers are distinguished by their filtered cohomology rings. We even show that any filtered cohomology ring isomorphism between two Bott towers is induced by an isomorphism of the Bott towers.

1. Introduction

A *Bott tower* of height n is a sequence of $\mathbb{C}\mathrm{P}^1$-bundles

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{1}$$

where each fibration $\pi_i \colon B_i \to B_{i-1}$ for $i = 1, \ldots, n$ is the projectivization of a rank 2 decomposable complex vector bundle over B_{i-1}. A decomposable complex vector bundle is the Whitney sum of line bundles. Each B_i is called a *Bott manifold*. As is known, a Bott manifold B_n is a smooth projective toric variety. We denote the Bott tower (1) of height n by $B_\bullet = (\{B_i\}_{i=0}^n, \{\pi_i\}_{i=1}^n)$.

Two Bott towers $B_\bullet = (\{B_i\}_{i=0}^n, \{\pi_i\}_{i=1}^n)$ and $B'_\bullet = (\{B'_i\}_{i=0}^n, \{\pi'_i\}_{i=1}^n)$ are isomorphic if there is a collection of diffeomorphisms $\varphi_\bullet = \{\varphi_i \colon B_i \to B'_i\}_{i=0}^n$ such that the following diagram is commutative:

$$\begin{tikzcd}
B_n \arrow[r, "\pi_n"] \arrow[d, "\varphi_n"] & B_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r, "\pi_2"] & B_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi_1"] & B_0 \arrow[d, "\varphi_0"] \\
B'_n \arrow[r, "\pi'_n"] & B'_{n-1} \arrow[r, "\pi'_{n-1}"] & \cdots \arrow[r, "\pi'_2"] & B'_1 \arrow[r, "\pi'_1"] & B'_0.
\end{tikzcd}$$

Since $\pi_i \colon B_i \to B_{i-1}$ has a cross section, the induced homomorphism $\pi_i^* \colon \mathrm{H}^*(B_{i-1}) \to \mathrm{H}^*(B_i)$ is injective for each i. Therefore the cohomology ring $\mathrm{H}^*(B_{i-1})$ can be regarded as a subring of $\mathrm{H}^*(B_i)$ via π_i^*. We define a filtered graded $\mathbb{Z}$-algebra $\mathrm{F}_\bullet\mathrm{H}^*(B_\bullet)$ by

- For $i \geq n$, $\mathrm{F}_i\mathrm{H}^*(B_\bullet) := \mathrm{H}^*(B_n)$.
- For $0 \leq j \leq n-1$, $\mathrm{F}_j\mathrm{H}^*(B_\bullet) := \pi_n^* \circ \pi_{n-1}^* \circ \cdots \circ \pi_{j+1}^*(\mathrm{H}^*(B_j))$.

2010 Mathematics Subject Classification. 57R19, 57R20, 57S25, 14M25.

We call $F_\bullet H^*(B_\bullet)$ the *filtered cohomology ring* of the Bott tower $B_\bullet$.

The following is our main result:

Theorem 1.1. *Isomorphism classes of Bott towers are distinguished by their filtered cohomology rings. Moreover, any filtered cohomology ring isomorphism between two Bott towers are induced by an isomorphism between the Bott towers.*

Our study is motivated by the so-called *cohomological rigidity problem* for toric manifolds. A *toric manifold* is a smooth (complete) toric variety. Bott manifolds are examples of toric manifolds. The cohomological rigidity problem asks whether topological types or diffeomorphism types of toric manifolds are distinguished by their cohomology rings or not. This problem is open and we do not know any counterexamples.

This paper is organized as follows. In Section 2, we recall some preliminary facts about Bott manifolds. In Section 3, we see that rank 2 decomposable vector bundles over Bott manifolds are distinguished by their total Chern classes. In Section 4, we study $\mathbb{C}P^1$-bundles over Bott manifolds by using the results obtained in Section 3. In Section 5, we give a proof of Theorem 1.1.

Throughout this paper, all cohomology groups are taken with $\mathbb{Z}$-coefficient.

2. Preliminaries

In this section, we recall some preliminary facts (see [1, Section 2] and [2, Section 2] for more details). For a complex vector bundle η over a topological space, we denote its projectivization by $P(\eta)$.

Lemma 2.1 (Lemma 2.1, [2]). *Let B be a smooth manifold, ξ be a complex line bundle over B and η be a complex vector bundle over B. Then, the projectivizations $P(\eta)$ and $P(\xi \otimes \eta)$ are isomorphic as bundles.*

Let B_{i-1} be a Bott manifold and $\eta_i = \xi_i \oplus \xi_i'$ be the Whitney sum of complex line bundles ξ_i and ξ_i' over B_{i-1}. Since $\xi_i^* \otimes \eta_i = \underline{\mathbb{C}} \oplus (\xi_i^* \otimes \xi_i')$, the projectivization $B_i = P(\eta_i)$ is isomorphic to $P(\underline{\mathbb{C}} \oplus (\xi_i^* \otimes \xi_i'))$ as fiber bundles by Lemma 2.1, where $\underline{\mathbb{C}}$ denotes the trivial line bundle over B_{i-1} and ξ_i^* denotes the dual line bundle of ξ_i. Hence we can assume that η_i is the Whitney sum of $\underline{\mathbb{C}}$ and a complex line bundle, denoted ξ_i again, in the definition of Bott towers.

Let $B_\bullet = (\{B_i\}_{i=0}^n, \{\pi_i\}_{i=1}^n)$ be a Bott tower of height n. We suppose that each fibration $B_i \to B_{i-1}$ is the projectivization $P(\underline{\mathbb{C}} \oplus \xi_i)$. Let γ_i be the tautological line bundle over $P(\underline{\mathbb{C}} \oplus \xi_i) = B_i$. By Borel–Hirzebruch formula, the cohomology ring $H^*(B_i)$ viewed as an $H^*(B_{i-1})$-algebra via π_i^* is of the form

$$H^*(B_i) \cong H^*(B_{i-1})[X]/(X^2 - c_1(\xi_i)X)$$

where $c_1(\xi_i)$ denotes the first Chern class of ξ_i and X represents the first Chern class of the tautological line bundle γ_i. Using this formula inductively on i, we see that the cohomology ring $\mathrm{H}^*(B_n)$ is generated by

$$x_i := \pi_n^* \circ \cdots \circ \pi_{i+1}^*(c_1(\gamma_i))$$

for $i = 1, \ldots, n$. More precisely, $\mathrm{H}^*(B_n)$ is of the form

$$\mathrm{H}^*(B_n) = \mathbb{Z}[x_1, \ldots, x_n]/(x_i^2 - \alpha_i x_i\,;\ i = 1, \ldots, n)$$

as rings, where $\alpha_i := \pi_n^* \circ \cdots \circ \pi_{i+1}^*(c_1(\xi_i))$ for all i.

With this understood, we may regard $\mathrm{H}^*(B_k)$ as a subring of $\mathrm{H}^*(B_n)$ generated by x_j's with $j = 1, \ldots, k$. Namely, we have

$$\mathrm{F}_k\mathrm{H}^*(B_\bullet) = \mathbb{Z}[x_1, \ldots, x_k]/(x_j^2 - \alpha_j x_j\,;\ j = 1, \ldots, k).$$

A *vanishing pair* is an ordered pair $(z, \bar{z})$ of elements in $\mathrm{H}^2(B_n)$ such that $z\bar{z} = 0$. We say that a vanishing pair is *primitive* if both elements are primitive. The following lemma is very helpful for our purpose.

Lemma 2.2 (Lemma 2.3, [1]). *A primitive vanishing pair is of the form*

$$(ax_j + u, \pm(a(x_j - \alpha_j) - u))$$

for some j, where a is a non-zero integer, u is a linear combination of x_i's with $i < j$, and $u(u + a\alpha_j) = 0$.

3. Rank 2 decomposable vector bundles over Bott manifolds

For an element $\alpha \in \mathrm{H}^2(B_n)$, we denote by γ^α the complex line bundle over a Bott manifold B_n whose first Chern class is α. The purpose of this section is to prove the following.

Theorem 3.1. *Rank 2 decomposable vector bundles over a Bott manifold are distinguished by their total Chern classes.*

We prepare two lemmas.

Lemma 3.2. $\gamma^{x_j} \oplus \gamma^{-x_j+\alpha_j} \cong \underline{\mathbb{C}} \oplus \gamma^{\alpha_j}$.

Proof. Since γ^{x_j} is the pull-back bundle of the tautological line bundle over $P(\underline{\mathbb{C}} \oplus \xi_j) = B_j$, γ^{x_j} is a subbundle of $\underline{\mathbb{C}} \oplus \gamma^{\alpha_j}$ where $\alpha_j = c_1(\xi_j)$. It follows that the first Chern class of the orthogonal complement of γ^{x_j} in $\underline{\mathbb{C}} \oplus \gamma^{\alpha_j}$ (with an Hermitian metric) is $-x_j + \alpha_j$. Since the orthogonal complement is a complex line bundle and complex line bundles are classified by their first Chern classes, it must be isomorphic to $\gamma^{-x_j+\alpha_j}$, proving the lemma. $\square$

Lemma 3.3. *If $\gamma^\alpha \oplus \gamma^\beta \cong \underline{\mathbb{C}} \oplus \gamma^{\alpha+\beta}$, then $\gamma^{a\alpha} \oplus \gamma^{b\beta} \cong \underline{\mathbb{C}} \oplus \gamma^{a\alpha+b\beta}$ for any integers a and b.*

Proof. There is a nowhere zero cross section f of $\gamma^\alpha \oplus \gamma^\beta$ since $\gamma^\alpha \oplus \gamma^\beta$ contains a trivial complex line bundle as a subbundle by assumption. Let f_α and f_β be the projections of f on γ^α and γ^β, which are cross sections of γ^α and γ^β respectively. Their zero loci do not intersect. If a (respectively, b) is positive, we define $f_{a\alpha} := f_\alpha^a = f_\alpha \otimes \cdots \otimes f_\alpha$ (respectively, $f_{b\beta} := f_\beta^b$) which is a cross section of $\gamma^{a\alpha}$ (respectively, $\gamma^{b\beta}$). Note that there is a map $\varphi_{a\alpha}\colon \gamma^{a\alpha} \to \gamma^{-a\alpha}$ (respectively, $\varphi_{b\beta}\colon \gamma^{b\beta} \to \gamma^{-b\beta}$) which is an isomorphism as real 2-plane bundles but anti-$\mathbb{C}$-linear on fibers. If a (respectively, b) is negative, we define a cross section $f_{a\alpha}$ of $\gamma^{a\alpha}$ (respectively, $f_{b\beta}$ of $\gamma^{b\beta}$) to be the pull-back of f_α^{-a} (respectively, f_β^{-b}) by $\varphi_{a\alpha}\colon \gamma^{a\alpha} \to \gamma^{-a\alpha}$ (respectively, $\varphi_{b\beta}\colon \gamma^{b\beta} \to \gamma^{-b\beta}$). The pair $(f_{a\alpha}, f_{b\beta})$ determines a nowhere zero cross section of $\gamma^{a\alpha} \oplus \gamma^{b\beta}$. Thus $\gamma^{a\alpha} \oplus \gamma^{b\beta}$ contains a trivial complex line bundle as a subbundle. By computing the first Chern class of the orthogonal complement of the trivial subbundle in $\gamma^{a\alpha} \oplus \gamma^{b\beta}$, we get the lemma. $\square$

Proposition 3.4. *Let ξ and ξ' be complex line bundles over B_n. Then, the second Chern class $c_2(\xi \oplus \xi')$ is zero if and only if $\xi \oplus \xi'$ is isomorphic to $\underline{\mathbb{C}} \oplus (\xi \otimes \xi')$.*

Proof. The "if" part is clear. So we show the "only if" part.

We assume that $c_2(\xi \oplus \xi') = 0$. Then $c_1(\xi)c_1(\xi') = 0$. Note that if either $c_1(\xi) = 0$ or $c_1(\xi') = 0$, then ξ or ξ' is trivial and Proposition 3.4 holds. Thus, by Lemma 3.3, it suffices to show that $\xi \oplus \xi' \cong \mathbb{C} \oplus (\xi \otimes \xi')$ only when the vanishing pair $(c_1(\xi_i), c_1(\xi_i'))$ is primitive.

By Lemmas 2.2 and 3.3, we may assume that

$$(c_1(\xi), c_1(\xi')) = (ax_j + u, a(x_j - \alpha_j) - u)$$

for some j, where a is a non-zero integer, u is a linear combination of x_i's with $i < j$, and $u(u + a\alpha_j) = 0$. Then, j is the minimal integer such that $c_1(\xi)$ and $c_1(\xi')$ can be written as linear combinations of x_k's with $k \le j$. We will show that $\xi \oplus \xi' \cong \underline{\mathbb{C}} \oplus (\xi \otimes \xi')$ by induction on j. If $j = 1$, then $(c_1(\xi), c_1(\xi')) = \pm(x_1, x_1)$ and we are done by Lemma 3.2 and 3.3. Now we assume that $\gamma^\alpha \oplus \gamma^{\alpha'} \cong \underline{\mathbb{C}} \oplus (\gamma^\alpha \otimes \gamma^{\alpha'})$ for any vanishing pair (α, α') such that α and α' can be written as linear combinations of x_k's with $k < j$. Then,

$$\begin{aligned}\gamma^{ax_j+u} \oplus \gamma^{a(x_j-\alpha_j)-u} &\cong \gamma^{ax_j} \otimes (\gamma^u \oplus \gamma^{-(u+a\alpha_j)})\\ &\cong \gamma^{ax_j} \otimes (\underline{\mathbb{C}} \oplus \gamma^{-a\alpha_j}) \quad \text{(by hypothesis of induction)}\\ &\cong \gamma^{ax_j} \oplus \gamma^{a(x_j-\alpha_j)}\\ &\cong \underline{\mathbb{C}} \oplus \gamma^{a(2x_j-\alpha_j)} \quad \text{(by Lemma 3.2 and 3.3).}\end{aligned}$$

This completes the induction step, proving the proposition. □

Proof of Theorem 3.1. Let α, β, α' and β' be elements in $\mathrm{H}^2(B_n)$ such that

$$c(\gamma^\alpha \oplus \gamma^\beta) = c(\gamma^{\alpha'} \oplus \gamma^{\beta'}),$$

that is, $(1+\alpha)(1+\beta) = (1+\alpha')(1+\beta')$. Then, we have

$$\beta - \alpha = (\alpha' - \alpha) + (\beta' - \alpha) \quad \text{and} \quad (\alpha' - \alpha)(\beta' - \alpha) = 0.$$

Therefore, it follows from Proposition 3.4 that

$$\underline{\mathbb{C}} \oplus \gamma^{\beta-\alpha} \cong \gamma^{\alpha'-\alpha} \oplus \gamma^{\beta'-\alpha}.$$

By tensoring the both sides above with γ^α, we obtain the theorem. □

4. $\mathbb{C}\mathbf{P}^1$-bundles over Bott manifolds

In this section, we study $\mathbb{C}\mathrm{P}^1$-bundles over Bott manifolds.

Theorem 4.1. *Let B_n be a Bott manifold. Let γ^α and γ^β be complex line bundles over B_n. If $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\alpha))$ and $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\beta))$ are isomorphic as $\mathrm{H}^*(B_n)$-algebras, then $P(\underline{\mathbb{C}} \oplus \gamma^\alpha)$ and $P(\underline{\mathbb{C}} \oplus \gamma^\beta)$ are isomorphic as bundles.*

Proof. By Borel–Hirzebruch formula, $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\alpha))$ and $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\beta))$ are of the forms

$$\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\alpha)) = \mathrm{H}^*(B_k)[X]/(X^2 - \alpha X)$$

and

$$\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\beta)) = \mathrm{H}^*(B_k)[Y]/(Y^2 - \beta Y).$$

Let $\Phi\colon \mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\beta)) \to \mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^\alpha))$ be an $\mathrm{H}^*(B_n)$-algebra isomorphism. We write $\Phi(Y) =: sX + \alpha'$, $s = \pm 1$, $\alpha' \in \mathrm{H}^2(B_k)$. Since $\Phi(Y^2 - \beta Y) = 0$ and $X^2 = \alpha X$, we have

$$\begin{aligned}0 &= (sX + \alpha')^2 - \beta(sX + \alpha')\\ &= (\alpha + 2s\alpha' - s\beta)X + \alpha'(\alpha' - \beta).\end{aligned}$$

Therefore $\alpha = s(-2\alpha' + \beta)$ and $\alpha'(\alpha' - \beta) = 0$. Thus we obtain

$$\begin{aligned} P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}) &= P(\underline{\mathbb{C}} \oplus \gamma^{s(-2\alpha'+\beta)}) \\ &\cong P(\underline{\mathbb{C}} \oplus \gamma^{-2\alpha'+\beta}) \quad \text{(by taking complex conjugation when } s = -1\text{)} \\ &\cong P(\gamma^{\alpha'} \oplus \gamma^{-\alpha'+\beta}) \quad \text{(by Lemma 2.1)} \\ &\cong P(\underline{\mathbb{C}} \oplus \gamma^{\beta}) \quad \text{(by Theorem 3.1).} \end{aligned}$$

This proves the theorem. □

Lemma 4.2. *The group of* $\mathrm{H}^*(B_n)$*-algebra automorphisms of* $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}))$ *is of order* 2 *and generated by the automorphism induced by the bundle map* $p\colon P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}) \to P(\underline{\mathbb{C}} \oplus \gamma^{\alpha})$ *which assigns a line* l *in* $\underline{\mathbb{C}} \oplus \gamma^{\alpha}$ (*with an Hermitian metric*) *to its orthogonal complement* $l^{\perp}$.

Proof. Let $\Phi\colon \mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^{\alpha})) \to \mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}))$ be an $\mathrm{H}^*(B_n)$-algebra automorphism. The same argument as in the proof of Theorem 4.1 shows that $\Phi(X) = X$ or $-X + \alpha$. Thus the group of $\mathrm{H}^*(B_n)$-algebra automorphisms of $\mathrm{H}^*(P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}))$ is of order 2. We have to show that $p^*(X) = -X + \alpha$. Note that the total space of the tautological line bundle γ^X over $P(\underline{\mathbb{C}} \oplus \gamma^{\alpha})$ is the set

$$\{(l, v) \in P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}) \times E(\underline{\mathbb{C}} \oplus \gamma^{\alpha});\ l \ni v\}$$

and the total space of the pull-back $p^*\gamma^X$ of γ^X by p is given by

$$\{(l, v) \in P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}) \times E(\underline{\mathbb{C}} \oplus \gamma^{\alpha});\ l^{\perp} \ni v\},$$

so $p^*\gamma^X \oplus \gamma^X \cong \pi^*(\underline{\mathbb{C}} \oplus \gamma^{\alpha})$, where $\pi\colon P(\underline{\mathbb{C}} \oplus \gamma^{\alpha}) \to B_n$ is the projection. Therefore we obtain $p^*(X) = -X + \alpha$. □

The following corollary follows from Theorem 4.1 and Lemma 4.2.

Corollary 4.3. *Let* B_n *be a Bott manifold. Let* η *and* η' *be rank* 2 *decomposable vector bundles over* B_n. *Then, their projectivizations* $P(\eta)$ *and* $P(\eta')$ *are isomorphic as bundles if and only if their cohomology rings are isomorphic as* $\mathrm{H}^*(B_n)$*-algebras. Moreover, any* $\mathrm{H}^*(B_n)$*-algebra isomorphism between* $\mathrm{H}^*(P(\eta))$ *and* $\mathrm{H}^*(P(\eta'))$ *is induced by a bundle isomorphism.*

5. Proof of Theorem 1.1

Let $B_\bullet = (\{B_i\}_{i=0}^n, \{\pi_i\}_{i=1}^n)$ and $B'_\bullet = (\{B'_i\}_{i=0}^n, \{\pi'_i\}_{i=1}^n)$ be Bott towers of height n. Let $\Phi_\bullet\colon \mathrm{F}_\bullet\mathrm{H}^*(B'_\bullet) \to \mathrm{F}_\bullet\mathrm{H}^*(B_\bullet)$ be an isomorphism. We will show Theorem 1.1 by induction on the height. In the case when the height $k = 0$, Theorem 1.1 clearly holds.

Assume that Theorem 1.1 holds for Bott towers of height k. For $\Phi_k\colon \mathrm{F}_k\mathrm{H}^*(B'_\bullet) = \mathrm{H}^*(B'_k) \to \mathrm{H}^*(B_k) = \mathrm{F}_k\mathrm{H}^*(B_\bullet)$, there is a diffeomorphism $\varphi_k\colon B_k \to B'_k$ such that $\varphi_k^* = \Phi_k$ by the hypothesis of induction. Then, the pull-back bundle $\varphi_k^* B'_{k+1} \to B_k$ is isomorphic to $B_{k+1} \to B_k$. In fact, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^*(B_{k+1}) & & \mathrm{H}^*(B'_{k+1}) \arrow[ll, "\Phi_{k+1}"'] \arrow[dl, "\tilde{\varphi}_k^*"] \\
& \mathrm{H}^*(\varphi_k^* B'_{k+1}) & \\
\mathrm{H}^*(B_k) \arrow[uu, "\pi_{k+1}^*"] \arrow[ur, "(\varphi_k^*\pi'_{k+1})^*"] & & \mathrm{H}^*(B'_k) \arrow[ll, "\varphi_k^*=\Phi_k"] \arrow[uu, "\pi'^*_{k+1}"']
\end{tikzcd}$$

where $\tilde{\varphi}_k^*$ is the homomorphism induced from the bundle map $\tilde{\varphi}_k\colon \varphi_k^* B'_{k+1} \to B'_{k+1}$ and $(\varphi_k^*\pi'_{k+1})^*$ is the homomorphism induced from the projection $\varphi^*\pi'_{k+1}\colon \varphi_k^* B'_{k+1} \to B_k$. Since $\tilde{\varphi}_k$ is a diffeomorphism, the induced homomorphism $\tilde{\varphi}_k^*$ is a ring isomorphism. The composition $\Phi_{k+1} \circ (\tilde{\varphi}_k^*)^{-1}\colon \mathrm{H}^*(\varphi_k^* B'_{k+1}) \to \mathrm{H}^*(B_{k+1})$ is an $\mathrm{H}^*(B_k)$-algebra isomorphism. By Corollary 4.3, there is a bundle isomorphism $b\colon B_{k+1} \to \varphi_k^* B'_{k+1}$ such that $b^* = \Phi_{k+1} \circ (\tilde{\varphi}_k^*)^{-1}$.

By the construction of b, the bundle map $\varphi_{k+1} := \tilde{\varphi}_k \circ b$ satisfies $\varphi_{k+1}^* = \Phi_{k+1}$. $\square$

ACKNOWLEDGEMENT. The author would like to thank Professor Mikiya Masuda for stimulating discussion about toric topology.

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Department of Mathematics
Graduate School of Science
Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka 558-8585
Japan

Current address:
Osaka City University Advanced Mathematical Institute
Sugimoto, Sumiyoshi-ku, Osaka 558-8585
Japan
e-mail: ishida@sci.osaka-cu.ac.jp