Illman S.
**Osaka J. Math.**
**23** (1986), 881–927

# ACTIONS OF COMPACT LIE GROUPS AND THE EQUIVARIANT WHITEHEAD GROUP

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(Received August 20, 1985)

**1. Introduction.** Let $G$ be a compact Lie group and $X$ a finite $G$-CW complex. By $Wh_G(X)$ we denote the equivariant Whitehead group of $X$ as defined in [5]. The group $Wh_G(X)$ is defined in a geometric way and $Wh_G(X)$ is an abelian group. If $f\colon X\to Y$ is a $G$-homotopy equivalence between finite $G$-CW complexes then the (geometric) equivariant Whitehead torsion of $f$ is an element $\tau(f)\in Wh_G(X)$ and $f$ is a simple $G$-homotopy equivalence if and only if $\tau(f)=0$, see Theorem II.3.6′ in [5].

Let us here first state the two main results of this paper and then describe some earlier work on the subject, and also give a quick outline of some other results and constructions contained in this paper which are of independent interest. The first main result has to do with the algebraic determination of $Wh_G(X)$.

**Theorem A.** *There exists an isomorphism*

$$\Phi\colon Wh_G(X)\xrightarrow{\cong}\sum_{C(X)}\oplus Wh(\pi_0(WH)^*_\alpha)\,.$$

In the above formula the right hand side is a direct sum of ordinary (algebraically) defined Whitehead groups of discrete groups $\pi_0(WH)^*_\alpha$ which will be defined below. The direct sum is over the set $C(X)$ of equivalence classes of connected components $X^H_\alpha$ of arbitrary fixed point sets $X^H$, where $H$ is any closed subgroup of $G$. The components $X^H_\alpha$ and $X^K_\beta$ of the fixed point sets $X^H$ and $X^K$, respectively, are defined to be in relation, denoted

$$X^H_\alpha\sim X^K_\beta$$

if there exists $n\in G$ such that $nHn^{-1}=K$ and $n(X^H_\alpha)=X^K_\beta$. Given a component $X^H_\alpha$ of $X^H$ we define

$$(WH)_\alpha=\{w\in WH\mid wX^H=X^H\}\,.$$

Here $WH=NH/H$, and $NH$ denotes the normalizer of $H$ in $G$. The group $(WH)^*_\alpha$ is a Lie group (not necessarily compact) which acts on the universal covering space $\tilde{X}^H_\alpha$ of $X^H_\alpha$ by an action which covers the action of $(WH)_\alpha$ on $X^H_\alpha$. There is a short exact sequence of topological groups

$$e \to \Delta \to (WH)^*_\alpha \to (WH)_\alpha \to e$$

where $\Delta$ denotes the group of deck transformations of $\tilde{X}^H_\alpha$ and we therefore have $\Delta \cong \pi_1(X^H_\alpha)$. By $\pi_0(WH)^*_\alpha$ we denote the group of components of $(WH)^*_\alpha$.

The isomorphism $\Phi$ is constructed in an explicit way. In particular this allows us to determine the equivariant Whitehead torsion $\tau(f)$ of a $G$-homotopy equivalence $f\colon X \to Y$ on the algebraic side, i.e., it allows us to determine the element $\Phi(\tau(f)) \in \sum_{C(X)} \oplus Wh(\pi_0(WH)^*_\alpha)$.

Our methods also enable us to prove that equivariant Whitehead torsion is a combinatorial invariant, i.e., we prove the following. (See Section 12.)

**Theorem B.** *Let $X^*$ be an equivariant subdivisoin of a finite G-CW complex X. Then the identity map $h\colon X^* \to X$ is a simple G-homotopy equivalence.*

This result is important for further developments of equivariant simple homotopy theory, and one should also in this connection recall the well-known fact that equivariant Whitehead torsion is *not* a topological invariant.

In the case when $G$ is a finite abelian group (in fact discrete abelian group) and each component $X^H_\alpha$ of any fixed point set $X^H$ is simply connected we determined $Wh_G(X)$ algebraically in [5], Theorem III. 1.4. Although our algebraic determination of $Wh_G(X)$ in the present paper, when $G$ is a compact Lie group and $X$ is an arbitrary finite $G$-CW complex, is a much more complicated task than the special case treated in [5] it is interesting to notice that the proof of Theorem III.1.4 in [5] carries over quite well to the present general case. In fact some parts of it carries over word by word.

The first algebraic description of $Wh_G(X)$ in the general case when $G$ is a compact Lie group is due to H. Hauschild [3]. His method is completely different from ours. Hauschild's approach gives $Wh_G(X)$ as a direct sum, over the same indexing set as in Theorem A, of Whitehead groups of fundamental groups of appropriate Borel constructions. In order to be more specific we have that Hauschild's result, Satz IV.1 in [3], shows that $Wh_G(X)$ is isomorphic to a direct sum of groups of the form $Wh(\pi_1(EK_\alpha \times_{K_\alpha} X^H_\alpha))$, where we have denoted $K_\alpha = (WH)_\alpha$ and $EK_\alpha$ is the total space of the universal principal $K_\alpha$-bundle $EK_\alpha \to BK_\alpha$, and $EK_\alpha \times_{K_\alpha} X^H_\alpha$ is the Borel construction applied to the $K_\alpha$-space $X^H_\alpha$. It is easy to see that we have an isomorphism $\pi_1(EK^*_\alpha \times_{K^*_\alpha} \tilde{X}^H_\alpha) \xrightarrow{\cong} \pi_1(EK_\alpha \times_{K_\alpha} X^H_\alpha)$, where $K^*_\alpha = (WH)^*_\alpha$, and since $\tilde{X}^H_\alpha$ is simply connected we also have $\pi_1(EK^*_\alpha \times_{K^*_\alpha} \tilde{X}^H_\alpha) = \pi_1(BK^*_\alpha) \cong \pi_0(K^*_\alpha)$. Thus the groups occurring in the direct sum in Hauschild's result are isomorphic to the groups occurring in the direct sum in Theorem A. But conceptually Hauschild's approach and ours are very different from each other. Our method has the advantage that it also provides the chain complexes from which the torsion invariants are to be computed, and moreover it leads to a proof of the combinatorial invariance of equivariant Whitehead torsion.

In the case when $G$ is a finite group we also have the paper [1] by D. Anderson, which gives a parallel treatment of equivariant Wall obstructions and the equivariant Whitehead group $Wh_G(X)$. Unfortunately Theorem A in [1], describing $Wh_G(X)$ as a direct sum of ordinary (algebraically defined) Whitehead groups, is not quite correct. In [1] a connected component $X^H_\alpha$ of a fixed point set $X^H$ is used to represent the whole $G$-component $GX^H_\alpha$, whereas $X^H_\alpha$ should only be used to represent the $NH$-component $(NH)X^H_\alpha$, where $NH$ denotes the normalizer of $H$ in $G$. This mistake causes problems with the well-definition of the isomorphism $\Phi$ in Theorem A in [1], and also has the effect that the summation in the direct sum of ordinary Whitehead groups occurring in Theorem A in [1] is done over the wrong indexing set. In the double direct sum occurring in Theorem A in [1] the first sum is over the set of all conjugacy classes $(H)$ for which $X^H \neq \emptyset$ and for a fixed $H$ representing $(H)$ the second direct sum *ought to be* over the set of all $NH$-components of $X^H$, i.e., over a set consisting of one representing connected component $X^H_\alpha$ for each $NH$-component $(NH)X^H_\alpha$ of $X^H$.

In Part I of this paper we present various constructions and results that are used in Part II where the main results are proved. Much of the material in Part I is of independent interest. The contents of Part I are as follows.

3. An acyclic chain complex.
4. Equivariant CW complex structure on $X^H$.
5. Covering actions.
6. Lifting equivariant CW complex structure to universal coverings.
7. Induced actions on homotopy groups of equivariant spaces and pairs.

In Part II we give the main work that leads to a proof of Theorem A. In section 12 we make use of Theorem A in establishing the combinatorial invariance of equivariant Whitehead torsion. The headings for the sections in Part II are as follows.

8. The component structure of a $G$-CW complex.
9. Definition of torsion invariants.
10. Definition of the homomorphism $\Phi$.
11. $\Phi$ is an isomorphism.
12. Combinatorial invariance of equivariant Whitehead torsion.

## 2. Preliminaries and notations

Recall from [5] that the elements of $Wh_G(X)$ are equivalence classes $s(V, X)$, where $(V, X)$ is a finite $G$-CW pair such that $i: X \to V$ is a $G$-homotopy equivalence, and two such pairs $(V, X)$ and $(W, X)$ are in relation if there exists an equivariant formal deformation from $V$ to $W$ rel. $X$. We refer the reader to [5] for further information about $Wh_G(X)$ and equivariant simple homotopy theory.

Our notations are (hopefully) the standard ones. For any closed subgroup $H$ of $G$ and any $G$-space $X$ we let $X^H$ denote the fixed point set of $H$ and we define

$$X^{>H} = \{x \in X \mid G_x \supsetneqq H\} .$$

Furthermore we denote $X^{(H)} = GX^H$ and $X^{>(H)} = GX^{>H}$. A $G$-isotropy type is the same thing as a conjugacy class $(H)$ of a closed subgroup $H$ of $G$. We define a partial order in the set of all $G$-isotropy types by

$$(H) \geq (K) \Leftrightarrow \text{there exists } g \in G \text{ such that } gHg^{-1} \supset K$$

Furthermore we define $(H) > (K)$ to mean $(H) \geq (K)$ and $(H) \neq (K)$. We then also have that $X^{>(H)} = \{x \in X \mid (G_x) > (H)\}$.

PART I

## 3. An acyclic chain complex

In this section $P$ denotes an arbitrary locally compact group. We shall prove the following result.

**Proposition 3.1.** *Let $(X, A)$ be a $P$-equivariant relative CW complex such that the inclusion $i: A \to X$ is a $P$-homotopy equivalence. Then the chain complex*

$$(1) \quad \cdots \to H_{n+1}((X, A)^{n+1}, (X, A)^n; \boldsymbol{Z}) \to H_n((X, A)^n, (X, A)^{n-1}; \boldsymbol{Z}) \to H_{n-1}((X, A)^{n-1}, (X, A)^{n-2}; \boldsymbol{Z}) \to \cdots$$

*is acyclic.*

Here $H_m(Y, B; \boldsymbol{Z})$ denotes ordinary singular homology with integer coefficients of the pair $(Y, B)$. In the applications of Proposition 3.1 in this paper $P$ will be a Lie group and $(X, A)$ will be a $P$-equivariant CW pair, where $X$ moreover is a finite $P$-equivariant CW complex.

To prove Proposition 3.1 we first show that the chain complex (1) is isomorphic to a similarly defined chain complex using equivariant singular homology (see [6]), with a specific coefficient system $k$, instead of ordinary singular homology. This much will in fact hold without the assumption that $i: A \to X$ is a $P$-homotopy equivalence. By then invoking the assumption that $i: A \to X$ is a $P$-homotopy equivalence it follows that this other chain complex is acyclic and hence also the chain complex (1). We now proceed to give the relevant steps for the proof of Proposition 3.1.

We define a covariant coefficient system $k$, (see [6], Definition 1.2, we are now taking the family $\mathcal{F}$ to be the family of all closed subgroups of $P$), over the ring $Z$, as follows: For any closed subgroup $Q$ of $P$ we set

$$k(P/Q) = H_0(P/Q; Z)$$

and for any $P$-map $\alpha: P/Q \to P/R$, where $Q$ and $R$ are closed subgroups of $P$, we define

$$\alpha_* = k(\alpha): k(P/Q) \to k(P/R)$$

to be the induced map $\alpha_*: H_0(P/Q; \boldsymbol{Z}) \to H_0(P/R; \boldsymbol{Z})$. Here $H_0(\ ; \boldsymbol{Z})$ is the 0:th ordinary singular homology group.

We shall now define a natural transformation of equivariant homology theories (defined on the category of all $P$-pairs and $P$-maps)

$$(2) \qquad \varphi_*: H_*(\ ; \boldsymbol{Z}) \to H_*^P(\ ; k).$$

Here $H_*(\ ; \boldsymbol{Z})$ denotes ordinary singular homology. Observe that ordinary singular homology also is an equivariant homology theory, but the equivariant dimension axiom is not satisfied in general. Thus ordinary singular homology gives us a generalized equivariant homology theory. The other functor $H_*^P(\ ; k)$ is equivariant singular homology with coefficients in $k$, see [6].

We first define a natural transformation on the chain level. Recall from [6] that the equivariant singular homology functor $H_*^P(\ ; k)$ is defined as the homology of a chain complex $S^P(\ ; k) = \{C_n^P(\ ; k), \partial_n\}$ and that $S^P(\ ; k)$ is a quotient of the chain complex $\hat{S}^P(\ ; k) = \{\hat{C}_n^P(\ ; k), \partial_n\}$, see Section 3 in Chapter I of [6]. For any $P$-space $X$ we define

$$(3) \qquad \varphi': C_n(X; \boldsymbol{Z}) \to \hat{C}_n^P(X; k), \qquad n \geq 0,$$

as follows. If $S: \Delta_n \to X$ is an ordinary singular $n$-simplex in $X$ we set

$$\varphi'(S) = \hat{S} \otimes c_0 \in \boldsymbol{Z}_{\hat{S}} \otimes_{\boldsymbol{Z}} k(G) \subset \hat{C}_n^P(X; k).$$

Here $\hat{S}: \Delta_n \times P \to X$ is the equivariant singular $n$-simplex, of type $\{e\}$, in $X$ defined by $\hat{S}(z, p) = pS(z)$, for $(z, p) \in \Delta_n \times P$, and $c_0 \in k(P) = H_0(P; \boldsymbol{Z})$ represents the identity component of $P$. Extending linearly we obtain the homomorphism $\varphi'$ in (3). It follows directly from the definition of $\varphi'$ that $\varphi'$ commutes with the boundary operators and hence gives us a chain map

$$\varphi: S(X; \boldsymbol{Z}) \to \hat{S}^P(X; k).$$

Composing with the natural projection $p: \hat{S}^P(X; k) \to S^P(X; k)$ we obtain a chain map

$$\varphi\colon S(X; \boldsymbol{Z}) \to S^P(X; k).$$

If $(X, A)$ is a $P$-pair we have the chain map $\varphi\colon S(A; Z) \to S^P(A; Z)$ and hence also

$$(4) \qquad \varphi\colon S(X, A; \boldsymbol{Z}) \to S^P(X, A; k).$$

Then the induced map in homology gives us

$$(5) \qquad \varphi_*\colon H_n(X, A; \boldsymbol{Z}) \to H_n^P(X, A; \boldsymbol{Z}), \qquad \text{for all} \quad n \geq 0.$$

If $f\colon (X, A) \to (Y, B)$ is a $P$-map it follows directly from the definition that the chain map $\varphi$ in (4) commutes with the chain maps induced by $f$ and hence also $\varphi_*$ commutes with the homomorphisms induced by $f$.

The following simple lemma is a key fact for us.

**Lemma 3.2.** *Let $Q$ be a closed subgroup of $P$. Then*

$$\varphi_*\colon H_0(P/Q; \boldsymbol{Z}) \to H_0^P(P/Q; k)$$

*is an isomorphism.*

Proof. Recall that equivariant singular homology $H_*^P(\ ; k)$ satisfies the equivariant dimension axiom, i.e., the $P$-space $P/Q$ has non-trivial equivariant singular homology only in dimension zero, and in this dimension the homology is isomorphic to $k(P/Q)$. The explicit isomorphism

$$\alpha\colon H_0^P(P/Q; k) \xrightarrow{\cong} k(P/Q)$$

is given as follows, (see [6], Section 7 in Chapter I, p. 39–40). If $T\colon \Delta_0 \times P/R = P/R \to P/Q$ is an equivariant singular 0-simplex in $P/Q$ and $a \in k(P/R)$ then $T \otimes a \in \hat{C}_0^P(P/Q; k)$ and by $\{T \otimes a\}$ we denote the corresponding class in $C_0^P(P/Q; k) = H_0^P(P/Q; k)$. Then

$$\alpha(\{T \otimes a\}) = T_*(a) \in k(P/Q).$$

Direct verification now shows that the composite

$$(6) \qquad H_0(P/Q; \boldsymbol{Z}) \xrightarrow{\varphi_*} H_0^P(P/Q; k) \xrightarrow[\cong]{\alpha} k(P/Q) = H_0(P/Q; \boldsymbol{Z})$$

is the identity map. Thus $\varphi_*$ is an isomorphism. □

**Corollary 3.3.** *For all $n \geq 0$ the map*

$$\varphi_*\colon H_n(D^n \times P/Q, S^{n-1} \times P/Q; \boldsymbol{Z}) \to H_n^P(D^n \times P/Q, S^{n-1} \times P/Q; k)$$

*is an isomorphism.*

Proof. For both ordinary singular homology and equivariant singular homology we have natural isomorphisms

$$H_n(D^n\times P/Q,\ S^{n-1}\times P/Q;\ \boldsymbol{Z})\xrightarrow{\cong} H_0(P/Q;\ \boldsymbol{Z})$$

and

$$H_n^P(D^n\times P/Q,\ S^{n-1}\times P/Q;\ k)\xrightarrow{\cong} H_0^P(P/Q;\ \mathcal{Y})$$

If we use the same procedure in defining both of these isomorphisms we obtain a strictly commutative diagram (not only commutative up to sign, which in fact would be sufficient here)

$$\begin{array}{ccc} H_n(D^n\times P/Q,\ S^{n-1}\times P/Q;\ \boldsymbol{Z}) & \xrightarrow{\cong} & H_0(P/Q;\ \boldsymbol{Z}) \\ \varphi_*\downarrow & & \cong\downarrow\varphi_* \\ H_n^P(D^n\times P/Q,\ S^{n-1}\times P/Q;\ k) & \xrightarrow{\cong} & H_0^P(P/Q;\ k)\,. \end{array} \quad \square$$

**Coorollary 3.4.** *Let $(X, A)$ be a P-equivariant relative CW complex. Then*

$$\varphi_*\colon H_n((X,^n A), (X, A)^{n-1};\ \boldsymbol{Z})\to H_n^P(X, A)^n, (X, A)^{n-1};\ k)$$

*is an isomorphism for all $n\geq 0$.*

Proof. We have

$$\begin{aligned} H_n((X, A)^n, (X, A)^{n-1};\ \boldsymbol{Z}) &\cong \sum_i \oplus H_n(c_i^n,\ \dot{c}_i^n;\ \boldsymbol{Z}) \\ &\cong \sum_i \oplus H_n(D^n\times P/Q_i,\ S^{n-1}\times P/Q_i;\ \boldsymbol{Z}) \end{aligned}$$

and

$$\begin{aligned} H_n^P((X, A)^n, (X, A)^{n-1};\ k) &\cong \sum_i \otimes H_n^P(c_i^n,\ \dot{c}_i^n;\ k) \\ &\cong \sum_i \oplus H_n^P(D^n\times P/Q_i,\ S^{n-1}\times P/Q_i;\ k) \end{aligned}$$

where in both cases the direct sum is over all equivariant $n$-cells of $(X, A)$. The isomorphisms above commute with $\varphi_*$ and hence Corollary 3.4 follows from Corollary 3.3. □

The isomorphisms $\varphi_*$ given by Corollary 3.4 form an isomorphism from the chain complex (1) to the chain complex

$$(6)\quad \cdots\to H_{n+1}^P((X, A)^{n+1}, (X, A)^n;\ k)\xrightarrow{\partial} H_n^P((X, A)^n, (X, A)^{n-1};\ k)\to H_{n-1}^P((X, A)^{n-1}, (X, A)^{n-2};\ k)\to\cdots.$$

The above chain complex (6) is the cellular chain complex with coefficients in $k$ of $(X, A)$ and the $n$: th homology group of this chain complex is isomorphic to $H_n^P(X, A;\ k)$. See [4], Theorem 2.7.

We can now complete the proof of Prcposition 3.1 by observing that if the $P$-equivariant relative CW pair $(X, A)$ is such that $i: A \to X$ is a $G$-homotopy equivalence then $H_n^P(X, A; k)=0$ for all $n \geq 0$, and thus the chain complex (6) is acyclic and hence so is the chain complex (1). □

## 4. Equivariant CW complex structure on $X^H$

In this section $G$ denotes an arbitrary compact Lie group. The purpose of this section is to establish the following two results.

**Theorem 4.1.** *Let $G$ be a compact Lie group and $X$ a $G$-equivariant CW complex. Then $X^H$ is locally contractible for every closed subgroup $H$ of $G$.*

**Theorem 4.2.** *Let $G$ be a compact Lie group and $H$ a closed subgroup of $G$, and let $(X, A)$ be a $G$-equivariant relative CW complex such that all equivariant cells in $X-A$ have type $(H)$. Then $(X^H, A^H)$ is a $WH$-equivariant relative CW complex and the $WH$-action is free in $X^H - A^H$.*

Although the results in Theorem 4.1 and 4.2, respectively are of quite different nature their proofs have a lot in common. A main part of the proof of Theorem 4.1 consists of showing that $X^H$ is a $WH$-equivariant CW complex. Theorem 4.2 is just the relative version of a special case of the result that the fixed point set $X^H$ is a $WH$-equivariant CW complex.

The basic fact that we need is given by the following lemma.

**Lemma 4.3.** *Let $G$ be a compact Lie group and let $H$ be a closed subgroup of $G$. Assume that the $G$-space $X$ is obtained from the $G$-space $A$ by adjoining one equivariant $n$-cell. Then the $NH$-space $X^H$ is obtained from the $NH$-space $A^H$ by adjoining a finite number of $NH$-equivariant $n$-cells.*

Proof. We have $X = A \cup c$, where $c$ is a $G$-equivariant $n$-cell. Let

$$f: (D^n \times G/K, S^{n-1} \times G/K) \to (c, \dot{c}) \to (X, A)$$

be a characteristic $G$-map for $c$. Here $\dot{c} = c \cap A$ and $f(D^n \times G/K) = c$ and $f: |\mathring{D}^n \times G/K \xrightarrow{\cong} \mathring{c}$, is a $G$-homeomorphism, where $\mathring{c} = c - \dot{c}$. Restricting to the fixed point set of $H$ we obtain an $NH$-equivariant map

$$f^H: (D^n \times (G/K)^H, S^{n-1} \times (G/K)^H) \to (X^H, A^H).$$

We have $(\mathring{c})^H = \phi$, and hence also $X^H = A^H$, unless $(K) \geq (H)$. Thus it is enough to consider the case $(K) \geq (H)$. Furthermore we may assume that we have chosen the subgroup $K$ so that $H \subset K$. The group $NH$ acts on $(G/K)^H$ by multiplication on the left. By Corollary II. 5.7. in [9] the orbit space of this $NH$-action on $(G/K)^H$ is finite. Thus the $NH$-space $(G/K)^H$ consists of a finite number of disjoint $NH$-orbits $NHa_1, \cdots, NHa_t$, i.e., we have

$$(G/K)^H = NHa_1 \cup \cdots \cup NHa_t$$

where $a_i \in (G/K)^H$, $1 \leq i \leq t$, as $NH$-spaces. Let $P_i$ denote the isotropy subgroup of $NH$ at $a_i$, i.e.,

$$P_i = (NH)_{a_i}, \qquad 1 \leq i \leq t.$$

Using the canonical $NH$-homeomorphism

$$\alpha \colon NH/P_i \xrightarrow{\cong} NHa_i$$

to identify $NH/P_i$ with $NHa_i$, we can write

$$(G/K)^H = NH/P_1 \cup \cdots \cup NH/P_t.$$

Thus the map $f^H$ gives us an $NH$-map

$$f^H \colon \bigcup_{i=1}^{t} (D^n \times NH/P_i, S^{n-1} \times NH/P_i) \to (X^H, A^H).$$

Denote $f^H(D^n \times NH/P_i) = b_i$, $1 \leq i \leq t$ and let $\dot{b}_i = b_i \cap A^H$. Then

$$f^H| \colon \mathring{D}^n \times NH/P_i \to \mathring{b}_i = b_i - \dot{b}_i$$

is an $NH$-homeomorphism. Furthermore we have

$$(\mathring{c})^H = \mathring{b}_1 \cup \cdots \cup \mathring{b}_t.$$

Thus we have that

$$X^H = A^H \cup b_1 \cup \cdots \cup b_t$$

and that

$$f^H| \colon (D^n \times NH/P_i, S^{n-1} \times NH/P_i) \to (b_i, \dot{b}_i) \to (X^H, A^H)$$

is a characteristic $NH$-map for $b_i$, $1 \leq i \leq t$. This completes the proof. □

REMARK. Since the subgroup $H$ acts trivially on the pair $(X^H, A^H)$ we may as well consider $(X^H, A^H)$ as a $WH = NH/H$-pair, and the conclusion of Lemma 4.3 is then that $X^H$ is obtained from $A^H$ by adjoining a finite number of $WH$-equivariant $n$-cells.

**Theorem 4.4.** *Let $(X, A)$ be a $G$-equivariant relative CW complex. Then $(X^H, A^H)$ is a $WH$-equivariant relative CW complex and the equivariant $n$-skeleton $(X^H, A^H)^n$ of $(X^H, A^H)$ is given by*

$$(X^H, A^H)^n = ((X, A)^n)^H.$$

*Furthermore we have that if $(X, A)$ has a finite number of $G$-equivariant cells in* *each dimension then* $(X^H, A^H)$ *also has a finite number of WH-equivariant cells in each dimension.*

Proof. This follows directly from Lemma 4.3 and the above remark. □

Let $X^H_\alpha$ be a component of $X^H$ and define

$$(WH)_\alpha = \{w \in WH \mid wX^H_\alpha = X^H_\alpha\}$$

We denote $A^H_\alpha = A^H \cap X^H_\alpha$. Then $(X^H_\alpha, A^H_\alpha)$ is a $(WH)_\alpha$-pair and it now follows easily that we have:

**Corollary 4.5.** $(X^H_\alpha, A^H_\alpha)$ *is a* $(WH)_\alpha$*-equivariant relative CW complex.* □

We will also need the following fact about $G$-equivariant $CW$ complexes.

**Proposition 4.6.** *A G-equivariant CW complex is locally contractible.*

Proof. This is proved in a similar way as the same result for ordinary CW complexes, see for example [10], Theorem 11.6.6. We leave the details to the reader. □

Now observe that Theorem 4.1 is an immediate consequence of Theorem 4.4 and Proposition 4.6.

## 5. Covering actions

In this section $K$ denotes an arbitrary Lie group. Let $X$ be a $K$-space, where $X$ is a connected, locally path-connected and semilocally 1-connected space. We are *not* assuming that the action of $K$ on $X$ is effective. Let $p: \tilde{X} \to X$ be a universal covering of $X$. Then there exists a Lie group $K^*$ and a continuous surjective homomorphism $\pi: K^* \to K$, such that $K^*$ acts on $\tilde{X}$ by an action which covers the action of $K$ on $X$, via $\pi: K^* \to K$. The purpose of this section is to give the construction of the group $K^*$ and its action on $\tilde{X}$, and to establish some properties of this construction that will be used later on in this paper.

First we consider the case of an *effective* action. Suppose that $J$ is a Lie group acting effectively on $X$. Let $J^*$ denote the subgroup of Homeo($\tilde{X}$) defined as follows,

$$J^* = \{h: \tilde{X} \to \tilde{X} \mid h \text{ is a homeomorphism which covers some } j: X \to X, \text{ where } j \in J\}.$$

The group $J^*$ is topologized as described on a page 65 in [9], see also the appendix to this section. Then $J^*$ is a Lie group and we have a continuous effective action of $J^*$ on $\tilde{X}$. Moreover there is a well defined continuous homomorphism $\pi: J^* \to J$, given by $\pi(h) = j$ if and only if $h: \tilde{X} \to \tilde{X}$ covers

$j\colon X\to X$. Clearly $\pi$ is surjective and the kernel of $\pi$ is the discrete group $\Delta$ of all deck transformations, i.e., we have a short exact sequence

$$(1)\qquad e\to\Delta\to J^*\xrightarrow{\pi}J\to e$$

of topological groups.

We shall now consider the general case, where the given action on $X$ *need not be effective*. Let $K$ be a Lie group acting on $X$ and let $N$ denote the kernel of this action. Then $N$ is a normal closed subgroup of $K$ and $J=K/N$ is a Lie group which acts effectively on $X$. Let $\rho\colon K\to J$ be the natural projection and consider the pull-back diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi'} & K \\ {\scriptstyle\rho'}\downarrow & & \downarrow{\scriptstyle\rho} \\ J^* & \xrightarrow{\pi} & J \end{array}$$

Thus $P$ is defined by

$$(2)\qquad P=\{(h,k)\in J^*\times K\mid \pi(h)=\rho(k)\}$$

and the homomorphisms $\rho'$ and $\pi'$ are given by $\rho'(h,k)=h$ and $\pi'(h,k)=k$, for all $(h,k)\in P$. Then $P$ is a Lie group and the homomorphisms $\rho'$ and $\pi'$ are continuous.

We now define an action of $P$ on $\tilde{X}$

$$(3)\qquad \theta^*\colon P\times\tilde{X}\to\tilde{X}$$

by $\theta^*((h,k),\tilde{x})=h\tilde{x}$. It is then immediately verified that the diagram

$$(4)\qquad \begin{array}{ccc} P\times\tilde{X} & \xrightarrow{\theta^*} & \tilde{X} \\ {\scriptstyle\pi'\times p}\downarrow & & \downarrow{\scriptstyle p} \\ K\times X & \xrightarrow{\theta} & X \end{array}$$

commutes, i.e., the action of $P$ on $\tilde{X}$ covers the action of $K$ on $X$, via $\pi'\colon P\to K$.

The kernel of the action of $P$ on $\tilde{X}$ equals $\{e\}\times N\cong N$, and we have a commutative diagram

$$(5)\qquad \begin{array}{ccccccccc} & & & & e & & e & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & N & \xrightarrow{id} & N & & \\ & & & & \downarrow & & \downarrow & & \\ e & \longrightarrow & \Delta & \longrightarrow & P & \xrightarrow{\pi'} & K & \longrightarrow & e \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle\rho'}\downarrow & & \downarrow{\scriptstyle\rho} & & \\ e & \longrightarrow & \Delta & \longrightarrow & J^* & \xrightarrow{\pi} & J & \longrightarrow & e \\ & & & & \downarrow & & \downarrow & & \\ & & & & e & & e & & \end{array}$$

of short exact sequences.

Now suppose that we have an inclusion $i$: $X \to V$ of $K$-spaces, where $V$ is also assumed to be connected, locally path-connected and semilocally 1-connected. Furthermore assume that $X$ is a $K$-equivariant strong deformation retract of $V$, and that the action of $K$ on $V$ is effective but the action of $K$ on $X$ *need not* be effective. In this paper we will naturally encounter such situations, but it should also be observed that starting with the given $K$-space $X$ one can easily construct a $K$-space $V$ containing $X$ such that the action of $K$ on $V$ is effective and $X$ is a $K$-equivariant strong deformation retract of $V$. For this one simply adds to $X$ an equivariant free 1-cell $I \times K$.

Let $p$: $\tilde{V} \to V$ be a universal covering of $V$. Then $p|: \tilde{X} = p^{-1}(X) \to X$ is a universal covering of $X$. Since the action of $K$ on $V$ is effective we have the group $K^*$ acting effectively on $\tilde{V}$. This action of $K^*$ on $\tilde{V}$ restricts to an action of $K^*$ on $\tilde{X}$ which need not be effective. On the other hand the action of $K$ on $X$ induces an effective action of $J = K/N$ on $X$, where $N$ denotes the kernel of the action of $K$ on $X$. We then obtain an effective action of $J^*$ on $\tilde{X}$ and a corresponding non-effective action of the Lie group $P$, defined in (2), on $\tilde{X}$. We shall now prove that $K^*$ is isomorphic to $P$ and that the two corresponding non-effective actions on $\tilde{X}$ agree.

**Proposition 5.1.** *Let $(V, X)$ be a $K$-pair as above. Then there exists a natural isomorphism $\alpha$: $K^* \xrightarrow{\cong} P$ of Lie groups, such that the action of $K^*$ on $\tilde{X}$ corresponds via the isomorphism $\alpha$ to the action of $P$ on $\tilde{X}$.*

Proof. Let $k^* \in K^*$. Then $k^*$: $\tilde{X} \to \tilde{X}$ is a homeomorphism which covers the homeomorphism $\rho\pi^*(k^*)$: $X \to X$, where $\pi^*$: $K^* \to K$ and $\rho$: $K \to K/N = J$ denote the projections. Thus $k^*$: $\tilde{X} \to \tilde{X}$ gives us a unique element in $J^*$ and we denote this element by $\rho^*(k^*) \in J^*$. Then $\rho^*$: $K^* \to J^*$ is a homomorphism and by Lemma 5.3 in the appendix to this section $\rho^*$ is continuous. Therefore the map $\alpha$: $K^* \to P$ defined by $\alpha(k^*) = (\rho^*(k^*), \pi^*(k^*))$, for all $k^* \in K^*$, is a continuous homomorphism. Since it is easily seen that $\alpha$ is both surjective and injective it follows that $\alpha$ is an isomorphism of Lie groups. Moreover we have $k^*\tilde{x} = \rho^*(k^*)\tilde{x} = (\rho^*(k^*), \pi^*(k^*))\tilde{x} = \alpha(k^*)\tilde{x}$, for any $k^* \in K^*$ and $\tilde{x} \in \tilde{X}$. □

Observe that the group $P$ is defined in terms of data coming only from the $K$-space $X$. In particular Proposition 5.1 shows that the group $K^*$ does not depend on $V$ but only on $X$, i.e., if $(V, X)$ and $(V', X)$ are two $K$-pairs as above with the action of $K$ on $V$ and $V'$ being effective, then the extension $K^*$ of $K$ obtained by considering the action of $K$ on $V$ is the same as the extension $K^{*\prime}$ obtained by considering the action of $K$ on $V'$.

Since we have established Proposition 5.1 we will in the sequel use $K^*$ to also denote the group acting possibly non-effectively on $\tilde{X}$ covering the possibly non-effective action of $K$ on $X$.

**Proposition 5.2.** *Let* $(V, X)$ *be as in Proposition* 1. *Then* $\tilde{X}$ *is a* $K^*$-*equivariant strong deformation retract of* $\tilde{V}$.

Proof. Let $r$: $V\to X$ be a $K$-retraction of $V$ to $X$, and let $H$: $V\times I\to V$ be a $K$-homotopy rel $X$ from $\mathrm{id}_V$ to $i\circ r$, where $i$: $X\to V$ denotes the inclusion. Choose some $x_0\in X$ and $\tilde{x}_0\in\tilde{X}_0$ such that $p(\tilde{x}_0)=x_0$. Let $\tilde{r}$: $\tilde{V}\to\tilde{X}$ be the lifting of $r$ which satisfies $\tilde{r}(\tilde{x}_0)=\tilde{x}_0$. Then $\tilde{r}$ is a retraction of $\tilde{V}$ to $\tilde{X}$. Let $k^*\in K^*$ be an arbitrary element in $K^*$. We claim that

$$\tilde{r}(k^*\tilde{v})=k^*\tilde{r}(\tilde{v})\,, \qquad \text{for all} \quad \tilde{v}\in\tilde{V}\,.$$

This holds since the two maps

$$\tilde{r}\circ k^*,\ k^*\circ\tilde{r}\colon\ \tilde{V}\to\tilde{X}$$

both cover the same map $r\circ k=k\circ r$: $V\to X$, where $k=\pi(k^*)$, and moreover $(\tilde{r}\circ k^*)(\tilde{x}_0)=k^*\tilde{x}_0=(k^*\circ\tilde{r})(\tilde{x}_0)$. This shows that $\tilde{r}$ is a $K^*$-map.

Let $\tilde{H}$: $\tilde{V}\times I\to\tilde{V}$ be the lifting of $H$ which satisfies $\tilde{H}(\tilde{x}_0, 0)=\tilde{x}_0$. In the same way as above it then follows that $\tilde{H}$ is a $K^*$-homotopy rel $\tilde{X}$ from $\mathrm{id}_{\tilde{V}}$ to $\tilde{i}\circ\tilde{r}$, where $\tilde{i}$: $\tilde{X}\to\tilde{V}$ denotes the inclusion. □

**Appendix to section 5.**

The purpose of this appendix is to prove the following fact, which was used in the proof of Proposition 5.1.

**Lemma 5.3.** $\rho^*$: $K^*\to J^*$ *is continuous.*

Recall that the situation here is the following. We are given a $K$-pair $(V, X)$, where $X$ is a strong $K$-deformation retract of $V$ and both $V$ and $X$ are connected, locally path-connected and semilocally 1-connected spaces. The action of $K$ on $V$ is effective but the restricted action of $K$ on $X$ need not be effective and $J=K/N$ denotes the corresponding group acting effectively on $X$. The group $K^*$ is the extension of $K$ that acts (effectively) on $\tilde{V}$, and $J^*$ is the extension of $J$ that acts (effectively) on $\tilde{X}$. By restricting the action of $K^*$ on $\tilde{V}$ to $\tilde{X}$ we obtain an induced homomorphism $\rho^*$: $K^*\to J^*$.

In order to show that $\rho^*$: $K^*\to J^*$ is continuous we need to recall how the groups $K^*$ and $J^*$ are topologized, see [9] p. 65. Let $K_0\subset K$ be the identity component of $K$. Let $\tilde{\pi}_0$: $\tilde{K}_0\to K_0$ be the universal covering of $K_0$. The action of $K_0$ on $V$ lifts to an action of $\tilde{K}_0$ on $\tilde{V}$. Let $Q\subset\tilde{K}_0$ be the kernel of this action and define $\hat{K}_0=\tilde{K}_0/Q$. Then $\hat{K}_0$ acts effectively on $\tilde{V}$ and $\tilde{\pi}_0$ induces a homomorphism $\hat{\pi}_0$: $\hat{K}_0\to K_0$ such that the action of $\hat{K}_0$ on $\tilde{V}$ covers the action of $K_0$ on $V$ via $\hat{\pi}_0$. Now we have $\hat{K}_0\subset K^*$ and $K^*$ is topologized by making $\hat{K}_0$ into the identity component of $K^*$. The corresponding construction for $J$ topologizes $J^*$.

Now the continuous homomorphism $\rho: K \to K/N = J$ induces $\rho_0: K_0 \to J_0$ and $\rho_0$ lifts to a continuous homomorphism $\tilde{\rho}_0: \tilde{K}_0 \to \tilde{J}_0$ that covers $\rho_0$. Let $r: V \to X$ be a $K$-retraction. Then $r$ can also be considered as a $\rho$-map when we consider $X$ to be a $J$-space. Considering $V$ as a $K_0$-space and $X$ as a $J_0$ space we then have that $r: V \to X$ is a $\rho_0$-map. Let $\tilde{r}: \tilde{V} \to \tilde{X}$ be the retraction of $\tilde{V}$ onto $\tilde{X}$ that covers $r$. We claim that $\tilde{r}$ is a $\tilde{\rho}_0$-map. This is seen as follows. The diagram

$$\begin{array}{ccc} \tilde{K}_0 \times \tilde{V} & \xrightarrow{\tilde{\theta}} & \tilde{V} \\ {\scriptstyle \tilde{\rho}_0 \times \tilde{r}} \downarrow & & \downarrow {\scriptstyle \tilde{r}} \\ \tilde{J}_0 \times \tilde{X} & \xrightarrow{\tilde{\theta}'} & \tilde{X} \end{array}$$

covers the commutative diagram

$$\begin{array}{ccc} K_0 \times V & \xrightarrow{\theta} & V \\ {\scriptstyle \rho_0 \times r} \downarrow & & \downarrow {\scriptstyle r} \\ J_0 \times X & \xrightarrow{\theta'} & X \end{array}$$

i.e., the maps $\tilde{r} \circ \tilde{\theta}$ and $\tilde{\theta}' \circ (\tilde{\rho}_0 \times \tilde{r})$ cover the same map. Since $\tilde{r} \circ \tilde{\theta}(\tilde{e}_0, \tilde{v}_1) = \tilde{r}(\tilde{v}_1)$ and $\tilde{\theta}' \circ (\tilde{\rho}_0 \times \tilde{r})(\tilde{e}_0, \tilde{v}_1) = \tilde{r}(\tilde{v}_1)$, where $\tilde{v}_1 \in \tilde{V}_1$ it follows by the uniqueness of liftings that

$$\tilde{r} \circ \tilde{\theta} = \tilde{\theta}' \circ (\tilde{\rho}_0 \times \tilde{r}) .$$

Thus $\tilde{r}: \tilde{V} \to \tilde{X}$ is a $\tilde{\rho}_0$-map. Let $q \in Q$, where $Q \subset \tilde{K}_0$ is the kernel of the $\tilde{K}_0$-action on $\tilde{V}$. Then

$$\tilde{r}(q\tilde{v}) = \tilde{\rho}_0(q)\tilde{r}(\tilde{v}) , \qquad \text{for all} \quad \tilde{v} \in V .$$

Hence $\tilde{\rho}_0(q)$ belongs to the kernel of the $\tilde{J}_0$-action on $\tilde{X}$. Thus we have an induced continuous homomorphism $\hat{\rho}_0: \hat{K}_0 \to \hat{J}_0$ such that $\tilde{r}: \tilde{V} \to \tilde{X}$ is a $\hat{\rho}_0$-map. We have $\hat{J}_0 \subset J^*$ so we may also consider $\hat{\rho}_0$ as a continuous homomorphism $\hat{\rho}_0: \hat{K}_0 \to J^*$. Now let us return to our original homomorphism $\rho^*: K^* \to J^*$. In order to prove that $\rho^*$ is continuous it is enough to show that $\rho^*| : \hat{K}_0 \to J^*$ is continuous. We do this by proving that

$$(*) \qquad \rho^*| = \hat{\rho}_0: \hat{K}_0 \to J^*.$$

We first show that $\tilde{r}: \tilde{V} \to \tilde{X}$ is a $\rho^*|$-map. The following diagram (1) covers diagram (2)

$$(1) \quad \begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{X} \\ {\scriptstyle k^*} \downarrow & & \downarrow {\scriptstyle \rho^*(k^*)} \\ \tilde{V} & \xrightarrow{\tilde{r}} & \tilde{X} \end{array} \qquad\qquad (2) \quad \begin{array}{ccc} V & \xrightarrow{r} & X \\ {\scriptstyle \pi^*(k^*)} \downarrow & & \downarrow {\scriptstyle \pi(\rho^*(k^*))} \\ V & \xrightarrow{r} & X . \end{array}$$

Since

$$\begin{array}{ccc} K^* & \xrightarrow{\rho^*} & J^* \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{\rho} & J \end{array}$$

commutes we have

$$\pi(\rho^*(k^*)) = \rho\pi^*(k^*)\,, \qquad \text{for all} \quad k^* \in K^*$$

Since $r\colon V \to X$ is a $\rho$-map we see that the diagram (2) commutes. Thus the maps $\rho^*(k^*)\circ\tilde{r}$ and $\tilde{r}\circ k^*$ cover the same map. Since moreover

$$(\rho^*(k^*)\circ\tilde{r})(\tilde{x}) = \rho^*(k^*)(\tilde{x})$$

and

$$(\tilde{r}\circ k^*)(\tilde{x}) = \tilde{r}(k^*\tilde{x}) = k^*\tilde{x} = \rho^*(k^*\tilde{x})$$

for any $\tilde{x}\in\tilde{X}$, it follows by the uniqueness of liftings that $\rho^*(k^*)\circ\tilde{r} = \tilde{r}\circ k^*$. This shows that (1) commutes i.e., $\tilde{r}$ is a $\rho^*\colon K^*\to J^*$ map, and hence of course in particular a $\rho^*|\colon \hat{K}_0 \to J^*$ map.

Now let $k^*\in\hat{K}_0\subset K^*$. For any $\tilde{v}\in\tilde{V}$ we have

$$\tilde{r}(k^*\tilde{v}) = \hat{\rho}_0(k^*)\tilde{r}(\tilde{v})$$

and

$$\tilde{r}(k^*\tilde{v}) = \rho^*(k^*)\tilde{r}(\tilde{v})\,.$$

Thus we have

$$\hat{\rho}_0(k^*)\tilde{x} = \rho^*(k^*)\tilde{x}\,, \qquad \text{for all} \quad \tilde{x}\in X^*\,.$$

Since $\hat{\rho}_0(k^*)$, $\rho^*(k^*)\in J^*$ and the action of $J^*$ on $\tilde{X}$ is effective we have $\hat{\rho}_0(k^*) = \rho^*(k^*)$. This proves that $(*)$ holds and thus $\rho^*|\colon\hat{K}_0\to J^*$ is continuous and hence $\rho^*\colon K^*\to J^*$ is continuous. □

## 6. Lifting equivariant CW complex structures to universal coverings

In this section $K$ denotes an arbitrary Lie group. We prove that if $X$ is a connected $K$-CW complex then $\tilde{X}$ is in a natural way a $K^*$-CW complex, where $\tilde{X}$ is a universal covering of $X$ and $K^*$ is as in § 5. We are *not* assuming that the action of $K$ on $X$ is effective.

In the Lemmas and Corollaries below $X$ denotes an arbitrary $K$-space, where $X$ is connected, locally path-connected and semilocally 1-connected. Furthermore $p\colon\tilde{X}\to X$ is a universal covering of $X$, and the group $K^*$ and the projection $\pi$: $K^* \to K$ are as in §5.

**Lemma 6.1.** *For any $y \in \tilde{X}$ the homomorphism $\pi$: $K^* \to K$ induces an isomorphism*

$$\pi| : K_y^* \xrightarrow{\cong} K_{p(y)} .$$

Proof. Let $y \in X$ and denote $p(y)=x$. For every $l \in K_y^*$ we have $ly=y$ and hence also $\pi(l)x=x$. Thus $\pi(K_y^*) \subset K_x$, and hence we have an induced homomorphism

$$(1) \qquad \pi| : K_y^* \to K_x .$$

We shall first show that the homomorphism (1) is surjective. Let $k \in K_x$ and choose $l \in K^*$ such that $\pi(l)=k$. Then $l$: $\tilde{X} \to \tilde{X}$ is a homeomorphism which covers $k$: $X \to X$. Since $kx=x$ it follows that $p(ly)=p(y)$. Let $\gamma \in \Delta$ be the deck transformation for which $\gamma(ly)=y$. Then $\gamma l \in K^*$ and $(\gamma l)y=y$, and hence $\gamma l \in K_y^*$. Furthermore $\pi(\gamma l)=\pi(l)=k$. This shows that (1) is surjective.

We now show that the homomorphism (1) is injective. Let $l_1$, $l_2 \in K_y^*$ such that $\pi(l_1)=\pi(l_2)=k$. Then there exists a deck transformation $\gamma \in \Delta$ such that $l_1=\gamma l_2$. Since $l_1 y=y=l_2 y$ we have $\gamma y=y$. Since $\gamma$ is a deck transformation this implies $\gamma=\text{id}$ and hence $l_1=l_2$. □

**Lemma 6.2.** *Let $C \subset X$ be a path-connected subset of $\tilde{X}$ such that $p(C) \subset F(X; K_x)$, for some $x \in p(C)$. Then we have $C \subset F(X; K_y^*)$, for any $y \in C$ with $p(y)=x$.*

Proof. Let $y \in C$ be such that $p(y)=x$ and let $z \in C$ be an arbitrary point in $C$. Choose a path $\tilde{\omega}$ from $y$ to $z$ in $C$. For any $l \in K_y^*$ we have that $l\tilde{\omega}$ is a path in $X$ starting at $y$ and covering the path $\pi(l)\omega$, where $\omega=p\circ\tilde{\omega}$. Since $\omega$ lies in $p(C) \subset F(X; K_x)$ and $\pi(l) \in K_x$ it follows that $\pi(l)\omega=\omega$. By the unique path lifting property of covering projections it follows that $l\tilde{\omega}=\tilde{\omega}$. In particular we have $l\tilde{\omega}(1)=\tilde{\omega}(1)$, i.e., $lz=z$. This shows that $C \subset F(\tilde{X}; K_y^*)$. □

**Corollary 6.3.** *Let $D$ be a path-connected space and $f$: $D \to X$ a map such that $f(D) \subset F(X; K_{f(d)})$, for some $d \in D$. Then, for any lifting $\tilde{f}$: $D \to \tilde{X}$ of $f$ we have $\tilde{f}(D) \subset F(\tilde{X}; K_{\tilde{f}(d)}^*)$.* □

**Lemma 6.4.** *Let $C$ be a path-connected subset of $\tilde{X}$ such that the isotropy subgroups of $K$ at all points of $p(C)$ are the same. Then we have $K_y^*=K_z^*$ for any $y$, $z \in C$.*

Proof. Let $y, z \in C$. The argument given in the proof of Lemma 6.2 shows that $K_y^* \subset K_z^*$. Since the situation here is completely symmetric in $y$ and $z$ we also get $K_z^* \subset K_y^*$, and hence $K_y^*=K_z^*$. □

**Corollary 6.5.** *Let $B$ be a path-connected space and let $f: B \to X$ be a map such that $f(B) \subset I(X; K_{f(b)})$, for some $b \in B$. Then, for any lifting $\tilde{f}: B \to \tilde{X}$ of $f$ we have $\tilde{f}(B) \subset I(\tilde{X}; K^*_{\tilde{f}(b)})$.*

**Theorem 6.6.** *Let $X$ be a connected $K$-CW complex and $p: \tilde{X} \to X$ a universal covering of $X$. Then $\tilde{X}$ is a $K^*$-CW complex.*

Proof. First of all observe that by Proposition 4.6 every $K$-CW complex is locally contractible and thus semilocally 1-connected and hence the existence of a universal covering $p: \tilde{X} \to X$ of $X$ is guaranteed.

Let $X^0 \subset X^1 \subset \cdots \subset X^m \subset \cdots$ be the equivariant skeletons of the $K$-CW complex $X$. We claim that $\tilde{X}$ is a $K$-CW complex with equivariant $m$-skeleton equal to

$$p^{-1}(X^m) = \tilde{X}^m, \qquad m \geq 0.$$

Let $X = {}^m X^{m-1} \cup \bigcup_{j \in J} c_j^n$, where $J$ is the set of all equivariant $m$-cells of $X$, and denote

$$p^{-1}(c_j^m) = d_j^m, \qquad j \in J.$$

By Lemma 6.7 in the appendix to this section the topology of $\tilde{X}$ is coherent with the family $\{\tilde{X}^m\}_{m \geq 0}$, and the topology of $\tilde{X}^m$ is coherent with the family $\{\tilde{X}^{m-1}, d_j^m\}_{j \in J}$.

Thus it only remains to exhibit a $K^*$-equivariant characteristic map for each $d_j^m$. Let $c = c_j^m$ be an equivariant $m$-cell of $X$, and let

$$f: (D^m \times K/H, S^{m-1} \times K/H) \to (c, \dot{c})$$

be a characteristic $K$-map for $c$. Here $\dot{c} = c \cap X^{m-1}$. We shall construct a characteristic $K^*$-map for $d = p^{-1}(c) \subset \tilde{X}$. Let

$$f_e: D^m \to X$$

be defined by $f_e(z) = f(z, eH)$. Denote $f_e(0) = x_0$. Then we have $K_{x_0} = H$ and $f_e(D^m) \subset X^H$. Let

$$\tilde{f}_e: D^m \to \tilde{X}$$

be a lifting of $f_e$. We denote $\tilde{f}_e(0) = y_0$ and $K^*_{y_0} = L$. By Corollary 6.3. We have $\tilde{f}_e(D^m) \subset \tilde{X}^L$. Thus we get a well defined $K^*$-map.

$$\tilde{f}: D^m \times K^*/L \to \tilde{X}$$

by defining $\tilde{f}(z, k^*L) = k^* \tilde{f}_e(z)$.

Moreover the diagram

$$\begin{array}{ccc} D^m \times K^*/L & \xrightarrow{\tilde{f}} & d \\ {\scriptstyle \mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle p} \\ D^m \times K/H & \xrightarrow{f} & c \end{array}$$

commutes. It is easy to see that $\tilde{f}(D^m \times K^*/L)=d$ and that the above diagram in fact is a pull-back diagram. Now Lemma 6.8 in the appendix to this section implies that $\tilde{f}$: $D^m \times K^*/L \to d$ is a quotient map.

Furthermore we have $\tilde{f}(S^{m-1}\times K^*/L)\subset \dot{d}=p^{-1}(\dot{c})$ and $\tilde{f}(\mathring{D}^m \times K^*/L)=\mathring{d}$, where $\mathring{d}=d-\dot{d}$. We claim that

$$(1) \qquad \tilde{f}\colon \mathring{D}^m \times K^*/L \to \mathring{d}$$

is a $K^*$-homeomorphism. We already know that this map is a surjective $K^*$-map. To see that $\tilde{f}$ in (1) is injective one uses the fact that $f$: $\mathring{D}^m \times K/H \to \mathring{c}$ is injective and that by Corollary 6.5 we have $\tilde{f}_e(\mathring{D}^m)\subset \tilde{X}_L$. By Lemma 6.9 in the appendix tc this section $\tilde{f}$: $\mathring{D}^m \times K^*/L \to \mathring{d}$ is an open map. Thus $\tilde{f}$ in (1) is a $K^*$-homeomorphism. We have shown that

$$\tilde{f}\colon (D^m \times K^*/L,\ S^{m-1}\times K^*/L) \to (d, \dot{d})$$

is a characteristic $K^*$-map for $d$, and this completes the proof. □

A somewhat sketchy description of the lifting of a finite $K$-CW complex structure of $X$ to a finite $K^*$-CW complex structure on the universal covering $\tilde{X}$ of $X$ is given in Lemma 2.2 in [2]. In [2] $K$ is taken to be a compact Lie group and it is also implicitely assumed that the action of $K$ on $X$ is effective.

## Appendix to section 6

**Lemma 6.7.** *Let $p$: $Y \to X$ be a local homeomorphism and let $\{X_j\}_{j\in J}$ be a family of subsets of $X$ such that the topology of $X$ is coherent with $\{X_j\}_{j\in J}$. Then the topology of $Y$ is coherent with $\{p^{-1}(X_j)_j\}_{j\in J}$.*

Proof. We denote $Y_j=p^{-1}(X_j)$, $j\in J$. Let $V\subset Y$ such that $V\cap Y_j$ is open in $Y_j$ for all $j\in J$. We shall prove that $V$ is open in $Y$.

For each $v\in V$ let $W(v)$ be an open neighqorhood of $v$ in $Y$ such that $p(W(v))$ is open in $X$ and $p|$: $W(v)\to p(W(v))$ is a homeomorphism. Since $p$: $Y\to X$ is an open map it follows that $p|$: $Y_j \to X_j$ is open. The set $W(v)\cap V\cap Y_j$ is open in $Y_j$ and hence $p(W(v)\cap V\cap Y_j)=p(W(v)\cap V)\cap X_j$ is open in $X_j$, for every $j\in J$. It follows that $p(W(v)\cap V)$ is open in $X$. Therefore $W(v)\cap V$ is open in $W(v)$, and hence also in $Y$. Since $V=\bigcup_{v\in V}(W(v)\cap V)$ it follows that $V$ is open in $Y$. □

**Lemma 6.8.** *Let*

$$\begin{array}{ccc} B & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & X \end{array}$$

*be a commutative diagram of maps and spaces such that the natural map from B to the pull-back,*

$$(*) \qquad \varphi\colon B \to P = \{(a, y) \in A \times Y \mid f(a) = p(y)\}$$

*defined by* $\varphi(b)=(p'(b), \tilde{f}(b))$, *is surjective. Assume that* $p$ *is a local homeomorphism and that* $p'$ *is an open map and that* $f$ *is a quotient map. Then* $\tilde{f}$ *is a quotient map.*

Proof. Let $V \subset Y$ such that $\tilde{f}^{-1}(V)$ is open in $B$. We claim that $V$ is open in $Y$. For each $v \in V$ let $W(v)$ be an open neighborhood of $v$ in $Y$ such that $p(W(v))$ is open in $X$ and $p| : W \to p(W(v))$ is a homeomorphism. Then $\tilde{f}^{-1}(W(v) \cap V) = \tilde{f}^{-1}(W(v)) \cap \tilde{f}^{-1}(V)$ is open in $B$. Since the map $\varphi$ in $(*)$ is surjective it follows that

$$p'\tilde{f}^{-1}(W(v) \cap V) = f^{-1}p(W(v) \cap V)\,.$$

Thus $f^{-1}p(W(v) \cap V)$ is open in $A$ and hence $p(W(v) \cap V)$ is open in $X$. It follows that $W(v) \cap V$ is open in $W(v)$ and hence also in $Y$. Since $V = \cup(W(v) \cap V)$ it follows that $V$ is open in $Y$. □

**Lemma 6.9.** *Let*

$$\begin{array}{ccc} B & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & X \end{array}$$

*be a commutative diagram of maps and spaces, where* $p$ *is a local homeomorphism and* $p'$ *and* $f$ *are open maps. Then* $\tilde{f}$ *is an open map.*

Proof. The proof is obvious. □

## 7. Induced actions on homotopy groups of equivariant spaces and pairs

In this section $K$ denotes an arbitrary Lie group. Assume that $K$ acts on a space $X$ which is connected, locally path-connected and semilocally 1-connected. We shall then define an action of $\pi_0(K^*)$ on $\pi_n(X, x_0)$, where $x_0 \in X$ is an arbitrary point in $X$ and $n \geq 2$ and $K^*$ is as in §5. This action of $\pi_0(K^*)$ on $\pi_n(X, x_0)$ makes $\pi_n(X, x_0)$ into a module over the group ring $\boldsymbol{Z}[\pi_0(K^*)]$.

Let $(X, A)$ be a $K$-pair, where both $X$ and $A$ are connected, locally path-connected and semilocally 1-connected spaces and the inclusion $i: A \to X$ induces an isomorphism

$$i_{\sharp}: \pi_1(A, a_0) \xrightarrow{\cong} \pi_1(X, a_0), \qquad a_0 \in A,$$

between the fundmaental groups. We then also define an action action of $\pi_0(K^*)$ on $\pi_n(X, A, a_0)$, for $n \geq 2$, which makes $\pi_n(X, A, a_0)$ into a $\boldsymbol{Z}[\pi_0(K^*)]$-module. (Observe that $\pi_2(X, A, a_0)$ is abelian under the above assumptions.)

*The simply connected case*

We shall first treat the special case of a $K$-pair $(Y, B)$, where both $Y$ and $B$ are simply connected. Let $b_0, b_1 \in B$ and let $\omega: I \to B$ be a path in $B$ from $b_0$ to $b_1$. Then we have the isomorphism $h_{[\omega]}: \pi_n(Y, B, b_1) \xrightarrow{\cong} \pi_n(Y, B, b_0)$, $n \geq 2$. Since $B$ is simply connected the isomorphism $h_{[\omega]}$ is independent of the choice of the path $\omega$ from $b_0$ to $b_1$, and hence gives us a canonical isomorphism which we shall denote by

$$I(b_0, b_1): \pi_n(Y, B, b_1) \to \pi_n(Y, B, b_0), \qquad n \geq 2.$$

Similarly we have for any two points $y_0, y_1 \in Y$ a uniquely determined natural isomorphism $I(y_0, y_1): \pi_n(Y, y_1) \xrightarrow{\cong} \pi_n(Y, y_0)$, $n \geq 2$.

Each element $k \in K$ gives us a homeomorphism $k: (Y, B, b_0) \to (Y, B, kb_0)$ and hence an induced isomorphism

$$k_{\sharp}: \pi_n(Y, B, b_0) \xrightarrow{\cong} \pi_n(Y, B, kb_0).$$

We define

$$k_*: \pi_n(Y, B, b_0) \to \pi_n(Y, B, b_0).$$

to be the composite

$$k_* = I(b_0, kb_0) \circ k_{\sharp}.$$

For the identity element $e \in K$ we have $e_* = \mathrm{id}$ and for any $k, k' \in K$ it is easily seen that $k'_* \circ k_* = (k'k)_*$. Thus we have defined an action of the (discrete) group $K$ on $\pi_n(Y, B, b_0)$, $n \geq 2$. We shall often use the notation $kz$ instead of $k_*(z)$, i.e., we set $kz = k_*(z)$ for any $k \in K$ and $z \in \pi_n(Y, B, b_0)$.

For later use we note here that since the diagram

$$\begin{array}{ccc} \pi_n(Y, B, b_0) & \xrightarrow{k_\sharp} & \pi_n(Y, B, kb_0) \\ {\scriptstyle I(k^{-1}b_0,\, b_0)}\downarrow & & \downarrow{\scriptstyle I(b_0,\, kb_0)} \\ \pi_n(Y, B, k^{-1}b_0) & \xrightarrow{k_\sharp} & \pi_n(Y, B, b_0) \end{array}$$

commutes we also have $k_*=k_\sharp\circ I(k^{-1}b_0, b_0)$.

The same procedure as above gives an action of $K$ on the absolute groups $\pi_n(Y, y_0)$ and $\pi_n(B, b_0)$, $n\geq 2$.

Now assume that $k$ and $k'$ are two elements of $K$ that belong to the same component of $K$. We claim that

$$(1) \qquad k_* = k'_*\colon \pi_n(Y, B, b_0) \to \pi_n(Y, B, b_0)\,.$$

This is seen as follows. Let $\tau\colon I\to K$ be a path in $K$ from $k$ to $k'$. Then $\omega\colon I\to B$, defined by $\omega(t)=\tau(t)b_0$, is a path in $B$ from $kb_0$ to $k'b_0$. The homotopy $F\colon I\times(Y, B)\to(Y, B)$ given by $F(t, y)=\tau(t)y$ is a $\omega$-homotopy from the map $k\colon (Y, B, b_0)\to(Y, B, kb_0)$ to the map $k'\colon (Y, B, b_0)\to(Y, B, k'b_0)$. Thus the diagram

$$\begin{array}{ccc} \pi_n(Y, B, b_0) & \xrightarrow{k'_\sharp} & \pi_n(Y, B, k'b_0) \\ & {\scriptstyle k_\sharp}\searrow & \downarrow{\scriptstyle h_{[\omega]}=I(kb_0,\, k'b_0)} \\ & & \pi_n(Y, B, kb_0) \end{array}$$

commutes. Since $I(b_0, k'b_0)=I(b_0, kb_0)\circ I(kb_0, k'b_0)$ it follows that $k'_*=k_*$ i.e., we have showed that (1) is valid.

Thus every element $k\in K_0$, where $K_0$ denotes the identity component of $K$, acts trivially on $\pi_n(Y, B, b_0)$. Hence the action of $K$ on $\pi_n(Y, B, b_0)$ induces an action

$$\theta\colon \pi_0(K)\times\pi_n(Y, B, b_0)\to\pi_n(Y, B, b_0)$$

of $\pi_0(K)=K/K_0$, the group of components of $K$, on $\pi_n(Y, B, b_0)$ given by $\theta(kK_0, z)=kz$. In the same way it is seen that the action of $K$ on $\pi_n(Y, y_0)$, where $y_0\in Y$ and $n\geq 2$, induces an action of $\pi_0(K)$ on $\pi_n(Y, y_0)$.

The exact homotopy sequence

$$\to\pi_n(B;\, b_0)\xrightarrow{i_\sharp}\pi_n(Y;\, b_0)\xrightarrow{j_\sharp}\pi_n(Y, B;\, b_0)\xrightarrow{\partial}\pi_{n-1}(B;\, b)\to\cdots\to\pi_2(Y, B;\, b_0)$$

is a sequence of $\boldsymbol{Z}[\pi_0(K)]$-modules and homomorphisms of $\boldsymbol{Z}[\pi_0(K)]$-modules.

Also observe that the Hurewicz homomorphisms

$$\phi\colon \pi_n(Y, y_0)\to H_n(Y)\,, \qquad n\geq 2\,,$$

and

$$\phi\colon \pi_n(Y, B, b_0) \to H_n(Y, B), \qquad n \geq 2,$$

are homomorphism between $\boldsymbol{Z}[\pi_0(K)]$-modules.

*The general case*

Let us now consider the case of a $K$-pair $(X, A)$, where $X$ and $A$ are connected, locally-connected and semilocally 1-connected, and we assume that

$$(*) \qquad i_\sharp\colon \pi_1(A, a_0) \xrightarrow{\cong} \pi_1(X, a_0), \qquad a_0 \in A,$$

is an isomorphism. Let $p\colon \tilde{X} \to X$ be a universal covering of $X$. Since $(*)$ holds we have that $p|\colon \tilde{A} = p^{-1}(A) \to A$ is a universal covering of $A$.

Let $K^*$ be the extension of $K$ by $\Delta_{\tilde{X}}$ acting on the pair $(\tilde{X}, \tilde{A})$ as in § 5. For any $a_0 \in A$ and $n \geq 2$ we shall define an action of $\pi_0(K^*)$ on $\pi_n(X, A, a_0)$. Choose $b_0 \in \tilde{A}$ such that $p(b_0) = a_0$. As we saw above the group $K^*$ acts on $\pi_n(\tilde{X}, \tilde{A}, b_0)$, and in fact this action induces an action of $\pi_0(K^*)$ on $\pi_n(\tilde{X}, \tilde{A}, b_0)$. Since we have the isomorphism

$$(3) \qquad p_\sharp\colon \pi_n(\tilde{X}, \tilde{A}, b_0) \xrightarrow{\cong} \pi_n(X, A, a_0), \qquad n \geq 2.$$

we also obtain an action of $K^*$ on $\pi_n(X, A, a_0)$ and a corresponding induced action of $\pi_0(K^*)$ on $\pi_n(X, A, a_0)$. That is, for any $l \in K^*$ and $z \in \pi_n(X, A, a_0)$ we have

$$(4) \qquad lz = p_\sharp(l p_\sharp^{-1}(z)),$$

where $p_\sharp$ is as in (3), and similarly with $l$ replaced by an element $\bar{l} \in \pi_0(K^*)$.

Observe that the above definition of an action of $\pi_0(K^*)$ on $\pi_n(X, A, a_0)$ involved a choice of a point $b_0 \in \tilde{A}$ such that $p(b_0) = a_0$. In order to be able to be very specific let us denote the action obtained above, by choosing the point $b_0 \in \tilde{A}$, by

$$\theta[b_0]\colon \pi_0(K^*) \times \pi_n(X, A, a_0) \to \pi_n(X, A, a_0).$$

For any $l \in K^*$ and $z \in \pi_n(X, A, a_0)$ we then rewrite (4) in the more precise form

$$(4') \qquad (l[b_0])z = p_\sharp(l p_\sharp^{-1}(z)).$$

and similarly with $l$ replaced by an element $\bar{l} \in \pi_0(K^*)$.

Let $b_0' \in \tilde{A}$ be another point such that $p(b_0') = a_0$. We claim that the action $\theta[b_0]$ is weakly equivalent to $\theta[b_0']$ by an inner automorphism of $\pi_0(K^*)$. This is seen as follows.

Let $T\colon \tilde{X} \to \tilde{X}$ be the covering transformation for which $T(b_0') = b_0$. We have $T \in K^*$. The commutative diagram

$$
\begin{array}{ccccccccc}
 & & & \overset{l'_{\#}}{\longrightarrow} & \pi_n(\tilde{X}, \tilde{A}; lb'_0) & \overset{I(b'_0,\, lb_0)}{\longrightarrow} & & & \\
\pi_n(\tilde{X}, \tilde{A}; b'_0) & \xleftarrow{I(b'_0,\, b_0)} & \pi_n(\tilde{X}, \tilde{A}; b_0) & \xrightarrow{l_{\#}} & \pi_n(\tilde{X}, \tilde{A}; lb_0) & \xrightarrow{I(b_0,\, lb_0)} & \pi_n(\tilde{X}, \tilde{A}; b_0) & \xrightarrow{I(b'_0,\, b_0)} & \pi_n(\tilde{X}, \tilde{A}; b'_0) \\
p_{\#}\downarrow\cong & & \cong\downarrow p_{\#} & & & & p_{\#}\downarrow\cong & & \cong\downarrow p_{\#} \\
\pi_n(X, A; a_0) & \xleftarrow{T[b_0]} & \pi_n(X, A; a_0) & & \xrightarrow{\quad l[b_0] \quad} & & \pi_n(X, A; a_0) & \xrightarrow{T[b_0]} & \pi_n(X, A; a_0)
\end{array}
$$

shows that

$$l[b'_0] = T[b_0]\circ l[b_0]\circ T[b_0]^{-1} = (TlT^{-1})[b_0]$$

Here the left- and righthand squares commute since the following diagram commutes

$$
\begin{array}{ccccc}
\pi_n(\tilde{X}, \tilde{A}; b_0) & \xrightarrow{T_{\#}} & \pi_n(\tilde{X}, \tilde{A}; b'_0) & \xrightarrow{I(b'_0,\, b_0)} & \pi_n(\tilde{X}, \tilde{A}; b_0) \\
p_{\#}\downarrow\cong & & \cong\downarrow p_{\#} & & \cong\downarrow p_{\#} \\
\pi_n(X, A; a_0) & \xrightarrow{\text{id}} & \pi_n(X, A; a_0) & \xrightarrow{T[b_0]} & \pi_n(X, A; a_0)\,.
\end{array}
$$

Thus we have

$$\bar{l}[b'_0] = (\bar{T}\bar{l}\bar{T}^{-1})[b_0]$$

for every $\bar{l}\in\pi_0(K^*)$, where $\bar{T}\in\pi_0(K^*)$ is the component of $T\in\Delta_{\tilde{X}}\subset K^*$, $T(b'_0)=b_0$.

The choice of $b_0\in\tilde{X}$ such that $p(b_0)=a_0$ will not concern us anymore so we will from now on drop the $b_0$ from the notation.

The exact homotopy sequence

$$\rightarrow \pi_n(A; a_0) \xrightarrow{i_{\#}} \pi_n(X; a_0) \xrightarrow{j_{\#}} \pi_n(X, A; a_0) \xrightarrow{\partial} \pi_{n-1}(A; a_0) \rightarrow \cdots \rightarrow \pi_2(X, A; a_0)$$

is a sequence of $\boldsymbol{Z}[\pi_0(K^*)]$-modules and homomorphisms of $\boldsymbol{Z}[\pi_0(K^*)]$-modules.

Consider the commutative diagram

$$
\begin{array}{ccc}
\pi_n(\tilde{X}, \tilde{A}; b_0) & \xrightarrow{\phi'} & H_n(\tilde{X}, \tilde{A}) \\
p_{\#}\downarrow\cong & & \downarrow p_* \\
\pi_n(X, A; a_0) & \xrightarrow{\phi} & H_n(X, A)
\end{array}
$$

where the maps $\phi$ and $\phi'$ are Hurewicz homomorphisms. We already know that $\phi$ is a homomorphism of $\boldsymbol{Z}[\pi_0(K^*)]$-modules. The isomorphism $p_{\#}\colon \pi_n(\tilde{X}, \tilde{A}; b_0)\xrightarrow{\cong}\pi_n(X, A; a_0)$ is a $\boldsymbol{Z}[\pi_0(K^*)]$-isomorphism, and $p_*\colon H_n(\tilde{X}, \tilde{A})\rightarrow H_n(X, A)$ is a $\pi_0(\pi)$-homomorphism, where $\pi_0(\pi)\colon \pi_0(K^*)\rightarrow\pi_0(K)$ is the homomorphism induced by the natural projection $\pi\colon K^*\rightarrow K$. It follows that

$\phi: \pi_n(X, A; a_0) \to H_n(X, A)$ is a $\pi_0(\pi)$-homomorphism.

We conclude this section by pointing out the following fact, which we shall use later on in the paper in the absolute case. Let

$$f: (I^n, I^{n-1}, J^{n-1}) \to (X, A, a_0)$$

be a map representing the element $[f] \in \pi_n(X, A; a_0)$. Let $l \in K^*$ and let

$$h: (I^n, I^{n-1}, J^{n-1}) \to (X, A, a_0)$$

be such that $l[f] = [h] \in \pi_n(X, A; a_0)$. Let $k = \pi(l)$, where $\pi: K^* \to K$. Then the maps

$$kf, h: (I^n, I^{n-1}) \to (X, A)$$

are homotopic, where $(kf)(z) = kf(z)$, for all $z \in I^n$. This is seen directly from the way the action $K^*$ on $\pi_n(X, A; a_0)$ is defined.

PART II

## 8. The component structure of a $G$-CW complex

Let $G$ be a compact Lie group and $X$ a finite $G$-CW complex. For any closed subgroup $H$ of $G$ the fixed point set $X^H$ is an $NH$-space and since $H$ acts trivially on $X^H$ we may as well consider $X^H$ as a $WH$-space, where $WH = NH/H$ is the Weyl group of $H$. Observe that the action of $WH$ on $X^H$ need not be effective. By Theorem 4.4 we know that $X^H$ is a finite $WH$-CW complex.

Now let $H'$ be another closed subgroup of $G$ which is conjugate to $H$ in $G$. We denote

$$N(H', H) = \{g \in G \mid gHg^{-1} = H'\} \tag{1}$$

Even in the case when $H'$ is not conjugate to $H$ in $G$ we may define $N(H', H)$ by (1), and we then have that $N(H', H) = \phi$ as soon as $H$ and $H'$ are two closed subgroups of $G$ that are not conjugate to each other in $G$. It is easily seen that we have

$$N(H, H) = NH$$
$$N(H', H)^{-1} = N(H, H')$$

and

$$N(H'', H') \circ N(H', H) = N(H'', H)$$

for any three closed subgroup $H$, $H'$ and $H''$ of $G$.

For any $g \in N(H', H)$ we get a homeomorphism

$$g: X^H \to X^{H'} \tag{2}$$

Moreover we have that $X^H$ is a $WH$-space and $X^{H'}$ is a $WH'$-space and the homeomorphism (2) is a $\gamma(g)$-map. Here

$$\gamma(g): WH \to WH' \tag{3}$$

is the isomorphism given by

$$\gamma(g)(nH) = (gng^{-1})H', \qquad \text{for any} \quad nH \in WH.$$

For another choice of an element $g_1 \in N(H', H)$ we get another homeomorphism

$$g_1: X^H \to X^{H'} \tag{4}$$

and this homeomorphism is a $\gamma(g_1)$-map, where

$$\gamma(g_1): WH \to WH'$$

is given by $\gamma(g_1)(nH) = (g_1 n g_1^{-1})H'$, for all $nH \in WH$.

What should be observed here is that the composite $g_1^{-1} \circ g: X^H \to X^H$ is a $\gamma(g_1^{-1}g): WH \to WH$ map, where $g_1^{-1}g \in NH$ and we have

$$\gamma(g_1^{-1}g)(nH) = g_1^{-1}gn(g_1^{-1}g)^{-1}H = (g_1^{-1}gH)(nH)((g_1^{-1}g)^{-1}H),$$

i.e., $\gamma(g_1^{-1}g): WH \to WH$ is an inner automorphism of $WH$.

Furthermore observe that for any $g \in N(H', H)$ the homeomorphism $g: X^H \to X^{H'}$ is a $\gamma(g)$-isomorphism from the $WH$-CW complex $X^H$ to the $WH'$-CW complex $X^{H'}$.

Now let $X^H_\alpha$ be a connected component of $X^H$. Recall that we denote

$$(NH)_\alpha = \{n \in NH \mid nX^H_\alpha = X^H_\alpha\}$$

and

$$(WH)_\alpha = \{w \in WH \mid wX^H_\alpha = X^H_\alpha\}.$$

Both $(NH)_\alpha$ and $(WH)_\alpha$ are compact Lie groups and moreover we have $(WH)_\alpha = (NH)_\alpha/H$. The group $(WH)_\alpha$ acts on $X^H_\alpha$ and by Corollary 4.5 we know that $X^H_\alpha$ is a finite $(WH)_\alpha$-CW complex. The set $(WH)X^H_\alpha = (NH)X^H_\alpha$ is called a $WH$-component (or $NH$-component) of $X^H$.

Let $X^H_\alpha$ be a component of $X^H$ and let $X^H_\beta$ be a component of $X^{H'}$, where $H$ and $H'$ are two arbitrary closed subgroups of $G$. We then define

$$N(X^{H'}_\beta, X^H_\alpha) = \{n \in N(H', H) \mid nX^H_\alpha = X^{H'}_\beta\}$$

Thus in case $H$ and $H'$ are not conjugate in $G$ we have $N(X^{H'}_\beta, X^H_\alpha) = \phi$. Of course we may have $N(X^{H'}_\beta, X^H_\alpha) = \phi$ even when $H$ and $H'$ are conjugate in $G$.

In the case when $H=H'$ we have that $N(X^H_\beta, X^H_\alpha)\neq\phi$ if and only if $X^H_\alpha$ and $X^H_\beta$ belong to the same $NH$-component of $X^H$.

It is easy to verify that we have

$$N(X^H_\alpha, X^H_\alpha) = (NH)_\alpha$$
$$N(X^{H'}_\beta, X^H_\alpha)^{-1} = N(X^H_\alpha, X^{H'}_\beta)$$

and

$$N(X^{H''}_\gamma, X^{H'}_\beta)\circ N(X^{H'}_\beta, X^H_\alpha) = N(X^{H''}_\gamma, X^H_\alpha)$$

for any three components $X^H_\alpha$, $X^{H'}_\beta$ and $X^{H''}_\gamma$ of $X^H$, $X^{H'}$ and $X^{H''}$, respectively.

For any $n\in N(X^{H'}_\beta, X^H_\alpha)$ we have the homeomorphism

$$n\colon X^H_\alpha \to X^{H'}_\beta\,. \tag{5}$$

We have that $X^H_\alpha$ is a $(WH)_\alpha$-space and $X^{H'}_\beta$ is a $(WH')_\beta$-space and the map (5) is a $\gamma(n)$-map, where $\gamma(n)$ is the isomorphism

$$\gamma(n)\colon (WH)_\alpha \to (WH')_\beta\,.$$

defined by

$$\gamma(n)(n_\alpha H) = (nn_\alpha n^{-1})H\,, \qquad \text{for all} \quad n_\alpha H\in(WH)_\alpha\,.$$

For another choice of an element $n_1\in N(X^{H'}_\beta, X^H_\alpha)$ we get another homeomorphism

$$n_1\colon X^H_\alpha \to X^{H'}_\beta \tag{6}$$

and this map is a $\gamma(n_1)\colon (WH)_\alpha\to(WH')_\beta$ map, where $\gamma(n_1)(n_\alpha H)=(n_1n_\alpha n_1^{-1})H'$, for every $n_\alpha H\in(WH)_\alpha$.

The composite $n_1^{-1}\circ n\colon X^H_\alpha\to X^H_\alpha$ is a $\gamma(n_1^{-1}n)$-map, where $n_1^{-1}n\in N(X^{H'}_\beta, X^H_\alpha)^{-1}\circ N(X^{H'}_\beta, X^H_\alpha)_\alpha=(NH)_\alpha$ and

$$\gamma(n_1^{-1}n)(n_\alpha H) = ((n_1^{-1}n)n_\alpha(n_1^{-1}n))^{-1}H = [(n_1^{-1}n)H][n_\alpha H][(n_n^{-1}n)H]^{-1}$$

i.e., $\gamma(n_1^{-1}n)\colon (WH)_\alpha\to(WH)_\alpha$ is an inner automorphism.

Moreover observe that for any $n\in N(X^{H'}_\beta, X^H_\alpha)$ the map $n\colon X^H_\alpha\to X^{H'}_\beta$ is a $\gamma(n)\colon (WH)_\alpha\to(WH')_\beta$ isomorphism from the $(WH)_\alpha$-CW complex $X^H_\alpha$ to the $(WH')_\beta$-CW complex $X^{H'}_\beta$.

We define a relation $\sim$ in the set of all components $X^H_\alpha$ of all possible fixed point set $X^H$, all closed subgroups of $G$, by defining

$$X^H_\alpha\sim X^{H'}_\beta \Leftrightarrow N(X^{H'}_\beta, X^H_\alpha)\neq\phi\,.$$

We denote the set of equivalence class of this relation by $\mathcal{C}(X)$.

**Lemma 8.1.** *Let $(V, X)$ be a finite G-CW pair such that $i: X\to V$ is a G-homotopy equivalence. Then we have*
*a) The inclusion $i$ induces a one-to-one correspondence between the components of $X^H$ and $V^H$, for any closed subgroup $H$ of $G$.*
*b) The Weyl group of the component $V^H_\alpha$ corresponding to the component $X^H_\alpha$ equals the Weyl group $(WH)_\alpha$ of $X^H_\alpha$.*
*c) The induced inclusion $i: X^H_\alpha\to V^H_\alpha$ is a $(WH)_\alpha$-homotopy equivalence.*

Proof. The proof is obvious. □

Let $(V, X)$ be as above and denote $V^{>H}_\alpha = V^{>H}\cap V^H_\alpha$. We shall show in Corollary 8.5 below that the inlcusion

$$j_\alpha: X^H_\alpha \cup V^{>H}_\alpha \to V^H_\alpha$$

is a $(WH)_\alpha$-homotopy equivalence. It then follows, see Corollary 8.6, that the inclusion

$$\tilde{j}_\alpha: \widetilde{X^H_\alpha \cup V^{>H}_\alpha} \to \tilde{V}^H_\alpha$$

is a $(WH)^*_\alpha$-homotopy equivalence. Here $\tilde{V}^H$ and $\widetilde{X^H_\alpha \cup V^{>H}_\alpha}$ denote universal coverings of $V^H$ and $X^H_\alpha \cup V^{>H}_\alpha$, respectively, and $(WH)^*_\alpha$ is the extension of $(WH)_\alpha$ which acts on $\tilde{V}^H_\alpha$ and $\widetilde{X^H_\alpha \cup V^{>H}_\alpha}$, as in §5.

The main lemma for the proof of these results is Lemma 8.2 below, which is identical with Lemma III.1.1 in [5], except that in [5] we only stated it for $G$ a finite group. We also used Lemma III.1.1 in [5] in exactly the same way as we will use Lemma 8.2 in this paper.

**Lemma 8.2.** *Let $(Y, B)$ be a G-CW pair and let $\{Y_1, \cdots, Y_m\}$ be a finite collection closed under intersection of equivariant subcomplexes of $Y$. If the inclusions $i: B\to Y$ and $i_k: B\cap Y_k\to Y_k$, $k=1, \cdots, m$, are G-homotopy equivalences then so is the inclusion $j: B\cup(\bigcup_{k=1}^m Y_k)\to Y$.*

Proof. We shall prove by induction in $m$ that $\hat{i}: B\to B\cup(\bigcup_{k=1}^m Y_k)$ is a $G$-homotopy equivalence, and since $i: B\to Y$ is a $G$-homotopy equivalence the claim follows from this. Let $m=1$. Since $i_1: B\cap Y_1\to Y_1$ is a $G$-homotopy equivalence it follows that $B\cap Y_1$ is a strong $G$-deformation retract of $Y_1$ and hence $B$ is a strong $G$-deformation retract of $B\cup Y_1$. This shows that $\hat{i}: B\to B\cup Y_1$ is a $G$-homotopy equivalence.

Now let $m\geq 2$ and assume that our claim is true in the case when the finite collection has at most $m-1$ elements. Let $\{Y_1, \cdots, Y_m\}$ be a collection of $m$ equivariant subcomplexes of $Y$ which is closed under intersection, and assume that each inclusion $i_k: B\cap Y_k\to Y_k$, $k=1, \cdots, m$, is a $G$-homotopy equivalence. Furthermore we may assume that the enumeration $Y_1, \cdots, Y_m$ is chosen such that

$Y_j \subset Y_i \Rightarrow j \leq i$. First of all we claim that the collection $\{Y_1, \cdots, Y_{m-1}\}$ is closed under intersection. If

$$Y_k \cap Y_l = Y_m, \qquad \text{where} \quad 1 \leq k \leq m-1 \quad \text{and} \quad 1 \leq l \leq m-1,$$

we would have $Y_m \subset Y_k$ and $Y_m \subset Y_l$ and hence $m \leq k$ and $m \leq l$, both of which are contradictions. Since $\{Y_1, \cdots, Y_m\}$ is closed under intersection this shows that $\{Y_1, \cdots, Y_{m-1}\}$ is closed under intersection. Thus by the inductive assumption we have that the inclusion

$$\text{(1)} \qquad \hat{i}_1 \colon B \to B \cup (\bigcup_{k=1}^{m-1} Y_k) = B \cup Y^*_{m-1}$$

is a $G$-homotopy equivalence. Here we have denoted

$$Y^*_{m-1} = \bigcup_{k=1}^{m-1} Y_k .$$

Since $\{Y_1, \cdots, Y_{m-1}\}$ is closed under intersection it follows that also the family $\{Y_1 \cap Y_m, \cdots, Y_{m-1} \cap Y_m\}$ is closed under intersection. Moreover we have $Y_k \cap Y_m = Y_{m(k)}$, where $1 \leq m(k) \leq m-1$, for $k=1, \cdots, m-1$, and hence the inclusions

$$i'_k \colon B \cap (Y_k \cap Y_m) \to Y_k \cap Y_m, \quad 1 \leq k \leq m-1,$$

are $G$-homotopy equivalences. Thus by the inductive assumption the inclusion

$$\text{(2)} \qquad \hat{i}_2 \colon B \to B \cup (\bigcup_{k=1}^{m-1} (Y_i \cap Y_m)) = B \cup (Y^*_{m-1} \cap Y_m)$$

is a $G$-homotopy equivalence. Since (1) and (2) are $G$-homotopy equivalences it follows that the inclusion

$$\text{(3)} \qquad i_1 \colon B \cup (Y^*_{m-1} \cap Y_m) \to B \cup Y^*_{m-1}$$

is a $G$-homotopy equivalence. Since the inclusion $i \colon B \to B \cup Y_m$ is a $G$-homotopy equivalence (by the case $m=1$ established at the very beginning of the proof) and (2) is a $G$-homotopy equivalence it follows that the inclusion

$$\text{(4)} \qquad i_2 \colon B \cup (Y^*_{m-1} \cap Y_m) \to B \cup Y_m$$

is a $G$-homotopy equivlaence. By (3) and (4) we have that $B \cup (Y^*_{m-1} \cap Y_m)$ is a strong $G$-deformation retract of both $B \cup Y^*_{m-1}$ and of $B \cup Y_m$. Since $(B \cup Y^*_{m-1}) \cap (B \cap Y_m) = B \cup (Y^*_{m-1} \cap Y_m)$ it now follows that $B \cup (Y^*_{m-1} \cap Y_m)$ is a strong $G$-deformation retract of $(B \cup Y^*_{m-1}) \cup (B \cup Y_m) = B \cup (\bigcup_{k=1}^{m} Y_k)$. This fact combined with the fact that (2) is a $G$-homotopy equivalence gives us the result that $\hat{i} \colon B \to B \cup (\bigcup_{k=1}^{m} Y_k)$ is a $G$-homotopy equivalence. □

Let $(V, X)$ be a finite $G$-CW pair such that the inclusion

$$i: X \to V \tag{1}$$

is a $G$-homotopy equivalence. Let

$$(H_1), \cdots, (H_s) \tag{2}$$

be all the $G$-isotropy types occurring in $V$, ordered in such a way that

$$(H_i) > (H_j) \Rightarrow i < j\,. \tag{3}$$

**Lemma 8.3.** *Let $(V, X)$ be as above and let $(H_{i_1}), \cdots, (H_{i_r})$ be $r$ distinct $G$-isotropy types occurring in $V$. Then the inclusion*

$$i| : \bigcup_{j=1}^{r} X^{(H_{i_j})} \to \bigcup_{j=1}^{r} V^{(H_{i_j})}$$

*is a $G$-homotopy equivalence.*

Proof. Assume inductively that we have proved our claim in case $i_j \leq s_0 - 1$, for all $j=1, \cdots, r$, where $2 \leq s_0 \leq s$. In case $i_j = 1$ the inclusion $i| : X^{(H_1)} \to V^{(H_1)}$ is a $G$-homotopy equivalence, so this fact starts the induction.

Now let $(H_{i_1}), \cdots, (H_{i_r})$ be $r$ distinct $G$-isotropy types such that $i_j \leq s_0$ for $j=1, \cdots, r$. We shall prove that

$$i| : \bigcup_{j=1}^{r} X^{(H_{i_j})} \to \bigcup_{j=1}^{r} V^{(H_{i_j})} \tag{4}$$

is a $G$-homotopy equivalence. This fact we shall establish by induction in $r$. If $r=1$ the inclusion (4) is a $G$-homotopy equivalence. Let $r \geq 2$ and assume that the inclusion (4) is a $G$-homotopy equivalence when the two unions are unions of at most $r-1$ sets $X^{(H_{i_j})}$. Then

(5) $\quad i: \bigcup_{j=1}^{r-1} X^{(H_{i_j})} \to \bigcup_{j=1}^{r-1} V^{(H_{i_j})}$ and (6) $\quad i: X^{(H_{i_r})} \to V^{(H_{i_r})}$

are $G$-homotopy equivalences. Moreover we claim that

$$i: \left(\bigcup_{j=1}^{r-1} X^{(H_{i_j})}\right) \cap X^{(H_{i_r})} \to \left(\bigcup_{j=1}^{r-1} V^{(H_{i_j})}\right) \cap V^{(H_{i_r})} \tag{7}$$

is a $G$-homotopy equivalence. Each intersection $V^{(H_{i_j})} \cap V^{(H_{i_r})}$, where $1 \leq j \leq r-1$, has a presentation of the form

$$V^{(H_{i_j})} \cap V^{(H_{i_r})} = \bigcup V^{(K)} \tag{8}$$

where the union is over the set of all $G$-isotropy types $(K)$ occurring in $V$ such that $(K) \geq (H_{i_j})$ and $(K) \geq (H_{i_r})$. Since we here have $(H_{i_j}) \neq (H_{i_r})$ it follows that $(K) > (H_{i_j})$ and $(K) > (H_{i_r})$. We have $i_j \leq s_0$ and $i_r \leq s_0$ and hence it now follows that each $(K)$ occurring in (8) is a $G$-isotropy type that appears in (2) among the $s_0-1$ first ones. By intersecting both sides of the formula (8) with $X$ we see that (8) also holds with $V$ replaced by $X$. It now follows by the first inductive assumption that (7) is a $G$-homotopy equivalence.

Since (5), (6) and (7) are $G$-homotopy equivalences it follows that (4) is a $G$-homotopy equivalence. This completes the second induction and hence also the first induction and the proof of the lemma. □

**Proposition 8.4.** *Let $(V, X)$ be a finite G-CW pair such that the inclusion $i: X \to V$ is a G-homotopye equivalence. Let H be an arbitrary closed subgroup of G. Then the inclusion*

$$i: X^{(H)} \cup V^{>(H)} \to V^{(H)}$$

*is a G-homotopy equivalence.*

Proof. First of all observe that if the $G$-isotropy type $(H)$ does not occur in $V^{(H)}$ we have $V^{>(H)}=V^{(H)}$ and the claim in the proposition is trivially true. Thus we may assume that the $G$-isotropy type $(H)$ occurs in $V^{(H)}$.

Let $(K_1), \cdots, (K_r)$ be all the $G$-isotropy types, except $(H)$, that occur in $V^{(H)}$. Then

$$V^{>(H)} = \bigcup_{r=1}^{r} V^{(K_i)} .$$

Let $\mathcal{F}$ be the finite family consisting of all the $G$-equivariant sub-complexes $V^{(K_i)}$, $i=1, \cdots, r$, of $V^{(H)}$ and all finite intersections of these subcomplexes. We have

$$V^{(K_{i_1})} \cap \cdots \cap V^{(K_{i_t})} = \bigcup V^{(K)}$$

where the union is over all $G$-isotropy types occurring in $V^{(H)}$ such that $(K) \geq (K_{i_j})$ for $j=1, \cdots, t$. The same formula holds with $V$ replaced by $X$. Thus by Lemma 8.3 we have that

$$i: X^{(K_{i_1})} \cap \cdots \cap X^{(K_{i_t})} \to V^{(K_{i_1})} \cap \cdots \cap V^{(K_{i_t})}$$

is a $G$-homotopy equivalence. Thus by applying Lemma 8.2 to the finite $G$-CW pair $(V^{(H)}, X^{(H)})$ and the family $\mathcal{F}$ we have that the inclusion

$$j: X^{(H)} \cup \bigcup_{i=1}^{r} V^{(K_i)} \to V^{(H)}$$

is a $G$-homotopy equivalence, i.e., the inclusion

$$j: X^{(H)} \cup V^{>(H)} \to V^{(H)}$$

is a $G$-homotopy equivalence. □

**Corollary 8.5.** *Let $(V, X)$ be a finite G-CW pair such that the inclusion $i$: $X \to V$ is a G-homotopy equivalence. Let $H$ be an arbitrary closed subgroup of $G$, and let $V_\alpha^H$ be a component of $V^H$ then:*
(a) $j_1\colon X^H \cup V^{>H} \to V^H$ *is a WH-homotopy equivalence*
(b) $j_\alpha\colon X_\alpha^H \cup V_\alpha^{>H} \to V_\alpha^H$ *is a $(WH)_\alpha$-homotopy equivalence.*

Proof. By Proposition 8.4 $i$: $X^{(H)} \cup V^{>(H)} \to V^{(H)}$ is a $G$-homotopy equivalence so by taking the $H$-fixed point set of this inclusion we obtain (a).

Since (a) is a homotopy equivalence it induces a one-to-one correspondence between the components of $X^H \cup V^{>H}$ and $V^H$. Let $(X^H \cup V^{>H})_\alpha$ be the component of $X^H \cup V^{>H}$ that corresponds to the component $V_\alpha^H$ of $V^H$. Then in fact

$$(X^H \cup V^{>H})_\alpha = (X^H \cup V^{>H}) \cap V_\alpha^H = X_\alpha^H \cup V_\alpha^{>H}$$

where $X_\alpha^H$ denotes the component of $X^H$ that corresponds to $V_\alpha^H$ under the homotopy equivalence $i$: $X^H \to V^H$, and $V_\alpha^{>H}$ is the part of $V^{>H}$ that lies in $V_\alpha^H$.

Since (a) is a $(WH)$-homotopy equivlaence it now follows that it induces a $(WH)_\alpha$-homotopy equivalence

$$j_\alpha\colon X_\alpha^H \cup V_\alpha^{>H} \to V_\alpha^H . \quad \square$$

**Corollary 8.6.** *Let $(V, X)$ be as in Corollary* 8.5. *Then*

$$\tilde{j}_\alpha\colon \widetilde{X_\alpha^H \cup V_\alpha^{>H}} \to \tilde{V}_\alpha^H$$

*is a $(WH)_\alpha^*$-homotopy equivalence.*

Proof. This follows from Corollary 8.5 and Proposition 5.2. $\square$

## 9. Definition of torsion invariants

In this section we describe how to associate with any finite $G$-CW pair $(V, X)$, such that $i$: $X \to V$ is a $G$-homotopy equivalence, and any component $X_\alpha^H$ of a fixed point set an element in $Wh(\pi_0(WH)_\alpha^*)$. We also show that this is independent of the choice of representative $X_\alpha^H$ for an equivalence class $[X_\alpha^H]$ of components of fixed point sets.

Let $(V, X)$ be a finite $G$-CW pair such that $i$: $X \to V$ is a $G$-homotopy equivalence. Let $H$ be a closed subgroup of $G$. The inclusion $i$: $X^H \to V^H$ is a $WH$-homotopy equivalence, and it induces a one-to-one correspondence between the components of $X^H$ and $V^H$. Let $X_\alpha^H$ be a component of $X^H$ and let $V_\alpha^H$ be the correpsonding component of $V^H$. As before we let $(WH)_\alpha$ denote the Weyl group of the component $X_\alpha^H$ and we know by Lemma 8.1 that the Weyl group of $V_\alpha^H$ also equals $(WH)_\alpha$. Moreover the inclusion $i$: $X_\alpha^H \to V_\alpha^H$ is a $(WH)_\alpha$-homotopy equivalence.

By Corollary 8.5.b we have that the inclusion $j\colon X_\alpha^H \cup V_\alpha^{>H} \to V_\alpha^H$ is a $(WH)_\alpha$-homotopy equivalence, and by Corollary 8.6 $\tilde{j}\colon \widetilde{X_\alpha^H \cup V_\alpha^{>H}} \to \tilde{V}_\alpha^H$ is a $(WH)_\alpha^*$-homotopy equivalence. For simplicity let us denote $Y = V_\beta^H$ and $B = X_\alpha^H \cup V_\alpha^{>H}$. Then $(Y, B)$ is a finite $(WH)_\alpha$-CW pair such that $j\colon B \to Y$ is a $(WH)_\alpha$-homotopy equivalence, and $(\tilde{Y}, \tilde{B})$ is a finite $(WH)_\alpha^*$-CW pair such that $\tilde{j}\colon \tilde{B} \to \tilde{Y}$ is a $(WH)_\alpha^*$-homotopy equivalence. Moreover $(WH)_\alpha^*$ acts freely on $\tilde{Y} - \tilde{B}$.

Now filter the finite $(WH)_\alpha^*$-CW pair $(\tilde{Y}, \tilde{B})$ by equivariant skeletons and define

$$C(\tilde{Y}, \tilde{B})\colon \qquad \cdots \to C_n(\tilde{Y}, \tilde{B}) \xrightarrow{\partial} C_{n-1}(\tilde{Y}, \tilde{B}) \to \cdots$$

to be the chain complex with

$$C_n(\tilde{Y}, \tilde{B}) = H_n(\tilde{Y}^n \cup \tilde{B}, \tilde{Y}^{n-1} \cup \tilde{B}; \boldsymbol{Z})$$

where $H_n(\ ,\ ; \boldsymbol{Z})$ denotes ordinary singular homology with integer coefficients, and $\partial$ is the boundary homomorphism in the exact homology sequence of the corresponding triple.

Since $(WH)_\alpha^*$ acts freely on $\tilde{Y} - \tilde{B}$ it follows that we have

$$\begin{aligned} C_n(\tilde{Y}, \tilde{B}) &\cong \sum_{i \in J} \oplus H_n(D_i^n \times (WH)_\alpha^*, S_i^{n-1} \times (WH)_\alpha^*) \\ &\cong H_0((WH)_\alpha^*) \oplus \cdots \oplus H_0((WH)_\alpha^*) \\ &\cong \boldsymbol{Z}[\pi_0(WH)_\alpha^*] \oplus \cdots \oplus \boldsymbol{Z}[\pi_0(WH)_\alpha^*] \end{aligned}$$

where $J$ is the set of equivariant $n$-cells of $(\tilde{Y}, \tilde{B})$ and $(D_i^n, S_i^{n-1}) = (D^n, S^{n-1})$ for $i = 1, \cdots, |J|$, and both the last two direct sums have $r = |J|$ summands. Thus each $C_n(\tilde{Y}, \tilde{B})$ is a finitely generated free $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-module. Moreover a basis $a_1, \cdots, a_r$ for the $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-module $C_n(\tilde{Y}, \tilde{B})$ can be obtained as follows. For each equivariant $n$-cell $d_i$ of $(\tilde{Y}, \tilde{B})$ let

$$f_i\colon (D^n \times (WH)_\alpha^*, S^{n-1} \times (WH)_\alpha^*) \to (d_i, \dot{d}_i), \quad i = 1, \cdots, r,$$

be a characteristic $(WH)_\alpha^*$-map for $d_i$. Consider the composite

$$(D^n, S^{n-1}) \xrightarrow{j_e} (D^n \times (WH)_\alpha^*, S^{n-1} \times (WH)_\alpha^*) \xrightarrow{f_i} (d_i, \dot{d}_i) \hookrightarrow (\tilde{Y}^n \cup \tilde{B}, \tilde{Y}^{n-1} \cup \tilde{B})$$

where $j_e$ denotes the inclusion given by $j_e(z) = (z, e)$ for all $z \in D^n$ and $e$ is the identity element of $(WH)_\alpha^*$. Let

$$a_i = (f_i \circ j_e)_*(g_n) \in H_n(\tilde{Y}^n \cup \tilde{B}, \tilde{Y}^{n-1} \cup \tilde{B}), \quad i = 1, \cdots, r,$$

where $g_n \in H(D^n, S^{n-1}) \cong \boldsymbol{Z}$ is a generator. Then $a_1, \cdots, a_r$ is a basis for the $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-module $C_n(\tilde{Y}, \tilde{B})$, and any two bases obtained in this way differ from each other only in the order of the basis elements and by multiplication of the bases elements by $\pm$ elements from the group $\pi_0(WH)^*_\alpha$. Thus it follows than any two bases obtained in the above way generate the same family of preferred bases. Thus each $C_n(\tilde{Y}, \tilde{B})$ becomes a finitely generated free and based $\boldsymbol{Z}[\pi_0(WH)^*_\alpha]$-module in this way. By Proposition 3.1 the chain complex $C(\tilde{Y}, \tilde{B})$ is acyclic and hence we have the torsion of the chain complex $C(\tilde{Y}, \tilde{B})=C(\tilde{V}^H_\alpha, \widetilde{X^H_\alpha \cup V^{>H}_\alpha})$, which we denote by

$$\tau(C(V, X)^H_\alpha) \in Wh(\pi_0(WH)^*_\alpha)\,. \tag{1}$$

Compare with Proposition III. 1.2 in [5] and the definition following Proposition III. 1.2 in [5]

Now let $X^{H'}_\beta$ be component of a fixed point set $X^{H'}$ such that $X^{H'}_\beta$ is in relation with $X^H_\alpha$. Recall that this means that there exists $n \in N(H', H)$ such that $n\, X^H_\alpha = X^{H'}_\beta$. Moreover the map $n\colon X^H_\alpha \to X^{H'}_\beta$ is a $\gamma(n)$-isomorphism from the $(WH)_\alpha$-CW complex $X^H_\alpha$ to the $(WH')_\beta$-CW complex $X^H_\beta$. Here $\gamma(n)\colon (WH)_\alpha \to (WH')_\beta$ is the isomorphism defined by $\gamma(n)\ (n_\alpha H)=(nn_\alpha n^{-1})\, H'$.

Furthermore $\gamma(n)$ induces the canonical isomorphism

$$I(H', \beta; H, \alpha)\colon Wh(\pi_0(WH)^*_\alpha) \to Wh(\pi_0(WH')^*_\beta)$$

which is independent of the choices involved. Since

$$n\colon (V^H_\alpha, X^H_\alpha \cup V^{>H}_\alpha) \to (V^{H'}_\beta, X^{H'}_\beta \cup V^{>H'}_\beta)$$

is a $\gamma(n)\colon (WH)_\alpha \to (WH')_\beta$ isomorphism we get a $\gamma(n)^*\colon (WH)^*_\alpha \to (WH)^*_\beta$ isomorphism of based chain complexes

$$\tilde{n}_*\colon C(\tilde{V}^H_\alpha, \widetilde{X^H_\alpha \cup V^{>H}_\alpha}) \to C(\tilde{V}^{H'}_\beta, \widetilde{X^{H'}_\beta \cup V^{>H'}_\beta})$$

and from this it follows that

$$I(H', \beta; H, \alpha)\ (\tau(C(V, X)^{H'}_\alpha)) = \tau(C(V, X)^{H'}_\beta)\,,$$

i.e., the elements $\tau(C(V,X)^H_\alpha) \in Wh(\pi_0(WH)^*_\alpha)$ and $\tau(C(V,X)^{H'}_\beta) \in Wh(\pi_0(WH')^*_\beta)$ correspond to each other under the canonical isomorphism between the groups $Wh(\pi_0(WH)^*_\alpha)$ and $Wh(\pi_0(WH')^*_\beta)$. This shows that the torsion defined in (1) is independent of the choice of representing component $X^H_\alpha$ from an equivalence class of components.

## 10. Definition of the homomorphism $\Phi$

In this section we first prove that if $s(V, X)=s(W, X) \in Wh_G(X)$ then the torsion invariants, defined in §9, of $(V, X)$ and of $(W, X)$ are the same. This allows us to define the homomorphism $\Phi\colon Wh_G(X) \to \sum_{C(X)} \oplus Wh(\pi_0(WH)^*_\alpha)$. Proposition 10.1 below corresponds to Proposition III.1.3 in [5].

**Proposition 10.1** *Let $(V, X)$ and $(W, X)$ be finite G-CW pairs such that the inclusions $i: X\rightarrow V$ and $i: X\rightarrow W$ are G-homotopy equivalences. Assume that $V\ s\ W$ rel. $X$. Then we have*

$$\tau(C(V, X)^H_\alpha) = \tau(C(W, X)^H_\alpha)\in Wh(\pi_0(WH)^*_\alpha)$$

*for every closed subgroup $H$ of $G$ and any component $X^H_\alpha$ of $X^H$.*

Proof. It is enough to prove this in the case when $W$ is an equivariant elementary expansion of $V$. Thus assume that this is the case and denote $W=V\cup b^{n-1}\cup b^n$. Assume that the type of this equivariant expansion is $(K)$ and let $\sigma: I^n\times G/K\rightarrow W$ be a characteristic simple $G$-map for $(b^n, b^{n-1})$. That is $\sigma: I^n\times G/K\rightarrow W$ is a characteristic $G$-map for $\bar{b}^n$ and $\sigma|: I^{n-1}\times G/K\rightarrow W$ is a characteristic $G$-map for $\bar{b}^{n-1}$ and $\sigma(J^{n-1}\times G/K)\subset V^{n-1}$.

Since the set $\sigma(J^{n-1}\times\{eK\})$ is connected it lies in one component, say $V^K_0$, of $V^K$. Let $X^K_0$ be the corresponding component of $X^K$ and $W^K_0$ the corresponding component of $W^K$.

In case $(K)\not\geq(H)$ we have $W^H=V^H$, and the claim is obvious. Thus we need only consider the case $(K)\geq(H)$.

First assume that $(K)>(H)$. In this case we have

$$(b^{n-1})^H = (b^{n-1})^{>H} \quad\text{and}\quad (b^n)^H = (b^n)^{>H}\,.$$

Thus

$$W^H = V^H\cup(b^{n-1})^H\cup(b^n)^H$$

and

$$W^{>H} = V^{>H}\cup(b^{n-1})^H\cup(b^n)^H$$

and hence the inclusion $i: (V^H, X^H\cup V^{>H})\rightarrow(W^H, X^H\cup W^{>H})$ is an excision. Similarly, for any component $X^H_\alpha$ of $X^H$ and corresponding components $V^H_\alpha$ and $W^H_\alpha$ of $V^H$ and $W^H$, respectively, we have that the inclusion $i: (V^H_\alpha, X^H_\alpha\cup V^{>H}_\alpha)\rightarrow(W^H_\alpha, X^H_\alpha\cup W^{>H}_\alpha)$ is an excision. Hence the inclusion $\tilde{i}: (\tilde{V}^H_\alpha, \widetilde{X^H_\alpha\cup V^{>H}_\alpha})\rightarrow(\tilde{W}^H_\alpha, \widetilde{X^H_\alpha\cup W^{>H}_\alpha})$ is an excision, and therefore the induced map $\tilde{i}_*: C(\tilde{V}^H_\alpha, \widetilde{X^H_\alpha\cup V^{>H}_\alpha})\rightarrow C(\tilde{W}^H_\alpha, \widetilde{X^H_\alpha\cup W^{>H}_\alpha})$ is an isomorphism of chain complexes which in each degree is an isomorphism of based $\boldsymbol{Z}[\pi_0(WH)^*_\alpha]$-modules, and hence the desired conclusion follows.

It remains to consider the case $(K)=(H)$. We may as well assume that $K=H$. First assume that the components $X^H_0$ and $X^H_\alpha$ are not in relation, i.e.,

$$X^H_0 \not\sim X^H_\alpha$$

which means that $X_0^H$ and $X_\alpha^H$ do not belong to the same $WH$-component of $X^H$. Thus we have

$$(WH)\, W_0^H \cap (WH)\, W_\alpha^H = \phi\,.$$

Since $\mathring{b}^{n-1} \cong \mathring{I}^{n-1} \times G/H$ and $\mathring{b}^n \cong \mathring{I}^n \times G/H$ we have

$$(\mathring{b}^{n-1})^H = WH(\mathring{b}^{n-1})_0^H \quad \text{and} \quad (\mathring{b}^n)^H = WH(\mathring{b}^n)_0^H\,.$$

Hence $$(b^{n-1})_\alpha^H = \phi \quad \text{and} \quad (b^n)_\alpha^H = \phi\,,$$

and therefore

$$W_\alpha^H = V_\alpha^H\,.$$

Thus our claim is trivially true in this case.

Now assume that $H = K$ and that the components $X_0^H$ and $X_\alpha^H$ belong to the same $WH$-component of $X^H$. We may then as well assume that $X_\alpha^H = X_0^H$. Then we have $W_\alpha^H = V_\alpha^H \cup (b^{n-1})_\alpha^H \cup (b^n)_\alpha^H$. Let $\sigma\colon I^n \times G/H \to W$ be a characteristic simple $G$-map for $(b^n, b^{n-1})$. Then the restriction $\sigma|\colon I^n \times (NH)_\alpha/H \to W_\alpha^H$ is a characteristic simple $(WH)_\alpha$-map for $((b^n)_\alpha^H, (b^{n-1})_\alpha^H)$. Let $\tilde{\sigma}|\colon I^n \times (WH)_\alpha^* \to \tilde{W}_\alpha^H$ be a lifting of $\sigma|$. Then $\tilde{\sigma}|$ is a $(WH)_\alpha^*$-map. We have

$$\tilde{W}_\alpha^H = \tilde{V}_\alpha^H \cup d^{n-1} \cup d^n$$

where $d^{n-1}$ and $d^n$ are $(WH)_\alpha^*$-equivariant cells of free type and $\tilde{\sigma}|$ is a characteristic simple $(WH)_\alpha^*$-map for $(d^n, d^{n-1})$.

Denote $C' = C(\tilde{V}_\alpha^H, \widetilde{X_\alpha^H \cup V_\alpha^{>H}})$ and $C = C(\tilde{W}_\alpha^H, \widetilde{X_\alpha^H \cup W_\alpha^{>H}})$ and $C'' = C(\tilde{W}_\alpha^H, \tilde{V}_\alpha^H)$. Then we have a short exact sequence of chain complexes

$$(*) \qquad 0 \to C' \to C \to C'' \to 0\,.$$

Moreover $C''$ has the form

$$(⁑) \qquad 0 \to C_n'' \overset{\partial}{\underset{\cong}{\to}} C_{n-1}'' \to 0$$

and both $C_n''$ and $C_{n-1}''$ are based $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules of rank 1, corresponding to the free $(WH)_\alpha^*$-equivariant cells $d_n$ and $d_{n-1}$, respectively. Moreover the existence of a characteristic simple $(WH)_\alpha^*$-map $\tilde{\sigma}|$ for $(d^n, d^{n-1})$ shows that $\partial$ in (⁑) takes a preferred basis element of $C_n''$ to a preferred basis element of $C_{n-1}''$ and hence $\tau(\partial) = 0$. Since the short exact sequence of chain complexes (*), in each dimension splits as a sequence of based $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules we have

$$\tau(C)=\tau(C')+\tau(C'')=\tau(C')+(-1)^{n-1}\tau(0)=\tau(C')\,,$$

which is exactly what we wanted to prove. □

Thus we have a well-defined map

$$\Phi\colon Wh_G(X)\to\sum_{C(X)}\oplus Wh(\pi_0(WH)^*_\alpha)$$

by defining

$$\Phi(s(V,X))=\{\tau(C(V,X)^H_\alpha\}_{C(X)}\,.$$

Here the direct sum is over the set $C(X)$ of equivalence classes of non-empty components of arbitrary fixed point sets, and the above definition says that for each $[X^H_\alpha]\in C(X)$ the $[X^H_\alpha]$-factor of $\Phi(s(V,X))$ is defined to be the element $\tau(C(V,X)^H_\alpha)\in Wh(\pi_0(WH)^*_\alpha)$, where the component $X^H_\alpha$ is any representative for the equivalence class $[X^H_\alpha]\in C(X)$. Moreover for any $(V,X)$ we have $\tau(C(V,X)^H_\alpha)\neq 0$ for only a finite number of $[X^H_\alpha]\in C(X)$.

The above direct sum can also be expressed in the form

$$\sum_{C(X)}\oplus Wh(\pi_0(WH)^*_\alpha)=\sum_{\substack{(H)\\ X^H\neq\phi}}\sum_\alpha\oplus Wh(\pi_0(WH)^*_\alpha)$$

where the first direct sum on the right hand side is over each $G$-isotropy type $(H)$, i.e., conjugacy class $(H)$, such that $X^H\neq\phi$ and for a fixed $H$ the second direct sum is over the set of $WH$-components of $X^H$, i.e., over a indexing set consisting of one connected component $X^H_\alpha$ for each $WH$-component $(WH)X^H_\alpha=(NH)X^H_\alpha$ of $X^H_\alpha$.

It follows from the appropriate short exact sequence of chain complexes that

$$\tau(C(V\cup_X W,X)^H_\alpha)=\tau(C(V,X)^H_\alpha)+\tau(C(W,X)^H_\alpha)$$

for any component $X^H_\alpha$. Thus

$$\Phi(s(V\cup_X W,X))=\Phi(s(V,X))+\Phi(s(W,X))\,,$$

that is, $\Phi$ is a homomorphism.

## 11. $\Phi$ is an isomorphism

In this section we prove the result.

**Theorem 11.1.** *The homomorphism* $\Phi\colon Wh_G(X)\to\sum_{C(X)}\oplus Wh(\pi_0(WH)^*_\alpha)$ *is an isomorphism.*

Proof. Let $s(W,X)\in Wh_G(X)$ be such that $\Phi(s(W,X))=0$. By Corollary

II.4.4 in [5] (and the fact that $\Phi$ is well-defined, i.e. Proposition 10.1) we can assume that $(W, X)$ is in simplified form. Thus we have

$$W = X \cup \cup b_i^{n-1} \cup \cup b_i^n\,, \quad \text{where } n-1 \geq 2\,.$$

We denote $Y = X \cup \cup b_i^{n-1}$.

Let $H$ be a closed subgroup of $G$ such that the $G$-isotropy type $(H)$ occurs as the type of some of the equivariant cells $b_i^n$. Let $X_\alpha^H$ be a component of $X^H$ and let $Y_\alpha^H$ and $W_\alpha^H$ be the corresponding components of $Y^H$ and $W^H$, respectively. By the second part of Corollary II.4.4 in [5] the number of equivariant $n$-cells $b_i^n$ in $GW_\alpha^H$ which have type $(H)$ equals the number of equivariant $(n-1)$-cells $b_i^{n-1}$ in $GW_\alpha^H$ (and hence in $GY_\alpha^H$) which have type $(H)$. Let us denote these by $b_s^n$ and $b_s^{n-1}$, $s=1, \cdots, m$. Thus we have

$$GW_\alpha^H = G(Y_\alpha^H \cup W_\alpha^{>H}) \cup \bigcup_{s=1}^m b_s^n\,, \quad GY_\alpha^H = G(X_\alpha^H \cup Y_\alpha^{>H}) \cup \bigcup_{s=1}^m b_s^{n-1}\,.$$

Now consider the $(WH)_\alpha^*$-CW pair $(\widetilde{W}_\alpha^H, \widetilde{X_\alpha^H \cup W_\alpha^{>H}})$. We have the commutative diagram

$$\begin{array}{ccccc}
0 \to H_n(\widetilde{W}_\alpha^H, \widetilde{Y_\alpha^H \cup W_\alpha^{>H}}) & \underset{\cong}{\overset{\partial}{\to}} & H_{n-1}(\widetilde{Y_\alpha^H \cup W_\alpha^{>H}}, \widetilde{X_\alpha^H \cup W_\alpha^{>H}}) \to 0 \\
\phi \uparrow \cong & & \phi \uparrow \cong \\
\pi_n(\widetilde{W}_\alpha^H, \widetilde{Y_\alpha^H \cup W_\alpha^{>H}}) & \overset{\bar{\partial}}{\to} & \pi_{n-1}(\widetilde{Y_\alpha^H \cup W_\alpha^{>H}}, \widetilde{X_\alpha^H \cup W_\alpha^{>H}}) \\
\cong \downarrow p_\sharp & & \cong \downarrow p_\sharp \\
\pi_n(W_\alpha^H, Y_\alpha^H \cup W_\alpha^{>H}) & \overset{\bar{\partial}}{\to} & \pi_{n-1}(Y_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H \cup W_\alpha^{>H})
\end{array}$$

where $\phi$ denotes the Hurewicz homomorphism. First of all observe that the upper row equals the chain complex $C(\widetilde{W}_\alpha^H, \widetilde{X_\alpha^H \cup W_\alpha^{>H}})$. Thus we have

$$(-1)^{n-1}\, \tau(\partial) = \tau(C(W, X)_\alpha^H) = 0 \in Wh(\pi_0(WH)_\alpha^*)\,.$$

In the above diagram all four vertical maps are isomorphisms. The homology groups $H_n(\widetilde{W}_\alpha^H, \widetilde{Y_\alpha^H \cup W_\alpha^{>H}})$ and $H_{n-1}(\widetilde{Y_\alpha^H \cup W_\alpha^{>H}}, \widetilde{X_\alpha^H \cup W_\alpha^{>H}})$ are based $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules, i.e., finitely generated free $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules together with a preferred basis. The homotopy groups $\pi_n(\widetilde{W}_\alpha^H, \widetilde{Y_\alpha^H \cup W_\alpha^{>H}})$ and $\pi_n(\widetilde{Y_\alpha^H \cup W_\alpha^{>H}}, \widetilde{X_\alpha^H \cup W_\alpha^{>H}})$ are also $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules, and the Hurewicz isomorphisms in the above diagram are $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-homomorphisms. (See section 7.) Likewise the two maps $p_\sharp$, induced by the projection $p$, are isomorphisms of $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules. Thus all the homotopy groups in the above diagram are finitely generated free $\boldsymbol{Z}[\pi_0(WH)^*_\alpha]$-modules, and they have preferred bases given as follows. Let

$$f_s\colon (I^n\times G/H,\ \partial I^n\times G/H)\to(\bar{b}^n_s,\ \mathring{b}^n_s)\to(GW^H_\alpha,\ G(Y^H_\alpha\cup W^{>H}_\alpha))$$

$$h_s\colon (I^{n-1}\times G/H,\ \partial I^{n-1}\times G/H)\to(\bar{b}^{n-1}_s,\ \mathring{b}^{n-1}_s)\to(G(Y^H_\alpha\cup W^{>H}_\alpha),\ G(X^H_\alpha\cup W^{>H}_\alpha))$$

be characteristic $G$-maps for $\bar{b}^n_s$ and $\bar{b}^{n-1}_s$, $s=1, \cdots, m$, as in Corollary II.4.4 in [5], and moreover chosen such that

$$\begin{array}{l} f_s(I^n\times\{eH\})\subset W^H_\alpha \\ h_s(I^{n-1}\times\{eH\})\subset Y^H_\alpha\subset Y^H_\alpha\cup W^{>H}_\alpha\,. \end{array}\qquad s=1,\cdots,m\,.$$

Define

$$\bar{f}_s\colon (I^n,\ \partial I^n)\to(W^H_\alpha,\ Y^H_\alpha\cup W^{>H}_\alpha)$$
$$\bar{h}_s\colon (I^{n-1},\ \partial I^{n-1})\to(Y^H_\alpha\cup W^{>H}_\alpha,\ X^H_\alpha\cup W^{>H}_\alpha)$$

by $\bar{f}_s=f_s\,|\,I^n\times\{eH\}$ and $\bar{h}_s=h_s\,|\,I^{n-1}\times\{eH\}$, $s=1, \cdots, m$. We may assume that $\bar{f}_s(J^{n-1})=\{x_\alpha\}$, $\bar{h}_s(\partial I^{n-1})=\{x_\alpha\}$, where $x_\alpha\in X^H_\alpha$ is any (chosen) basepoint. Thus we have

(1) $\quad [\bar{f}_s]\in\pi_n(W^H_\alpha,\ Y^H_\alpha\cup W^{>H}_\alpha)\,,\quad [\bar{h}_s]\in\pi_{n-1}(Y^H_\alpha\cup W^{>H}_\alpha,\ X^H_\alpha\cup W^{>H}_\alpha)\,,$

for $s=1, \cdots, m$.

Let

$$\tilde{\bar{f}}_s\colon I^n\to\tilde{W}^H_\alpha\,,\quad s=1,\cdots,m\,,$$

be liftings of $\bar{f}_s\colon I^n\to W^H_\alpha$. Then $\tilde{\bar{f}}_s(J^{n-1})=\{\tilde{x}_\alpha\}$, for $s=1, \cdots, m$, where $\tilde{x}_\alpha\in\tilde{X}^H_\alpha$ and $p(\tilde{x}_\alpha)=x_\alpha$. Then we know by Theorem 6.6 that the $(WH)^*_\alpha$-equivariant extension of $\tilde{\bar{f}}_s$ gives rise to a characteristic $(WH)^*_\alpha$-map for the $(WH)^*_\alpha$-equivariant cell $(\tilde{b}^n_s)^H_\alpha$, lying over the $(WH)_\alpha$-equivariant cell $(b^n_s)^H_\alpha$. The elements

(2) $\quad [\tilde{\bar{f}}_s]\in\pi_n(\tilde{W}^H_\alpha,\ \widetilde{Y^H_\alpha\cup W^{>H}_\alpha})\,,\quad [\tilde{\bar{h}}_s]\in\pi_{n-1}(\widetilde{Y^H_\alpha\cup W^{>H}_\alpha},\ \widetilde{X^H_\alpha\cup W^{>H}_\alpha})$

correspond under the Hurewicz isomorphisms to a preferred basis of the homology groups, and under the isomorphisms $p_\#$ the elements in (2) are mapped to the elements in (1). Thus all put together we have that the elements in (1) form a preferred basis for $\pi_n(W^H_\alpha,\ Y^H_\alpha\cup W^{>H}_\alpha)$ and $\pi_{n-1}(Y^H_\alpha\cup W^{>H}_\alpha,\ X^H_\alpha\cup W^{>H}_\alpha)$, respectively.

Now let

$$\partial[\bar{f}_s]=\sum_{t=1}^{m}a_{st}[\bar{h}_t]\,,\quad a_{st}\in\boldsymbol{Z}[\pi_0(WH)^*_\alpha]$$

and denote $A=(a_{st})$. Thus we have

$$\tau(A) = 0 \in Wh(\pi_0(WH)^*_\alpha)$$

and hence the matrix $A$ can be transformed into an identity matrix by a finite sequence of operations of the following four types. (We denote $\Gamma=\pi_0(WH)^*_\alpha$.)

(1) Multiply a row by $(-1)$
(2) Multiply a row by an element of $\Gamma$.
(3) Change a row by adding to it some other row.
(4) Expand to $\begin{pmatrix} A & 0 \\ 1 & 0 \end{pmatrix} \in GL(n+1, \boldsymbol{Z}[\Gamma])$.

1. To realize geometrically the operation that multiplies the $r$: th row of $A$ by $(-1)$ we simply change the characteristic $G$-map $f_r$ for $b^n_r$ by using a map $t\colon (I^n, I^{n-1}) \to (I^n, I^{n-1})$ of degree $-1$.
2. Let $1 \leq r \leq m$ and let $\eta \in \pi_0(WH)^*_\alpha$. We shall show how to geometrically realize the operation that multiplies the $r$-th row of $A$ by the element $\eta$.

Consider the element $[\bar{f}_r] \in \pi_n(W^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha)$ where

$$\bar{f}_r\colon (I^n, I^{n-1}, J^{n-1}) \to (W^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha, x_\alpha)\,.$$

We have

$$\bar{\partial}[\bar{f}_r] = \sum_{t=1}^{m} a_{rt}[\bar{h}_t] \in \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha, X^H_\alpha \cup W^{>H}_\alpha; x_\alpha)$$

and

$$\begin{array}{ccc} \pi_n(W^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha) & \xrightarrow{\ \bar{\partial}\ } & \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha, X^H_\alpha \cup W^{>H}_\alpha; x_\alpha) \\ & \bar{\partial}' \searrow \qquad \nearrow i_* & \\ & \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha) & \end{array}$$

The map $i_*$ in the above diagram is a homomorphism of $\boldsymbol{Z}[\pi_0(WH)^*_\alpha]$-modules. Consider the element

$$\bar{\partial}'[\bar{f}_r] = [\bar{f}_r|] \in \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha)$$

and let

$$\bar{v}|\colon (I^{n-1}, \partial I^{n-1}) \to (Y^H_\alpha \cup W^{>H}_\alpha, x_\alpha)$$

be such that

$$[\bar{v}|] = \eta[\bar{f}_r|] \in \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha)\,.$$

Extend $\bar{v}|$ to a map

$$\bar{v}\colon \partial I^n \to Y_\alpha^H \cup W_\alpha^{>H}$$

by defining $\bar{v}(J^{n-1})=x_\alpha$. Now define

$$W' = (W-b_r^n)\cup_v(I^n\times G/H) = (W-b_r^n)\cup \hat{b}_r^n$$

where $v\colon \partial I^n\times G/H\to G(Y_\alpha^H\cup W_\alpha^{>H})\subset W-b_r^n$ is the $G$-equivariant extension of $\bar{v}$. Let $\hat{\eta}\in(WH)_\alpha^*$ represent $\eta\in\pi_0(WH)_\alpha^*$, and denote $\pi(\hat{\eta})=\hat{\xi}\in(WH)_\alpha$, where $\pi\colon (WH)_\alpha^*\to(WH)_\alpha$ is the natural projection. Then

$$[\bar{v}|] = \eta[\bar{f}_r|] = \hat{\eta}[\bar{f}_r|]\in\pi_{n-1}(Y_\alpha^H\cup W_\alpha^{>H};\, x_\alpha)$$

and we know from section 7 (see the concluding remark) that the maps

$$\hat{\xi}(\bar{f}_r|)\colon (I^{n-1},\partial I^{n-1})\to(Y_\alpha^H\cup W_\alpha^{>H},\,\hat{\xi}x_\alpha)$$

and

$$\bar{v}|\colon (I^{n-1},\partial I^{n-1})\to(Y_\alpha^H\cup W_\alpha^{>H};\, x_\alpha)$$

are $\omega$-homotopic for some path $\omega$ from $\hat{\xi}x_\alpha$ to $x_\alpha$, i.e., the maps

$$\hat{\xi}(\bar{f}_r|),\ \bar{v}\colon \partial I^n\to Y_\alpha^H\cup W_\alpha^{>H}$$

are homotopic.

Thus the corresponding $G$-maps

$$f'_r| = (f_r|)\circ(id\times\hat{\xi})\colon \partial I^n\times G/H \xrightarrow{id\times\hat{\xi}} \partial I^n\times G/H \xrightarrow{f_r|} G(Y_\alpha^H\cup W_\alpha^{>H})$$

and

$$v\colon \partial I^n\times G/H\to G(Y_\alpha^H\cup W_\alpha^{>H})$$

are $G$-homotopic. Thus, by Lemma II.4.1 in [5], we have

$$W' \simeq ((W-b_r^n)\cup_{f'_r|}(I^n\times G/H)) \qquad \text{rel. } (W-b_r^n)$$

But

$$W = (W-b_r^n)\cup_{f'_r|}(I^n/\times G/H)$$

since adjoining $I^n\times G/H$ by $f'_r|$ is just adjoining the equivariant cell $b_r^n$ back again by a different choice of characteristic map.

For $W'$ we have that

$$\bar{\partial}[\hat{b}_r^n] = \eta\bar{\partial}[b_r^n] = \eta\sum_{t=1}^m a_{rt}[h_t] = \sum_{t=1}^m \eta\, a_{rt}[h_t]\,.$$

3. Let $1\leq r\leq m$ and $1\leq p\leq m$, where $r\neq p$. Let

$$\bar{v}\colon (I^n, \partial I^n, J^{n-1}) \to (W^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha, \{x_\alpha\})$$

be such that

$$[\bar{v}] = [\bar{f}_r] + [\bar{f}_p] \in \pi_n(W^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha; x_\alpha)\,.$$

Since the map

$$\bar{f}_p| : \partial I^n \to Y^H_\alpha \cup W^{>H}_\alpha \cup (b^n_p)^H$$

is null-homotopic it follows that the maps

$$\bar{f}_r|, \bar{v}| : \partial I^n \to Y^H_\alpha \cup W^{>H}_\alpha \cup (b^n_p)^H$$

are homotopic. Let

$$v| : \partial I^n \times G/H \to G(Y^H_\alpha \cup W^{>H}_\alpha) \cup b^n_p$$

be the $G$-equivariant extension of $\bar{v}|$. Then $v|$ is $G$-homotopic to $f_r| : \partial I^n \times G/H \to G(Y^H_\alpha \cup W^{>H}_\alpha) \cup b^n_p$. Now define

$$W' = (W - b^n_r) \cup_{v|} (I^n \times G/H)$$

(where $v|$ is considered as a $G$-map into $W - b^n_r$). By Lemma II.4.1 in [5] we have $W' \; s \; W$ rel. $(W - b^n_r)$ and hence in particular rel. $X$. Observe that $(W')^{>H}_\alpha = W^{>H}_\alpha$. Moreover the matrix of the boundary homomorphism

$$\bar{\partial}\colon \pi_n((W')^H_\alpha, Y^H_\alpha \cup W^{>H}_\alpha) \to \pi_{n-1}(Y^H_\alpha \cup W^{>H}_\alpha, X^H_\alpha \cup W^{>H}_\alpha)$$

is the one obtained from $A$ by changing the $r$: th row by adding to it the $p$: th row.

4. An expansion of the matrix $A$ to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is realized geometrically by performing an equivariant elementary expansion of type $(H)$, that is, by adjoining $I^n \times G/H$ to $W$ by a $G$-map

$$\sigma_+ \colon J^{n-1} \times G/H \to GX^H_\alpha \subset W$$

defined by $\sigma_+(y, gH) = gx_\alpha$, for all $(y, gH) \in J^{n-1} \times G/H$.

Thus we see by 1–4 above that there exists a $G$-CW complex $V$ such that

$$V \; s \; W \quad \text{rel. } W - (\bigcup_{s=1}^{m} b^n_s)$$

and $(V, X)$ is in simplified form and there are $m+q$, where $q \geq 0$, equivariant $n$-cells $e^n_1, \cdots, e^n_{m+q}$ and $q$ equivariant $(n-1)$-cells $e^{n-1}_{m+1}, \cdots, e^{n-1}_{m+q}$ in $V - (W - (\bigcup_{s=1}^{m} b^n_s))$, and characteristic $G$-maps $u_s\colon I^n \times G/H \to \bar{e}^n_s$, $s = 1, \cdots, m+q$, and $h_s\colon I^{n-1} \times G/H \to \bar{e}^{n-1}_s$, such that

$$\bar{\partial}[\bar{u}_s] = [\bar{h}_s], \quad s = 1, \cdots, m+q .$$

Here

$$\bar{\partial}: \pi_n(V_\alpha^H, U_\alpha^H \cup W_\alpha^{>H}) \to \pi_{n-1}(U_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H \cup W_\alpha^{>H})$$

where we have denoted $U = Y \cup e_{m+1}^{n-1} \cup \cdots \cup e_{m+q}^{n-1}$. Observe that $W_\alpha^{>H} = V_\alpha^{>H}$. The maps $\bar{u}_s$ and $\bar{h}_s$, $s=1, \cdots, m+q$, are maps

$$\bar{u}_s: (I^n, I^{n-1}, J^{n-1}) \to (V_\alpha^H, U_\alpha^H, x_\alpha)$$

and

$$\bar{h}_s: (I^{n-1}, \partial I^{n-1}) \to (U_\alpha^H, x_\alpha)$$

It follows from Lemma 8.1.c and Corollary 8.5.b that the inclusion $i: X_\alpha^H \to X_\alpha^H \cup W_\alpha^{>H}$ is a homotopy equivalence. Thus $i_*: \pi_{n-1}(U_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H) \to \pi_{n-1}(U_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H \cup W_\alpha^{>H})$ is an isomorphism and hence we in fact have

$$\bar{\partial}[\bar{u}_s] = [\bar{h}_s] \quad \text{in} \quad \pi_{n-1}(U_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H)$$

Thus the maps

$$\bar{u}_s|, \bar{h}_s: (I^{n-1}, \partial I^{n-1}) \to (U_\alpha^H \cup W_\alpha^{>H}, X_\alpha^H)$$

are homotopic. The homotopy between the maps $\bar{u}_s|$, $\bar{h}_s$: $\partial I^{n-1} \to X_\alpha^H$ can, by the homotopy extension theorem, be extended to a homotopy from $\bar{u}_s|$: $J^{n-1} \to X_\alpha^H$ to a map $\bar{w}_s$: $J^{n-1} \to X_\alpha^H$. Thus $\bar{w}_s | \partial I^{n-1} = \bar{h}_s | \partial I^{n-1}$, and we can extend $\bar{w}_s$ to a map

$$\bar{w}_s: \partial I^n \to U_\alpha^H \cup W_\alpha^{>H}$$

by setting $\bar{w}_s | I^{n-1} = \bar{h}_s$. We now have that $\bar{w}_s$ is homotopic to $\bar{u}_s|: \partial I^n \to U_\alpha^H \cup W_\alpha^{>H}$ and $\bar{w}_s | I^{n-1} = \bar{h}_s$ and $w_s(J^{n-1}) \subset X_\alpha^H$. Furthermore since $\bar{w}_s(\partial I^{n-1}) = \bar{h}_s(\partial I^{n-1}) = \{x_\alpha\}$ we can by applying the skeletal approximation theorem to the map $\bar{w}_s$: $(J^{n-1}, \partial I^{n-1}) \to (X_\alpha^H, x_\alpha)$ homotop $\bar{w}_s$: $J^{n-1} \to X_\alpha^H$ rel. $\partial I^{n-1}$ to a skeletal map, i.e., we may as well assume that $\bar{w}_s$ in fact satisfies $w_s(J^{n-1}) \subset (X_\alpha^H)^{n-1}$.

The corresponding $G$-map $w_s$: $\partial I^n \times G/H \to G(U_\alpha^H, \cup W_\alpha^{>H})$ is $G$-homotopic to $u_s | \partial I^n \times G/H$. Now form the $G$-CW complex $V'$ by attaching equivariant $n$-cells $I^n \times G/H$ to $V - (\bigcup_{s=1}^{m+q} e_s^n)$ by the attaching $G$-maps $w_s$, $s=1, \cdots, m+q$. It follows from Lemma II.4.1 in [5] that

$$V' \; s \; V \quad \text{rel. } V - (\bigcup_{s=1}^{m+q} e_s^n) .$$

Moreover it follows directly from the properties of the attaching $G$-maps $w_s$ that $V'$ collapses equivariantly to

$$V'-(\bigcup_{s=1}^{m} b_s^{n-1}\cup \bigcup_{s=m}^{m+q} e_s^{n-1}\cup \bigcup_{s=1}^{m+q} e_s^n) = W-(\bigcup_{s=1}^{m} b^{n-1}\cup \bigcup_{s=1}^{m} b^n)\,.$$

We have shown that

$$W\, s\, (W-(\bigcup_{s=1}^{m} b_s^{n-1}\cup \bigcup_{s=1}^{m} b_s^n) \quad \text{rel. } W-(\bigcup_{s=1}^{m} b_s^{n-1}\cup \bigcup_{s=1}^{m} b_s^n)\,.$$

That is, all equivariant cells in $W-X$ which have type $(H)$ and belong to the component $W_\alpha^H$ have been "removed". Applying this procedure for every $G$-isotropy type $(K)$ that occurs as the type of some equivariant cell in $W-X$ and to one representative $X_\beta^K$ for each equivalence class $[X_\beta^K]$ i.e., one representative $X_\beta^K$ from each $WK$-component $(WK)X_\beta^K$ of $X^K$, we get

$$W\, s\, X \text{ rel. } X\,.$$

That is $s(W, X)=0\in Wh_G(X)$ and we have proved that $\Phi$ is injective.

The surjectivity of $\Phi$ is proved as follows. Let $(H)$ be a $G$-isotropy type such that $X^H\neq\phi$, and let $X_\alpha^H$ be a component of $X^H$. Let $(WH)_\alpha$ be the group of the component $X_\alpha^H$ and let $(WH)_\alpha^*$ be the extension of $(WH)_\alpha$ by $\pi_1(X_\alpha^H)$ that acts on $\tilde{X}_\alpha^H$. Let $A=(a_{st})$ be any non-singular $m\times m$ matrix over $\boldsymbol{Z}[\pi_0(WH_\alpha^*)]$. Let $x_0\in(X_\alpha^H)^0$ and define

$$h| : \partial I^2\times G/H \to X$$

by $(h|)\,(z, gH)=gx_0$, for all $(z, gH)\in\partial I^2\times G/H$. Let

$$GY_\alpha^H = GX_\alpha^H\cup b_1^2\cup\cdots\cup b_m^2$$

be the $G$-CW complex obtained by adjoining $m$ different equivariant 2-cells $I^2\times G/H$ to $GX_\alpha^H$ by the attaching $G$-map $h|$, and let $h_t$ denote the corresponding characteristic $G$-map for $b_t^2$, $t=1, \cdots, m$. We have

$$H_2(\tilde{Y}_\alpha^H, \tilde{X}_\alpha^H) \overset{\phi}{\leftarrow} \pi_2(\tilde{Y}_\alpha^H, \tilde{X}_\alpha^H) \overset{p_\sharp}{\rightarrow} \pi_2\,(Y_\alpha^H, X_\alpha^H)$$

where both the Hurewicz homomorphism $\phi$ and the induced homomorphism $p_\sharp$ are isomorphisms of $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-modules. Since $H_2(\tilde{Y}_\alpha^H, \tilde{X}_\alpha^H)$ is a free $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-module on $m$ generators, the same holds for $\pi_2(Y_\alpha^H, X_\alpha^H)$. Moreover a bases for the free $\boldsymbol{Z}[\pi_0(WH)_\alpha^*]$-module $\pi_2(Y_\alpha^H, X_\alpha^H)$ is given by $[\bar{h}_t]\in\pi_2(Y_\alpha^H, X_\alpha^H)$ where $\bar{h}_t$: $(I^2, \partial I^2)\to(Y_\alpha^H, X_\alpha^H)$.

Now let $v_s$: $(I^2, \partial I^2)\to(Y_\alpha^H, X_\alpha^H)$ be such that $[v_s]=\sum_{t=1}^{m} a_{st}[\bar{h}_t]$. Since we in fact have $[\bar{h}_t]\in\pi_2(Y_\alpha^H, x_0)$ and the map $\pi_2(Y_\alpha^H, x_0)\to\pi_2(Y_\alpha^H, X_\alpha^H)$ is a map of $\boldsymbol{Z}$ $[\pi_0(WH)_\alpha^*]$-modules we can form the element $\sum_{t=1}^{m} a_{st}[\bar{h}_t]\in\pi_2(Y_\alpha^H, x_0)$ and we then, in fact, let $v_s$: $(I^2, \partial I^2)\to(Y_\alpha^H, x_0)$ be such that $[v_s]=\sum_{t=1}^{m} a_{st}[\bar{h}_t]\in\pi_g(Y_\alpha^H, x_0)$.

Now let $\bar{f}_s| : \partial I^3 \to Y^H_\alpha$ be the extension of $v_s$ defined by $(\bar{f}_s|)(J^2)=\{x_0\}$, and let $f_s| : \partial I^3 \times G/H \to GY^H_\alpha$ be the corresponding $G$-map. Let $GW^H_\alpha = GY^H_\alpha \cup b^3_1 \cup \cdots \cup b^3_m$ be obtained by adjoining equivariant 3-cells $I^3 \times G/H$ to $GY^H_\alpha$ by $f_s|$, $s=1, \cdots, m$, and let $f_s$ denote the corresponding characteristic $G$-maps. Now the boundary homomorphism

$$\bar{\partial}: \pi_3(W^H_\alpha, Y^H_\alpha) \to \pi_2(Y^H_\alpha, X^H_\alpha)$$

is given by $\bar{\partial}[\bar{f}_s]=\sum_{t=1}^{m} a_{st}[\bar{h}_t]$, i.e., $\bar{\partial}$ has matrix $A$ in these bases. Thus $\bar{\partial}$ is an isomorphism and it follows easily that $\pi_n(W^H_\alpha, X^H_\alpha)=0$ for all $n$. Hence the inclusion $i$: $X^H_\alpha \to W^H_\alpha$ induces isomorphisms between homotopy groups in all degrees. Thus, by Lemma 11.1 below the inclusion $i$: $GX^H_\alpha \to GW^H_\alpha$ is a $G$-homotopy equivalence. Thus $X$ is a strong $G$-deformation retract of $GW^H_\alpha \cup X$ and hence $s(GW^H_\alpha \cup X, X) \in Wh_G(X)$. We now have $\Phi(s(GW^H_1 \cup X, X))= \tau(C(GW^H_\alpha \cup X, X)^H_\alpha)=\tau(A) \in Wh(\pi_0(WH)^*_\alpha)$. Since $\Phi$ is a homomorphism this shows that $\Phi$ is surjective. □

**Lemma 11.1.** *Let $(Y, B)$ be a G-CW pair such that each point in Y-B has G-isotropy type equal to $(H)$. Assume that the map $i^H$: $B^H \to Y^H$ induces isomorphisms in homotopy in all degrees. Then $i$: $B \to Y$ is a G-homotopy equivalence.*

Proof. The map $i^H$: $B^H \to Y^H$ is map between $WH$-CW complexes and the $WH$-action is free on $Y^H - B^H$. If $Q \subset WH$ is a closed subgroup of $W$ we have $(Y^H)^Q=(B^H)^Q$ unless $Q=\{e\} \subset WH$, and the inclusion $i$: $(B^H)^{\{e\}} \to (Y^H)^{\{e\}}$ induces isomorphisms in homotopy. Thus, by the equivariant Whitehead theorem, (see [7] Theorem 5.3, or [4] Proposition 2.5), $i$: $B^H \to Y^H$ is a $(WH)$-homotopy equivalence. Hence $B^H$ is a strong $WH$-equivariant deformation retract of $Y^H$. Let $r$: $Y^H \to B^H$ be a $WH$-retraction. Now define

$$\hat{r}: Y \to B$$

as follows. We set $\hat{r}| Y=id_B$. Let $y \in Y-B$, then $y=gx$ where $x \in Y^H - B^H$ and we define $\hat{r}(y)=gr(x)$. If $y=gx=g_1 x_1$, where $x, x_1 \in Y^H - B^H$ we have $(g_1)^{-1} g \in NH$ and hence $r(x_1)=r(g_1^{-1} g\, x)=g_1^{-1} gr(x)$ and therefore $g_1 r(x_1)=gr(x)$. Which shows that $\hat{r}$ is a well-defined $G$-retraction. It is also immediately seen that $i \circ \hat{r}$ is $G$-homotopic to $id_Y$. □

## 12. Combinatorial invariance of equivariant Whitehead torsion

DEFINITION 12.1. Let $X$ be a $G$-CW complex. We say that a $G$-CW complex $X^*$ is an *equivariant subdivision* of $X$ if $X$ and $X^*$ as $G$-spaces are identical and the following two conditions hold:

1. For each open equivariant $n$-cell $\mathring{b}^n$ of $X^*$ there exists an open equivariant

$n$-cell $\mathring{c}^n$ of $X$ such that $\mathring{b}^n \subset \mathring{c}^n$.

2. Let $c^n$ be an equivariant $n$-cell of $X$ and let $b_1^n, \cdots, b_k^n$ be all the equivariant $n$-cells of $X^*$ such that $\mathring{b}_i^n \subset \mathring{c}^n$, $1 \leq i \leq k$. Then there is an equivariant characteristic map

$$f\colon D^n \times G/H \to c^n$$

and equivariant characteristic maps

$$f_i\colon D^n \times G/H \to b_i^n\,, \quad i=1, \cdots, k$$

such that $f_i(D^n \times \{eH\}) \subset f(D^n \times \{eH\})$.

**Theorem 12.2.** *Let $X^*$ be an equivariant subdivision of the finite G-CW complex $X$. Then the identity map $h\colon X^* \to X$ is a simple G-homotopy equivalence.*

Proof. The identity map $f\colon X \to X^*$ is a skeletal $G$-map and $f$ is the inverse of $h$. By Corollary II.3.9 in [5] we have $\tau(h) = -f_* \tau(f) \in Wh_G(X^*)$. We shall show that $\tau(f) = 0 \in Wh_G(X)$, then $\tau(h) = 0$ and hence $h$ is a simple $G$-homotopy equivalence by Theorem II.3.6′. in [5].

Let $c_1, \cdots, c_m$ be all the equivariant cells of $X$ ordered such that $\dim c_i \leq \dim c_{i+1}$. We denote $X_j = c_1 \cup \cdots \cup c_j$, where $1 \leq j \leq m$. Consider the map $f| \colon X_j \to f(X_j)$, and let $M_j$ be the mapping cylinder of this map. Then $X$ is a strong $G$-deformation retract of $M_j \cup X$, for $j=1, \cdots, m$, and hence we also have that $M_{j-1} \cup X$ is a strong $G$-deformation retract of $M_j \cup X$. By Lemma II.2.2 in [5] we therefore have

$$(1) \quad s(M_j \cup X, X) = r_* s(M_j \cup X, M_{j-1} \cup X) + s(M_{j-1} \cup X, X) \in Wh_G(X)\,,$$

where $r\colon M_{j-1} \cup X \to X$ is a skeletal $G$-retraction. We shall prove that $s(M_j \cup X, M_{j-1} \cup X) = 0 \in Wh_G(M_{j-1} \cup X)$, $0 \leq j \leq m$, and then we have by (1) and induction that $\tau(f) = s(M_m, X) = s(M_1, X) = s(X, X) = 0$.

We shall now prove that $s(M_j \cup X, M_{j-1} \cup X) = 0$. By Theorem 11.1 we may as well show that

$$(2) \qquad \Phi(s(M_j \cup X, M_{j-1} \cup X)) = 0\,.$$

Assume that the equivariant cell $c_j$ is of type $(H)$ and that it belongs to the component $X_\alpha^H$. That is there is a characteristic $G$-map $k\colon D^n \times G/H \to c_j$ such that $k(D^n \times \{eH\}) \subset X_\alpha^H$.

In order to prove that (2) is valid we must prove that for any $G$-isotropy type $(K)$, such that $X^K \neq \phi$, and any component $X_\beta^K$ of $X^K$, representing the $WK$-component $(WK)X_\beta^K$, we have

$$\tau(C(M_j \cup X, M_{j-1} \cup X)_\beta^K) = 0 \in Wh(\pi_0(WK)_\beta^*)$$

Exactly as in the proof of Proposition 10.1 we see that it is enough to consider the case $K=H$ and $X^K_\beta = X^H_\alpha$. Now observe that

$$(M_j \cup X)-(M_{j-1} \cup X)=\mathring{c}_j \times (0, 1] \cong (\mathring{D}^n \times G/H) \times (0, 1] \, .$$

Let us denote $k(D^n \times \{eH\})=(c_j)_e$ and $k(\mathring{D}^n \times \{eH\})=(\mathring{c}_j)_e$. Then we have

$$(\mathring{c}_j)_e \times (0, 1] \subset ((M_j \cup X)-(M_{j-1} \cup X))^H_\alpha \, .$$

The equivariant cells of $(M_j \cup X, M_{j-1} \cup X)$ consist of the equivariant cell $c_j \times [0, 1]$ and all the equivariant cells $b_i$ (of various dimensions) of $X^*$ whose interiors are contained in $\mathring{c}_j$. Each of these equivariant cells has a characteristic $G$-map with open $e$-section lying in $(\mathring{c}_j)_e \times (0, 1]$.

Let $\widetilde{M_j \cup X}$ and $\widetilde{M_{j-1} \cup X}$ be the universal coverings of $M_j \cup X$ and $M_{j-1} \cup X$ respectively. Since the subset $C=(\mathring{c}_j)_e \times (0, 1]$ of $M_j \cup X$ is simply connected there is a lifting $\tilde{C}$ of it in $\widetilde{M_j \cup X}$. The different liftings of $C$ are pairwise disjoint from each other. It follows that all the $(WH)^*_\alpha$-equivariant cells of $(\widetilde{M_j \cup X}, \widetilde{M_{j-1} \cup X})$ have $(WH)^*_e$-equivariant characteristic maps with open $e$-sections lying in $\tilde{C}$, see section 6. Hence in the chain complex

$$(3) \qquad \to C_q(\widetilde{M_j \cup X}, \widetilde{M_{j-1} \cup X}) \xrightarrow{\partial} C_{q-1}(\widetilde{M_j \cup X}, \widetilde{M_{j-1} \cup X}) \to \cdots$$

the boundary homomorphisms are of the form

$$\partial [e^q_s] = \sum_{t=1}^{t_0} \eta_{st} [e^{q-1}_t]$$

with $\eta_{st} \in \boldsymbol{Z} \subset \boldsymbol{Z}[\pi_0(WH)^*_\alpha]$. (Here $[e^q_1], \cdots, [e^q_{s_0}]$ and $[e^{q-1}_1], \cdots, [e^{q-1}_{t_0}]$ are the preferred bases of $C_q(\widetilde{M_j \cup X}, \widetilde{M_{j-1} \cup X})$ and $C_{q-1}(\widetilde{M_j \cup X}, \widetilde{M_{j-1} \cup X})$, respectively, coming from the $(WH)^*_\alpha$-equivariant characteristic maps described above.) Thus the torsion of the chain complex (3) comes from $Wh(\boldsymbol{Z})=0$. Thus

$$\tau(C(M_j \cup X, M_{j-1} \cup X)^H_\alpha) = 0 \in Wh(\pi_0(WH)^*_\alpha) \, .$$

We have proved that (2) is valid and, as we already observed above, this completes the proof. □

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