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ON STRONGLY INVARIANT COEFFICIENT RINGS

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Let A and B be rings with an identity. After P. Eakin and W. Heinzer we shall say that A and B are stably equivalent if there is an integer n such that polynomial rings $A[X_1, \cdots, X_n]$ and $B[Y_1, \cdots, Y_n]$ are isomorphic [3]. A number of recent investigations have been published concerning the study of this equivalence. One of the interesting question is the one, called the cancellation problem for rings, which asks when the stably equivalence implies the isomorphism. A ring with this property will be called an invariant ring. This paper contains some contribution to this problem.

In §1 we shall take up the notions of strongly invriant rings which are defined by several authors in their own way and make it clear the relationship among them. In §2 we shall consider the following problem: Let A be a strongly invariant ring. What conditions on A guarantees the invariance of the polynomial ring $A[X]$? Several conditions will be given. In the last section we shall give some examples of strongly invariant rings which have not such a nice property. As a result we can give examples of non invariant rings which are two dimensional affine domains over a field of positive characteristic.

1. Strongly invariant rings

In this paper $A[X_1, \cdots, X_n]$ and $B[Y_1, \cdots, Y_n]$ denote always polynomial rings in n indeterminates $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ over rings A and B, respectively. In this paper all rings are assumed to have an identity.

1.1. DEFINITIONS. (i) A ring A is said CE-strongly invariant provided that an isomorphism of rings $\sigma; A[X_1, \cdots, X_n] \rightarrow B[Y_1, \cdots, Y_n]$ yields $B[Y_1, \cdots, Y_n] = B[\sigma(X_1), \cdots, \sigma(X_n)]$ ([2]).
(ii) A ring A is said AHE-strongly invariant provided that an isomorphism of rings $\sigma; A[X_1, \cdots, X_n] \rightarrow B[Y_1, \cdots, Y_n]$ yields $\sigma(A) = B$ [1].

Let A be a commutative ring. We denote by $N(A)$ the nil radical of A and let A_{red} be the reduced ring $A/N(A)$. Let $R = A[X_1, \cdots, X_n]$. Then $N(R) = N(A)[X_1, \cdots, X_n]$ and $R_{\text{red}} = A_{\text{red}}[\bar{X}_1, \cdots, \bar{X}_n]$ where $\bar{X}_i \equiv X_i \pmod{N(R)}$. If $\sigma; A[X_1, \cdots, X_n] \rightarrow B[Y_1, \cdots, Y_n]$ is an isomorphism, then $\sigma(N(A)[X_1, \cdots, X]) = N(B)[Y_1, \cdots, Y_n]$. Hence we have the induced isomorphism $\bar{\sigma}; A_{\text{red}}[\bar{X}_1, \cdots, \bar{X}_n] \rightarrow$

$B_{\mathrm{red}}[\bar{Y}_1, \cdots, \bar{Y}_n]$.

(iii) A commutative ring A is said EK-strongly invariant if an isomorphism of rings σ; $A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n]$ yields $\bar{\sigma}(A_{\mathrm{red}}) = B_{\mathrm{red}}$ [4].

(iv) A ring A is said invariant if any isomorphism $A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n]$ implies an isomorphism $A \to B$ [1], [2].

1.2. **Theorem.** *Let A be a commutative ring. Then following conditions* (i) *and* (ii) *are equivalent to each other*:

(i) *A is CE-strongly invariant.*

(ii) *A is EK-strongly invariant.*

Moreover if A is reduced then above conditions are equvalent to:

(iii) *A is AHE-strongly invariant.*

Proof. In above circumstances one may consider the case $A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n]$ instead of considering an isomorphism σ; $A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n]$.

(i)⇨(ii): If $R = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n]$, then we have $B[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n]$ by assumption. Therefore $X_1, \cdots, X_n$ are algebraically independent over B[1, p. 315]. Set $B_s = B[X_1, \cdots, X_{s-1}, X_{s+1}, \cdots, X_n]$. Then $B_s[X_s]$ is a polynomial ring in one variable over B_s. Any element of A can be written in the form $\alpha = \beta_0 + \beta_1 X_s + \cdots + \beta_m X_s^m$ with $\beta_i \in B_s$. Assume that a polynomial $f(X_s) = \gamma_0 + \gamma_1 X_s + \cdots + \gamma_l X_s^l$ in $B_s[X_s]$ satisfy the condition that $B_s[X_s] = B_s[f(X_s)]$. Then γ_1 should be a unit in B_s and $\gamma_2, \cdots, \gamma_l$ should be nilpotent [5, p. 329]. Since $R = A[X_1, \cdots, X_{s-1}, X_s + \alpha, X_{s+1}, \cdots, X_n] = A[X_1, \cdots, X_{s-1}, X_s + \alpha^2, X_{s+1}, \cdots, X_n]$, the condition (i) implies $B_s[X_s] = B_s[X_s + \alpha] = B_s[X_s + \alpha^2]$. A simple calculation shows that $\beta_1, \cdots, \beta_m$ are nilpotent in B_s. Hence $A \subset B_s + N(R)$ and we have $A \subset \bigcap_{s=1}^{n} (B_s + N(R)) = B + N(R)$. Since $R_{\mathrm{red}} = A_{\mathrm{red}}[\bar{X}_1, \cdots, \bar{X}_n] = B_{\mathrm{red}}[\bar{Y}_1, \cdots, \bar{Y}_n]$ and $A \subset B + N(R)$, we have $A_{\mathrm{red}} = B_{\mathrm{red}}$ [1, p. 315].

(ii)⇨(i) is proved in [4, p. 334]. The rest is immediate from definitions.

1.3. EXAMPLES. If A is not commutative or not reduced, a CE-strongly invariant ring is not necessarily an AHE-strongly invariant ring as is shown by the following examples:

EXAMPLE 1. Let k be a field and let $A = k[x, y]/(x^2 = xy = y^2 = 0)$. Let $B = k[x, y + Zx]$where Z is indeterminate over A. Then $B[Z]$ is a polynomial ring over B and $A[Z] = B[Z]$. Hence A is not AHE-strongly invariant. However $A_{\mathrm{red}} = k$, hence A is CE-strongly invariant.

EXAMPLE 2. Let k be a field and let $A = k[V, W]$ be the free k algebra generated by symbols V, W. Let Z be an indeterminate over A and let $B = k[V, W + ZV]$. Then $A[Z] = B[Z]$. Hence A is not AHE-strongly invariant, but A is CE-strongly invariant by [2].

2. On a property of strongly invariant rings

In this section all rings are assumed to be commutative and contains an identity.

2.1. Let D be a ring and let A be a D-algebra. We say that A is D-invariant if for any D-algebra B such that $A[X_1, \cdots, X_n]$ is D-isomorphic to $B[Y_1, \cdots, Y_n]$, A is D-isomorphic to B (cf. [1, p. 329]).

The main result in this section is to prove the following.

2.2. **Theorem.** *Let D be an integrally closed domain and let $A=D[X]$ be a polynomial ring in one variable over D. Then A is D-invariant.*

Proof. By virtue of [1, p. 329], we may assume D an integrally closed local domain. Let B be D-algebra such that $R=A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$. Define a D-algebra homomorphism σ; $B[Y_1, \cdots, Y_n]\to B$ by $\sigma(f(Y_1, \cdots, Y_n))=f(0, \cdots, 0)$ for $f(Y_1, \cdots, Y_n)$ in $B[Y_1, \cdots, Y_n]$. Then σ is surjection and we see $B=\sigma(R)=D[\sigma(X_0), \cdots, \sigma(X_n)]$ where $X_0=X$. Therefore we have $B=D[f_0, \cdots, f_n]$ with $f_i\in D[X_0, \cdots, X_n]$ having no constant terms. For any polynomial f of $D[X_0, \cdots, X_n]$, $f^{(i)}$ denotes the i-homogeneous part of f. It is easy to see that $D[X_0, \cdots, X_n]=D[f_0^{(1)}, \cdots, f_n^{(1)}, Y_1^{(1)}, \cdots, Y_n^{(1)}]$, where f_i and Y_i are regarded as elements in $D[X_0, \cdots, X_n]$. Set $D^*=Df_0^{(1)}+\cdots+Df_n^{(1)}$. We have then $D^{(n+1)}=DX_0+\cdots+DX_n=D^*+DY_1^{(1)}+\cdots+DY_n^{(1)}\cong D^*\oplus D^{(n)}$ as D-modules where $D^{(n)}$ and $D^{(n+1)}$ are free D-modules of rank n and $n+1$, respectively. Hence D^* is a projective and hence a free D-module of rank 1, since D is local. Therefore there is an element α in D^* such that $D^*=D\alpha$. This implies that $D[X_0, \cdots, X_n]=D[\alpha, Y_1^{(1)}, \cdots, Y_n^{(1)}]$. Let δ be the D-algebra automorphism of R defined by $\delta(\alpha)=X_0$ and $\delta(Y_i^{(1)})=X_i$. Replacing B by $\delta(B)$ we may assume that B has a polynomial h in $D[X_0, \cdots, X_n]$ such that $h^{(1)}=X_0$. Since D is integrally closed, $D=\bigcap_v K_v$ where v runs over all valuations of the quotient field of K of D such that those valuation rings K_v contain D. Let v^* be the canonical extention of v to $K(X_0, \cdots, X_n)$ i.e. for any element θ of $K[X_0, \cdots, X_n]$, $v^*(\theta)=\min\{v(c_i); c_i$ runs over all coeficients of $\theta\}$. On the other hand since $K[X_0]$ is invariant [1, p. 321], there is an element ϕ in $K[X_0, \cdots, X_n]$ such that $K[f_0, \cdots, f_n]=K[\phi]$ and $K[\phi]\supset B$. As a polynomial in $K[X_0, \cdots, X_n]$, we may assume that $\phi^{(0)}=0$ and $\phi^{(1)}=X_0$. We shall show that $v^*(\phi)\geqq 0$ for any valuation v. Since $h\in B$ and $h^{(1)}=X_0$, $v^*(h)=0$. On the other hand h can be written as $h=\phi+a_2\phi^2+\cdots+a_s\phi^s$ with $a_i\in K$. Hence $v^*(\phi+a_2\phi^2+\cdots+a_s\phi^s)=v^*(\phi)+v^*(1+a_2\phi+\cdots+a_s\phi^{s-1})=0$. Since $\phi^{(0)}=0$, $v^*(1+a_2\phi+\cdots+a_s\phi^{s-1})\leqq 0$. Therefore we have $v^*(\phi)\geqq 0$. This implies that $\phi\in D[X_0, \cdots, X_n]$. Now $B\subset K[\phi]$ implies that one can write any element λ in B in the form $\lambda=\beta_0+\beta_1\phi+\cdots+\beta_t\phi^t$ with $\beta_i\in K$. Let β_i be the first coefficient with $\beta_i \notin D$. Then the coefficient of X^i in λ has the form β_i+(terms in D). Hence for some valuation v of K we have $v^*(\lambda)<0$. This is a contradiction. Hence β's are all in D and $B\subseteq D[\phi]$. Therefore $B[Y_1, \cdots, Y_n]=D[\phi, Y_1, \cdots, Y_n]$ and it is easy to see that $\phi, Y_1, \cdots, Y_n$ are algebraically independent over D. Hence we have $B=D[\phi]$ [1, p. 315].

2.3. We shall say that a ring D has the property (G) provided that an equality of polynomial rings $D[X_1, \cdots, X_m]=B[Y_1, \cdots, Y_n]$ (not necessarily $m=n$) implies $D\subset B$. If D is an integrally closed domain having the property (G), then Theorem 2.1 shows that $D[X]$ is invariant. Some examples of rings with property (G) are those given in [1, p. 317].

Another example is $D=\boldsymbol{R}[X_1, \cdots, X_n]/(X_1^2+\cdots+X_n^2-1)$ where $\boldsymbol{R}$ is the real number field. It is straightforward to prove that D has the property (G) because D is formally real. Moreover D is an integrally closed domain. Hence $D[X]$ is invariant.

2.4. **Theorem.** *Let D be a ring such that the polynomial ring $D_{\mathrm{red}}[X]$ in one variable is D_{red}-invariant. Then the polynomial ring $D[X]$ in one variable is D-invariant.*

Proof. If B is a D-algebra and $R=D[X_0, X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$, then $R_{\mathrm{red}}=D_{\mathrm{red}}[\bar{X}_0, \cdots, \bar{X}_n]=B_{\mathrm{red}}[\bar{Y}_1, \cdots, \bar{Y}_n]$ where $X_0=X$. By assumption there is an element h in B shuch that $B_{\mathrm{red}}=D_{\mathrm{red}}[\bar{h}]$. Hence $B\subset D[h]+N(R)$. Therefore X_i has the form $X_i=\phi_i+\psi_i$ where ϕ_i's are elements of $D[h, Y_1, \cdots, Y_n]$ and ψ_i's are elements of $N(R)$. Hence $\phi_i=X_i-\psi_i$, considered as element in the polynomial ring $D[X_0, \cdots, X_n]$, has the form $\phi_i=\sum_{j=0}^{n} a_{ij}X_j+$ (terms of higher degree in $N(R)$) where a_{ij}'s are nilpotent if $i\neq i$ and a_{ii}'s are units in D. If we denote by M the matrix (a_{ij}), the determinate det M of M is a unit in D. Hence there exists the inverse matrix $M^{-1}=(b_{ij})$ in D. Let us set $\phi_i'=\sum_{j=0}^{n} b_{ij}\phi_j$. Then $\phi_i'=X_i+$(terms of higher degree in $N(R)$). We shall show that $R=D[h, Y_1, \cdots, Y_n]$. For this purpose it is sufficient to prove that $R=D[\phi_0', \cdots, \phi_n']$. We can therefore assume from the first that $\phi_i=X_i+$(terms of higher degree in $N(R)$). By Gilmer ([5, p. 329]) a nilpotent ideal $\mathfrak{A}$ is said to have the order of nilpotence s if $\mathfrak{A}^{s-1}\neq 0$ and $\mathfrak{A}^s=0$. Let $\mathfrak{A}(\phi_0, \cdots, \phi_n)$ be the nilpotent ideal generated by all coefficients of terms of degree ≥ 2 of $\phi_0, \cdots, \phi_n$. By induction on the order of nilpotence s of $\mathfrak{A}(\phi_0, \cdots, \phi_n)$, we shall show that $R=D[\phi_0, \cdots, \phi_n]$. If $s=1$, it is clear that we have $R=D[\phi_0, \cdots, \phi_n]$. Assume that $R=D[\phi_0, \cdots, \phi_n]$ if the order s of nilpotence of the ideal $\mathfrak{A}(\phi_0, \cdots, \phi_n)<r$. Now consider polynomials $\phi_i''=\phi_i(X_0, \cdots, X_n)-\sum_{j=2}^{n_i}\phi_i^{(j)}(\phi_0, \cdots, \phi_n)$ where n_i is the total degree of polynomial $\phi_i(X_0, \cdots, X_n)$ in $X_0, \cdots, X_n$

and $\phi_i^{(j)}(X_0, \cdots, X_n)$ is the j-homogeneous part of ϕ_i. Then it is easy to see that $\mathfrak{A}(\phi_0'', \cdots, \phi_n'') \subset \mathfrak{A}(\phi_0, \cdots, \phi_n)^2$. Hence the order of nilpotence of $\mathfrak{A}(\phi_0'', \cdots, \phi_n'')$ is $<r$. By induction hypothesis, we have $R=D[\phi_0'', \cdots, \phi_n'']$. Therefore $D[\phi_0, \cdots, \phi_n]=D[\phi_0'', \cdots, \phi_n'']$. Hence $R=D[\phi_0, \cdots, \phi_n]$. Consequently we have $R=D[X_0, \cdots, X_n]=D[h, Y_1, \cdots, Y_n]=B[Y_1, \cdots, Y_n]$. Under these circumstances it is seen that $h, Y_1, \cdots, Y_n$ are algebraically independent over D and $B \supset D[h]$. This shows that $B=D[h]$ [1, p. 315]. Hence B is a polynomial ring in one variable over D.

2.5. **Corollary.** *Let D be a ring such that the polynomial ring $D_{\mathrm{red}}[X]$ in one variable is D_{red}-invariant, and D_{red} has the property (G). Then the polynomial ring $D[X]$ in one variable is D-invariant.*

Proof. If B is a ring such that $R=D[X_0, \cdots, X_n]=B[Y_1, \cdots, Y_n]$ where $X_0=X$, then $R_{\mathrm{red}}=D_{\mathrm{red}}[\bar{X}_0, \cdots, \bar{X}_n]=B_{\mathrm{red}}[\bar{Y}_1, \cdots, \bar{Y}_n]$. By assumption there is an element h in B such that $B_{\mathrm{red}}=D_{\mathrm{red}}[\bar{h}]$. Then from the proof of Theorem 2.4 we have $R=B[Y_1, \cdots, Y_n]=D[h, Y_1, \cdots, Y_n]$ and $h, Y_1, \cdots, Y_n$ is algebraically independent over D. Hence B is isomorphic to $D[X]$.

2.6. It is straightforward to prove that if a ring D is a direct sum of integrally closed domains, then the polynomial ring $D[X]$ is D-invariant. Hence Theorem 2.4 shows that if D is a ring such that D_{red} is a direct sum of integrally closed domains, then the polynomial ring $D[X]$ is D-invariant.

Moreover by Corollary 2.5 if D_{red} has the property (G), then $D[X]$ is invariant. In particular, if D is an Artin ring, then $D[X]$ is D-invariant and invariant.

3. Non invariant $D[X]$

In this section, we shall show some examples of non D-invariant and non invariant polynomial ring $D[X]$ in case of positive characteristic. In this section all rings are assumed to be a commutative reduced ring. Hence we say simply that a ring A is strongly invariant if A satisfies one of the conditions (i), (ii) and (iii) of Theorem 1.2.

3.1. **Theorem.** *Let $\tilde{D}$ be a reduced ring with positive characteristic p and let D be a subring of $\tilde{D}$. Assume that there is an element $t \in \tilde{D}$ such that $t \notin D$ and $t^i \in D$ for any $i \geqq n$ where n is some integer. Then the polynomial ring $D[X]$ in one variable is not D-invariant. Moreover if D is strongly invariant, then $D[X]$ is not invariant.*

Proof. Let Y be an indeterminate over $D[X]$. Set ϕ, ψ, f and g as follows: $\phi=X+t^{p^n+1}Y^{p^n}-tX^{p^{2n}}$, $\psi=Y+t^{p^{2n}+p^n-1}Y^{p^{2n}}-t^{p^n-1}X^{p^{3n}}$, $f=\phi+t\phi^{p^{2n}}$,

$g=\phi^{p^n}$. By assumption, we have $\psi, f, g\in D[X, Y]$ and $\phi\notin D[X, Y]$. If we set $B=D[f, g]$, then $D[X, Y]=B[\psi]$, because $X=f-t^{p^n+1}\psi^{p^n}$ and $Y=\psi-t^{p^{2n}+p^n-1}\psi^{p^{2n}}+t^{p^n-1}g^{p^{2n}}$. X and Y are algebraically independent over $D[t]$. Hehce ϕ and ψ are algebraically independent over $D[t]$. Hence ϕ and ψ are algebraically independent over $D[t]$. This shows that $B[\psi]$ is a polynomial ring over B in one variable. Now we shall show that $B=D[f, g]$ is not a polynomial ring over D. If B is a polynomial ring over D, then there is an element $\theta\in D[X, Y]$ such that $B=D[\theta]$. We may assume that θ has no constant term and the 1-homogeneous part is X. Since $\phi=f-tg^{p^n}$, we have $D[t][f, g]=D[t, \phi]=D[t, \theta]$. Since $D[t]$ is reduced and ϕ and θ have no constant term and have the same 1 homogeneous part X, then we have $\theta=\phi$ [5, p. 329]. This is a contradiction to the fact $B\not\ni\phi$. Hence $D[X]$ is not D-invariant.

We shall next show that $D[X]$ is not invariant provided that this D is strongly invariant. For if $D[X]$ is invariant, there is an isomorphism σ; $D[X]\rightarrow B$. Then we have $B=\sigma(D)[\sigma(X)]$ and $D[X, Y]=\sigma(D)[\sigma(X), \psi]$. Since D is assumed to be strongly invarinat we must have $\sigma(D)=D$. This contrdicts to the preceding result that B is not a polynomial ring over D. Therefore B is not isomorphic to $D[X]$ and $D[X]$ is not invariant.

3.2. **Corollary.** *Let k be a field with non zero characteristic and let t be an indeterminate over k. If $\tilde{D}=k[t]$ and D is a proper subring of $\tilde{D}$ containing k such that the integral closure of D is $\tilde{D}$ and the conductor $C(\tilde{D}/D)$ of D in $\tilde{D}$ contains t^n for some integer n, then the polynomial ring $D[X]$ over D is neither D-invariant nor invariant.*

Proof. As is known D is strongly invariant [1, p. 325]. Hence by Theorem 3.1 $D[X]$ is neither D-invariant nor invariant.

3.3. **Corollary.** *Let k be a field with positive characteristic and $D=k[t^n,t^{n+1}]$ $(n>1)$ be a subring of a polynomial ring $k[t]$ and let $p=(t^n, t^{n+1})$ be the maximal ideal of D. Then both $D[X]$ and $D_p[X]$ are not invariant.*

This is an immediate consequence of 3.2. It should be noted that for any other prime ideal q of D, $D_q[X]$ is an invariant ring since D_q is a normal ring. (cf [1, p. 342])

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