Nagasaki, I.
Osaka J. Math.
29 (1992), 595–606

LINEARITY OF HOMOTOPY REPRESENTATIONS

IKUMITSU NAGASAKI

(Received October 3, 1991)

0. Introduction

In the theory of transformation groups, it is an important problem to distinguish whether or not a group action is linear. In this paper we would like to consider linearity of homotopy representations of finite groups in the G-homotopy category.

Since the study of homotopy representations of finite groups G due to tom Dieck-Petrie [6], it is known that there exist many homotopy representations which are not linear (i.e., not G-homotopy equivalent to linear G-spheres). On the other hand, in [5], tom Dieck proved that any homotopy representation of a cyclic p-group C_{p^m} is linear under a restricted situation. For its proof, tom Dieck used the stable theory of homotopy representations.

We first consider the following problem under the general setting. We use the unstable theory of homotopy representations.

Problem. *When is a homotopy representation of G linear?*

If a homotopy representation is linear, its dimension function must be linear at least. Therefore we mainly discuss homotopy representations with linear diemsnion functions. (For the linearity of dimension functions, see [1], [2], [3], [6].)

In Section 1 we recall some definitions and well-known results, in particular, the unstable Picard group $\mathrm{Pic}(G;\underline{n})$ and Laitinen's invariant ([7]), which are the main tools in this paper.

In Section 2 we introduce subgroups $jO(G;\underline{n})$ and $\mathrm{Pic}^f(G;\underline{n})$ of $\mathrm{Pic}(G;\underline{n})$ for any linear dimension function $\underline{n}$, and put

$$LH^\infty(G;\underline{n}) = \mathrm{Pic}(G;\underline{n})/jO(G;\underline{n}),$$
$$LH(G;\underline{n}) = \mathrm{Pic}^f(G;\underline{n})/jO(G;\underline{n}).$$

For any homotopy representation X with dimension function $\underline{n}$, we define a G-homotopy invariant $l(X)$ in $LH^\infty(G;\underline{n})$ by using Laitinen's invariant, and

This work was partially supported by Grant-in-Aid for Scientific Research.

anwser the problem above.

Theorem 0.1. *A homotopy representation X with linear diemsnion function $\underline{n}$ is linear if and only if $l(X)$ vanishes in $LH^{\infty}(G;\underline{n})$.*

In Section 3 we compute $LH^{\infty}(G;\underline{n})$ and $LH(G;\underline{n})$ for any abelian group G. We also show the following result by using Theorem 0.1.

Theorem 0.2. *Let G be an abelian group. The following are equivalent.*

(1) *Any (finite) homotopy representation of G with linear dimension function $\underline{n}$ is linear.*

(2) $LH^{\infty}(G;\underline{n})$ $(LH(G;\underline{n}))=1$.

In Section 4 we determine finite abelian groups such that $LH^{\infty}(G;\underline{n})$ $(LH(G;\underline{n}))=1$ for any linear dimension function $\underline{n}$. We show

Theorem 0.3. *Let G be an abelian group.*

(1) $LH^{\infty}(G;\underline{n})=1$ *for any linear dimension function $\underline{n}$ if and only if G is isomorphic to C_{p^m}(p: prime), C_6 or $C_2\times C_2$, where C_m denotes a cyclic group of order m.*

(2) $LH(G;\underline{n})=1$ *for any linear dimension function $\underline{n}$ if and only if G is isomorphic to C_{p^m}, G_2(abelian 2-group), G_3(abelian 3-group) or $(C_2)^n\times(C_3)^m$ $(n\geq 1, m\geq 1)$.*

Since the dimension function of any homotopy representation of G is linear if and only if G is a p-group ([6], [3]), we get the following corollary which includes tom Dieck's result mientioned above.

Corollary 0.4. *Let G be an abelian group.*

(1) *Any homotopy representation of G is linear if and only if G is a cyclic p-group $C_{p^m}(m\geq 0)$ or $C_2\times C_2$.*

(2) *Any finite homotopy representation of G is linear if and only if G is C_{p^m}, G_2 (abelian 2-group) or (G_3 (abelian 3-group).*

1. Homotopy representations and Picard groups

In this section, we recall some definitions and well-knonwn results from [6], [7], [9].

DEFINITION. A finite dimensional G-CW complex X is called a *homotopy representation* of G if, for any subgroup H of G, the H-fixed point set X^H is homotopy equivalent to a ($\dim X^H$)-dimensional sphere or empty. Furthermore if X is G-homotopy equivalent to a finite G-CW complex, it is called *finite*, and if X is G-homotopy equivalent to a linear G-sphere (i.e., a sphere of a real representation of G), it is called *linear*.

Let $S(G)$ denote the set of subgroups of G and $\phi(G)$ the set of conjugacy classes of subgroups of G. The dimension function is defined by

$$\operatorname{Dim} X(H) = \dim X^H + 1 .$$

We call a dimension function $\operatorname{Dim} X$ *linear* if there exists a linear G-sphere $S(V)$ such that $\operatorname{Dim} X = \operatorname{Dim} S(V)$.

Let $V^\infty(G)$ be the homotopy representation group of G, which is defined by the Grothendieck group of G-homotopy types of homotopy representations of G under join operator. Similarly $V(G)$ and $JO(G)$ are defined for finite and linear homotopy representations respectively. It is known ([6]) that there are the natural inclusions

$$JO(G) \subset V(G) \subset V^\infty(G) .$$

We define subgroups $v^\infty(G)$, $v(G)$, $jO(G)$ as follows.

$$\begin{aligned} v^\infty(G) &= \{X - Y \in V^\infty(G) \mid \operatorname{Dim} X = \operatorname{Dim} Y\}, \\ v(G) &= \{X - Y \in V(G) \mid \operatorname{Dim} X = \operatorname{Dim} Y\}, \\ jO(G) &= \{X - Y \in JO(G) \mid \operatorname{Dim} X = \operatorname{Dim} Y\}. \end{aligned}$$

Theorem 1.1 ([6]). *The subgroups $v^\infty(G)$, $v(G)$, $jO(G)$ are the torsion subgroups of $V^\infty(G)$, $V(G)$, $JO(G)$ respectively, and $v^\infty(G)$ is isomorphic to the Picard group* Pic(G). ■

The Picard group Pic(G) is defined as follows. Let $C(G)$ be the set of integer-valued functions on $\phi(G)$ and $A(G)$ the Burnside ring of G. We regard $A(G)$ as a subring of $C(G)$ by the usual way. We define finite rings

$$\begin{aligned} \bar{C}(G) &= C(G)/|G|C(G) , \\ \bar{A}(G) &= A(G)/|G|C(G) . \end{aligned}$$

and denote their unit groups by $\bar{C}(G)^*$, $\bar{A}(G)^*$.

DEFINITION ([4], [6]).

$$\operatorname{Pic}(G) = \bar{C}(G)^*/C(G)^* \bar{A}(G)^* .$$

Homotopy representation groups and Picard groups play important roles in the stable theory for homotopy representations.

E. Laitinen introduced the unstable Picard group to study the unstable theory for homotopy representations. For any dimension function $\underline{n} = \operatorname{Dim} X$, the following lemma holds.

Lemma 1.2 ([7, Lemma 2.1]). *For any subgroup H, there exists a unique maximal subgroup $\bar{H}$ including H such that $\underline{n}(\bar{H}) = \underline{n}(H)$.* ■

If $\underline{n}(H)>0$ and $H=\bar{H}$, then H is called an *essential isotropy subgroup*. We denote by $\mathrm{Iso}(\underline{n})$ the set of essential isotropy subgroups.

DEFINITION ([7]). We say that a function $d\in C(G)$ satisfies the unstability conditions for $\underline{n}$ if d satisfies the following conditions:

(1) $d(H)=1$ when $\underline{n}(H)=0$,

(2) $d(H)=-1, 0, 1$ when $\underline{n}(H)=1$,

(3) $d(\bar{H})=d(H)$ for any $(H)\in\phi(G)$.

We call $d\in C(G)$ *invertible* if $d(H)$ is prime to $|G|$ for any (H) The Picard group $\mathrm{Pic}(G;\underline{n})$ is defined as follows Let $C(G;\underline{n})$ (resp. $A(G;\underline{n})$) denote the subset of $C(G)$ (resp. $A(G)$) which consists of all functions satisfying the unstability conditions. Let $\bar{C}^*(G;\underline{n})$ (resp. $\bar{A}^*(G;\underline{n})$) denote the subgroup of $\bar{C}(G)^*$ (resp. $\bar{A}(G)^*$) which consists of all elements represented by invertible functions in $C(G;\underline{n})$ (resp. $A(G;\underline{n})$). Similarly $C^*(G;\underline{n})\subset C(G)^*$ is defined.

DENINITION ([7]).

$$\mathrm{Pic}(G;\underline{n})=\bar{C}^*(G;\underline{n})/C^*(G;\underline{n})\bar{A}^*(G;\underline{n}).$$

Laitinen's invariant that distinguishes G-homotopy types of two homotopy representations is defined in $\mathrm{Pic}(G;\underline{n})$. For convenience we recall this here.

Let X, Y be homotopy representations with the same dimension function $\underline{n}$. There is a G-map $f\colon Y\to X$ such that $\deg f^H$ is prime to $|G|$ for any H. If we choose orientations of X and Y in the sense of Laitinen, the degree function $d(f)$ defined by $d(f)(H)=\deg f^H$ is well-defined, and satisfies the unstability conditions. Laitinen defines the invariant by

$$D_{\underline{n}}(X, Y)=[d(f)]\in\mathrm{Pic}(G;\underline{n}).$$

Theorem 1.3 ([7]). *X and Y are G-homotopy equivalent if and only if $D_{\underline{n}}(X, Y)=1$ in $\mathrm{Pic}(G;\underline{n})$.* ■

2. The groups $jO(G;\underline{n})$ and $LH^\infty(G;\underline{n})$

We assume that $\underline{n}$ is linear throughout this section.

We first introduce $jO(G;\underline{n})$, which is considered as the unstable version of $jO(G)$. We define $jO(G;\underline{n})$ as the subset of $\mathrm{Pic}(G;\underline{n})$ which consists of all $D_{\underline{n}}(S(V), S(W))$ for linear G-spheres $S(V)$, $S(W)$ with dimension function $\underline{n}$.

We need the following lemma in order to show that $jO(G;\underline{n})$ is a subgroup of $\mathrm{Pic}(G;\underline{n})$.

Lemma 2.1. *For any $x\in jO(G;\underline{n})$ and for any $S(V)$ with dimension function $\underline{n}$, there exists $S(W)$ with diemension function $\underline{n}$ such that $D_{\underline{n}}(S(V), S(W))=x$.*

Proof. Take linear G-spheres $S(T)$ and $S(U)$ with dimension function $\underline{n}$ such that $D_{\underline{n}}(S(T), S(U))=x$. We set $y=D_{\underline{n}}(S(V), S(U))$. Let

$$U = U_1 \oplus \cdots \oplus U_r$$

be the irreducible decomposition. Since $S(V)$ and $S(U)$ have the same dimension function, by [14, Proposition 1.11]

$$V = \psi^{k_1} U_1 \oplus \cdots \oplus \psi^{k_r} U_r$$

for some $k_1, \cdots, k_r \in \boldsymbol{Z}$ which are prime to $|G|$, where ψ^{k_i} are the Galois conjugations. We may assume that $k_i \equiv 1$ (4) since $\psi^k = \psi^{-k}$ on $RO(G)$. By [14, Theorem 4.1], there exists a G-map f_i: $S(\psi^{k_i} U_i) \to S(U_i)$ such that

$$d(f_i)(H) = k_i^{[(1/2)\mathrm{Dim}\, S(U_i)(H)]},$$

where [ ] indicates integer part. It is easy to see that $d(f_i)$ is invertible and satisfies the unstability conditions for $\underline{n_i} = \mathrm{Dim}\, S(U_i)$. Hence $D_{\underline{n_i}}(S(U_i), S(\psi^{k_i} U_i)) = [d(f_i)]$. Similarly

$$T = \psi^{t_1} U_1 \oplus \cdots \oplus \psi^{t_r} U_r,$$

for some $t_1, \cdots, t_r \in \boldsymbol{Z}$, $(t_i, |G|)=1$, $t_i \equiv 1$ (4), and

$$D_{\underline{n_i}}(S(U_i), S(\psi^{t_i} U_i)) = [d(g_i)],$$
$$d(g_i) = t_i^{[(1/2)\mathrm{Dim}\, S(U_i)]}.$$

We set

$$\begin{aligned} W &= \psi^{k_1 t_1} U_1 \oplus \cdots \oplus \psi^{k_r t_r} U_r \\ &= \psi^{t_1} V_1 \oplus \cdots \oplus \psi^{t_r} V_r \ (V_i = \psi^{k_i} U_i). \end{aligned}$$

Since $\mathrm{Dim}\, S(U_i) = \mathrm{Dim}\, S(V_i)$, it follows that

$$D_{\underline{n_i}}((S(V_i), S(\psi^{t_i} V_i)) = D_{\underline{n_i}}(S(U_i), S(\psi^{t_i} U_i)).$$

Therefore

$$\begin{aligned} D_{\underline{n}}(S(V), S(W)) &= \prod_{i=1}^{r} \alpha_{\underline{n_i}, \underline{n}}(D_{\underline{n_i}}(S(V_i), S(\psi^{t_i} V_i)) \\ &= \prod_{i=1}^{r} \alpha_{\underline{n_i}, \underline{n}}(D_{\underline{n_i}}(S(U_i), S(\psi^{k_i} U_i)) \\ &= D_{\underline{n}}(S(U), S(T)) = x, \end{aligned}$$

where $\alpha_{\underline{n_i}, \underline{n}}$: $\mathrm{Pic}(G; \underline{n_i}) \to \mathrm{Pic}(G; \underline{n})$ are the natural maps. This shows Lemma 2.1. ■

Proposition 2.2. *The subset $jO(G, \underline{n})$ forms a subgroup of* $\mathrm{Pic}(G; \underline{n})$.

Proof. Take any $x, y \in jO(G; \underline{n})$. By Lemma 2.1, there are $S(V)$, $S(W)$

and $S(U)$ with dimension function $\underline{n}$ such that $D_{\underline{n}}(S(V), S(W))=x$ and $D_{\underline{n}}(S(V), S(U))=y$. Using [11, Lemma 1.6],

$$\begin{aligned} xy^{-1} &= D_{\underline{n}}(S(V), S(W))D_{\underline{n}}(S(V), S(U))^{-1} \\ &= D_{\underline{n}}(S(U), S(W)) \in jO(G; \underline{n}) . \qquad \blacksquare \end{aligned}$$

We next define the group $\mathrm{Pic}^f(G; \underline{n})$.

It is known ([6], [8]) that there exists the finiteness obstruction homomorphism

$$\rho\colon \mathrm{Pic}(G) \to \kappa(G) := \bigoplus_{(H)} \tilde{K}_0(\boldsymbol{Z}WH) .$$

Here $WH=NH/H$ and NH is the mormalizer of H in G. $\tilde{K}_0(\boldsymbol{Z}WH)$ denotes the reduced projective group of $\boldsymbol{Z}WH$. Let ρ_n be the composite of ρ and the natural homomorphism $\mathrm{Pic}(G; \underline{n}) \to \mathrm{Pic}(G)$. We define $\mathrm{Pic}^f(G; \underline{n})$ as $\mathrm{Ker}\,\rho_n$.

Since a linear G-sphere is a finite homotopy representation, it follows that

$$jO(G; \underline{n}) \subset \mathrm{Pic}^f(G; \underline{n}) .$$

We define groups $LH^\infty(G; \underline{n})$ and $LH(G; \underline{n})$ for any linear dimension function $\underline{n}$.

DEFINITION.

$$\begin{aligned} LH^\infty(G; \underline{n}) &= \mathrm{Pic}(G; \underline{n})/jO(G; \underline{n}) , \\ LH(G; \underline{n}) &= \mathrm{Pic}^f(G; \underline{n})/jO(G; \underline{n}) . \end{aligned}$$

Let X be any homotopy representation with linear dimension function $\underline{n}$. We define $l(X) \in LH^\infty(G; \underline{n})$ by

$$l(X) = [D_{\underline{n}}(S(V), X)] ,$$

where $S(V)$ is any linear G-sphere with dimension function $\underline{n}$. This definition is independent of the choice of $S(V)$. Indeed let $S(W)$ be another linear G-sphere with dimension function $\underline{n}$. Then

$$D_{\underline{n}}(S(V), X)D_{\underline{n}}(S(W), X)^{-1} = D_{\underline{n}}(S(V), S(W)) \in jO(G; \underline{n}) .$$

We now prove Theorem 0.1.

Proof of Theorem 0.1. If $l(X)=1$, then $D_{\underline{n}}(S(V), X) \in jO(G; \underline{n})$. By Lemma 2.1, we can take $S(W)$ such that $D_{\underline{n}}(S(V), S(W))=D_{\underline{n}}(S(V), X)$. This implies that $D_{\underline{n}}(S(W), X)=1$. By Theorem 1.3, X is G-homotopy equivalent to $S(W)$. The converse is trivial. $\blacksquare$

We state the following theorem which follows from [11, Theorem 2.1].

Theorem 2.3. *Suppose that $\underline{n}$ satisfies the following condition* (H).

(H): $\underline{n}(H)\equiv\underline{n}(G)$ *mod* 2 *for any* $H\in\mathcal{S}_0:=\{H\in S(G)\,|\,\underline{n}(H)\leq 3\}$.

(1) *For any* $\alpha\in LH^{\infty}(G;\underline{n})$, *there exixts a homotopy representation* X *with dimension function n such that* $l(X)=\alpha$.

(2) *Furthermore if* $\alpha\in LH(G;\underline{n})$, *then* X *can be taken to be finite.*

Proof. (1): Let $x\in\mathrm{Pic}(G;\underline{n})$ be a representative of α. By [11, Theorem 2.1], there exist X and $S(V)$ with dimension function $\underline{n}$ such that $D_{\underline{n}}(S(V),X)=x$. (Note that conditions of [11, Theorem 2.1] are satisfied because of the condition (H) and linearity of $\underline{n}$.) This implies that $l(X)=\alpha$.

(2): Since the finiteness obstructions of x and $S(V)$ vanish, the finiteness obstruction of X also vanishes. Hence X is finite. ■

3. Computation of $LH^{\infty}(G;\underline{n})$ for abelian groups

Throughout this section, a dimension function $\underline{n}$ is linear and G is abelian.

We first compute $jO(G;\underline{n})$ and $\mathrm{Pic}^f(G;\underline{n})$. From [11], there is the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{Pic}(G;\underline{n}) & \longrightarrow & \mathrm{Pic}(G) & \xrightarrow{\rho} & \kappa(G) \\
\big\downarrow \bar{\mu}_{\underline{n}} & & \big\downarrow \bar{\mu} & & \Pi_H S_{G/H} \\
\prod_{H\in\mathrm{Iso}(\underline{n})} \boldsymbol{Z}/|G/H|^*/\pm 1 & \longrightarrow & \prod_{H\in S(G)} \boldsymbol{Z}/|G/H|^*/\pm 1 & &
\end{array}$$

Here $S_{G/H}:\boldsymbol{Z}/|G/H|^*/\pm 1\to\tilde{K}_0(\boldsymbol{Z}[G/H])$ is the Swan homomorphism and ρ is the finiteness obstruction homomorphism. The maps $\bar{\mu}$ and $\bar{\mu}_{\underline{n}}$ are isomorphisms which are defined as follows. The H-part $u(H)$ of $\bar{\mu}([d])$ is defined by $u(H)=\prod_{H\leq K\leq G} d(K)^{\mu(H,K)}$ (or equivalently $d(K)=\prod_{K\leq H\leq G} u(H)$), where $\mu(H,K)$ denotes the Möbius function on the subgroup lattice. The Möbius function is characterized as integers satisfiying the equations:

$$\sum_{H\leq K\leq L}\mu(H,K)=\begin{cases}1, & H=L\\ 0, & H\neq L,\end{cases}$$

for any H, L $(H\leq L)$. The isomorphism $\bar{\mu}_{\underline{n}}$ is defined as the restriction of $\bar{\mu}$ to $\mathrm{Pic}(G;\underline{n})$. Hence we have

Proposition 3.1. *The isomorphism* $\bar{\mu}_{\underline{n}}$ *induces an isomorphism*

$$\begin{aligned}\mathrm{Pic}^f(G;\underline{n}) &\cong \prod_{H\in\mathrm{Iso}(\underline{n})}\mathrm{Ker}\,S_{G/H}\\ &=\prod_{\substack{H\in\mathrm{Iso}(\underline{n})\\ G/H\,:\,\text{cyclic}}}\boldsymbol{Z}/|G/H|^*/\pm 1\times\prod_{\substack{H\in\mathrm{Iso}(\underline{n})\\ G/H\,:\,\text{non-cyclic}}}\mathrm{Ker}\,S_{G/H}\,.\end{aligned}$$

Proof. Since $S_G=0$ for a cyclic group G ([12]), the second equality follows. ■

Let $S(V)$ have the dimension function $\underline{n}$. Let $V(K)$ denote the direct sum of irreducible subrepresentations of V with kernel K. Then V decomposes into

$$V = V(K_1) \oplus \cdots \oplus V(K_r)\,.$$

for some K_i.

Let $\mathcal{K}(\underline{n})$ denote the set of such K_i's. We notice that $\mathcal{K}(\underline{n})$ is independent of the choice of V with dimension function $\underline{n}$. One can also see the following.

Lemma 3.2.

(1) *If $K \in \mathcal{K}(\underline{n})$, then G/K is cyclic.*

(2) $\mathcal{K}(\underline{n}) \subset \mathrm{Iso}(\underline{n})$.

(3) *For any set $\mathcal{K} = \{K_1, \cdots, K_r\}$ of subgroups such that G/K_i's are cyclic, there exists a linear dimension function $\underline{n}$ such that $\mathcal{K}(\underline{n}) = \mathcal{K}$.* ■

We next show

Proposition 3.3. *The homomorphism $\bar{\mu}_{\underline{n}}$ induces an isomorphsim*

$$jO(G; \underline{n}) \cong \prod_{K \in \mathcal{K}(\underline{n})} \boldsymbol{Z}/|G/K|^* / \pm 1\,.$$

Proof. Let

$$\begin{aligned} V &= V(K_1) \oplus \cdots \oplus V(K_r)\,, \\ W &= W(K_1) \oplus \cdots \oplus W(K_r)\,, \end{aligned}$$

where $\mathrm{Dim}\, S(V) = \mathrm{Dim}\, S(W) = \underline{n}$. There are G-maps $f_i \colon S(V(K_i)) \to S(W(K_i))$ such that $(d(f_i)(H), |G|) = 1$ for any H. Then the degree function $d(f)$ of $f = f_1 * \cdots * f_r$ represents $D_{\underline{n}}(S(W), S(V))$, and

$$D_{\underline{n}}(S(W), S(V)) = [d(f_1)] \cdots [d(f_r)]\,.$$

It is seen that

$$d(f_i)(H) = \begin{cases} d(f_i)(1) & \text{if } H \leq K_i \\ 1 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \bar{\mu}_{\underline{n}}([d(f_i)])(H) &= \prod_{H \leq K \leq K_i} d(f_i)(1)^{\mu(H,K)} \\ &= d(f_i)(1)^{\Sigma \mu(H,K)} \\ &= \begin{cases} d(f_i)(1) & \text{if } H = K_i \\ 1 & \text{if } H \neq K_i\,. \end{cases} \end{aligned}$$

It follows that

$$\bar{\mu}_{\underline{n}}(jO(G;\underline{n}))\subset \prod_{K\in\mathcal{K}(\underline{n})} \boldsymbol{Z}/|G/K|^*/\pm 1\,.$$

Next, take any $(a_K)\in \prod_{K\in\mathcal{K}(\underline{n})} \boldsymbol{Z}/|G/K|^*/\pm 1$. Let W be a representation of G with $\operatorname{Dim} S(W)=\underline{n}$. Then $S(W(K))$ has a free G/K-action and G/K is cyclic. It is seen that there exists a G/K-map $f_K\colon S(V_K)\to S(W(K))$ (for some V_K) such that $\deg f_K=a_K$. Then $\bar{\mu}_{\underline{n}}(d(\underset{K}{*}f'_K))=(a_K)$. Thus $\bar{\mu}_{\underline{n}}$ induces an isomorphism. ∎

Consequently we obtain

Corollay 3.4.

$$LH^\infty(G;\underline{n})\cong \prod_{H\in\mathrm{Iso}(\underline{n})\setminus\mathcal{K}(\underline{n})} \boldsymbol{Z}/|G/H|^*/\pm 1\,,$$

$$LH(G;\underline{n})\cong M(G;\underline{n})\times(G;\underline{n})\,,$$

where

$$M(G;\underline{n})=\prod_{\substack{H\in\mathrm{Iso}(\underline{n})\setminus\mathcal{K}(\underline{n})\\ G/H:\,\text{cyclic}}} \boldsymbol{Z}/|G/H|^*/\pm 1\,,$$

$$N(G;\underline{n})=\prod_{\substack{H\in\mathrm{Iso}(\underline{n})\\ G/H:\,\text{non-cyclic}}} \operatorname{Ker} S_{G/H}\,. \qquad ∎$$

We now prove Theorem 0.2.

Proof of Theorem 0.2. (2)⇒(1): This follows from Theorem 0.1.

(1)⇒(2): Let X be a homotopy representation with linear dimension function $\underline{n}$. When $\underline{n}$ satisfies (H) in Theorem 2.3, this direction follows from Theorem 2.3. When $\underline{n}$ does not satisfy (H), we put $\underline{n}'=\underline{n}+4$. Then $\underline{n}'$ is linear and satisfies (H). Furthermore $\mathrm{Iso}(\underline{n}')=\mathrm{Iso}(\underline{n})$ or $\mathrm{Iso}(\underline{n})\cup\{G\}$, and $\mathcal{K}(\underline{n}')=\mathcal{K}(\underline{n})$ or $\mathcal{K}(\underline{n})\cup\{G\}$. Therefore, by Corollary 3.4, $LH^{(\infty)}(G;\underline{n}')$ and $LH^{(\infty)}(G;\underline{n})$ are isomorphic, and so $LH^{(\infty)}(G;\underline{n})=1$. ∎

4. Abelian groups with $LH^\infty(G;\underline{n})=1$

Throughout this section, a dimension function $\underline{n}$ is linear and G is abelian. In this section, we prove Theorem 0.3.

In order to prove Theorem 0.3, we show several lemmas.

Lemma 4.1. *For any $H\in\mathrm{Iso}(\underline{n})$, there exist $K_i\in\mathcal{K}(\underline{n})$ such that $H=\bigcap_i K_i$. In particular* $\mathrm{Iso}(\underline{n})=\{\bigcap_{i=1}^{r} K_i\,|\,K_i\in\mathcal{K}(\underline{n}),\ r\geq 1\}$.

Proof. Since $\underline{n}(H)>0$, it is seen that there exists at least one $K_i\in\mathcal{K}(\underline{n})$ including H. Let H' be the intersection of subgroups in $\mathcal{K}(\underline{n})$ including H.

Then H' is in $\mathrm{Iso}(\underline{n})$ since $\mathrm{Iso}(\underline{n})$ is closed under intersection. Furthermore one can see that $\underline{n}(H)=\underline{n}(H')$. In fact, let us choose V such that $\underline{n}=\mathrm{Dim}\, S(V)$. Then

$$\begin{aligned}\underline{n}(H) = \dim V^H &= \sum_{\substack{H\leq K\\ K\in\mathcal{K}(n)}} \dim V(K)\\ &= \sum_{\substack{H'\leqq K\\ K\in\mathcal{K}(n)}} \dim V(K)\\ &= \dim V^{H'} = \underline{n}(H')\,.\end{aligned}$$

Therefore $H=H'$. ■

Lemma 4.2. *If $K_1, K_2\in\mathcal{K}(\underline{n})$ and if $G/K_1\cap K_2$ is a cyclic p-group, then $K_1\leq K_2$ or $K_1\geq K_2$. In particular $K_1\cap K_2\in\mathcal{K}(\underline{n})$.*

Proof. There are two subgroups $K_1/K_1\cap K_2$ and $K_2/K_1\cap K_2$ of $G/K_1\cap K_2$. Since $G/K_1\cap K_2$ is a cyclic p-group, it follows that $K_1/K_1\cap K_2\geq K_2/K_1\cap K_2$ or $K_1/K_1\cap K_2\leq K_2/K_1\cap K_2$. Hence $K_1\leq K_2$ or $K_1\geq K_2$. ■

Corollary 4.3. *If $H\in\mathrm{Iso}(\underline{n})$ and G/H is a cyclic p-group, then $H\in\mathcal{K}(\underline{n})$. In particular if G is a cyclic p-group, then $\mathrm{Iso}(\underline{n})=\mathcal{K}(\underline{n})$.*

Proof. This follows from Lemmas 4.1 and 4.2. ■

Proof of Theorem 0.3. (1): In the case where $G=C_{p^m}$, $LH^\infty(G;\underline{n})=1$ by Corollaires 3.4 and 4.3.

In the case where $G=C_6$ or $C_2\times C_2$, $LH^\infty(G;\underline{n})=1$ by Corollary 3.4.

In the case where G is cyclic and $|G|$ is neither prime power nor 6, there exists K such that $G/K\cong C_{pq}$, where p, q are distinct primes, and there exist $K_1, K_2>K$ such that $G/K_1\cong C_p$ and $G/K_2\cong C_q$. By Lemma 3.2 (3), there exists $\underline{n}$ such that $\mathcal{K}(\underline{n})=\{K_1, K_2\}$. Since $K_1\cap K_2=K\in\mathrm{Iso}(\underline{n})$, it follows from Corollary 3.4 that $1\neq \boldsymbol{Z}/|G/K|^*/\pm 1\subset LH^\infty(G;\underline{n})$.

In the case where G is neither a cyclic group nor $C_2\times C_2$, let $\mathcal{L}$ be the set of subgroups K such that G/K are cyclic and $\underline{n}$ a the dimension function such that $\mathcal{K}(\underline{n})=\mathcal{L}$. Since $1\in\mathrm{Iso}(\underline{n})\backslash\mathcal{K}(\underline{n})$ by Lemma 4.1, it follows that $1\neq\boldsymbol{Z}/|G|^*/\pm 1\subset LH^\infty(G;\underline{n})$.

(2): In the case where $G=C_{p^m}$, since $LH(G;\underline{n})\subset LH^\infty(G;\underline{n})$, it follows that $LH(G;\underline{n})=1$.

In the case where G is G_2 or G_3, since $\mathrm{Ker}\, S_{G/H}=1$ if G/H is cyclic ([10, Lemmas 3.9 and 3.10]), it follows from Corollary 3.4 that $N(G;\underline{n})=1$. By Corollary 4.3, $M(G;\underline{n})=1$. Therefore $LH(G;\underline{n})=1$.

In the case where $G=(C_2)^n\times(C_3)^m$, since $\mathrm{Ker}\, S_{G/H}=1$ if G/H is cyclic ([10, Lemma 3.11]), it follows from Corollary 3.4 that $N(G;\underline{n})=1$. If G/H is cyclic, then $G/H\cong 1$, C_2, C_3 or C_6. Therefore it follows that $M(G;\underline{n})=1$ and $LH(G;\underline{n})=1$.

In the case where G is cyclic and $|G|$ is neither prime power nor 6, by the same argument as in (1), one can see that $LH(G;\underline{n})\neq 1$.

In the case where G is a non-cyclic p-group ($p\neq 2, 3$), by the same argument as in (1), one can see that $\operatorname{Ker} S_G\subset N(G;\underline{n})$. By [13], the order of $\operatorname{Im} S_G$ is $|G|/p$. Hence $|\boldsymbol{Z}/|G|^*/\pm 1|>|\operatorname{Im} S_G|$ and so $\operatorname{Ker} S_G\neq 1$.

Finally we consider the case where G is neither cyclic nor of prime power order and furthermore G is not $(C_2)^n\times(C_3)^m$. Assume that a prime p (≥ 5) divides $|G|$. Then there exists K such that $G/K\cong C_{pq}(p, q$ are distinct primes) and there exist $K_1, K_2>K$ such that $G/K_1\cong C_p$ and $G/K_2\cong C_q$. As in (1) one can see that $LH(G;\underline{n})\neq 1$. Next assume that the order of G is 2^n3^m. Since G is not $(C_2)^n\times(C_3)^m$, there exists K such that $G/K\cong C_4\times C_3$ or $C_2\times C_9$. Assume that $G/K\cong C_4\times C_3$. Then there exist $K_1, K_2>K$ such that $G/K_1\cong C_4$ and $G/K_2\cong C_3$. Take $\underline{n}$ such that $\mathcal{K}(\underline{n})=\{K_1, K_2\}$. Since $K=K_1\cap K_2\in\operatorname{Iso}(\underline{n})\backslash\mathcal{K}(\underline{n})$, if follows that $1\neq\boldsymbol{Z}/|G/K|^*/\pm 1\subset LH(G;\underline{n})$. In the case where $G/K\cong C_2\times C_9$, by the same argument, one can see that $LH(G;\underline{n})\neq 1$.

Thus the proof is completed. ■

Proof of Corollary 0.4. Since the dimension function of any homotopy representation (not necessarily finite) is linear if and only if G is a p-group ([6], [3]). Therefore the first assertion is clear from Theorem 0.3. For the second assertion, it suffices to show that there exists a finite homotopy representation of $(C_2)^n\times(C_3)^m$ with nonlinear dimension function. By the theorem mentioned above, there exists a homotopy representation X of C_6 with nonlinear dimension function. Since the finiteness obstruction of a homotopy representation of a cyclic group vanishes, the homotopy representation X is finite. We consider X as a finite homotopy representation of $(C_2)^n\times(C_3)^m$ via a surjective homomorphism from $(C_2)^n\times(C_3)^m$ to C_6. ■

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan