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Stonian Spaces and the Second Conjugate Spaces of AM Spaces

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Let $X$ be a compact space and let $C(X)$ be the set of all real-valued continuous functions on $X$. If any non-void subset of $C(X)$ with an upper bound has a least upper bound in $C(X)$, such a compact space $X$ is called a stonian space. Stone [10] has shown that a compact space $X$ is stonian if and only if it is extremally disconnected, that is to say, if for any open set $U$ in $X$ its closure $\bar{U}$ is open. While, Kelley [9] has proved that if for any Banach space $F$ containing a Banach space $E$ there exists a projection of $F$ on $E$ whose norm is 1, $E$ is isometric to $C(X)$, where $X$ is stonian. Also Dixmier [4] has considered a compact space $X$ such that $C(X)$ is isomorphic to an $L^\infty(R, \mu)$ as Banach algebras, where $R$ is a locally compact space and $\mu$ is a positive measure on $R$. He called such a space $X$ a hyperstonian space. A hyperstonian space is stonian. We shall see that a compact space $X$ is hyperstonian if and only if $C(X)$ is lattice-isomorphic and isometric to a conjugate space of an AL space.

In §1 we state some general properties of stonian spaces, and in §2 we consider an AM space $C(X)$ which is the second conjugate space of an AM space. Such a space $X$ is hyperstonian, and if the character of $X$ is countable, then $X$ is the space $\beta N_0$, where $N_0$ is a discrete space whose cardinal number is at most countable (cf. Theorem 3, Corollary).

§ 1. Stonian spaces

For a completely regular space $X$, let $\beta X$ denote the Čech compactification of $X$. (cf. Čech [2]). Dixmier [4] has shown that $\beta U = X$ for any open dense set $U$ in a stonian space $X$. Therefore we obtain easily the following:

2) See Kakutani [7] and [8].
(i) If \( X \) is stonian if and only if for any open set \( U \) in \( X \), \( \beta U \subset X \).

(ii) If \( X \) is a stonian space and if \( X \) has an open dense subspace which satisfies the 1st axiom of countability, \( X \) is a space \( \beta N \), where \( N \) is a discrete space.

(iii) If \( X \) is a stonian space and if a point \( x \) of \( X \) is not isolated, then \( \{x\} \) is not a \( G_\delta \) set.

We shall prove here the following theorem.

**Theorem 1.** Let \( X \) be a stonian space. If \( X \) is a product \( R \times R \) of a compact space \( R \), then \( X \) is finite.

Proof. Suppose that a stonian \( X \) be of the form \( R \times R \), where \( R \) is a compact space with infinite points. Let \( U \) be a dense open set in \( R \). Then \( R \times R = \beta U \times \beta U \supset U \times U \) and \( U \times U \) is a dense open set in \( R \times R \). Since \( R \times R \) is stonian, \( \beta(U \times U) = R \times R = \beta U \times BU \).

Now if for a fully normal space \( S \) \( \beta(S \times S) = \beta S \times \beta S \), then \( S \) is compact. (Ishiwata [6]). Therefore it is sufficient to prove that there exists an open dense subset \( U \) in \( R \) which is not compact but fully normal. We shall construct such an open set \( U \). Since \( R \) is an infinite set, there exists a countable family \( \{U_i\}_{i=1}^\infty \) of mutually disjoint non void sets which are both open and closed. Let \( V = R - \sum_{i=1}^\infty U_i \) and let \( U = V \cup \sum_{i=1}^\infty U_i \). Since the set \( U \) is the union of a countable family of open and compact sets in \( R \) which are mutually disjoint, \( U \) is fully normal. Clearly, \( U \) is not compact and is dense in \( R \).

**REMARK.** Theorem 1 shows that there exists no stonian space \( S \) with infinite points of the form \( R \times R \). But we can find easily a totally disconnected compact space \( S \) which is a product space \( R \times R \), where \( R \) is compact and infinite.

§ 2. **Second conjugate spaces of \( AM \) spaces.**

Let \( X \) be a stonian space and let \( M(X) \) be the set of all measures on \( X \). A positive measure \( \mu \) on \( X \) is called a *normal measure* if for any nowhere dense set \( A \), \( \mu(A) = 0 \). A real measure \( \mu \) on \( X \) is called *normal* if its positive part and its negative part are both normal. Let \( M'(X) \)

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3) Henrikson and Isbell announced the following theorem (Bull. Amer. Soc. Vol. 63. 1957 Abstract): if \( X \) and \( Y \) are infinite completely regular spaces such that \( \beta(X \times Y) = \beta X \times \beta Y \), then \( X \times Y \) is pseudo-compact, that is, any continuous function on \( X \times Y \) is always bounded. If we make use of this theorem, we obtain moreover that if a stonian space \( X \) is a product \( R \times S \) of compact spaces \( R \) and \( S \), then either \( R \) or \( S \) is finite.

denote the set of all normal measures on $X$. A stonian space $X$ is called \textit{hyperstonian} if it has positive normal measures, the union of whose carriers is dense in $X$. We shall see that a compact space $X$ is hyperstonian if and only if $C(X)$ is lattice-isomorphic and isometric to the conjugate space of an $AL$ space. Let $E$ be an $AM$ space. Then the second conjugate space of $E$ is lattice-isomorphic and isometric to $C(X)$, where $X$ is hyperstonian, and the conjugate space of $E$ is lattice-isomorphic and isometric to $M'(X)$. $M'(X)$ is also lattice-isomorphic and isometric to an $L'(\Omega, \mu)$, where $\Omega$ is an open dense set in $X$ and $\mu$ is a suitable positive measure on $\Omega$. (cf. [4]). We consider now an $AM$ space $C(X)$ which is the second conjugate space of an $AM$ space. Let $E$ be a Banach space and $E^*, E^{**}$ denote the conjugate space of $E$ and the second conjugate space of $E$ respectively. For any closed linear subspace $V$ in $E^*$ we define its \textit{characteristic} $r$ by $r = \inf \sup_{f \in E^*, x \in S} \frac{|f(x)|}{\|x\|}$, where $S$ is a unit sphere in $E^*$. A closed linear subspace $V$ in $E^*$ is called \textit{minimally weakly dense} if it is weakly dense in $E^*$ and if any other closed subspace in $V$ is not weakly dense in $E^*$.

The following lemma was proved by Dixmier [3].

\textbf{Lemma.} (i) Let $E$ be a Banach space. Then $E$ is a minimally weakly dense subspace in $E^{**}$ which is characteristic one.

(ii) If $V$ is a minimally weakly dense subspace in $E^*$ which is characteristic one, then $E^{**} = E \oplus V^*$ and $\|x\| \leq \|x + z\|$ for $x \in E$, $z \in V^*$, where $V^*$ denotes the set $\{z \mid z \in E^{**}, z(f) = 0 \text{ for any } f \in V\}$.

Let $K$ be an open set in a hyperstonian space. Then the \textit{character} of $K$ is said to be countable if any family of non-void open and closed sets in $K$ which are mutually disjoint is at most countable.

We can prove the following theorem. Hereafter $X$ denotes a hyperstonian space.

\textbf{Theorem 2.} $C(X)$ is lattice-isomorphic and isometric to the second conjugate space of an $AM$ space with a unit if and only if there exists a lattice-closed and (topologically) closed linear subspace $V$ in $C(X)$ which has constant functions such that

(I) for any $f \in C(X)$ and for any open and closed set $K$ (in $X$) whose character is countable, there exists a sequence $\{f_n\}$ in $V$ such that $f_n$ pointwise converges to $f$ on $K$ except a nowhere dense set.

(II) $V$ is a minimal closed linear space which has the property (I): any other closed subspace in $V$ does not satisfy (I).

Proof. (a) Let $C(X)$ be lattice-isomorphic and isometric to $E^{**}$
and let $E$ be an $AM$ space with a unit. Then, by Lemma, there exists a minimally weakly dense subspace $V$ in $C(X)$ which is lattice-closed. We see here that $E$ and $E^*$ are lattice-isomorphic and isometric to $V$ and $M'(X)$ respectively. Since $E$ has a unit, we can assume that $V$ has constant functions in $C(X)$. In order to prove (I) and (II), we are only to prove the equivalence of (I) and that $V$ is weakly dense. Now if the property (I) is satisfied, then we see easily that $V$ is weakly dense. Conversely, if $V$ is weakly dense, then we see easily that for any $f \in C(X)$ and for any open and closed set $K$ (in $X$) whose character is countable, there exist $f_n \in V$ such that

$$\int_{K \cap \Omega} |f(x) - f_n(x)| \, d\mu(x) < \frac{1}{n} \quad (n = 1, 2, \ldots),$$

where $\Omega$ is an open dense set in $X$ and $\mu$ is a suitable positive measure on $\Omega$. Therefore, as is well known, a subsequence $f_{n_j}$ of $\{f_n\}$ pointwise converges to $f$ almost everywhere on $K \cap \Omega$. Since any set of measure null on $K \cap \Omega$ is nowhere dense, $f_{n_j}$ pointwise converges to $f$ on $K$ except a nowhere dense set.

(b) If properties (I) and (II) are satisfied, we see easily that $V$ is a minimally weakly dense subspace in $C(X)$. (cf. (a)). We shall prove that $V$ is of characteristic one. For any $u \in M'(X)$, let $A$ and $B$ be carriers of the positive part $u^+$ of $u$ and of the negative part $u^-$ of $u$ respectively and let the function $f$ take the value 1 on $A$ and the value $-1$ on $B$. Then, by (I), there exists a sequence $\{f_n\}$ in $V$ such that $f_n$ pointwise converges to $f$ on $A \cup B$ except a nowhere dense set. We may assume here that for any $n$, $\|f_n\| \leq 1$, since $V$ has constant functions.\(^5\)

Since $u(f_n)$ converges to $\|u\|$, $V$ is of characteristic one. By Lemma, $M(X) = M'(X) \oplus V^+$ and $\|u\| \leq \|u + z\|$ for $u \in M'(X)$, $z \in V^+$. Therefore if $F$ is a linear functional on $V$, then there exists $u \in M'(X)$ such that $F(f) = u(f)$ for any $f \in V$ and $\|F\| = \|u\|$, that is, $M'(X)$ is lattice-isomorphic and isometric to $V^*$, and $C(X)$ is lattice-isomorphic and isometric to $V^{**}$. This concludes the proof.

We consider next an $AL$ space with an $F$-unit. Let $l'$ be the set of all sequences $\{\xi_i\}$ of real numbers with convergent $\sum_{i=1}^{\infty} |\xi_i|$. $l'$ is a Banach space where the norm of $x = \{\xi_i\} \in l'$ is $\sum_{i=1}^{\infty} |\xi_i|$. (cf. Banach [1]).

**Theorem 3.** If an $AL$ space $E$ with an $F$-unit\(^5\) is lattice-isomorphic
and isometric to a conjugate space of an AM space, E is lattice-isomorphic and isometric to $l'$. 

Proof. Let $E$ be of the form $L'(\Omega, \mu)$, where is an open set in a hyperstonian space. Since $E$ has an $F$-unit, the character of $\Omega$ is countable. If $L'(\Omega, \mu)$ is the conjugate space of an AM space $F$ and if $F$ is of the form of $C(Y, y', \lambda, \lambda) = \{f | f \in C(Y), f(y') = \lambda f'(y_a), 0 < \lambda_a < 1, a \in \mathbb{N}\}$, then function $g_{\alpha}$ in $L'(\Omega, \mu)$ which correspond to $\mu_{y_a}' \in F^*$ are mutually distinct, where $\mu_{y_a}'$ is a dirac measure, that is to say $\mu_{y_a}'(f) = f(y_a)$ for any $f \in F$. We see easily that the carrier of function $g_{\alpha}$ is a one-point set $x_{\alpha}$, and therefore, $x_{\alpha}$ is an isolated point in $\Omega$. Since the character of $\Omega$ is countable, the cardinal number of $Z_0 = \{y_a\}_{a \in \mathbb{N}}$ is at most countable. Since $C(Y, y') \supset F$, any linear functional $\xi$ on $F$ can be extended to a linear functional $\xi'$ on $C(Y)$. $\xi'$ is a measure on $Y$ and for any $f \in F$, $\xi(f) = \xi'(f) = \int_Y f(x) d\xi'(x)$. Since the cardinal number of $Y_0$ is countable, we can put $Z_0 = \{z_1, z_2, \ldots\}$. For any $n$, let $Y_n$ denote the set of $y_{\beta}$ with $Y_\beta = z_n$. Then we have $\xi(f) = \sum_{n=1}^{\infty} \int_{Y_n} f(y) d\xi'(y) = \sum_{n=1}^{\infty} (\int_{Y_n \supset Y_\beta} \lambda_{\beta} d\xi'(y_{\beta}) f(z_n)).$ If we put $p_n = \int_{Y_n \supset Y_\beta} \lambda_{\beta} d\xi'(y_{\beta})$, we obtain that $\xi(f) = \sum_{n=1}^{\infty} p_n f(z_n)$. We see easily that if $\xi$ is positive, any $p_n$ is non-negative and $\|\xi\| = \sum_{n=1}^{\infty} p_n$.

**Corollary.** If $C(X)$ is lattice-isomorphic and isometric to the second conjugate space of an AM space and if the character of $X$ is countable, then $X$ is the space $\beta N_0$, where $N_0$ is a discrete space whose cardinal number is at most countable.

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**Bibliography**


7) See Kakutani [8], Theorem 1.
8) We may assume that $g_{\alpha}$ is a continuous function on $\Omega$. See [4].


