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LIFTINGS OF IRREDUCIBLE CHARACTERS OF FINITE REDUCTIVE GROUPS

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Introduction. Let $\boldsymbol{G}$ be a connected linear algebraic group defined over a finite field $k=\boldsymbol{F}_q$ of characteristic p with Frobenius σ. For any set X on which σ acts, X_σ is the set of σ-fixed points. T. Shintani [8] constructed an intrinsic bijection of $(\boldsymbol{G}_\sigma)^\wedge$ onto $(\boldsymbol{G}_{\sigma^m})^\wedge_\sigma$ in the case of $\boldsymbol{G}=GL_n$, where $G^\wedge$ is the set of irreducible characters of G. In the case of $\boldsymbol{G}=U_n$, an analogous result is obtained by N. Kawanaka [4]. Let us give the construction of the above mentioned bijection due to Shintani in a slightly modified manner. Let m be a fixed natural number, put $G=\boldsymbol{G}_{\sigma^m}$ and let A be a cyclic group of order m with generator σ'. We suppose that A acts on G by $x^{\sigma'}=x^\sigma (x\in G)$. In the following we write σ for σ'. Define the semidirect product AG by $\sigma^{-1}x\sigma=x^\sigma\,(x\in G)$. For any integer i, we construct a *norm map* N_i from the subset $\sigma^i G$ of AG to the group $G_{\sigma^i}(=\boldsymbol{G}_{\sigma^{(m,i)}})$ which induces a bijection from the set of G-conjugacy classes of $\sigma^i G$ onto the set of conjugacy classes of G_{σ^i}. Moreover this bijection is compatible with the σ-action. (See 3.2.) Denote the set of complex valued class functions on G by $\mathcal{C}(G)$. For any integer i, we define the i-restriction map of $\mathcal{C}(AG)$ to $\mathcal{C}(G_{\sigma^i})_\sigma$ as follows:

$$(i\text{-res}\, f)\circ N_i = f\,|_{\sigma^i G},\ f\in\mathcal{C}(AG)\,.$$

These i-restrictions define an isomorphism

$$(*)\qquad \mathcal{C}(AG) \xrightarrow{\sim} \oplus_{i=0}^{m-1}\mathcal{C}(G_{\sigma^i})_\sigma\,.$$

Let $\psi\in(G_\sigma)^\wedge$ and $\chi\in(G^\wedge)_\sigma$. The character χ is called the lifting of ψ('lift ψ') if there exists an irreducible character $\chi^\sim$ of AG such that 0-res $\chi^\sim=\chi$ and 1-res $\chi^\sim=\pm\psi$. Shintani and Kawanaka have proved that the lifting map is a bijection from $(G_\sigma)^\wedge$ onto $(G^\wedge)_\sigma$ when $\boldsymbol{G}=GL_n$ or U_n respectively. (In section 9, we show that the defining domain of the lifting map is not necessarily the whole $(G_\sigma)^\wedge$ for general reductive $\boldsymbol{G}$.

Let $\boldsymbol{G}$ be reductive and $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$ defined over k. For $\theta\in(T_{\sigma^i})^\wedge$, let $R^\theta_{T,i}$ be the virtual character of G_{σ^i} corresponding to $(\boldsymbol{T},\theta)$. (See P. Deligne, G. Lusztig [1] and D. Kazhdan [5].) Let N^i be the norm map of

T_{σ^i} onto T_σ. For $\theta\in(T_\sigma)^\wedge$, the class function on AG corresponding to $(R_{T,i}^{\theta\circ N^i})_{0\le i\le m-1}$ via the above isomorphism (*) is denoted by AR_T^θ. Our main theorem is:

Assume that m is not divisible by p or a power of p and p, q are sufficiently large. Then AR_T^θ is a virtual character of AG.

This theorem implies that lift $(\pm R_{T,1}^\theta)=\pm R_{T,m}^{\theta\circ N^m}$ for $\theta\in(T_\sigma)^\wedge$ in general position.

This paper consists of 9 sections. Section 1 is a preliminary. In section 2, we modify the lifting theory of modular characters given by Kawanaka. In section 3, the notion of i-restriction is introduced, which is fundamental in our theory. In section 4, the lifting theory of exponential unipotent groups is studied. In section 5, we prove that any R_T^θ can be lifted to some virtual character of G, when p, q are not too small. In section 6, it is shown that the lifting of regular character (resp. semisimple character) is regular (resp. semisimple) if it exists. In sections 7 and 8, the main theorem is proved.

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NOTATION. Let X be a set. If σ is a transformation of X, X_σ denotes the set of σ-fixed points of X. If X is a finite set, $|X|$ means the number of its elements. For complex valued functions f and g on X, define $\langle f, g\rangle_X=|X|^{-1}\sum_{x\in X}f(x)\overline{g(x)}$.

Let G be a finite group. $\mathcal{C}(G)$ denote the set of class functions on G. $\mathcal{R}(G)$ denotes the Grothendieck group of G. Since we are mainly concerned with complex representations, 'representation' means 'complex representation' unless otherwise stated. $\mathcal{R}_+(G)$ is the set of proper characters. $G^\wedge$ means the set of irreducible characters of G. Let H be a subgroup of G. For an element x of G, $Z_H(x)$ denotes $\{y\in H\,|\,xy=yx\}$. and x^H denotes the H-orbit of x. When a prime number p is fixed, an element x of G is called semisimple (resp. unipotent) if the order of x is prime to p (resp. a power of p). An arbitrary element x of G can be represented as $x=su=us$ where s is semisimple and u is unipotent. This decomposition is called the Jordan decomposition.

We denote by $\boldsymbol{G}$, $\boldsymbol{H}$, $\cdots$ a connected linear algebraic group defined over the finite field $k=\boldsymbol{F}_q$ of characteristic p. The Lie algebras of $\boldsymbol{G}$, $\boldsymbol{H}$, $\cdots$ are denoted by the corresponding German letter $\mathfrak{G}$, $\mathfrak{H}$, $\cdots$. We use the same letter σ for the Frobenius endomorphisms of $\boldsymbol{G}$, $\mathfrak{G}$, $\cdots$. A natural number m is fixed through out the paper. We put $\zeta=\exp 2\pi\sqrt{-1}/m$. For an algebraic group $\boldsymbol{G}$ (resp. a Lie algebra $\mathfrak{G}$), G (resp. $\mathfrak{g}$) means $\boldsymbol{G}_{\sigma^m}$ (resp. $\mathfrak{G}_{\sigma^m}$). We denote the induced character of χ from H to G by $\mathrm{ind}_H^G\chi$ or $\mathrm{ind}(\chi|H\to G)$.

1. Preliminaries

1.1. We consider $\mathcal{R}(A)\subset\mathcal{R}(AG)$ via the projection $AG\to A$. In the following A (resp. A_i) is a cyclic group with generator σ (resp. σ^i), where the order of σ is m. Define a character ξ of A by

$$\xi(\sigma^i)=\zeta^i \qquad (\zeta=\exp 2\pi\sqrt{-1}/m)\,.$$

1.2. When σ acts on a set X, denote the cardinality of the orbit of $x\in X$ by $d(x,\sigma,X)$. If there is no fear of confusion we omit σ or X.

Let R be an irreducible representation of a finite group G and ψ be its character. Let

$$T=R\oplus(R\circ\sigma)\oplus\cdots\oplus(R\circ\sigma^{d-1})$$

where $d=d(\psi,\sigma,\mathcal{R}(G))$. Fix a matrix $L=L_\psi$ such that

$$R(x^{\sigma^d})=L^{-1}R(x)L \quad\text{and}\quad L^{m/d}=1\,.$$

Put

$$I=\begin{bmatrix} & & L\\ 1 & & \\ & \ddots & \\ & & 1\end{bmatrix}.$$

Then

$$I^{-1}T(x)I=T(x^\sigma) \quad\text{and}\quad I^m=1\ (x\in G)\,.$$

Hence by putting $T^\sim(\sigma^i x)=I^iT(x)$ $(i=0,1,\cdots,m-1)$ we obtain a representation $T^\sim$ of AG whose restriction to G is T. It is easy to see the irreducibility of $T^\sim$. Denote the character of T(resp. $T^\sim$) by $\chi=\chi_\psi$(resp. $\chi^\sim=\chi_{\tilde\psi}$). Putting $R^\sim(\sigma^{di}x)=L^iR(x)$, we obtain a representation of A_dG which is an extension of R. Denote the character of $R^\sim$ by $\psi^\sim$. Then by a direct computation we obtain the equality

$$\chi^\sim=\mathrm{ind}\,(\psi^\sim\,|\,A_dG\to AG)\,. \tag{1.2.1}$$

Since

$$\textstyle\sum_{j=0}^{e-1}(\chi^\sim\otimes\xi^j)\,(1)\,(\chi^\sim\otimes\xi^j)\,(\sigma^i x)=0\quad(0<i\le m-1)$$

and

$$\textstyle\sum_{j=0}^{e-1}(\chi^\sim\otimes\xi^j)\,(1)\,(\chi^\sim\otimes\xi^j)\,(x)=m\sum_{j=0}^{d-1}\psi^{\sigma^j}(1)\,\psi^{\sigma^j}(x)\,,$$

where $e=m/d$, we obtain

$$\sum_{\psi\in G^\wedge/\langle\sigma\rangle}\sum_{j=0}^{e-1}(\chi_{\tilde{\psi}}\otimes\xi^j)(1)(\chi_{\tilde{\psi}}\otimes\xi^j)(x)$$
$$= m\sum_{\psi\in G^\wedge}\psi(1)\psi(x) = \begin{cases} |AG| & \text{if } x=1 \\ 0 & \text{if } x\neq 1 \end{cases}$$

Thus we obtain the irreducible decomposition of regular representation of AG.

Lemma 1.3. *All the irreducible characters of AG are obtained as $\chi_{\tilde{\psi}}\otimes\xi^j$ with $\psi\in G^\wedge/\langle\sigma\rangle$ and $0\leq j<m/d(\psi)$ without repetition. If $d(\psi)\nmid i$, then $\chi_{\tilde{\psi}}\equiv 0$ on $\sigma^i G$.*

Lemma 1.4.

(1.4.1) $$\langle\chi_{\tilde{\psi}}, \chi_{\tilde{\psi}}\rangle_{\sigma^i G} = d(\psi) \qquad \text{if } d(\psi)\,|\, i\,.$$

If $\chi_{\tilde{1}}, \chi_{\tilde{2}}\in(AG)^\wedge$ and $\chi_{\tilde{1}}|_G\neq\chi_{\tilde{2}}|_G$, then

(1.4.2) $$\langle\chi_{\tilde{1}}, \chi_{\tilde{2}}\rangle_{\sigma^i G} = 0 \qquad (0\leq i\leq m-1)\,.$$

Proof. These can be easily obtained by [8, Lemmas 1.1 and 1.2] or [4, Lemma 1.4], and by 1.2.1,

Lemma 1.4.3. *If $\chi\in(A_iG)^\wedge$ and $\chi(\sigma^i)\neq 0$, then*

$$d(\chi|_G, \sigma) = d(\chi|_{\sigma^i G}, \sigma) = d(\chi, \sigma)\,.$$

Proof. Put $s=d(\chi|_G)$ and $t=d(\chi|_{\sigma^i G})$. Then $\langle\chi^{\sigma^t}, \chi\rangle_{\sigma^i G}=\langle\chi, \chi\rangle_{\sigma^i G}\neq 0$. Hence $(\chi|_G)^{\sigma^t}=\chi|_G$. Thus we get $s|t$. We get the equality $\chi^{\sigma^s}=\chi\otimes\xi^j$ for some j, but $\xi^j(\sigma^i)=1$ since $\chi(\sigma^i)\neq 0$. Hence $\xi^j\equiv 1$ on A_i. Hence $\chi^{\sigma^s}=\chi$ and $d(\chi)|s$. Since $t|d(\chi)$ and $s|d(\chi)$, we complete the proof.

Lemma 1.5. *Fix a divisor d of m and $\chi\in\mathcal{R}(A_dG)$. Suppose that integers a_i $(1\leq i\leq m)$ satisfy the conditions:*

(1.5.1) *if* $(m, i)=(m, j)$, $a_i=a_j$

(1.5.2) *if* $d\nmid i$, $a_i=0$

(1.5.3) *if* $de|m$, $e|\sum_{i|de}\mu(de/i)a_i$,

where μ is the usual Möbius function. Define a class function ψ on AG by $\psi= a_i(\chi+\chi^\sigma+\cdots+\chi^{\sigma^{d-1}})$ on $\sigma^i G$. Then $\psi\in\mathcal{R}(AG)$.

Proof. Define a class function ψ' on A_dG by putting $\psi'=a_{di}\chi$ on $\sigma^{di}G$. Then $\psi=\text{ind}\,(\psi'|A_dG\to AG)$ by 1.5.2. Hence we may suppose that $d=1$. For a divisor e of m, put $ec_e=\sum_{i|e}\mu(e/i)a_i$. Then c_e's are integers by 1.5.3, and $a_i=a_{(m,i)}=\sum_{e|(m,i)}ec_e$. Hence, on $\sigma^i G$ we have

$$\sum_{e|m} c_e \operatorname{ind}_{A_eG}^{AG}(\chi|_{A_eG}) = \sum_{e|(m,\iota)} e c_e \chi = a_i \chi = \psi .$$

Therefore $\psi = \sum_{e|m} c_e \operatorname{ind}_{A_eG}^{AG}(\chi|_{A_eG}) \in \mathcal{R}(AG)$.

DEFINITION 1.6. We define a $\boldsymbol{Z}$-valued function μ on a finite partially ordered set $\mathcal{H}$ with the maximum element G as follows:

$$\mu(G) = 1$$

and

$$\sum_{H \in \mathcal{H},\, H \geq H_0} \mu(H) = 0 \qquad \text{for } H_0 \neq G .$$

This function μ is called *the Möbius function of* $\mathcal{H}$. Occasionally we write $\mu(\cdot, \mathcal{H})$ for $\mu(\cdot)$.

Lemma 1.7. *Suppose that σ acts on $\mathcal{H}$. Extend $\mu(\cdot, \mathcal{H}_{\sigma^\iota})$ to all over $\mathcal{H}$ by equating* 0 *outside of $\mathcal{H}_{\sigma^i}$. Put $a_\iota = \mu(H, \mathcal{H}_{\sigma^i})$ for a fixed $H \in \mathcal{H}$. Then the a_i's satisfy the conditions* 1.5.1 *to* 1.5.3 *for $d = d(H)$.*

Proof. The conditions 1.5.1 and 1.5.2 are easily verified. We prove 1.5.3 by induction on $|\mathcal{H}|$. If $|\mathcal{H}| = 1$, there is nothing to prove. Assume $|\mathcal{H}| > 1$. Put $\mathcal{H}_0 = \{H' \in \mathcal{H} \mid H' \geq H\}$. If H is not the minimum element of $\mathcal{H}$, $|\mathcal{H}_0| < |\mathcal{H}|$. σ^d acts on $\mathcal{H}_0$ and $\mu(H, \mathcal{H}_{0\sigma^{d\iota}}) = a_{d\iota}$. If de divides m, then by induction hypothesis e divides the integer

$$\sum_{i|e} \mu(e/i) a_{di} = \sum_{\iota|de} \mu(de/i) a_\iota .$$

Hence we may suppose that $\mathcal{H}$ has the minimum element H_0 and that $H = H_0$. Note that $d(H_0) = 1$ in this case. Fix a divisor e of m. By definition

$$\sum_{H \in \mathcal{H}} \sum_{\iota|e} \mu(e/i) \mu(H, \mathcal{H}_{\sigma^i}) = 0 . \tag{1.7.1}$$

For $H > H_0$

$$\begin{aligned} \sum_{j=1}^{d(H)} \sum_{\iota|e} \mu(e/i) \mu(H^{\sigma^j}, \mathcal{H}_{\sigma^\iota}) \\ = \sum_{\iota|e} \mu(e/i) \mu(H, \mathcal{H}_{\sigma^\iota}) \times d(H) . \end{aligned} \tag{1.7.2}$$

If $d(H) \nmid e$, this equals 0. Suppose $e = d(H)e'$. 1.7.2 equals $d(H) \sum_{\iota|d(H)e'} \mu(d(H)e'/i) a_\iota$. Since $d(H)e' = e$ divides m, this is divisible by $d(H)e' = e$. With 1.7.1, this implies 1.5.3.

Corollary 1.8. *Let $\mathcal{H}$ be a family of subgroups of a group G with the order defined by inclusion. Suppose that $\mathcal{H}$ is invariant under σ-action. Assume that for each $H \in \mathcal{H}$ a character $\chi_H \in \mathcal{R}(A_d H)$ with $d = d(H)$ is given and satisfies $(\chi_H)^\sigma = \chi_{H^\sigma}$. Define a class function ψ on AG by*

$$\psi = \sum_{H\in\mathcal{H},\, d(H)|i}\mu(H,\ \mathcal{H}_{\sigma^i})\ \mathrm{ind}\ (\chi_H | A_d H \to A_d G)\ \ on\ \ \sigma^i G \qquad (0\leq i\leq m-1)$$

Then $\psi\in\mathcal{R}(AG)$. *If we define a class function* ψ' *on* AG *by*

$$\psi' = \sum_{H\in\mathcal{H},\, d(H)|i,\, H\neq G}\mu(H,\ \mathcal{H}_{\sigma^i})\ \mathrm{ind}\ (\chi_H | A_d H \to A_d G)\ \ on\ \ \sigma^i G \qquad (0\leq i\leq m-1)$$

we also have $\psi'\in\mathcal{R}(AG)$.

2. Liftings of modular characters of finite groups

2.1. Let $\phi\colon \bar{k}^\times\to \boldsymbol{C}^\times (k=F_q)$ be an injective homomorphism. For $R\in GL(n,\bar{k})$, put $\beta_\phi[R]=\sum_{i=1}^n\phi(r_i)$, where r_i's are the eigenvalues of R.

2.2. Let G be a finite group on which $A=\langle\sigma\rangle$ acts, R a $\bar{k}$-representation of G and V its representation space. Define a representation R_i of G by

$$R_i(x) = R(x)\otimes R(x^\sigma)\otimes\cdots\otimes R(x^{\sigma^{d-1}}) \quad (x\in G)\,,$$

where $d=(m, i)$. Define an automorphism I of $V\otimes\cdots\otimes V$ (m-times) by

$$I(v_0\otimes\cdots\otimes v_{m-1}) = v_{m-1}\otimes v_0\otimes\cdots\otimes v_{m-2}\,,$$

and a representation A_iR_i of A_iG by

$$A_iR_i(\sigma^{ij}x) = I^{dj}\cdot(R_i(x)\otimes R_i(x^{\sigma^i})\otimes\cdots\otimes R_i(x^{\sigma^{i(e-1)}})) \qquad (0\leq j\leq e-1,\ x\in G)\,,$$

where $e=m/(m, i)$. We write AR for A_1R_1. Define an element J of the symmetric group S_m acting on $\boldsymbol{Z}/(m)$ by

$$J=\begin{pmatrix}0, 1, \cdots, d-1, d, d+1, \cdots 2d-1, 2d, 2d+1, \cdots\\ 0, 1, \cdots, d-1, i, i+1, \cdots i+d-1, 2i, 2i+1, \cdots\end{pmatrix},$$

and put $J(v_0\otimes\cdots\otimes v_{m-1})=v_{J(0)}\otimes\cdots\otimes v_{J(m-1)}$. Then we have $J^{-1}I^iJ=I^d$ and

$$J^{-1}AR(\sigma^i x)J = A_iR_i(\sigma^i x)\,. \tag{2.2.1}$$

Theorem 2.3. *If* $(m, p)=1$, *we have*

$$\beta_\phi[AR(\sigma^i x)] = \beta_\phi[R_i((\sigma^i x)^{m/d})]\,, \tag{2.3.1}$$

where $d=(m, i)$.

Lemma 2.4. *Let* $V=\bar{k}^n$ *and* $A_0, \cdots, A_{m-1}\in E=\mathrm{End}\ V$. *Then, there exist polynomials* f_d (*depending on* $A_0, \cdots, A_{m-1}$) *such that*

(2.4.1) $\qquad \det\,(x-A_{m-1}\circ\cdots\circ A_0)^{-1}\det\,(x-I\circ(A_0\otimes\cdots\otimes A_{m-1}))$

$$= \prod_{d|m, d\geq 2} f_d(x^d) .$$

Proof. Let $\{e_1, \cdots, e_n\}$ be a basis of V and D be the set of endomorphisms of V which are represented by diagonal matrices with respect to $\{e_1, \cdots, e_n\}$. If $A_0, \cdots, A_{m-1} \in D$, 2.4.1 is proved in [4, Proof of Th. 3.6]. Let us consider the following diagram.

$$\begin{array}{ccc} E^m & \xrightarrow{q} & \boldsymbol{P}^n \\ p\downarrow & & \downarrow\psi \\ \boldsymbol{P}^{n^m} & \xrightarrow{\phi} & \boldsymbol{P}^{n^m} \end{array} \tag{2.4.2}$$

where

$$\begin{aligned} p(A_0, \cdots, A_{m-1}) &= \det(x - I\circ(A_0\otimes\cdots\otimes A_{m-1})) \\ q(A_0, \cdots, A_{m-1}) &= \det(x - A_{m-1}\circ\cdots\circ A_0) \\ \phi(\textstyle\prod_{j=1}^{n^m}(a_j x - \lambda_j)) &= \textstyle\prod_{j=1}^{n^m}(a_j^m x - \lambda_j^m) \\ \psi(\textstyle\prod_{i=1}^{n}(b_i x - \mu_i)) &= \textstyle\prod_{1\leq i_j\leq n}(b_{i_0}\cdots b_{i_{m-1}} x - \mu_{i_0}\cdots\mu_{i_{m-1}}) . \end{aligned}$$

Here we identify $a_0 + a_1 x + \cdots + a_n x^n$ with $(a_0, \cdots, a_n) \in \boldsymbol{P}^n$. Since

$$\begin{aligned} &(I\circ(A_0\otimes\cdots\otimes A_{m-1}))^m \\ &= (A_{m-1}\circ\cdots\circ A_1\circ A_0)\otimes(A_0\circ A_{m-1}\circ\cdots\circ A_1) \\ &\qquad \cdots\otimes(A_{m-2}\circ\cdots\circ A_0\circ A_{m-1}) , \end{aligned}$$

2.4.2 is commutative. Put $\psi(\boldsymbol{P}^n) = X$. The morphisms ψ: $\boldsymbol{P}^n \to X$ and ϕ: $\phi^{-1}(X) \to X$ are both quasi finite, hence finite. (See [EGA. IV Th. 8.11.1].) In the following we assume the knowledge of the materials in [6, Chapter 1]. Put $p(E^m) = Y$ and $p(D^m) = Y'$. Then $\phi(\bar{Y}') = \overline{\phi p(D^m)} = \overline{\psi q(D^m)} = \overline{\psi(\boldsymbol{A}^n)}$. Here $\boldsymbol{A}^n = \{(a_0, \cdots, a_n) \in \boldsymbol{P}^n \mid a_n \neq 0\}$. Hence $\dim \bar{Y}' = n$. On the other hand, $\dim \phi^{-1}(X) = \dim X = n$, $\bar{Y}' \subset \bar{Y} \subset \phi^{-1}(X)$. Hence

$$\bar{Y}' = \bar{Y} . \tag{2.4.3}$$

Let us consider the following mappings.

$$\begin{array}{ccccccc} E^m & \xrightarrow{\Delta} & E^m \times E^m & \xrightarrow{p\times q} & \boldsymbol{P}^{n^m} \times \boldsymbol{P}^n & \xrightarrow{\pi} & \boldsymbol{P}^{n^m} \\ x \mapsto (x, x) & & & & (x, y) \mapsto x . & & \end{array}$$

Put $Z = (p\times q)\circ\Delta(E^m)$. Then $\pi(Z) = Y$. Let Y_0 (resp. Z_0) be a subset of Y (resp. Z) which is open and dense in $\bar{Y}$ (resp. $\bar{Z}$). Then each fibre of π: $\pi^{-1}(Y_0) \cap Z_0 \to Y_0$ is 0-dimensional. Hence

$$\dim \bar{Y} = \dim \bar{Z} .$$

By the commutativity of 2.4.2, the following commutative diagram can be completed with some r.

$$\begin{array}{ccccc} E^m \times E^m & \xrightarrow{p\times q} & Y \times \boldsymbol{P}^n & \xrightarrow{\phi\times\psi} & X \times X \\ \Delta \uparrow & & r & & \Delta \uparrow \\ E^m & & \longrightarrow & & X \end{array}$$

Then we have

$$\begin{aligned} \dim \overline{r(E^m)} &= \dim \overline{\Delta \circ r(E^m)} \\ &= \dim \overline{(\phi\times\psi)\circ(p\times q)\circ\Delta(E^m)} \\ &= \dim \overline{(\phi\times\psi)\,(Z)} \\ &= \dim \bar{Z} = \dim \bar{Y}\,. \end{aligned}$$

By the same reason, we get

$$\dim \overline{r(D^m)} = \dim \overline{Y'}\,.$$

Hence by 2.4.3, we get

$$\overline{r(D^m)} = \overline{r(E^m)}\,. \tag{2.4.4}$$

Further more $\dim \overline{(p\times q)\circ\Delta(E^m)} = \dim \overline{(\phi\times\psi)\circ(p\times q)\circ\Delta(E^m)} = \dim \overline{\Delta\circ r(E^m)} =$ $\dim \overline{r(E^m)}$.

By the same reason, we get

$$\dim \overline{(p\times q)\circ\Delta(D^m)} = \dim \overline{r(D^m)}$$

Hence by 2.4.4,

$$\overline{(p\times q)\circ\Delta(E^m)} = \overline{(p\times q)\circ\Delta(D^m)}\,.$$

Take a subset U of $(p\times q)\Delta(D^m)$ which is open and dense in $\overline{(p\times q)\circ\Delta(D^m)}$, and put $U'=((p\times q)\circ\Delta)^{-1}(U)$. For any element $(A_0, \cdots, A_{m-1})$ of U', there exists an element $(D_0, \cdots, D_{m-1})$ of D^m such that

$$\begin{aligned} p(A_0, \cdots, A_{m-1}) &= p(D_0, \cdots, D_{m-1}) \\ q(A_0, \cdots, A_{m-1}) &= q(D_0, \cdots, D_{m-1})\,. \end{aligned}$$

Since 2.4.1 holds for $(D_0, \cdots, D_{m-1})$, we get 2.4.1 for such an $(A_0, \cdots, A_{m-1})$. Since U' is open and dense in E^m, 2.4.1 holds in general.

2.5. Proof of 2.3. By 2.2.1. we get

$$\beta_\phi[AR(\sigma^i x)] = \beta_\phi[A_i R_i(\sigma^i x)]\,.$$

Hence it suffices to prove that

$$\beta_\phi[AR(\sigma x)] = \beta_\phi[R((\sigma x)^m)] .$$

Put $R(x^{\sigma^i})=A_i$. Then this can be rewritten as

$$\beta_\phi[I\circ(A_0\otimes\cdots\otimes A_{m-1})] = \beta_\phi[A_{m-1}\circ\cdots\circ A_0] . \tag{2.5.1}$$

By lemma 2.4 the left hand side of 2.5.1 is equal to $\sum\phi(\alpha)+\beta_\phi[A_{m-1}\circ\cdots\circ A_0]$, where α runs over the roots of $f_d(x^d)$. If α is a root of $f_d(x^d)$, then $\eta\alpha$ is also a root of $f_d(x^d)$ for any d'th root of unity η. Since $(d, p)=1$, the first summand is zero. Thus we obtain 2.5.1.

3. Preliminaries for lifting theory of finite algebraic groups

In the following, $\boldsymbol{G}$ is a connected linear algebraic group defined over a finite field $k=\boldsymbol{F}_q$ of characteristic p and σ is the Frobenius endomorphism. Let G be $\boldsymbol{G}_{\sigma^m}$ and write σ for $\sigma|_G$.

3.1. We define *the norm map* N_i of the subset $\sigma^i G$ of AG to the group $\boldsymbol{G}$ as follows:

$$N_i(\sigma^i x) = \alpha(x)^{-1}(\sigma^i x)^{m/d}\alpha(x) ,$$

where $\alpha(x)$ is an element of $\boldsymbol{G}$ such that

$$\alpha(x)^{\sigma^d}\alpha(x)^{-1} = \sigma^{-it}(\sigma^i x)^t$$

and d, t are integers given as follows:

$$d = (m, i) \quad ti \equiv d \pmod{m} .$$

Lemma 3.2. (1) *The norm map N_i induces a bijection from the set of G-conjugacy classes of $\sigma^i G$ onto the set of conjugacy classes of $\boldsymbol{G}_{\sigma^i}$. This bijection is independent of the choice of α.*

(2) *The norm map N_i is compatible with the σ-action. Here σ acts on $\sigma^i G$ by $(\sigma^i x)^\sigma=\sigma^i x^\sigma$.*

(3) $|Z_G(\sigma^i x)| = |Z_{\boldsymbol{G}_{\sigma^i}}(N_i(\sigma^i x))|$.

Proof. Denote the free cyclic group generated by the symbol σ by $\langle\sigma\rangle$. This group $\langle\sigma\rangle$ acts on $\boldsymbol{G}$ by $\sigma^{-1}x\sigma=x^\sigma$. By this action we define the semidirect product $\langle\sigma\rangle\boldsymbol{G}$. Then

$$N_i(\sigma^i x) = \alpha(x)^{-1}(\sigma^{-mi/d}(\sigma^i x)^{m/d})\alpha(x)$$
$$\alpha(x)^{\sigma^d}\alpha(x)^{-1} = \sigma^{-it}(\sigma^i x)^t .$$

For $x\in G$,

$$\begin{aligned} N_i(\sigma^i x)^{\sigma^d} &= \alpha(x)^{-\sigma^d}\sigma^{-d}(\sigma^{-mi/d}(\sigma^i x)^{m/d})\sigma^d\alpha(x)^{\sigma^d} \\ &= \alpha(x)^{-\sigma^d}\sigma^{-it}(\sigma^{-mi/d}(\sigma^i x)^{m/d})\sigma^{it}\alpha(x)^{\sigma^d} \\ &= \alpha(x)^{-1}(\sigma^i x)^{-t}(\sigma^{-mi/d}(\sigma^i x)^{m/d})\,(\sigma^i x)^t\alpha(x) \\ &= N_i(\sigma^i x)\,. \end{aligned}$$

Therefore $N_i(\sigma^i x)\in \boldsymbol{G}_{\sigma^d}=G_{\sigma^i}$.
If $\alpha(x)^{\sigma^d}\alpha(x)^{-1}=\beta(x)^{\sigma^d}\beta(x)^{-1}$, then $\alpha(x)^{-1}\beta(x)\in G_{\sigma^d}$. Hence $\alpha(x)^{-1}(\sigma^{-mi/d}(\sigma^i x)^{m/d})$ $\alpha(x)$ is conjugate to $\beta(x)^{-1}(\sigma^{-mi/d}(\sigma^i x)^{m/d})\beta(x)$ in G_{σ^d}.
For $y\in G$,

$$\begin{aligned} \text{(3.2.1)} \qquad \alpha(y^{-\sigma^i}xy)^{\sigma^d}\alpha(y^{-\sigma^i}xy)^{-1} &= \sigma^{-it}(y^{-1}(\sigma^i x)y)^t \\ &= y^{-\sigma^{it}}\alpha(x)^{\sigma^d}\alpha(x)^{-1}y \\ &= y^{-\sigma^d}\alpha(x)^{\sigma^d}\alpha(x)^{-1}y\,. \end{aligned}$$

Hence

$$\begin{aligned} N_i(y^{-1}\sigma^i xy) &= \alpha(y^{-\sigma^i}xy)^{-1}\sigma^{-mi/d}(y^{-1}\sigma^i xy)^{m/d}\alpha(y^{-\sigma^i}xy) \\ &= \alpha(y^{-\sigma^i}xy)^{-1}y^{-1}(\sigma^{-mi/d}(\sigma^i x)^{m/d})y\alpha(y^{-\sigma^i}xy)\,, \end{aligned}$$

which is conjugate to $N_i(\sigma^i x)$ in G_{σ^d} by 3.2.1.
Hence we obtain a mapping from the set of G-conjugacy classes of $\sigma^i G$ to the set of conjugacy classes of G_{σ^i} which does not depend on the choice of α. If $g\in Z_G(\sigma^i x)$, then

$$g\in Z_G(\sigma^{-mi/d}(\sigma^i x)^{m/d}) \quad \text{and} \quad \alpha(x)^{-1}g\alpha(x)\in Z_{\boldsymbol{G}}(N_i(\sigma^i x))\,.$$

Since

$$\begin{aligned} (\alpha(x)^{-1}g\alpha(x))^{\sigma^d} &= \alpha(x)^{-\sigma^d}\sigma^{-d}g\sigma^d\alpha(x)^{\sigma^d} \\ &= \alpha(x)^{-\sigma^d}\sigma^{-it}g\sigma^{it}\alpha(x)^{\sigma^d} \\ &= \alpha(x)^{-1}(\sigma^i x)^{-t}g(\sigma^i x)^t\alpha(x) \\ &= \alpha(x)^{-1}g\alpha(x)\,, \end{aligned}$$

we have

$$\alpha(x)^{-1}g\alpha(x)\in Z_{\boldsymbol{G}_{\sigma^i}}(N_i(\sigma^i x))\,.$$

Conversely, let g be an element of $\boldsymbol{G}$ such that

$$\alpha(x)^{-1}g\alpha(x)\in Z_{G_{\sigma^i}}(N_i(\sigma^i x))\,.$$

Then

$$\text{(3.2.2)} \qquad g\in Z_{\boldsymbol{G}}(\sigma^{-mi/d}(\sigma^i x)^{m/d})$$

$$\text{(3.2.3)} \qquad (\alpha(x)^{-1}g\alpha(x))^{\sigma^d} = \alpha(x)^{-1}g\alpha(x)\,.$$

By 3.2.3

$$
\begin{aligned}
g^{\sigma^d} &= \alpha(x)^{\sigma^d}\alpha(x)^{-1}g\alpha(x)\alpha(x)^{-\sigma^d} \\
&= \sigma^{-it}(\sigma^i x)^t g(\sigma^i x)^{-t}\sigma^{it} \\
g^{\sigma^{2d}} &= (\sigma^{-d}\sigma^{-it}(\sigma^i x)^t\sigma^d)g^{\sigma^d}(\sigma^{-d}(\sigma^i x)^{-t}\sigma^{it}\sigma^d) \\
&= (\sigma^{-it}\sigma^{-it}(\sigma^i x)^t\sigma^{it})\ (\sigma^{-it}(\sigma^i x)^t g(\sigma^i x)^{-t}\sigma^{it}) \\
&\quad \times(\sigma^{-it}(\sigma^i x)^{-t}\sigma^{it}\sigma^{it}) \\
&= \sigma^{-2it}(\sigma^i x)^{2t}g(\sigma^i x)^{-2t}\sigma^{2it}\,.
\end{aligned}
$$

Repeating this, we get

$$
(3.2.4) \qquad g^{\sigma^{jd}} = \sigma^{-jit}(\sigma^i x)^{jt}g(\sigma^i x)^{-jt}\sigma^{jit}\,.
$$

Substituting m/d for j in 3.2.4, we get

$$
\begin{aligned}
g^{\sigma^m} &= \sigma^{-mit/d}(\sigma^i x)^{mt/d}g(\sigma^i x)^{-mt/d}\sigma^{mit/d} \\
&= (\sigma^{-mi/d}(\sigma^i x)^{m/d})^t g((\sigma^i x)^{-m/d}\sigma^{mi/d})^t \\
&= g\,.
\end{aligned}
$$

Since $ti/d\equiv 1 \pmod{m/d}$, there exists an integer μ such that $ti/d+m\mu/d=1$. Substituting i/d for j in 3.2.4, we get

$$
\begin{aligned}
g^{\sigma^i} &= \sigma^{-i^2t/d}(\sigma^i x)^{it/d}g(\sigma^i x)^{-it/d}\sigma^{i^2t/d} \\
&= \sigma^{-i^2t/d}(\sigma^i x)^{it/d}\sigma^{-mi\mu/d}(\sigma^i x)^{m\mu/d}g \\
&\qquad (\sigma^i x)^{-m\mu/d}\sigma^{mi\mu/d}(\sigma^i x)^{-it/d}\sigma^{i^2t/d} \\
&= xgx^{-1}\,.
\end{aligned}
$$

Hence $g\in Z_G(\sigma^i x)$. Thus we obtain

$$
(3.2.5) \qquad \alpha(x)^{-1}Z_G(\sigma^i x)\alpha(x) = Z_{G_{\sigma^i}}(N_i(\sigma^i x))\,.
$$

This proves the part (3). The bijectivity of N_i can be proved as in [4]. Since $\alpha(x^\sigma)^{\sigma^d}\alpha(x^\sigma)^{-1}=\sigma^{-it}(\sigma^i x^\sigma)^t$, we get also the part (2).

Corollary 3.3. *For any $f, g\in\mathcal{C}(G_{\sigma^i})$,*

$$
\langle f, g\rangle_{G_{\sigma^i}} = \langle f\circ N_i, g\circ N_i\rangle_{\sigma^i G}\,.
$$

Corollary 3.4. $|(G_{\sigma^i})^\wedge/\langle\sigma\rangle| = |(G^\wedge)_{\sigma^i}/\langle\sigma\rangle|\,.$

Proof. By 1.3 and 1.4, the right hand side is equal to $\dim\{f|_{\sigma^i G}; f\in\mathcal{C}(AG)\}$. Since the left hand side is equal to $\dim\mathcal{C}(G_{\sigma^i})_\sigma$, we obtain the equality from lemma 3.2 (1).

DEFINITION 3.5. We define a map

$$
\mathcal{C}(AG)\xrightarrow{i\text{-res}}\mathcal{C}(G_{\sigma^i})_\sigma\longrightarrow 0
$$

by

$$(i\text{-res}\, f)\circ N_i = f\,|_{\sigma^i G} \qquad f \in \mathcal{C}(AG)\,.$$

The map is called *the i-restriction.*

REMARK 3.5.1. The equality 2.3.1 can be rewritten as follows. Let R be a rational representation of $\boldsymbol{G}$. If $(m, p)=1$, then

$$i\text{-res}\ \beta_\Phi[AR] = \beta_\Phi[R_i]\,,$$

where we consider R as a representation of G.

Lemma 3.6. *Let $\boldsymbol{H}$ be a connected closed subgroup of $\boldsymbol{G}$ defined over k. Then the following diagrams are commutative:*

(3.6.1)
$$\begin{array}{ccc} \mathcal{C}(AH) & \xrightarrow{\text{ind}} & \mathcal{C}(AG) \\ {\scriptstyle i\text{-res}}\downarrow & & \downarrow{\scriptstyle i\text{-res}} \\ \mathcal{C}(H_{\sigma^i})_\sigma & \xrightarrow{\text{ind}} & \mathcal{C}(G_{\sigma^i})_\sigma \end{array}$$

(3.6.2)
$$\begin{array}{ccc} \mathcal{C}(AH) & \xleftarrow{\text{res}} & \mathcal{C}(AG) \\ {\scriptstyle i\text{-res}}\downarrow & & \downarrow{\scriptstyle i\text{-res}} \\ \mathcal{C}(H_{\sigma^i})_\sigma & \xleftarrow{\text{res}} & \mathcal{C}(G_{\sigma^i})_\sigma \end{array}$$

where ind *and* res *means the usual induction map and restriction map respectively. Let $\boldsymbol{H}$ be normal, and π: $\boldsymbol{G}\to\boldsymbol{G}/\boldsymbol{H}$ the canonical homomorphism. Then the following diagrams are commutative:*

(3.6.3)
$$\begin{array}{ccc} \mathcal{C}(A(G/H)) & \xrightarrow{\pi^*} & \mathcal{C}(AG) \\ {\scriptstyle i\text{-res}}\downarrow & & \downarrow{\scriptstyle i\text{-res}} \\ \mathcal{C}((G/H)_{\sigma^i})_\sigma & \xrightarrow{\pi^*} & \mathcal{C}(A(G_{\sigma^i})_\sigma\,, \end{array}$$

(3.6.4)
$$\begin{array}{ccc} \mathcal{C}(AG_1)\otimes\cdots\otimes\mathcal{C}(AG_n) & \longrightarrow & \mathcal{C}(A(G_1\times\cdots\times G_n)) \\ \downarrow{\scriptstyle i\text{-res}\otimes\cdots\otimes i\text{-res}} & & \downarrow{\scriptstyle i\text{-res}} \\ \mathcal{C}(G_{1\sigma^i})\otimes\cdots\otimes\mathcal{C}(G_{n\sigma^i})_\sigma & \longrightarrow & \mathcal{C}((G_1\times\cdots\times G_n)_{\sigma^i})\,. \end{array}$$

Here the map π: $AG\to A(G/H)$ is defined by $\pi(\sigma^i x)=\sigma^i\pi(x)$ ($i=0, 1, \cdots, m-1$).

Proof. The commutativity of 3.6.2–3.6.4 are easy to verify. We shall prove only 3.6.1. Let $x_r\in H$ ($r=1, \cdots c$) be so chosen that

$$(\sigma^i x)^G \cap \sigma^i H = \cup_{r=1}^{c}(\sigma^i x_r)^H$$

is a disjoint union. Then by 3.2,

$$N_i(\sigma^i x)^{G_{\sigma^i}} \cap H_{\sigma^i} = \cup_{r=1}^{c} N_i(\sigma^i x_r)^{H_{\sigma^i}}$$

Hence for $f \in \mathcal{C}(AH)$,

$$\begin{aligned}
&\operatorname{ind}(f \mid AH \to AG)(\sigma^i x) \\
&= |AH|^{-1} \textstyle\sum_{j=0}^{m-1} \sum_{y \in G} f((\sigma^j y)^{-1}(\sigma^i x)(\sigma^j y)) \\
&= m^{-1} |H|^{-1} \textstyle\sum_{j=0}^{m-1} \sum_{y \in G} f(y^{-1}(\sigma^i x) y) \\
&= \textstyle\sum_{r=1}^{c} |Z_G(\sigma^i x_r)| \cdot |Z_H(\sigma^i x_r)|^{-1} f(\sigma^i x_r) \\
&= \textstyle\sum_{r=1}^{c} |Z_{G_{\sigma^i}}(N_i(\sigma^i x_r))| \cdot |Z_{H_{\sigma^i}}(N_i(\sigma^i x_r))|^{-1} \\
&\qquad\qquad \cdot (i\text{-res} f)(N_i(\sigma^i x_r)) \\
&= \operatorname{ind}(i\text{-res} f \mid H_{\sigma^i} \to G_{\sigma^i})(N_i(\sigma^i x)) .
\end{aligned}$$

Here we considered $f \equiv 0$ outside AH.

Lemma 3.7. *Let $\psi \in (G_{\sigma^i})^{\wedge}_{\sigma}$ be given. Suppose that there exists a virtual character $\chi^{\sim}$ of AG such that i-res $\chi^{\sim} = \psi$. Then there exists an irreducible character $\chi_{\tilde{\psi}}$ of AG such that i-res $\chi_{\tilde{\psi}} = \pm\psi$.*

Proof. Let

$$\chi^{\sim} = (c_0 \chi_{\tilde{\psi}} + c_1 \xi \otimes \chi_{\tilde{\psi}} + \cdots) + \cdots .$$

We may suppose that the right hand side does not contain any irreducible character which vanish identically on $\sigma^i G$. Since

$$i\text{-res}\, \chi^{\sim} = (c_0 + c_1 \zeta^i + \cdots)\, i\text{-res}\, \chi_{\tilde{\psi}} + \cdots \tag{3.7.1}$$

we get the inequality

$$|(c_0 + c_1 \zeta^i + \cdots)^{\tau}| \leq 1 \tag{3.7.2}$$

for each $\tau \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. (See 1.4.1 and 1.4.2.) If at least two terms appeared in 3.7.1, the strict inequality would hold in 3.7.2. Hence $|N_{\boldsymbol{Q}(\zeta)/\boldsymbol{Q}}(c_0 + c_1 \zeta^i + \cdots)| < 1$ and $c_0 + c_1 \zeta^i + \cdots = 0$. Hence only one term appears in 3.7.1, and $|c_0 + c_1 \zeta^i + \cdots| = 1$. The following lemma shows that $c_0 + c_1 \zeta^i + \cdots = \pm \zeta^{ij}$ ($j \in \boldsymbol{Z}$). Thus $\xi^{-j} \otimes \chi_{\psi}$ satisfies our condition.

Lemma 3.8. *If $c \in \boldsymbol{Z}[\zeta]$ has the absolute value one, then c is a root of unity.*

Proof. Put $K = \boldsymbol{Q}(\zeta)$ and $K_0 = \boldsymbol{Q}(\zeta + \zeta^{-1})$. Denote the unit group of K (resp. K_0) by E (resp. E_0). Since c is a unit of K and the rank of E and E_0 are the same, some power c^N of c is contained in E_0. Let $\varepsilon_0, \cdots, \varepsilon_r > 0$ be fundamental units of E_0. Let $c^N = w \varepsilon_0^{e_0} \cdots \varepsilon_r^{e_r}$, where w is a root of unity. Since $|c|^N = \varepsilon_0^{e_0} \cdots \varepsilon_r^{e_r} = 1$, we get $c^N = w$.

4. Lifting theory of exponential unipotent groups

4.1. Let $\boldsymbol{U}$ be a nilpotent Lie algebra over $\bar{k}$ defined over k. For $x, y \in \boldsymbol{U}$, let

$$H(x, y) = x+y+a[x, y]+b[x, [x, y]]+c[y, [x, y]]+\cdots , \tag{4.1.1}$$

where $a, b, c, \cdots$ are elements of k which is independent of x and y. Suppose that $\boldsymbol{U}$ is a group under the multiplication rule $x \cdot y = H(x, y)$ and denote this group by $\boldsymbol{U}$. Such $\boldsymbol{U}$ is called an exponential unipotent group. Denote an element $x \in \mathfrak{U}$ by exp x when x is considered as an element of $\boldsymbol{U}$. The inverse map of exp: $\mathfrak{U} \to \boldsymbol{U}$ is denoted by log. Occasionally exp and log are omitted.

4.2. Let $\mathfrak{U}'$ be the dual space of $\mathfrak{U}$. Fix a $\lambda \in \mathfrak{U}'$ and put $B(x, y) = \lambda[x, y]$. Then B is an alternating bilinear form on $\mathfrak{U}$. Let $\mathfrak{H}^\lambda$ be a subalgebra of $\mathfrak{U}$ such that

$$B(x, y) = 0 \qquad \text{for} \quad x, y \in \mathfrak{H}^\lambda , \tag{4.2.1}$$

$$\dim \mathfrak{H}^\lambda = \frac{1}{2} (\dim \mathfrak{U} + \dim \mathfrak{U}_B^\perp) , \tag{4.2.2}$$

where $\mathfrak{U}_B^\perp$ is the null space of B. Put $\boldsymbol{H}^\lambda = \exp \mathfrak{H}^\lambda$.

4.3. Let ψ_0 be an additive character of $\bar{k}$ such that $\psi_0|_{k_m}$ is σ-invariant and non-trivial. Then $\psi_0(s) \neq 1$ for some $s \in k_m^\times$. Let $\psi(x) = \psi_0(sx)$. Since $\psi(1) \neq 1$, the restriction of ψ to an arbitrary subfield of k_m is non-trivial. Since $\psi(s^{-1}x) = \psi(s^{-1}x^{\sigma^i})$ for $x \in k_m$,

$$\psi(x^{\sigma^{-i}}) = \psi(s^{-1}s^{\sigma^i}x) . \tag{4.3.1}$$

We define the σ-action on $\mathfrak{U}'$ by

$$\lambda^\sigma(x) = (\lambda(x^{\sigma^{-1}}))^\sigma \qquad \text{for} \quad \lambda \in \mathfrak{U}' .$$

For $\lambda \in \mathfrak{U}'$ we define a linear character ϕ_λ of $\boldsymbol{H}^\lambda$ by $\phi_\lambda = \psi_0 \circ \lambda \circ \log$. (See 4.1.1 and 4.2.1.) Let $\lambda \in \mathfrak{U}'_\sigma$ and choose $\boldsymbol{H}^\lambda$ to be σ-invariant. Since the restriction of ϕ_λ to H^λ is σ-invariant, we can define a linear character $A\phi_\lambda$ of AH by $A\phi_\lambda(\sigma^i x) = \phi_\lambda(x)$. Define Tr_i: $k_m \to k_d$ ($d=(m, i)$) by $Tr_i x = \sum_{j=0}^{(m/d)-1} x^{\sigma^{ij}}$ ($x \in k_m$, $i = 0, 1, \cdots, m-1$). If $Tr_i s = 0$, then s can be represented as $s = t - t^{\sigma^d}$, $d = (m, i)$ with some $t \in k_m$. Hence

$$\psi_0(s) = \psi_0(t - t^{\sigma^d}) = \psi_0(t - t) = 1 .$$

This contradicts the choice of s. Hence we can define an element $\lambda_i \in \mathfrak{U}'_{\sigma^i}$ by $\lambda = (Tr_i s)\lambda_i$. Note that we can take $\mathfrak{H}^{\lambda_i} = \mathfrak{H}^\lambda$. For an element $x \in \mathfrak{H}^\lambda$, by 4.1.1 and 4.2.1,

$$\begin{aligned}\psi\circ\lambda_i(N_i(\sigma^i x)) &= \psi\circ\lambda_i(\textstyle\sum_{j=0}^{m/d-1} x^{\sigma^{-ij}}) \\ &= \psi(\textstyle\sum_j \lambda_i(x)^{\sigma^{-ij}}) \\ &= \psi(\textstyle\sum_j s^{-1}s^{\sigma^{ij}}\lambda_i(x)) \\ &= \psi(s^{-1}\lambda(x))\,.\end{aligned}$$

On the other hand,

$$A\phi_\lambda(\sigma^i x) = \phi_\lambda(x) = \psi_0\circ\lambda(x) = \psi(s^{-1}\lambda(x))\,.$$

Hence we obtain

(4.3.2) $$i\text{-res}\, A\phi_\lambda = \phi_{\lambda,i}$$

where $\phi_{\lambda,i}$ is a linear character of $H_{\sigma^i}^\lambda$ defined by

$$\phi_{\lambda,i}(x) = \psi((Tr_i s)^{-1}\lambda(x))\,.$$

Let

$$\begin{aligned}\chi_{\lambda,i} &= \operatorname{ind}(\phi_{\lambda,i}\,|\,H_{\sigma^i}^\lambda \to U_{\sigma^i}) \\ A\chi_\lambda &= \operatorname{ind}(A\phi_\lambda\,|\,AH^\lambda \to AU)\,.\end{aligned}$$

Then by 3.6 and 4.3.2,

(4.3.3) $$i\text{-res}\, A\chi_\lambda = \chi_{\lambda,i}\,.$$

In general, if $\lambda\in\mathfrak{u}'$ satisfies $d=d(\lambda)\,|\,m$, then we can define a character $A_d\chi_\lambda$ of A_dU in the same manner. It is known (Kazhdan [5]) that every irreducible character of U can be obtained as $\chi_{\lambda,0}$ with some $\lambda\in\mathfrak{u}'/U$. Let

$$A\chi_\lambda = \operatorname{ind}(A_d\chi_\lambda\,|\,A_dU \to AU)\,.$$

Then every irreducible character of AU can be obtained as $A\chi_\lambda\otimes\xi^j$ with some $\lambda\in\mathfrak{u}'/AU$ and $0\le j<m/d(\lambda)$ without repetition. Thus by 3.6, we obtain

Proposition 4.4. *Suppose that* U *is an exponential unipotent group. Then for any* $\chi\in\mathcal{R}(G_{\sigma^i})$, *there exists a virtual character* $\chi^\sim$ *such that* i-res $\chi^\sim=\chi$.

5. Existence of lifting of R_T^θ

Lemma 5.1. *Let* G *be a finite group,* Z *a central subgroup of* G *and* $\theta\in Z^\wedge$. *Let* p *be aprime such that* $|G|=p^n l$, $(p, l)=(p, |Z|)=1$. *Let* U *be a* p*-Sylow subgroup of* G. *Suppose that a virtual character* $\chi\in\mathcal{R}(G)$ *satisfies the following conditions*:

(5.1.1) $$\chi(x) = 0 \qquad \text{if } x_s\notin Z\,,$$

(5.1.2) $$\chi(x) = \theta(x_s)\chi(x_u) \qquad \text{if } x_s\in Z\,,$$

(5.1.3) $$|Z|\cdot|Z_G(x)|^{-1}\chi(x)\in\mathcal{O}[p^{-1}],$$

where $\mathcal{O}$ is the ring of algebraic integers.
Then there exists a virtual character $\psi\in\mathcal{R}(U)$ such that

$$\chi=\operatorname{ind}(\theta\otimes\psi\,|\,Z\times U\to G).$$

Proof. For an integer n, define a class function n^* on G by

(5.1.4) $$n^*(x)=\begin{cases} n & \text{if } x_s\in Z \\ 0 & \text{if } x_s\notin Z \end{cases}$$

Then lemma 5.1.7 below shows that $l^*\in\mathcal{O}\otimes\operatorname{ind}_{Z\times U}^G\mathcal{R}(Z\times U)$. By 5.1.1, we obtain

(5.1.5) $$l\chi\in\operatorname{ind}_{Z\times U}^G\mathcal{R}(Z\times U).$$

Let $\{u_1,\cdots,u_n\}$ be a complete set of representatives of unipotent classes of G, and, for each i, $\{v_{ij}(j=1,\cdots,c_i)\}$ be a complete set of representatives of U-conjugacy classes of $u_i^G\cap U$. Define a class function ϕ on U by

$$\phi(v_{i1})=|Z_U(v_{i1})|\times|Z|\cdot|Z_G(u_i)|^{-1}\chi(u_i)$$

and

$$\phi(v_{ij})=0\qquad\text{for } j\neq 1.$$

Then $\chi=\operatorname{ind}_{Z\times U}^G(\theta\otimes\phi)$. Since ϕ is an $\mathcal{O}[p^{-1}]$-valued class function on a p-group U, $p^N\phi\in\mathcal{O}\otimes\mathcal{R}(U)$ for a large integer N. Hence

(5.1.6) $$p^N\chi\in(\mathcal{O}\otimes\operatorname{ind}_{Z\times U}^G\mathcal{R}(Z\times U))\cap\mathcal{R}(G)=\operatorname{ind}_{Z\times U}^G\mathcal{R}(Z\times U).$$

By 5.1.2, 5.1.5 and 5.1.6, there exists $\psi\in\mathcal{R}(U)$ such that

$$\chi=\operatorname{ind}_{Z\times U}^G\theta\otimes\psi\ .$$

Lemma 5.1.7. *Under the same assumptions as in* 5.1, *we get* $l^*\in\mathcal{O}\otimes\operatorname{ind}_{Z\times U}^G\mathcal{R}(Z\times U)$.

Proof. For a cyclic subgroup A of G, put

$$\theta_A(x)=\begin{cases}|A| & \text{if } \langle x\rangle=A\\ 0 & \text{if } \langle x\rangle\neq A.\end{cases}$$

Then

$$(\operatorname{ind}_A^G\theta_A)(x)=\sum_{\substack{y\in G\\ \langle y^{-1}xy\rangle=A}}1$$

and

$$\sum_{A\subset Z\times G_{\text{unipo}}}\operatorname{ind}_A^G\theta_A=g^*,$$

where G_{unipo} is the set of unipotent elements of G. (See [7, proof of Proposition

27].) Hence for every $\boldsymbol{Z}$-valued class function f, $fg^* \in \mathcal{O}\otimes \mathrm{ind}_{Z\times U}^{G}\mathcal{R}(Z\times U)$. (See [7, proof of lemma 6].) For each element $x\in Z$, there exists a $\boldsymbol{Z}$-valued function $\psi_x \in \mathcal{O}\otimes \mathrm{ind}_{\langle x\rangle\times U}^{G}\mathcal{R}(\langle x\rangle\times U)$ such that

$$\psi_x(x) \not\equiv 0 \qquad \mathrm{mod}\, p$$

and

$$\psi_x(y) \equiv 0 \qquad \mathrm{mod}\, p$$

if $x\neq y\in Z$. (See [7, lemma 8].) Put $\psi = \sum_{x\in Z}\psi_x$. Then ψ is $\boldsymbol{Z}$-valued, $\psi\in \mathcal{O}\otimes \mathrm{ind}_{Z\times U}^{G}\mathcal{R}(Z\times U)$ and $\psi(x)\not\equiv 0$ mod p for $x\in Z\times G_{\mathrm{unipo}}$. Hence, for some integer N, $l^*(\psi^N-1)$ can be written as fg^* with some $\boldsymbol{Z}$-valued class function f and $l^*(\psi^N-1^*)\in \mathcal{O}\otimes \mathrm{ind}_{Z\times U}^{G}\mathcal{R}(Z\times U)$. Since $l^*\psi^N = l\psi^N \in \mathcal{O}\otimes \mathrm{ind}_{Z\times U}^{G}\mathcal{R}(Z\times U)$, we obtain $l^*\in \mathcal{O}\otimes \mathrm{ind}_{Z\times U}^{G}\mathcal{R}(Z\times U)$.

DEFINITION 5.2 ([5]). Let $\boldsymbol{T}$ be a maximal torus defined over k. A reductive subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$ defined over k is called a distinguished subgroup if it can be represented as $\boldsymbol{H}=Z_{\boldsymbol{G}}^{\circ}(\boldsymbol{T}_0)$ with some subgroup $\boldsymbol{T}_0$ of $\boldsymbol{T}$. Denote the set of distinguished subgroups by $\boldsymbol{\mathcal{H}}=\boldsymbol{\mathcal{H}}_{\boldsymbol{T}}$. We define a partial order in $\mathcal{H}_{\sigma^i}$ by the inclusion and the Möbius function μ_i on it, where we put $\mathcal{H}=\boldsymbol{\mathcal{H}}_{\sigma^m}$. (See 1.6.) For $\theta\in (T_{\sigma^i})^{\wedge}$, let

$$R_{\boldsymbol{T},i}^{\theta} = R_{\boldsymbol{T},G_{\sigma^i}}^{\theta},$$

where $R_{\boldsymbol{T},G_{\sigma^i}}^{\theta}$ is the virtual character of G_{σ^i} corresponding to $(\boldsymbol{T},\theta)$ constructed by Deligne and Lusztig [1]. Let

$$\begin{aligned} K_{\boldsymbol{T},i}^{\theta} &= K_{\boldsymbol{T},G_{\sigma^i}}^{\theta} \\ &= \textstyle\sum_{\boldsymbol{H}\in \mathcal{H}_{\sigma^i}}\mu_i(\boldsymbol{H})\ \mathrm{ind}\ (R_{\boldsymbol{T},H_{\sigma^i}}^{\theta}\,|\,H_{\sigma^i}\to G_{\sigma^i})\,. \end{aligned}$$

Let N^i: $T_{\sigma^i}\to T_{\sigma}$ be the norm map. For $\theta\in (T_\sigma)^{\wedge}$, we define a class functions $AR_{\boldsymbol{T}}^{\theta}$ and $AK_{\boldsymbol{T}}^{\theta}$ on AG by

$$\begin{aligned} i\text{-res}\ AR_{\boldsymbol{T}}^{\theta} &= R_{\boldsymbol{T},i}^{\theta\circ N^i} \\ i\text{-res}\ AK_{\boldsymbol{T}}^{\theta} &= K_{\boldsymbol{T},i}^{\theta\circ N^i} \end{aligned}$$

Lemma 5.3 ([5; Propositions 4 and 5]). *Let $\boldsymbol{Z}$ be the center of $\boldsymbol{G}$. If the Jordan decomposition of $x\in G_{\sigma^i}$ is $x=x_s x_u$ where x_s (resp. x_u) is semisimple (resp. unipotent), then*

(5.3.1) $$K_{\boldsymbol{T},i}^{\theta}(x) = 0 \qquad \textit{if}\ \ x_s\notin Z_{\sigma^i}$$

(5.3.2) $$K_{\boldsymbol{T},i}^{\theta}(x) = \theta(x_s)K_{\boldsymbol{T},i}^{\theta}(x_u) \quad \textit{if}\ \ x_s\in Z_{\sigma^i}$$

Moreover there exist constants $p(l)$ and $q(l)$ which depend only on the semisimple rank l of $\boldsymbol{G}$ such that if $p>p(l)$ and $q>q(l)$, then

(5.3.3) $$|Z_{\sigma^i}|\cdot|Z_{G_{\sigma^i}}(x)|^{-1}K^{\theta}_{T,i}(x)\in\mathcal{O}[p^{-1}]\,.$$

By 5.1 and 5.3, we get

Corollary 5.4. *Let $\boldsymbol{Z}$ be the center of $\boldsymbol{G}$. If $p>p(l)$ and $q>q(l)$, then there exists a character $\psi\in\mathcal{R}(U_{\sigma^i})$ such that*

$$K^{\theta}_{T,i}=\operatorname{ind}\,(\theta\otimes\psi\,|\,Z_{\sigma^i}\times U_{\sigma^i}\to G_{\sigma^i})\,.$$

Theorem 5.5. *Let $\boldsymbol{T}$ be a maximal torus defined over k and $\theta\in(T_{\sigma^i})^{\wedge}_{\sigma}$. If $p>p(l)$ and $q>q(l)$, then there exist virtual characters $A\rho$, $A\rho'\in\mathcal{R}(AG)$ such that*

$$i\text{-res }A\rho=R^{\theta}_{T,i}$$
$$i\text{-res }A\rho'=K^{\theta}_{T,i}\,.$$

If $\langle R^{\theta}_{T,i}, R^{\theta}_{T,i}\rangle=1$, then we can choose $A\rho$ so that $\langle A\rho, A\rho\rangle_{AG}=1$.

Proof. We prove by induction on dim $\boldsymbol{DG}$, where $\boldsymbol{DG}$ is the derived group of $\boldsymbol{G}$. If dim $\boldsymbol{DG}=0$, the statement is clear. Let dim $\boldsymbol{DG}>0$. Since the statement about $R^{\theta}_{T,i}$ follows from that about $K^{\theta}_{T,i}$ by an induction argument and by 3.7, it suffices to prove the statement about $K^{\theta}_{T,i}$. By imbedding the group $\boldsymbol{G}$ into a group with a connected center and the same derived gıoup as $\boldsymbol{G}$, we may suppose that the center of $\boldsymbol{G}$ is connected. Hence we must prove the existence of a character $A\rho\in\mathcal{R}(A(Z\times U))$ such that i-res $A\rho=\theta\otimes\psi$. (See 3.6 and 5.4.) Such an $A\rho$ exists by 4.4.

6. Liftings of regular and semisimple characters

6.1. Let $\boldsymbol{G}$ be a reductive group with a connected center $\boldsymbol{Z}$. Let $\boldsymbol{B}$ and $\boldsymbol{T}$ be a Borel subgroup and a maximal torus both defined over k. Let I be the set of σ-orbits of the simple roots with respect to $\boldsymbol{T}\subset\boldsymbol{B}$. In the following we use the notations of [1; Chapter 10]. Let χ be a linear character of U in general position. Then

(6.1.1) $$\Gamma_G=\operatorname{ind}^G_U\chi$$

is independent of the choice of χ. Put

(6.1.2) $$\Delta_G=\textstyle\sum_{J\subset I}(-1)^{|J|}\operatorname{ind}^G_{P(J)}\Gamma_{L(J)}\,,$$

where $L(J)$ is the Levi subgroup of a parabolic subgroup $P(J)$. An irreducible component of Γ_G (resp. Δ_G) is called a regular character (resp. a semisimple character). Then the followings are known. (See [1], [3], [10].) For an arbitrary irreducible character ρ of G,

(6.1.3) $$\langle\Gamma_G, \rho\rangle=0\quad\text{or}\quad 1$$

(6.1.4) $$\langle \Delta_G, \rho \rangle = 0 \quad \text{or} \quad \pm 1 .$$

Let x be a geometric conjugacy class of $\boldsymbol{G}$. Put

$$\rho_x = \sum_{\substack{(T,\theta) \bmod G \\ [\theta]=x}} (-1)^{r(G)-r(T)} \langle R_T^\theta, R_T^\theta \rangle^{-1} R_T^\theta$$

and

$$\rho'_x = (-1)^{r(G)-\delta_x} \sum_{\substack{(T,\theta) \bmod G \\ [\theta]=x}} \langle R_T^\theta, R_T^\theta \rangle^{-1} R_T^\theta .$$

Then ρ_x and ρ'_x are irreducible characters of G and one has

(6.1.5) $$\Gamma_G = \sum_x \rho_x$$

and

(6.1.6) $$\Delta_G = \sum_x (-1)^{r(G)-\delta_x} \rho'_x ,$$

where $r(\boldsymbol{G})$ is the split rank of $\boldsymbol{G}$. Note that an irreducible character is regular and semisimple if and only if it is equal to some irreducible $\pm R_T^\theta$. Let l be the semisimple rank of $\boldsymbol{G}$, then

(6.1.7) $$\langle \Gamma_G, \Gamma_G \rangle = \langle \Delta_G, \Delta_G \rangle = |Z| q^l .$$

Denote $\Gamma_{G_{\sigma^i}}$ (resp. $\Delta_{G_{\sigma^i}}$, $St_{G_{\sigma^i}}$) by Γ_i (resp. Δ_i, St_i).

Lemma 6.2. (1) *Define a class function $A\Gamma = A\Gamma_G$ on AG by i-*res *$A\Gamma = \Gamma_i$. Then $A\Gamma \in \mathcal{R}_+(AG)$.*
(2) *Define a class function $A\Delta = A\Delta_G$ on AG by i-*res *$A\Delta = \Delta_i$. Then $A\Delta \in \mathcal{R}(AG)$.*
(3) *Define a class function $ASt = ASt_G$ on AG by i-*res *$ASt = St_i$. Then $ASt \in (AG)^\wedge$.*
(4) *Denote the $k_{(m,i)}$-split rank of $\boldsymbol{G}$ by $r(\boldsymbol{G}, i)$ and put $\varepsilon_T(i) = (-1)^{r(\boldsymbol{G},i)-r(\boldsymbol{T},i)}$. Define a class function $A\varepsilon_T$ on AG by i-*res *$A\varepsilon_T = \varepsilon_T(i)$. Then $A\varepsilon_T \in \mathcal{R}(AG)$.*

Proof. (1) Choose the character χ in 6.1.1 to be σ-invariant and extend χ to a linear character $A\chi$ of AU by $A\chi(\sigma^i x) = \chi(x)$. It suffices to prove that the linear character i-res $A\chi$ of U_{σ^i} is in general position. This can be proved by 3.6.3 and 3.6.4.
(2) We prove (2) by using lemma 1.5. Fix a subset $J \subset I$ and put $d = \min\{j > 0 \mid J^{\sigma^j} = J\}$. Let

$$a_i = \begin{cases} (-1)^{|J/\langle \sigma^i \rangle|} & \text{if } d \mid i \\ 0 & \text{if } d \nmid i . \end{cases}$$

If $de \mid m$, then it is easy to verify that $e \mid \sum_{i \mid de} \mu(de/i) a_i$. Hence $A\Delta \in \mathcal{R}(AG)$.
(3) The proof is similar to (2).
(4) If the Frobenius endomorphism of $\boldsymbol{T}$ is given by $q\tau w$, then $\varepsilon_T(i) = \det w^{(m,i)}$.

Here we assume that the Frobenius endomorphism of a maximally split torus is given by $q\tau$. (See [1; 1.1].) Hence $A\varepsilon_T \in \mathcal{R}(AG)$.

Lemma 6.3. *Let $\boldsymbol{G}$ be a reductive group with a connected center. Suppose that an irreducible character $\rho_\iota \in (\boldsymbol{G}_{\sigma^\iota})^\wedge_\sigma$ is regular and represented as i-res $A\rho = \varepsilon\rho_i$ with $\varepsilon = \pm 1$ and some $A\rho \in (AG)^\wedge$. Then by modifying $A\rho$, if needed, we can suppose that $\varepsilon = 1$ and*

$$\langle j\text{-res } A\rho,\, \Gamma_j\rangle = 1 \qquad 0 \le j \le m-1\,.$$

In particular $A\rho|_G = 0$-res $A\rho$ is regular. Moreover

$$(6.3.1) \qquad |\{\text{irreducible components of } \Gamma\}/\langle\sigma\rangle| = \langle A\Gamma,\, A\Gamma\rangle\,.$$

Proof. Let

$$A\Gamma = (c_0 A\rho + c_1 \xi \otimes A\rho + \cdots + c_{m-1}\xi^{m-1} \otimes A\rho) + \cdots\,.$$

Then c_l are non-negative and

$$\Gamma = (c_0 + c_1 + \cdots + c_{m-1})\rho + \cdots\,,$$

where $A\rho|_G = \rho$. Hence there is at most one non-zero c_l and, if exists, such a c_l equals one. Put $A\rho' = \xi^{-l} \otimes A\rho$ and $c_j' = c_{j+l}$. Here we identify $\{0, \cdots, m-1\}$ with $\boldsymbol{Z}/(m)$ naturally. Then $\zeta^{il}\langle i\text{-res } A\rho, \Gamma_\iota\rangle = c_0'$. Hence if we take such $A\rho'$ instead of $A\rho$, we have $\varepsilon = 1$, $c_0 = 1$ and $c_1 = \cdots = c_{m-1} = 0$. Since

$$\textstyle\sum_{j=0}^{m-1} \zeta^{lj}\langle j\text{-res } A\rho,\, \Gamma_j\rangle = m\langle \xi^l \otimes A\rho,\, A\Gamma\rangle_{AG} = mc_l\,,$$

we obtain

$$\langle j\text{-res } A\rho,\, \Gamma_j\rangle = 1 \qquad 0 \le j \le m-1\,.$$

Since, for each irreducible component χ of Γ,

$$\chi + \chi^\sigma + \cdots + \chi^{\sigma^{d-1}} \qquad (d = d(\chi))$$

is the restriction of some irreducible component $A\chi$ of $A\Gamma$ and the converse is also true, 6.3.1 holds.

Lemma 6.4. *Let $\boldsymbol{G}$ be a reductive group with a connected center. Suppose that an irreducible character $\rho_\iota \in (G_{\sigma^\iota})^\wedge_\sigma$ is semisimple and represented as i-res $A\rho = \varepsilon\rho_\iota$ with $\varepsilon = \pm 1$ and some $A\rho \in (AG)^\wedge$. Then $A\rho|_G = \rho$ is semisimple.*

Proof. Let

$$\Delta A = (c_0 A\rho + c_1 \xi \otimes A\rho + \cdots + c_{m-1}\xi^{m-1} \otimes A\rho) + \cdots\,.$$

If we can prove that there exists at most one non-zero c_j, then we can prove the semisimplicity of ρ by the same argument as in 6.3. Since, for each irreducible component χ of Δ, $\chi+\chi^{\sigma}+\cdots+\chi^{\sigma^{d-1}}$ $(d=d(\chi))$ is the restriction of some irreducible component $A\chi$ of $A\Delta$, we obtain

(6.4.1) $\qquad \langle A\Delta, A\Delta\rangle \geq |\{\text{irreducible components of } \Delta\}/\langle\sigma\rangle|$.

Let

$$A\Delta = (d_0 A\chi + d_1\xi\otimes A\chi+\cdots)+\cdots .$$

Then

$$\begin{aligned}&\text{the left hand side of 6.4.1}\\ &\geq \textstyle\sum_{\langle\chi,\Delta\rangle\neq 0}(d_0^2+d_1^2+\cdots)\geqq \text{the right hand side of 6.4.1.}\end{aligned}$$

Since

$$\text{the left hand side of 6.4.1} = m^{-1}\textstyle\sum_{i=0}^{m-1}\langle\Delta_i, \Delta_i\rangle_{G_{\sigma^i}}$$

and

$$\begin{aligned}&\text{the right hand side of 6.4.1}\\ &= |\{\text{irreducible components of } \Gamma\}/\langle\sigma\rangle| \text{ (by 6.1.5 and 6.1.6)}\\ &= \langle A\Gamma, A\Gamma\rangle_{AG}\\ &= m^{-1}\textstyle\sum_{i=0}^{m-1}\langle\Gamma_i, \Gamma_i\rangle_{G_{\sigma^i}},\end{aligned}$$

these two terms are equal by 6.1.7. Hence for each irreducible component χ of Δ, we have $d_0^2+d_1^2+\cdots=1$. Hence there exists at most one non-zero c_j.

6.5. If $\langle R_T^{\theta}, R_T^{\theta}\rangle=1$, a virtual character of the form R_T^{θ} is called a regular semisimple character. Denote the set of regular semisimple characters of G by $RS(G)$. Further, put $RS_+(G)=\{RS(G)\cup(-RS(G))\}\cap G^{\wedge}$.

Lemma 6.6. *If $R_T^{\theta}\in RS(G)_{\sigma}$, then there exists a σ-invariant pair $(\boldsymbol{T}_1, \theta_1)$ such that $R_T^{\theta}=R_T^{\theta_1}$.*

Proof. By Deligne and Lusztig [1, Chapter 5], a conjugacy class of $(\boldsymbol{T}, \theta)$ corresponds to some regular semisimple conjugacy class of the dual group $\boldsymbol{G}^*$. Since a σ-invariant regular semisimple class contains a σ-invariant element, the lemma is clear.

Lemma 6.7. *Let $\boldsymbol{G}$ be a reductive group. If $p>p(l)$ and $q>q(l)$, for each $\rho_i\in RS_+(G_{\sigma^i})_{\sigma}$, there exists an $A\rho\in(AG)^{\wedge}$ such that i-res $A\rho=\rho_i$ and $A\rho|_G=\rho\in RS_+(G)_{\sigma}$.*

Proof. By the same reason as in the proof of 5.5, we may suppose that the center of $\boldsymbol{G}$ is connected. By 6.6 and 5.5, there exist an irreducible character $A\rho$ of AG and $\varepsilon=\pm1$ such that i-res $A\rho=\varepsilon\rho_i$. Since ρ_i is regular, we may suppose that $\varepsilon=1$. By 6.3 and 6.4, $A\rho|_G$ is regular and semisimple. Hence $A\rho|_G \in RS_+(G)$.

6.8. Denote the mapping $RS_+(G_{\sigma^i})_\sigma \ni \rho_i \mapsto \rho \in RS_+(G)_\sigma$ by i-lift$_+$. Denote the mapping $RS(G_{\sigma^i})_\sigma \to RS(G)_\sigma$ induced by i-lift$_+$ by i-lift.

Lemma 6.9. *If $R_{T,1}^\theta \in RS(G_\sigma)$, then $R_{T,m}^{\theta\circ N^m} \in RS(G)_\sigma$, where N^m: $T \to T_\sigma$ is the norm map. Denote the mapping $RS(G_\sigma) \ni R_{T,1}^\theta \mapsto R_{T,m}^{\theta\circ N^m} \in RS(G)_\sigma$ by $*$-lift. This induces the mapping $RS_+(G_\sigma) \to RS_+(G)_\sigma$, which is denoted by* $*$-lift$_+$. *Then* $*$-lift *is well defined and bijective.*

The proof is clear from [1; 5.21.5].

Corollary 6.10 *The mapping*

$$i\text{-lift}_+ : RS_+(G_{\sigma^i})_\sigma \to RS_+(G)_\sigma$$

is bijective.

Proof. By 1.4, i-lift$_+$ is injective. By 6.9

$$|RS_+(G_{\sigma^i})_\sigma| = |RS_+(G_\sigma)| = |RS_+(G)_\sigma| .$$

Hence i-lift$_+$ is bijective.

Lemma 6.11. *Let $\mathbf{G}$ be a reductive group and $p > p(l)$, $q > q(l)$. For each $\rho_i \in RS_+(G_{\sigma^i})_\sigma$, there exists an $A\rho \in (AG)^\wedge$ such that*

$$i\text{-res } A\rho = \rho_i$$

and

$$j\text{-res } A\rho \in RS_+(G_{\sigma^j})_\sigma \qquad 0 \le j \le m-1.$$

Proof. Fix an integer j. Let i-lift$_+\rho_i=\rho$ and j-lift$_+\rho_j=\rho$ (See 6.10.). Then, by 6.3 and 6.7, there exist $A\rho$, $A\rho' \in (AG)^\wedge$ such that

$$\begin{aligned} &i\text{-res } A\rho = \rho_i && A\rho|_G = \rho , \\ &\langle l\text{-res } A\rho, \Gamma_l\rangle = 1 && 0 \le l \le m-1 , \\ &j\text{-res } A\rho' = \rho_j && A\rho'|_G = \rho , \end{aligned}$$

and

$$\langle l\text{-res } A\rho', \Gamma_l\rangle = 1 \quad 0 \le l \le m-1 .$$

Then $A\rho' = \xi^t \otimes A\rho$ for some t. Since

$$\langle l\text{-res } A\rho', \Gamma_l\rangle = \zeta^{tl}\langle l\text{-res } A\rho, \Gamma_l\rangle ,$$

$\xi^t=1$. Hence $A\rho'=A\rho$. This proves the lemma.

7. Main theorem (The case: $(m, p)=1$)

7.1. Let $\boldsymbol{G}$ be a reductive group defined over k and l be its semisimple rank. Let $\boldsymbol{T}$ be a maximal torus defined over k, let $\boldsymbol{W}$ be the Weyl group with respect to $\boldsymbol{T}$ and suppose that the Frobenius endomorphism of $\boldsymbol{T}$ is given by $\sigma=q\tau w_T$ with some $w_T\in\boldsymbol{W}$ (See the proof of 6.2 (4).). Let $X=X(\boldsymbol{T})$ be the lattice of characters of $\boldsymbol{T}$. Then X is a $\boldsymbol{W}$-module.

Theorem 7.2. *There exist constants $p(l)$ and q_1, where $p(l)$ is the same constant as in* 5.3, *and q_1 depends only on $(\boldsymbol{W},\sigma)$-module X and m, such that if $p>p(l)$, and $q>q_1$ and $(m,p)=1$, then AR_T^θ is a virtual character of AG for any $\theta\in(T_\sigma)^\wedge$.*

Corollary 7.2.1. *Under the same condition as in* 7.2, *the map* 1-lift *coincides with* $*$-lift.

In the remaining of this section, we prove theorem 7.2, and q_i, c_i $(i=1, 2, \cdots)$ are some positive constants depending only on $(\boldsymbol{W},\sigma)$-module X and m. The set of $n\times n$-matrices is denoted by $M_n(\boldsymbol{Z})$.

Lemma 7.3. *If $f(x), g(x)\in M_n(\boldsymbol{Z})[x]$ and $g(x)$ is monic, then one and only one of the followings holds.*

(1) $$[f(q)\boldsymbol{Z}^n: f(q)\boldsymbol{Z}^n\cap g(q)\boldsymbol{Z}^n]>c_0q \qquad \text{for } q>q_0,$$

where c_0 and q_0 are some positive constants depending only on f and g.

(2) $$f(x)=g(x)r(x) \qquad \text{for some } r(x)\in M_n(\boldsymbol{Z})[x].$$

Proof.

$$\begin{aligned}&[f(q)\boldsymbol{Z}^n: f(q)\boldsymbol{Z}^n\cap g(q)\boldsymbol{Z}^n]\\ &=[f(q)\boldsymbol{Z}^n+g(q)\boldsymbol{Z}^n: g(q)\boldsymbol{Z}^n].\end{aligned}$$

Choose $r(x)\in M_n(\boldsymbol{Z})[x]$ and put $s(x)=f(x)+g(x)r(x)$ so that $s(x)=0$ or $d=\deg s<\deg g$. Suppose that $s(x)\neq 0$. Then

$$\begin{aligned}&[f(q)\boldsymbol{Z}^n: f(q)\boldsymbol{Z}^n\cap g(q)\boldsymbol{Z}^n]\\ &=[s(q)\boldsymbol{Z}^n: s(q)\boldsymbol{Z}^n\cap g(q)\boldsymbol{Z}^n]\\ &=[q^{-d}s(q)\boldsymbol{Z}^n: q^{-d}s(q)\boldsymbol{Z}^n\cap q^{-d}g(q)\boldsymbol{Z}^n].\end{aligned}$$

Thus we obtain (1).

7.4. To prove 7.2, it suffices to prove that $AK_T^\theta\in\mathcal{R}(AG)$ by 1.8. Note that AK_T^θ depends only on $\theta|_{Z_\sigma}$. For a divisor d of m, $(w_0, \cdots, w_{d-1})\in\boldsymbol{W}^d-\Delta$ (Δ is the diagonal set) and $w\in\boldsymbol{W}$, denote by $Y_d(w, w_0, \cdots, w_{d-1})$ the set of μ's in X which satisfy the following condition:

(7.4.1) $$\sum_{i=0}^{d-1}(q\tau w)^i w_i\mu \in \left(\sum_{i=0}^{d-1}(q\tau w)^i\right)X.$$

For $(w_1, w_2)\in \boldsymbol{W}^2-\Delta$, $w\in \boldsymbol{W}$, denote by $Y_0(w, w_1, w_2)$ the set of μ's in X which satisfy the following condition:

(7.4.2) $$(w_1-w_2)\mu \in (q\tau w-1)X.$$

Put $S=\bigcup Y_d \cup \bigcup Y_0$. We claim that

(7.4.3) $$\mu+\sum_{w\in W_\sigma}(1-w)X \not\subset S$$

for $\mu\in X$, if $q>q_2$ for some q_2. Put $X_0=\sum_{w\in W_\sigma}(1-w)X$. For every Y,

(7.4.4) $$[\mu+X_0\colon Y\cap(\mu+X_0)]>c_1 q \qquad \text{for } q>q_3,$$

for some q_3 or $\mu+X_0\subset Y$ for some Y. Assume that $\mu+X_0$ is contained in S. Note that in 7.4.4, constants c_1 and q_3 can be chosen independently of μ. Since $\mu+X_0$ is contained in S, if $q>q_4$, for some q_4, 7.4.4 can not hold. Hence, if $q>q_5$, for some q_5, there exists an $r(x)\in \mathrm{End}(X)[x]$ such that one of the followings holds:

(7.4.5) $$\sum_{i=0}^{d-1}(x\tau w)^i w_i(1-w)=\sum_{i=0}^{d-1}(x\tau w)^i r(x)$$

(7.4.6) $$(w_1-w_2)(1-w)=(x\tau w-1)r(x).$$

Comparing the degree in x, one sees that 7.4.6 can not hold and that $r(x)$ in 7.4.5 is a constant. Put $r(x)=a$. Then for each i, $w_i(1-w)=a$. This contradicts $(w_0, \cdots, w_{d-1})\notin\Delta$. Hence our claim 7.4.3 is proved. Hence to prove 7.2, it suffices to prove $AK_T^{\phi\circ\mu}\in\mathcal{R}(AG)$ for $\mu\in X-S$. Here ϕ is chosen as in 2.1.

In the following we fix a $\mu\in X$, put $\theta=\phi\circ\mu$ and assume that $p>p(l)$ and $q>q_2$.

7.5. For $\lambda\in X$, we define a rational representation $R(\lambda)$ of $\boldsymbol{G}$ by $R(\lambda)|_{\boldsymbol{T}}=\sum_{\lambda'}\lambda'$, where λ' runs all over the class of λ mod $\boldsymbol{W}$.

Lemma 7.6. *If* $\lambda\in X-S$, *then*

$$\langle R_{T,i}^{\theta\circ N^i}, \beta_\phi[R(\lambda)_i]\rangle_{G_\sigma{}^i}=0 \quad \text{or} \quad 1.$$

This inner product equals 1, *iff* $\mu\equiv w\lambda$ *mod* $(q\tau w_T-1)X$ *for some* $w\in\boldsymbol{W}$.

Proof.
$$\begin{aligned}&\langle R_{T,i}^{\theta\circ N^i}, \beta_\phi[R(\lambda)_i]\rangle_{G_\sigma{}^i}\\ &=\langle\phi\circ\mu\circ N^i, \beta_\phi[R(\lambda)_i]|_{T_\sigma{}^i}\rangle_{T_\sigma{}^i}\\ &=\langle\phi\circ\sum_{j=0}^{d-1}(q\tau w_T)^j\mu, \sum_{(w_0,\cdots,w_{d-1})}\phi\circ\sum_{j=0}^{d-1}(q\tau w_T)^j w_j\lambda\rangle_{T_\sigma{}^i}.\end{aligned}$$

If $\sum_{j=0}^{d-1}(q\tau w_T)^j w_i\lambda$ mod $((q\tau w_T)^d-1)X$ is $q\tau w_T$-invariant, $\sum_{j=0}^{d-1}(q\tau w_T)^j w_j\lambda\in\left(\sum_{j=0}^{d-1}(q\tau w_T)^j\right)X$. By 7.4.1, $w_0=\cdots=w_{d-1}$. Hence the above inner product equals

$$\langle\phi\circ\sum_{j=0}^{d-1}(q\tau w_T)^j\mu, \sum_w\phi\circ(\sum_{j=0}^{d-1}(q\tau w_T)^j w\lambda)\rangle_{T_{\sigma^i}}$$
$$= \langle\phi\circ\mu\circ N^i, \sum_w\phi\circ w\lambda\circ N^i\rangle_{T_{\sigma^i}}$$
$$= \langle\phi\circ\mu, \sum_w\phi\circ w\rangle_{T_\sigma} .$$

If $w_1\lambda=w_2\lambda$ on T_σ, $(w_1-w_2)\lambda\in(q\tau w_T-1)X$. By 7.4.2, $w_1=w_2$. Thus we obtain the lemma.

Lemma 7.7. *Assume that* $(m, p)=1$ *and* $\mu\in X-S$. *Let* $\rho_i=R_{T,i}^{\theta\circ N^i}$, $\rho_0=\rho$ $=i$-lift ρ_i *and define* $\rho_j\in RS(G_{\sigma^j})_\sigma$ *by* $\rho=j$-lift ρ_j *for* $0\le j\le m-1$. *Define a class function* $A\rho$ *on* AG *by* j-res $A\rho=\rho_j$ $(0\le j\le m-1)$. *Then* $A\rho\in\mathcal{R}(AG)$.

Proof. Let $\varepsilon_j\rho_j\in RS_\pm(G_{\sigma^j})_\sigma$ with $\varepsilon_j=\pm1$. Then there exists an irreducible character $A\rho'$ such that j-res $A\rho'=\varepsilon_j\rho_j$. (See 6.11.) Let

$$\beta_\phi[AR(\mu)] = (c_0A\rho'+c_1\xi\otimes A\rho'+\cdots+c_{m-1}\xi^{m-1}\otimes A\rho')+\cdots$$

and

$$a_j = \langle\beta_\phi[AR(\mu)], A\rho'\rangle_{\sigma^j G} .$$

Then

$$\sum_{j=0}^{m-1}a_j\zeta^{jl} = mc_l .$$

But by 7.6,

$$a_j = \langle j\text{-res } \beta_\phi[AR(\mu)], j\text{-res } A\rho'\rangle_{G_{\sigma^j}}$$
$$= \langle\beta_\phi[R(\mu)_j], \varepsilon_j\rho_j\rangle$$
$$= 0 \text{ or } \varepsilon_j .$$

Hence, unless $a_j\zeta^{lj}$ $(0\le j\le m-1)$ are equal to each other

$$|mc_l| = |\sum a_j\zeta^{lj}| < m,\ c_l = 0 .$$

Since $a_j\neq0$, there exists an l such that $c_l\neq0$. Then $\zeta^l=\varepsilon=\pm1$. Since $a_j\zeta^{lj}=a_j\varepsilon^j=\varepsilon_j\varepsilon^j$ $(0\le j\le m-1)$ are equal to each other, $\varepsilon_j=\varepsilon_0\cdot\varepsilon^j$. Hence $A\rho\in\mathcal{R}(AG)$.

Lemma 7.8. *Assume that* $(m, p)=1$ *and* $\mu, \lambda\in X-S$. *Then we have the equality*

$$\langle R_{T,i}^{\theta\circ N^i}, \beta_\phi[R(\lambda)_i]\rangle_{G_{\sigma^i}}$$
$$= \langle i\text{-lift } R_{T,i}^{\theta\circ N^i}, \beta_\phi[R(\lambda)]\rangle_G = 0 \text{ or } 1 .$$

Proof. Let $\rho_i=R_{T,i}^{\theta\circ N^i}$ and define $A\rho$ as in 7.7. Let

$$\beta_\phi[AR(\lambda)] = (c_0A\rho+c_1\xi\otimes A\rho+\cdots+c_{m-1}\xi^{m-1}\otimes A\rho)+\cdots$$

and

$$a_j = \langle\beta_\phi[AR(\lambda)], A\rho\rangle_{\sigma^j G} .$$

Then

$$\sum_{j=0}^{m-1} a_j \zeta^{lj} = mc_l .$$

But $a_j = \langle \beta_\phi[R(\lambda)_j], \rho_j \rangle = 0$ or 1. Hence $c_1 = \cdots = c_{m-1} = 0$ and $a_0 = \cdots = a_{m-1}$.

7.9. Proof of 7.2. Assume $(m, p) = 1$ and $\mu \in X - S$. Then, by 7.6 and 7.8, for an arbitrary $\lambda \in X - S$,

$$\begin{aligned} &\langle R_{T,m}^{\theta \circ N^m}, \beta_\phi[R(\lambda)] \rangle_G \\ &= \langle R_{T,i}^{\theta \circ N^i}, \beta_\phi[R(\lambda)_i] \rangle_{G_\sigma^i} \\ &= \langle i\text{-lift } R_{T,i}^{\theta \circ N^i}, \beta_\phi[R(\lambda)] \rangle_G \\ &= 0 \text{ or } 1 . \end{aligned}$$

By this and 7.6, there exists a $w \in \boldsymbol{W}$ such that

$$i\text{-lift } R_{T,i}^{\theta \circ N^i} = R_{T,m}^{w\theta \circ N^m} . \tag{7.9.1}$$

Hence, it suffices to prove that the element w of $\boldsymbol{W}$ commutes with τw_T. (See 7.7)

If we take $\mu + (q\tau w_T - 1)\lambda$ instead of μ, $R_{T,i}^{\theta \circ N^i}$ does not change. Hence $R_{T,m}^{w\theta \circ N^m}$ does not change also. Hence for an arbitrary $\lambda \in X$, there exists an element $w(\lambda) \in W$ such that

$$\begin{aligned} (q\tau w_T - 1)^{-1}((q\tau w_T)^m - 1) w\mu \equiv w(\lambda)\, &(q\tau w_T - 1)^{-1}((q\tau w_T)^m - 1) \\ &\times w(\mu + (q\tau w_T - 1)\lambda) \bmod ((q\tau w_T)^m - 1)X . \end{aligned}$$

Then, dividing by $(q\tau w_T)^m - 1$, we obtain

$$(q\tau w_T - 1)^{-1} w\mu \equiv w(\lambda)\,(q\tau w_T - 1)^{-1} w(\mu + (q\tau w_T - 1)\lambda) \quad \bmod X .$$

If we put $\tau w' = w^{-1}(\tau w_T)w$,

$$(q\tau w' - 1)^{-1}\mu \equiv (w^{-1}w(\lambda)w)\,(q\tau w' - 1)^{-1}(\mu + (q\tau w_T - 1)\lambda) \quad \bmod X . \tag{7.9.2}$$

Put $X_z = \{\lambda \in X \mid w^{-1}w(\lambda)w = z\}$ for $z \in \boldsymbol{W}$, then

$$\cup_{z \in \boldsymbol{W}} X_z = X . \tag{7.9.3}$$

If $\lambda_1, \lambda_2 \in X_z$, then, by 7.9.2,

$$(q\tau w' - 1)^{-1}(q\tau w_T - 1)(\lambda_1 - \lambda_2) \equiv 0 \quad \bmod X .$$

Hence, if we put $S' = \{\lambda \in X \mid (q\tau w_T - 1)\lambda \in (q\tau w' - 1)X\}$, and if $\lambda \in X_z$, then $\lambda + S' \supset X_z$. Hence

$$[X : S'] \leq |\boldsymbol{W}| . \tag{7.9.4}$$

But

(7.9.5) $[X: S'] = [(q\tau w_T - 1)X : (q\tau w_T - 1)X \cap (q\tau w' - 1)X]$.

Hence, if $q > q_1$, for some q_1, $w_T = w'$ by 7.9.4 and 7.9.5. Hence w commutes with τw_T. Thus we complete the proof of 7.2.

8. Main theorem (The case: $m = a$ power of p)

8.1. Let G be reductive and T (resp. U) be a maximal torus (resp. a maximal unipotent subgroup) of G defined over k. Let l be the semisimple rank of G and $p(l)$, $q(l)$ the same constants as in 5.3. If $p > p(l)$, U is an exponential unipotent group. Let $Q_{T,i}$ be the Green function of G_{σ^i} corresponding to T ([1], [5]). Define a class function AQ_T on AU by i-res $AQ_T = Q_{T,i}$.

Theorem 8.2. *If $p > p(l)$, $AQ_T \in \mathfrak{R}(AU)$.*

Proof. Since U is an exponential unipotent group, all the irreducible characters of AU are known from 4.3. By 3.3 and 4.3.2, it suffices to prove

(8.2.1) $$m^{-1}\sum_{i=0}^{m-1}\langle Q_{T,i}, \phi_{\lambda,i}\rangle \zeta^{ij} \in Z$$

for $0 \le j < m$ and $\lambda \in \mathfrak{U}'_\sigma$. Take an element $t \in \mathfrak{G}_\sigma$ such that $Z_G(t) = T$. Put $X^\lambda = \{y \in t^G \mid B(\cdot, y) \equiv \lambda \text{ on } H^\lambda\}$. Note that $|X^\lambda| = |X^{a\lambda}|$ if $a \in k_m^\times$. To prove 8.2.1, it suffices to prove

(8.2.2) $$m^{-1}\sum_{i=0}^{m-1}|X^\lambda_{\sigma^i}| \cdot |U_{\sigma^i}|^{-1}\zeta^{ij} \in Z .$$

The proof of 8.2.2 can be reduced to the following lemma as in [5].

Lemma 8.3. *Let Z be an algebraic variety defined over a finite field k and $Z^\sim$ be the variety ovr $\bar{k}$ corresponding to Z. Suppose that $Z^\sim$ can be represented as a finite disjoint union $Z_j^\sim$ and each $Z_i^\sim$ is open in $\cup_{j \ge i} Z_j^\sim$. Moreover suppose that there exist a variety $Y_i^\sim$ and morphism $f_i: Z_i^\sim \to Y_i^\sim$ for each i such that each fibre is empty or isomorphic to a fixed affine space A^n. Let $K = k_m$ and ζ be an m-th root of unity. Then*

$$m^{-1}\sum_{i=0}^{m-1}|Z_{\sigma^i}| \cdot |K_{\sigma^i}|^{-n}\zeta^i \in Z .$$

(Note that $K_{\sigma^i} = K_{\sigma^{(m,i)}} = k_{(m,i)}$.)

Proof. Denote the eigenvalues of Frobenius σ on $H_c^{\text{even}}(Z, \bar{Q}_l)$ (resp. $H_c^{\text{odd}}(Z, \bar{Q}_l)$) by $|k|^n\alpha_j$ (resp. $|k|^n\beta_j$). Then α_j's and β_j's are algebraic integers. (See [5].) Put

$$\chi(i) = \sum \alpha_j^{(m,j)} - \sum \beta_j^{(m,i)} .$$

By Lefschetz fixed point theorem, it suffices to prove that χ is a character of $Z/(m)$. This follows from the following lemma.

Lemma 8.3.1. *Let $\alpha, \beta, \cdots$ be algebraic integers and $m(\alpha), m(\beta), \cdots$ be rational integers. Put*

$$\psi(i) = m(\alpha)\alpha^i + m(\beta)\beta^i + \cdots$$
$$\chi(i) = \psi((m, i)).$$

If $\psi(i) \in \boldsymbol{Z}$ for $i=1, 2, \cdots$, then χ is a character of $\boldsymbol{Z}/(m)$.

Proof. Since $\psi(i)^\tau = \psi(i)$ for $\tau \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, we get $m(\alpha)=m(\alpha^\tau)$. Hence we may suppose that $\alpha, \beta, \cdots$ are conjugate over $\boldsymbol{Q}$ and $m(\alpha)=m(\beta)=\cdots=1$. In general $f_i(x, y, \cdots)$, $(x, y, \cdots \in \mathcal{O})$, means the i-th fundamental symmetric polynomial of $\{x^\tau, y^\tau, \cdots | \tau \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})\}$, and

$$s_i(x, y, \cdots) = \sum (x')^i + \sum (y')^i + \cdots,$$

where $x', y', \cdots$ run all over the conjugacy classes of $x, y, \cdots$ over $\boldsymbol{Q}$ respectively. If there exist non-negative integers c_i, d_i such that

$$\prod(1-x^i)^{c_i}\prod(1+x^i)^{d_i} = 1+f_1(\alpha)x+\cdots+f_{r-1}(\alpha)x^{r-1}+a_r x^r+\cdots,$$

then

$$(1 \pm x^r)^{\pm(f_r(\alpha)-a_r)}\prod(1-x^i)^{c_i}\prod(1+x^i)^{d_i}$$
$$= 1+f_1(\alpha)x+\cdots+f_{r-1}(\alpha)x^{r-1}+f_r(\alpha)x^r+\cdots.$$

Hence there exist roots of unity $\zeta_1, \zeta_2, \cdots$ such that $f_i(\alpha)=f_i(\zeta_1, \zeta_2, \cdots)$ for $i \leq m$. Then

$$\psi(i) = s_i(\alpha) = s_i(\zeta_1, \zeta_2, \cdots)$$
$$= s_i(\zeta_1)+s_i(\zeta_2)+\cdots \quad \text{for } i \leq m.$$

Hence it suffices to prove that

$$\chi(i) = \sum_{j=0}^{r-1} \zeta^{(m,i)j}$$

gives a character of $\boldsymbol{Z}/(m)$ if ζ is an r-th root of unity. If $r|m$, then χ is the pullback of the regular character of $\boldsymbol{Z}/(r)$ by the projection $\boldsymbol{Z}/(m) \to \boldsymbol{Z}/(r)$. If $r \not| m$, then $\chi=0$.

Theorem 8.4. *If $p>p(l)$, $q>q(l)$ and m is a power of p, then AR_T^θ and AK_T^θ are virtual characters of AG.*

Proof. By 1.8, it suffices to prove that $AK_T^\theta \in \mathcal{R}(AG)$. We may suppose that the center of $\boldsymbol{G}$ is connected. By the Brauer's characterization of characters, it suffices to prove that $AK_T^\theta|_{G_s \times G_u}$ is a character. Here G_s (resp. G_u) is a subgroup of AG which consists of p'-elements (resp. p-elements). If $s \in G_s$ and $\sigma^i u \in G_u$, then by some $\alpha \in \boldsymbol{G}$

$$N_i(s\cdot\sigma^i u) = (\alpha^{-1}s^{m/d}\alpha)\cdot\alpha^{-1}(\sigma^i u)^{m/d}\alpha$$

with $d=(m, i)$. If $s\in Z$, then two elements $\alpha^{-1}(\sigma^i u)^{m/d}\alpha$ and $N_i(\sigma^i u)$ are conjugate in G_{σ^i}. Since m is a power of p, $\alpha^{-1}s^{m/d}\alpha$ belongs to Z if and only if $s\in Z$. Hence $AK_T^\theta|_{G_s\times G_u}$ is supported by $(G_s\cap Z)\times G_u$. Hence

$$AK_T^\theta|_{G_s\times G_u} = \mathrm{ind}(|G_s|^{-1}\cdot|G_s\cap Z|\cdot AK_T^\theta;(Z\cap G_s)\times G_u\to G_s\times G_u)\,.$$

If $s\in G_s\cap Z$ and $\sigma^i u\in G_u$, then

$$G_s\supset Z_G(\sigma^i u)\simeq Z_{G_{\sigma^i}}(N_i(\sigma^i u))\,. \tag{8.4.1}$$

Since $\sigma^i u\cdot s=s\cdot\sigma^i u=\sigma^i u\cdot s^{\sigma^i}$,

$$G_s\cap Z = G_s\cap Z_{\sigma^i}\,. \tag{8.4.2}$$

Moreover

$$\begin{aligned} AK_T^\theta(s\cdot\sigma^i u) &= K_{T,i}^{\theta\circ N^i}(s^{m/d}N_i(\sigma^i u)) \\ &= \theta(N^i(s^{m/d}))\cdot K_{T\,i}^1(N_i(\sigma^i u)) \\ &= \theta(N^m(s))\cdot AK_T^1(\sigma^i u)\,, \end{aligned}$$

Hence

$$AK_T^\theta|_{(G_s\cap Z)\times G_u} = (\theta\circ N^m|_{G_s\cap Z})\otimes(AK_T^1|_{G_u})\,.$$

Hence it suffices to prove

$$|G_s|^{-1}\cdot|G_s\cap Z|\cdot AK_T^1|_{G_u}\in\mathcal{R}(G_u)\,. \tag{8.4.3}$$

By the same argument as in [5], it suffices to prove that 8.4.3 is $\boldsymbol{Z}$-valued. If $\sigma^i u\in G_u$, we have

$$\begin{aligned} &|G_s|^{-1}\cdot|G_s\cap Z|\cdot AK_T^1(\sigma^i u) \\ =\ &|G_s|^{-1}\cdot|G_s\cap Z_{\sigma^i}|\cdot K_{T,i}^1(N_i(\sigma^i u)) \qquad \text{by 5.3.2} \\ =\ &|G_sZ_{\sigma^i}|^{-1}\cdot|Z_{\sigma^i}|K_{T,i}^1(N_i(\sigma^i u))\,. \end{aligned}$$

By this and 8.4.1, 8.4.3 is $\boldsymbol{Z}$-valued. Thus we complete the proof.

Corollary 8.4.4. *Under the some condition as in* 8.4, *the map* 1-lift *coincides with* $*$-lift.

9. A counter example

Let $\boldsymbol{G}=Sp_4$, $(x_{ij})^\sigma=(x_{ij}^{\ q})$, $m=2$ and p, q be sufficiently large. Le' us prove that the liftings of the irreducible characters θ_9, θ_{10}, θ_{11}, θ_{12} of $G_\sigma=Sp_4(q)$ do not exist. Here we follow the notations of [9]. (We denote by $\theta_i'(i=9, \cdots)$ the irreducible character of $G=Sp_4(q^2)$ 'corresponding' to $\theta_i\in(G_\sigma)^\wedge$ $(i=9, \cdots)$.) Let ρ_1 be one of the irreducible characters θ_i $(i=9, \cdots)$. Assume that the lifting of

ρ_1 exists and denote this by ρ_0. Then there exists an irreducible character ρ of AG such that i-res $\rho=\rho_i$ ($i=0, 1$). Since

$$\begin{aligned}
\chi_1(0) &= \theta_0-\theta_9+\theta_{10}+\theta_{13} \\
\chi_2(0) &= \theta_0+\theta_{11}-\theta_{12}-\theta_{13} \\
\chi_3(0, 0) &= \theta_0+2\theta_9+\theta_{11}+\theta_{12}+\theta_{13} \\
\chi_4(0, 0) &= \theta_0-2\theta_{10}-\theta_{11}-\theta_{12}+\theta_{13} \\
\chi_5(0, 0) &= \theta_0-\theta_{11}+\theta_{12}-\theta_{13},
\end{aligned}$$

and $\langle AR^{\theta}_{T,1}, \rho\rangle_{AG}=2^{-1}(\langle R^{\theta,N0}_{T,0}, \rho_0\rangle_G+\langle R^{\theta}_{T,1}, \rho_1\rangle_{G\sigma})$ is an integer, we have lift $\theta_9=\theta'_9$ or θ'_{10}, lift $\theta_{10}=\theta'_9$ or θ'_{10}, lift $\theta_{11}=\theta'_{11}$ or θ'_{12} and lift $\theta_{12}=\theta'_{11}$ or θ'_{12}. Since ρ is $\boldsymbol{Z}$-valued, by [7, proposition 3] we get

$$\rho(\sigma u) \equiv \rho((\sigma u)^2) \mod 2 .$$

Let c (resp. d) be a representative of the conjugacy class A_{31} (resp. A_{32}) of G_σ. Then by the above congruence relation, we get

$$\begin{aligned}
\rho_1(c) &\equiv \rho_0(c) \mod 2 \\
\rho_1(d) &\equiv \rho_0(d) \mod 2 .
\end{aligned}$$

Since c is conjugate to d in G, we get

$$\rho_1(c) \equiv \rho_1(d) \mod 2 .$$

This contradicts the known values of θ_i. The fact that the liftings of θ_9 and θ_{10} do not exist was first pointed by G. Lusztig.

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