Masamoto, K., Matsuhisa, T., Shimomura, K.
Osaka J. Math.
33 (1996), 69-82

THE HOMOTOPY GROUPS OF A SPECTRUM WHOSE BP_*-HOMOLOGY IS $v_2^{-1} BP_* / (2,v_1^\infty) [t_1] \otimes \Lambda (t_2)$

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(Received May 12, 1994)

1. Introduction

In [5], Mahowald gave some examples of ring spectra obtained as Thom spectra. One of them is X_2 in [5], which is a Thom spectrum associated to ω: $\Omega S^2 \to BO$, where ω is a mapping corresponding to the generator of $\pi_1(BO)$. Let BP denote the Brown-Peterson spectrum at the prime 2. Then the spectrum X_2 is also characterized by the BP_*-homology $BP_*(X_2)=BP_*/(2)[t_1]$ as a sub-comodule algera of $BP_*(BP)/(2)=BP_*/(2)[t_1, t_2, \cdots]$, where $BP_*=\boldsymbol{Z}_{(2)}[v_1, v_2, \cdots]$ over Hazewinkel's generators v_i (*cf*. [14]).

Relating to X_2, consider a spectrum X constructed as follows: Let C be a cofiber of the Bousfield localization map $X_2 \to L_1X_2$ with respect to the Johnson-Wilson spectrum $E(1)$ with $\pi_*(E(1))=\boldsymbol{Z}_{(2)}[v_1, v_1^{-1}]$. Then C is an X_2-module spectrum since X_2 is a ring spectrum. Consider the element $h_{20} \in \pi_5(X_2)$. Now the spectrum X is a cofiber of a map h_{20}: $\Sigma^5 C \to C$. By this definition, the BP_*-homology of X is $BP_*(X)=BP/(2, v_1^\infty)[t_1]\otimes \Lambda(t_2)$. Once we determined the homotopy groups $\pi_*(L_2X_2)$ in [17], the homotopy groups $\pi_*(L_2X)$ can be obtained from it. Here L_2 denotes the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$ with $\pi_*(E(2))=\boldsymbol{Z}_{(2)}[v_1, v_2, v_2^{-1}]$ as a subalgebra of $v_2^{-1}BP_*$. But, in this paper, we compute, independently of [17], the homotopy groups $\pi_*(L_2X)$ of the $E(2)_*$-localized spectrum of X by using the Adams-Novikov spectral sequence. The computation of the E_2-term is done in the same manner as that of [17], using the v_1-Bockstein spectral sequence. Different from the odd prime case, there may involve non-trivial differentials of the Adams-Novikov spectral sequence. On the other hand, different from the case for X_2, this case may support at most one family of non-trivial differentials. In this sense, it is a little easier to determine the homotopy groups of L_2X than those of L_2X_2. By using the results of [7], we show here that the differentials are all trivial, in a different fashion from that of [17], and have the E_∞-term of the spectral sequence. In order to state the result, consider the integers A_n defined by

$$A_0=1,\ A_{2n+1}=1+2A_{2n} \text{ and } A_{2n+2}=2A_{2n+1}$$

for $n\geq 0$, and use the notations:

$C_\infty\langle x\rangle$ is a $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to
$$\boldsymbol{Z}/2[v_1, v_1^{-1}, v_2, v_2^{-1}]/\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$$
generated by elements $\{x/v_1^j\}_{j>0}$ such that $v_1(x/v_1^j)=x/v_1^{j-1}$.

$C_j\langle x\rangle$ is a cyclic $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to
$$\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]/(v_1^j)$$
generated by an element x/v_1^j.

Theorem. *The E_∞-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$ is a $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module*

$$M_*\otimes\Lambda(\rho).$$

Here, the graded $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module M_* is given by:

$$\begin{aligned}
M_0&=C_\infty\langle 1\rangle\oplus\bigoplus_{n,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)}\rangle,\\
M_1&=\oplus_{t\geq 0}(C_1\langle v_3^{2t+1}h_{30}\rangle\oplus C_1\langle v_3^{2t+1}h_{31}\rangle\oplus C_3\langle v_3^{4t+2}h_{30}\rangle)\\
&\quad\oplus\bigoplus_{n>0,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)+1}h_{21}\rangle\\
&\quad\oplus\bigoplus_{t,k\geq 0}(C_{A_{2k+1}}\langle v_3^{4^k(4t+2)+b_{k+1}}h_{30}\rangle\oplus C_{A_{2k}}\langle x_3^{4^k(2t+1)+b_{k+1}/2}h_{31}\rangle),\\
M_2&=\oplus_{t\geq 0}C_1\langle v_3^{2t+1}h_{30}h_{31}\rangle\\
&\quad\oplus\bigoplus_{t,k\geq 0}(C_{A_{2k+1}}\langle v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}\rangle\\
&\quad\oplus C_{A_{2k}}\langle v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}\rangle) \text{ and}\\
M_n&=0 \text{ for } n>2.
\end{aligned}$$

Furthermore, the generators have the following degrees:

$$|v_3|=14,\ |h_{20}|=5,\ |h_{21}|=11,\ |h_{30}|=13, \text{ and } |h_{31}|=27.$$

In the theorem, an element x has a degree r if $x\in\pi_r(L_2X)$.

This paper is organized as follows: In the next section, we recall some facts known about the v_1-Bockstein spectral sequence. In §3, we define elements x_n, which will play the main role in the computation of the Bockstein spectral sequence. We compute E_2-terms of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(L_2X)$ in §4, by using the tools given in the previous sections. In section 5, we prepare some lemmas to compute the Adams-Novikov differentials in the last section using the results of [7].

2. The Bockstein spectral sequence

Let (A, Γ) denote a Hopf algebroid with Γ A-flat. Then it is known (*cf*. [14, Ch. A1]) that the category of Γ-comodules has enough injectives and so we can define the Ext groups as a cohomology of an injective resolution. Furthermore it is given by a cohomology of the cobar resolution. So we can define $\mathrm{Ext}_\Gamma^n(A, M) = H^n(\Omega_\Gamma^* M)$ for a Γ-comodule M, where $\Omega_\Gamma^* M$ is a cobar complex (*cf*. [14]). The cobar complex $\Omega_\Gamma^* M$ is a defferential graded module with

$$\Omega_\Gamma^s M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (s \text{ copies of } \Gamma),$$

and the differentials $d_r : \Omega_\Gamma^r M \to \Omega_\Gamma^{r+1} M$ defined inductively by

$$d_0(m) = \psi(m) - m \otimes 1 \text{ and } d_r(x \otimes y) = d_s(x) \otimes y + (-1)^s x \otimes d_t(y)$$

for $x \in \Omega_\Gamma^s M$ and $y \in \Omega_\Gamma^t A$. Here $\psi : M \to M \otimes_A \Gamma$ denotes the comodule structure map of M. In the following, every comodule is induced from A and so we use η_R for ψ.

Suppose that $A = \boldsymbol{Z}_{(2)}[v_1, v_2, \cdots]$ and $\Gamma = A[t_1, t_2, \cdots]$. Consider a Hopf algebroid $\Phi = A[t_1] \otimes \Lambda(t_2)$ and a coalgebroid $\Sigma = \Gamma \square_\Phi A$ over A. Then $\Sigma = A[t_2^2, t_3, \cdots]$ and we have the change of rings theorem:

Lemma 2.1. *For a comodule A, there is an isomorphism*

$$\mathrm{Ext}_\Gamma^*(A, M \otimes_A \Phi) \cong \mathrm{Ext}_\Sigma^*(A, M).$$

Proof. Consider a relative injective Γ-resolution of $M \otimes_A \Phi$:

$$M \otimes_A \Phi \longrightarrow I_0 \otimes_A \Gamma \longrightarrow I_1 \otimes_A \Gamma \longrightarrow \cdots,$$

which is split as A-modules. Then apply the cotensor product $- \square_\Phi A$ and we obtain a relative injective Σ-resolution of M:

$$M \longrightarrow I_0 \otimes_A \Sigma \longrightarrow I_1 \otimes_A \Sigma \longrightarrow \cdots,$$

since $\Sigma = \Gamma \square_\Phi A$. Thus the both Ext groups are obtained from the same complex $I_0 \to I_1 \to \cdots$. q.e.d.

In this paper, we will compute $\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1^\infty) \otimes_A \Phi)$. By virtue of this lemma, we will work in the category of Σ-comodules. In order to compute the Ext groups $\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1^\infty))$, we adopt the v_1-Bockstein spectral sequence with E_1-term

$$\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1)).$$

To compute the E_1-term we recall [7] the structure

(2.2) $$\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1)[t_1]) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This is shown by using the change of rings theorems

$$\begin{aligned}\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1)[t_1]) &= \mathrm{Ext}_{K(2)_*K(2)}^*(K(2)_*, K(2)_*[t_1]) \\ &= \mathrm{Ext}_{S(2,2)}^*(\boldsymbol{Z}/2, \boldsymbol{Z}/2) \otimes_{K(2)_*} K(2)_*[v_3],\end{aligned}$$

in which $K(2)_* = \boldsymbol{Z}/2[v_2, v_2^{-1}]$, $K(2)_*K(2) = K(2)_* \otimes_A \Gamma \otimes_A K(2)_*$ and $S(2,2) =$

$\boldsymbol{Z}/2[t_2, t_3, \cdots]/(t_i^4-t_i: i>1)$. Note here that the action of A on $K(2)_*$ is given by sending v_i to 0 for $i\neq 2$ and v_2 to v_2, and $(K(2)_*, K(2)_*K(2))$ becomes a Hopf algebroid induced from (A, Γ). The second equation follows from the $K(2)_*K(2)$-comodule structure $K(2)_*[t_1]=K(2)_*[t_1]/(v_2t_1^4+v_2^2t_1)\otimes_{K(2)_*}K(2)_*[v_3]$ which is obtained from Landweber's formula $\eta_R(v_3)\equiv v_3+v_2t_1^4+v_2^2t_1 \bmod (2, v_1)$.

Lemma 2.3. *The E_1-term is given by*

$$\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1))=K(2)_*[v_3]\otimes\Lambda(h_{21}, h_{30}, h_{31}, \rho),$$

where $K(2)_*=\boldsymbol{Z}/(2)[v_2, v_2^{-1}]$ and h_{21}, h_{30}, h_{31} and ρ are the homology classes represented by t_2^2, t_3, t_3^2 and $v_2^5t_4+t_4^2$ in the cobar complex, respectively.

Proof. Let H^*M for a Γ-comodule M denote the Ext group $\mathrm{Ext}_\Gamma^*(A, M)$, and E_* and D_* be Γ-comodules

$$E_*=v_2^{-1}A/(2, v_1)[t_1]\otimes\Lambda(t_2) \text{ and } D_*=v_2^{-1}A/(2, v_1)[t_1].$$

Then the short exact sequence $0\to D_*\subset E_*\to\Sigma^{-6}D_*\to 0$ of Γ-comodules yields the long exact sequence

$$\cdots\longrightarrow H^{s,t}D_*\longrightarrow H^{s,t}E_*\longrightarrow H^{s,t-6}D_*\xrightarrow{\delta} H^{s+1,t}D_*\longrightarrow\cdots$$

with $\delta(x)=h_{20}x$. By (2.2),

$$H^*D_*=K(2)_*[v_3, h_{20}]\otimes\Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This shows that $h_{20}: H^sD_*\to H^{s+1}D_*$ is a monomorphism and we have the lemma. q.e.d.

3. The elements x_n

In this section we will define elements x_n such that

$$x_n\equiv v_3^{2^n} \bmod(2, v_1) \text{ and } d_0(x_n)\equiv v_1^{e_n}g_n,$$

in which g_n repesents a generator of $\mathrm{Ext}_\Sigma^1(A, v_2^{-1}A/(2, v_1))$ and e_n to be taken as greate as possible. These elements play a central role in the Bockstein spectral sequence.

Hereafter we use the following abbreviation:

$\mathrm{Ext}^*(N)=\mathrm{Ext}_\Sigma^*(A, N)$ for a comodule N,
$M(j)=v_2^{-1}A/(2, v_1^j)$ and $M=\varinjlim_j M(j)=v_2^{-1}A/(2,v_1^\infty)$.

Then note that

$$BP_*(L_2X)=M\otimes_A\Phi \text{ and } \mathrm{Ext}^*(M)=\mathrm{Ext}_\Gamma^*(A, BP_*(L_2X)).$$

In $v_2^{-1}BP_*/(2)$, we define elements x_n, which will be used to define elements of $\mathrm{Ext}^*(M)$. From here on, we compute everything with setting $v_2=1$ for the sake of simplicity. We also write

$$x\equiv y \bmod(v_1^j)$$

for $x, y\in \Omega_\Sigma^* M$ if $x=y$ in the cobar complex $\Omega_\Sigma^* M(j)$.

We first introduce elements c_{3i} $(i=0, 1)$ and $\tilde{c}_{31}$ in $\Sigma=A[t_2^2, t_3, \cdots]$ defined by

$$\begin{aligned} v_1^2 c_{30} &= d_0(v_4^2+v_1^2 v_5)+t_2^8+t_2^2, \\ v_1 c_{31} &= d_0(v_4)+t_2^4 \text{ and} \\ \tilde{c}_{31} &= c_{31}+v_1(v_3^2 c_{31}+v_3 t_2^2). \end{aligned} \tag{3.1}$$

Lemma 3.2. *The cochains c_{30} and c_{31} are cocycles of the cobar complex $\Omega_\Sigma^1 M(j)$ for any $j>0$. Furthermore,*

$$c_{30}\equiv t_3+v_3 t_2^8 \bmod(v_1) \textit{ and } c_{31}\equiv t_3^2+v_1 v_3 t_2^2 \bmod(v_1^4).$$

Proof. Since $d_1 d_0=0$, $d_1(t_2)=0$ and $d_0(v_1)=0$, the first part of the lemma follows immediately from the definition, since the multiplication by v_1 on $\Omega_\Sigma^1 M(j)$ is monomorphic. The latter half is shown by the direct computation using

$$\begin{aligned} \eta_R(v_1^2) &= v_1^2,\ \eta_R(v_4)\equiv v_4+v_2 t_2^4+v_1 t_3^2+v_1^2 v_3 t_2^2 \bmod(v_1^5), \\ \eta_R(v_4^2) &\equiv v_4^2+v_2^2 t_2^8+v_2^8 t_2^2+v_1^2 t_3^4+v_1^4 v_3^2 t_2^4 \bmod(v_1^{10}), \text{ and} \\ \eta_R(v_5) &\equiv v_5+v_3 t_2^8+v_2 t_3^4+v_2^8 t_3 \bmod(v_1) \end{aligned} \tag{3.3}$$

in Σ, noticing that $d_0(x)=\eta_R(x)-x$. In fact, $d_0(v_4^2+v_1^2 v_5)\equiv t_2^8+t_2^2+v_1^2 t_3 +v_1^2 v_3 t_2^8 \bmod(v_1^3)$, by setting $v_2=1$, which gives c_{30}. For c_{31}, follows from $\eta_R(v_4)$.

q.e.d.

Lemma 3.4. *Put $\varphi_1=v_1 v_3^2(v_4+v_4^4)$, and we have*

$$d_0(\varphi_1)\equiv v_1(c_{30}^2+\tilde{c}_{31}) \bmod(v_1^3)$$

in $v_2^{-1}\Sigma=v_2^{-1}A[t_2^2, t_3, \cdots]$.

Proof. Since $d_0(x)=\eta_R(x)-x$ and η_R is a map of algebras, this is verified by Lemma 3.2 and the following facts on η_R:

$$\begin{gathered} \eta_R(v_1)=v_1,\ \eta_R(v_2)=v_2, \\ \eta_R(v_3^2)\equiv v_3^2 \bmod(v_1^2), \\ \eta_R(v_4)=v_4+t_2^4+v_1 c_{31} \text{ and } \eta_R(v_4^4)\equiv v_4^4+t_2^{16}+t_2^4 \bmod(v_1^4) \end{gathered}$$

in $v_2^{-1}\Sigma$. In fact, by Lemma 3.2, we see that

$$c_{30}^2+\tilde{c}_{31}\equiv v_3^2 t_2^{16}+v_1 v_3^2 c_{31}.$$

On the other hand, we compute

$$d_0(\varphi_1)\equiv v_1v_3^2d_0(v_4+v_4^4)\equiv v_1v_3^2(v_1c_{31}+t_2^{16}).$$

q.e.d.

Note that $v_2^{-1}\Sigma$ is not a Hopf algebroid and so (3.1) does not imply the above lemma. In fact, $d_0(v_4^2)=d_0(v_4)^2+t_2^2$. This with (3.1) yields the following

Lemma 3.5. *In* $v_2^{-1}\Sigma$,

$$d_0(v_1^6v_5)=v_1^6(c_{31}^2+c_{30}).$$

Lemma 3.6. *There exist elements* x_i *of* $v_2^{-1}A$ *with* $x_i\equiv v_3^{2^i}$ mod$(2, v_1)$ *such that*

$$\begin{aligned}
&d_0(x_0)=v_1t_2^2,\\
&d_0(x_1)=v_1^3c_{31},\\
&d_0(x_2)=v_1^6c_{30},\\
&d_0(x_{2n+1})\equiv v_1^{1+2a_n}v_3^{2b_n}(v_3^2c_{31}+v_3t_2^2) \bmod(v_1^{2+2a_n}) \text{ and}\\
&d_0(x_{2n+2})\equiv v_1^{a_{n+1}}v_3^{b_{n+1}}c_{30} \bmod(v_1^{1+a_{n+1}})
\end{aligned}$$

for $n>0$. *Here the integers* a_n *and* b_n *are given by*

$$a_0=1 \text{ and } a_n=4a_{n-1}+2\ (n>0)$$
$$b_0=0,\ b_1=0 \text{ and } b_n=4b_{n-1}+4\ (n>1).$$

Proof. Define the elements x_i inductively as follows:

$$(3.7)\qquad \begin{aligned}
x_0&=v_3,\\
x_1&=v_3^2+v_1^2v_4,\\
x_2&=x_1^2+v_1^6v_5,\\
x_{2n}&=x_{2n-1}^2+v_1^{a_n}v_3^{b_n}v_5 \text{ and}\\
x_{2n+1}&=x_{2n}^2+v_1^{2a_n-1}v_3^{2b_n}\varphi_1+v_1^{2a_n-3}v_3^{2b_n}x_1.
\end{aligned}$$

Then the lemma will be proved by induction. The first equation follows immediately from the Landweber formula: $\eta_R(v_3)=v_3+v_1t_2^2$. The second and the third are verified by (3.1). The others are inductively shown by Lemmas 3.4 and 3.5.

q.e.d.

4. The E_2-term

Put $L=v_2^{-1}BP_*/(2, v_1)$ and $M=v_2^{-1}BP_*/(2, v_1^\infty)$. Then we have the short exact sequence

$$0\longrightarrow L\xrightarrow{f} M\xrightarrow{v_1} M\longrightarrow 0,$$

which yields the long exact sequence

$$(4.1)\qquad \begin{aligned}
&0\longrightarrow \mathrm{Ext}^0(L)\xrightarrow{f_*}\mathrm{Ext}^0(M)\xrightarrow{v_1}\mathrm{Ext}^0(M)\xrightarrow{\delta_0}\\
&\cdots\xrightarrow{\delta_{n-1}}\mathrm{Ext}^n(L)\xrightarrow{f_*}\mathrm{Ext}^n(M)\xrightarrow{v_1}\mathrm{Ext}^n(M)\longrightarrow\cdots.
\end{aligned}$$

Here f is a Σ-comodule map given by $f(x)=x/v_1$,

$$\mathrm{Ext}^n(N)=\mathrm{Ext}^n_{\Sigma}(A, N)$$

for a Σ-comodule N, and note that the Ext group $\mathrm{Ext}^*(L)$ is determined in Lemma 2.3.

We here introduce some notations:

$$K(2)_*=\boldsymbol{Z}/2[v_2, v_2^{-1}],\ K=K(2)_*[v_1]=\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}].$$

For an element $x\in\mathrm{Ext}^*(L)$,

$C_n\langle x\rangle$ denotes a cyclic K-module isomorphic to $K/(v_1^n)$ generated by $\{x/v_1^n+z/v_1^{n-1}\}\in\mathrm{Ext}^*(M)$ for some $z\in\Omega_{\Sigma}^* v_2^{-1}BP_*/(2)$.

$C_\infty\langle x\rangle$ denotes a K-module isomorphic to $v_1^{-1}K/K$ with basis $\{x/v_1^j+z/v_1^{j-1}\}_{j>0}\subset\mathrm{Ext}^*(M)$ for some $z\in\Omega_{\Sigma}^* v_2^{-1}BP_*/(2)$.

Note that these $C_*\langle x\rangle$ are sub-K-module of $\mathrm{Ext}^*(M)$.

We compute $\mathrm{Ext}^*(M)=\mathrm{Ext}^*_{\Sigma}(A, v_2^{-1}A/(2, v_1^\infty))$ from $\mathrm{Ext}^*(L)=\mathrm{Ext}^*_{\Sigma}(A, v_2^{-1}A/(2, v_1))$ by using the following

Lemma 4.2. ([8, Remark 3.11]) *Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a set of generators of $K(2)_*$-module* $\mathrm{Ext}^i(L)$, *and $\{\xi_\lambda\}_{\lambda\in\Lambda_0}$ and $\{\xi_{\lambda,j}\}_{\lambda\in\Lambda_1}$ subsets of* $\mathrm{Ext}^i(M)$ *such that $\Lambda=\Lambda_0\coprod\Lambda_1$,*

1) *there exists a positive integer $a(\lambda)$ for each $\lambda\in\Lambda_0$ such that*

$$v_1^{a(\lambda)-1}\xi_\lambda=f_*(x_\lambda)\ \text{and}$$
$$\delta_i(\xi_\lambda)\neq 0,$$

2) *$\xi_{\lambda,1}=f_*(x_\lambda)$, $v_1\xi_{\lambda,j}=\xi_{\lambda,j-1}$ and $\delta_i(\xi_{\lambda,j})=0$ for $\lambda\in\Lambda_1$.*

Suppose that the set $\{\delta_i(\xi_\lambda)\}_{\lambda\in\Lambda_0}$ is linearly independent over $K(2)_*$. Then $\mathrm{Ext}^i(M)=\bigoplus_{\lambda\in\Lambda_0}C_{a(\lambda)}\langle x_\lambda\rangle\oplus\bigoplus_{\lambda\in\Lambda_1}C_\infty\langle x_\lambda\rangle$.

In this section, we will use Lemma 4.2 to compute $\mathrm{Ext}^*(M)$, which is the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$. Let ρ denote the homology class of $\mathrm{Ext}^1(L)$ given in Lemma 2.3.

Lemma 4.3. *There exist elements $\rho_i\in\Omega_{\Sigma}^1 v_2^{-1}A/(2)$ such that*

$$\rho_i\equiv\rho \bmod(2, v_1)$$

up to homology and

$$d_1(\rho_i)\equiv 0 \bmod(2, v_1^{2^i}).$$

Proof. In [9], Moreira constructed an element $u \in \Omega_\Sigma^1 L$ such that

$$d_0(u) = (\tilde{\rho} + \zeta) + (\tilde{\rho} + \zeta)^2$$
$$= (\tilde{\rho} + t_2^2) + \tilde{\rho}^2 + t_2^2 + t_2^4$$

in the cobar complex $\Omega_\Sigma^2 L$. Here ζ is represented by a cochain $t_2 + t_2^2$ in $\Omega_\Gamma^1 L$, and $\tilde{\rho}$ denotes a cocycle which represents the cohomology class ρ. Since t_2^4 is homologous to 0, so is $\tilde{\rho}$ to $\tilde{\rho}^2$. Hence define $\rho_i = \tilde{\rho}^{2^i}$ and we have the lemma.
q.e.d.

For each j, there is an integer i such that ρ_i/v_1^j is a cocycle. In this case, we write

$$x\rho/v_1^j = x\rho_i/v_1^j.$$

Such an abbreviation would not cause any confusion.

The main lemma of the last section implies

Lemma 4.4. *For the connecting homomorphism δ_0 in* (4.1),

$$\delta_0(v_3^{2t+1}/v_1) = v_3^{2t} h_{21},$$
$$\delta_0(v_3^{4t+2}/v_1^3) = v_3^{4t} h_{31},$$
$$\delta_0(v_3^{8t+4}/v_1^6) = v_3^{8t} h_{30},$$
$$\delta_0(v_3^{4^n(4t+2)}/v_1^{1+2a_n}) = v_3^{4^{n+1}t+2b_n}(v_3^2 h_{31} + v_3 h_{21}) \text{ and}$$
$$\delta_0(v_3^{4^{n+1}(2t+1)}/v_1^{a_{n+1}}) = v_3^{2\cdot 4^{n+1}t+b_{n+1}} h_{30}$$

for $t \geq 0$, $n > 0$.

Here v_3^s/v_1^j denotes a cocycle of the cobar complex whose leading term is v_3^s/v_1^j. Therefore, we obtain the lemma by setting $v_3^{2^n s}/v_1^j = x_n^s/v_1^j$ from Lemma 3.6. Now apply Lemma 4.2 to obtain

Proposition 4.5. *The* Ext *group* $\mathrm{Ext}^0(M)$ *is a direct sum of* $C_\infty\langle 1\rangle$ *and* $C_{A_n}\langle v_3^{2^n(2t+1)}\rangle$ *for* $n \geq 0$ *and* $t \geq 0$. *Here* $A_{2n} = a_n$ *and* $A_{2n+1} = 1 + 2a_n$.

These give us the cokernel of δ_0:

Corollary 4.6. *The cokernel of* δ_0: $\mathrm{Ext}^0(M) \to \mathrm{Ext}^1(L)$ *is a* $K(2)_*$*-free module generated by*

$$v_3^{2t+1} h_{21},\ v_3^{u'} h_{30},\ v_3^u h_{31} \text{ and } v_3^t \rho$$

for $t \geq 0$, $u \notin T$ *and* $u' \notin 2T$. *Here* T *is a subset of the natural numbers* $\boldsymbol{N}$:

$$T = \{n : 4 | n \text{ or } 4^{i+1} | (n - 2b_i - 2) \text{ for some } i > 0\},$$

for $b_i = 4(4^{i-1} - 1)/3$.

Lemma 4.7. *The complement* $U=N-T$ *is given as*

$$U=\{n:\ 2\nmid n \text{ or } n=2\cdot 4^k t+6\cdot 4^{k-1}+2(4^{k-1}-1)/3 \\ \text{for some } k>0 \text{ and } t\geq 0\}$$

For the computation of δ_1, we introduce other elements:

Lemma 4.8. *Consider an element* $\varphi=v_5+v_3v_4^2$. *Then there exist elements* H_{21} *and* H_{32} *in* Σ *such that*

$$d_0(\varphi)=H_{32}+t_3+H_{21},\ \ d_1(H_{21})=0=d_1(H_{32}),$$
$$H_{21}\equiv t_2^2 \quad \text{and} \quad H_{32}\equiv t_3^4 \mod(v_1)$$

in the cobar complex $\Omega_\Sigma^1 v_2^{-1}A/(2)$.

Proof. *For an element* $\psi=v_3^2+v_1^7v_3$, *we compute* $d_0(\psi)=v_1^2t_2^4$ *by* $\eta_R(v_3)=v_3+v_1t_2^2+v_1^4t_2$ *in* $BP_*[t_2, t_3, \cdots]$. *Now put*

$$H_{32}=t_3^4+v_1^2\psi t_2^4.$$

Then, the formula $\Delta(t_3^4)=t_3^4\otimes 1+1\otimes t_3^4+v_1^4t_2^4\otimes t_2^4$ yields

$$d_1(H_{32})=0 \quad \text{and} \quad H_{32}\equiv t_3^4 \mod(v_1).$$

Furthermore, we compute

$$d_0(\varphi)\equiv t_3^4+t_3+v_3t_2^2 \mod(v_1),$$

and so

$$d_0(\varphi)\equiv H_{32}+t_3+v_3t_2^2 \mod(v_1).$$

Put, then,

$$H_{21}=d_0(\varphi)+H_{32}+t_3$$

and we have

$$d_1(H_{21})=0 \quad \text{and} \quad H_{21}\equiv v_3t_2^2 \mod(v_1).$$

q.e.d.

Lemma 4.9. *For the connecting homomorphism* $\delta_1:\mathrm{Ext}^1(M)\longrightarrow \mathrm{Ext}^2(L)$, *we have*

$$\delta_1(v_3^{4t+3}h_{21}/v_1^3)=v_3^{4t+1}h_{21}h_{31},$$
$$\delta_1(v_3^{8t+5}h_{21}/v_1^6)=v_3^{8t+1}h_{21}h_{30},$$
$$\delta_1(v_3^{4^n(4t+2)+1}h_{21}/v_1^{1+2a_n})=v_3^{4^{n+1}t+2b_n+1}h_{21}(v_3^2h_{31}+v_3h_{21}),$$
$$\delta_1(v_3^{4^{n+1}(2t+1)+1}h_{21}/v_1^{a_{n+1}})=v_3^{2\cdot 4^{n+1}t+b_{n+1}+1}h_{21}h_{30}$$
$$\delta_1(v_3^{2t+1}h_{30}/v_1)=v_3^{2t}h_{21}h_{30},$$

$$\delta_1(v_3^{4t+2}h_{30}/v_1^3)=v_3^{4t}h_{30}h_{31},$$
$$\delta_1(v_3^{4^k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k})=v_3^{4^k(4t+2)-2}h_{30}(h_{31}+v_3^{-1}h_{21}),$$
$$\delta_1(v_3^{2t+1}h_{31}/v_1)=v_3^{2t}h_{21}h_{31} \quad and$$
$$\delta_1(v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k})=v_3^{4^k(2t+1)-2}h_{30}(h_{31}+v_3^{-1}h_{21}).$$

Proof. The first four equations follow immediately from Lemmas 4.4 and 4.8 with replacing v_3h_{21} by H_{21}. The fifth, sixth and eighth equations follow immediately from Lemmas 3.2 and 3.6. For the other equations, just put

$$v_3^{4^k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k}=v_3^{4^k(4t+2)}d_0(x_{2k+2})/v_1^{1+2a_k+a_{k+1}} \quad \text{and}$$
$$v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k}=v_3^{4^k(2t+1)}d_0(x_{2k+1})/v_1^{a_k+1+2a_k},$$

and we have the result by Lemma 3.6. q.e.d.

Now use Lemma 4.2, and we obtain

Proposition 4.10. $\mathrm{Ext}^1(M)$ *is a direct sum of* $\rho\mathrm{Ext}^0(M)$ *and*

$$e^1(M)=\bigoplus_{t\geq 0}(C_1\langle v_3^{2t+1}h_{30}\rangle\oplus C_1\langle v_3^{2t+1}h_{31}\rangle\oplus C_3\langle v_3^{4t+2}h_{30}\rangle)$$
$$\oplus\bigoplus_{n>0,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)+1}h_{21}\rangle$$
$$\oplus\bigoplus_{t,k\geq 0}(C_{1+2a_k}\langle v_3^{4^k(4t+2)+b_{k+1}}h_{30}\rangle\oplus C_{a_k}\langle v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}\rangle).$$

Corollary 4.11. *The cokernel of* $\delta_1:\mathrm{Ext}^1(M)\to\mathrm{Ext}^2(L)$ *is a direct sum of* $\rho\mathrm{Coker}\ \delta_0$ *and a* $K(2)_*$*-module generated by*

$$v_3^{2t+1}h_{30}h_{31},\ v_3^{2u+1}h_{21}h_{31}\ \textit{and}\ v_3^{2u'+1}h_{21}h_{30}$$

for $t\geq 0$, $2u\notin T$ *and* $u'\notin 2T$.

Lemma 4.12. *For the connecting homomorphism* $\delta_2:\mathrm{Ext}^1(M)\to\mathrm{Ext}^2(L)$, *we have*

$$\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1)=v_3^{2t}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4t+3}h_{21}h_{30}/v_1^3)=v_3^{4t+1}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}/v_1^{1+2a_k})=v_3^{4^k(4t+2)-1}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}/v_1^{a_k})=v_3^{4^k(2t+1)-1}h_{21}h_{30}h_{31}.$$

Proof. Note that $\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1)=\delta_0(v_3^{2t+1}/v_1)h_{30}h_{31}$ since $h_{3i}=c_{3i}$'s are cocycles by Lemma 3.2. Now the first equation follows from Lemmas 4.4 and 4.9. For the other equations, use Lemmas 4.8 and 4.9 since $\delta_2(v_3^{2t+1}h_{21}h_{3i}/v_1^j)=\delta_1(v_3^{2t}h_{3i}/v_1^j)v_3h_{21}$ if we use the representative H_{21} for the cohomology class v_3h_{21}.

Again by Lemma 4.2, we obtain q.e.d.

Proposition 4.13. $\mathrm{Ext}^2(M)$ *is a direct sum of* $\rho e^1(M)$ *and*

$$e^2(M)=\oplus_{t,k\geq 0}(C_{1+2a_k}\langle v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}\rangle$$
$$\oplus C_{a_k}\langle v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}\rangle)\oplus C_1\langle v_3^{2t+1}h_{30}h_{31}\rangle).$$

Corollary 4.14. *The cokernel of* δ_2: $\mathrm{Ext}^2(M) \to \mathrm{Ext}^3(L)$ *is a* $K(2)_*$*-module* $\rho\mathrm{Coker}\ \delta_1$.

Now the following proposition follows immediately, by the same argument as above.

Proposition 4.15. *For* $n>3$, $\mathrm{Ext}^n(M)=0$, *and*

$$\mathrm{Ext}^3(M)=\rho e^2(M).$$

5. On the map $j_*: E_2(X) \to E_2(C)$

As is stated in the introduction, C denotes the cofiber of $X_2 \to L_2X_2$. Then it is an X_2-module spectrum and $h_{20} \in \pi_5(X_2)$ induces a map $h_{20}: C \to C$. In fact, it is the composition

$$C=S^0\wedge C \xrightarrow{h_{20}\wedge C} X_2\wedge C \xrightarrow{\nu} C,$$

in which ν denotes the X_2-module structure. Then we have a cofiber sequence

$$\Sigma^5 C \xrightarrow{h_{20}} C \xrightarrow{i} X \xrightarrow{j} \Sigma^6 C.$$

Let $E_r^*(Y)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_2Y)$ for a spectrum Y, and d_r^{AN}, its differentials. Then this gives rise to the exact sequence

$$0 \longrightarrow E_2^{0,t}(C) \xrightarrow{i_*} E_2^{0,t}(X) \xrightarrow{j_*} E_2^{0,t-6}(C) \xrightarrow{\delta} E_2^{1,t}(C) \longrightarrow \cdots.$$

Here $E_2^{s,t}(X)=\mathrm{Ext}^{s,t}(M)$, whose structure is given in the previous section. We further consider a cofiber E of $h_{20}: C \to C$. Then we have a commutative diagram

$$\text{(5.1)}\qquad \begin{array}{ccccccc} C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C \\ \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 \\ C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C\ , \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ \Sigma D & \xrightarrow{h_{20}} & \Sigma D & \xrightarrow{i} & \Sigma E & \xrightarrow{j} & \Sigma^2 D \end{array}$$

in which rows and columns are cofibrations.

Lemma 5.2. *Let* v_3^t/v_1^A *denote a generator of* $E_2(X)$ *as a* $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$*-module. Then*

$$j_*(v_3^t/v_1^{A-1})=0.$$

Proof. If $t=2^n(2s+1)$ for some n, $s\geq 0$, then v_3^t/v_1^A is a homology class represented by $x_n^{2s+1}/v_1^{A_n}$. For $n=0$, the lemma is trivial. Now suppose that $j_*(x_n^{2s+1}/v_1^{A_n})=0$ for even $n=2m$. Then squaring this, we obtain

$$j_*(x_{n+1}^{2s+1}/v_1^{A_{n+1}})=v_3^w/v_1$$

for some $w \geq 0$. Consider the diagram

$$\begin{array}{ccccc} & & E_2^0(X) & \xrightarrow{j_*} & E_2^0(C) \\ & & \downarrow \delta & & \downarrow \delta \\ E_2^1(D) & \xrightarrow{i_*} & E_2^1(E) & \xrightarrow{j_*} & E_2^1(D) \end{array}$$

induced from (5.1). Since $\delta(x_{n+1}^{2s+1}/v_1^{An+1})$ is in the image of i_* by Lemma 4.4, $\delta(v_3^w/v_1)=0$ in $E_2^1(D)$ by the above diagram, and so $2|w$ since $\delta(v_3^w/v_1)=wv_3^{w-1}h_{21}$ by Landweber's formula $d_0(v_3)=v_1t_2^2+v_1^4t_2$ in $BP_*[t_2, t_3, \cdots]$. Thus we have

$$j_*(x_{n+1}^{2s+1}/v_1^{An+1})=v_3^{2u}/v_1.$$

Square this, and we have

$$j_*(x_{n+2}^{2s+1}/v_1^{An+2})=v_3^{4u}/v_1^2.$$

Notice that $j_*(x)=y$ if $d_0(x)=yt_2$, where $d_0(x)=\eta_R(x)-x$. A direct computation shows us $d_0(v_3^{4u}x_1/v_1^4)=v_3^{4u}t_2/v_1^2$ in the cobar complex $\Omega_\Sigma^2 M$. Thus we have shown inductively that $j_*(v_3^{2^n(2s+1)}/v_1^{An})$ equals to 0 if n is even, and to v_3^{2s}/v_1 for some u if n is odd. q.e.d.

6. The Adams-Novikov differential

Consider the cofiber E of $h_{20}: \Sigma^5 D \to D$. Then by [7, Th. 7.1], we immediately obtain the following

Proposition 6.1. *The Adams-Novikov spectral sequence for computing $\pi_*(L_2E)$ collapses from the E_2-term.*

Note that the E_2-term for our X is

$$E_2^*(X)=\mathrm{Ext}_\Gamma^*(A, v_2^{-1}BP_*(X))=\mathrm{Ext}^*(M).$$

Lemma 6.2. *For the Adams-Novikov differential $d_3^{AN}: E_2^0(X) \to E_2^3(X)$, $d_3^{AN}(v_3^t/v_1^A)$ is a sum of the elements of the form $v_3^{2u+1}h_{21}h_{3i}\rho/v_1^k$ for $i=0, 1$ and $k>1$. Here v_3^t/v_1^A is a generator of the $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$-module M_0.*

Proof. Consider the diagram (5.1). The third column induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^3(M) \xrightarrow{v_1} \mathrm{Ext}^3(M) \xrightarrow{\delta_3} \mathrm{Ext}^4(L) \longrightarrow \cdots$$

of the E_2-terms. If the δ_0 image of v_3^t/v_1^A is $x \neq 0$, then $\delta_3(d_3^{AN}(v_3^t/v_1^A))=d_3^{AN}(x)=0$ by Proposition 6.1. Thus $d_3^{AN}(v_3^t/v_1^A)$ is divisible by v_1. Furthermore it implies that $v_3^{2t+1}h_{30}h_{31}\rho/v_1$ cannot be a target of d_3^{AN}. In fact, it is not divisible by v_1 by Proposition 4.15. Now the lemma follows from Lemma 4.15. q.e.d.

Theorem 6.3. *The Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$ collapses from the E_2-term.*

Proof. By proposition 4.15, the Adams-Novikov differentials are all trivial except for $d_3^{AN}: E_2^0(X)\to E_2^3(X)$. So it is sufficient to show that $d_3^{AN}(v_3^t/v_1^j)=0$ for each $v_3^t/v_1^j\in E_2^0(X)$. By Lemma 6.2,

(6.4) $$d_3^{AN}(v_3^t/v_1^{A-k})=\sum_{u,i}\lambda_{u,i}v_3^{2u+1}h_{21}h_{3i}\rho/v_1^2$$

for some $k\geq 0$, where $\lambda_{u,i}\in \boldsymbol{Z}/2$. Since

$$d_3(v_3^{2u+1}h_{21}h_{3i}\rho/v_1^2)=v_3^{2u}h_{20}^2h_{3i}\rho/v_1\neq 0$$

in the cobar complex $\Omega_\Gamma^4 BP_*(C)$, we see that

(6.5) $$j_*(\sum_{u,i}\lambda_{u,i}v_3^{2u+1}h_{21}h_{3i}\rho/v_1^2)=\sum_{u,i}\lambda_{u,i}v_3^{2u}h_{20}h_{3i}\rho/v_1\neq 0.$$

Now send (6.4) by j_* and we have a contradiction to Lemma 5.2, which says $j_*(v_3^t/v_1^{A-k})=0$ if $k>0$. If $k=0$ and $j_*(v_3^t/v_1^A)\neq 0$, then

$$j_*(v_3^t/v_1^A)=v_3^{2u}/v_1$$

for some $u\geq 0$ as is seen in the proof of Lemma 5.2. Therefore, (6.4) and (6.5) yield

$$d_3^{AN}(v_3^{2u}/v_1)=\sum_{u,i}\lambda_{u,i}v_3^{2u}h_{20}h_{3i}\rho/v_1\neq 0$$

in $E_2^*(C)$ for some $\lambda_{u,i}\in\boldsymbol{Z}/2$. Now pull this back to $E_2^*(D)$ under the map $i_*: E_2^*(D)\to E_2^*(C)$ to obtain the non-trivial differential

$$d_3^{AN}(v_3^{2u})=\sum_{u,i}\lambda_{u,i}v_3^{2u}h_{20}h_{3i}\rho\neq 0$$

in $E_2^*(D)$, which again contradicts to a result of [7] which says $d_3^{AN}(v_3^{4k})=0$ and $d_3^{AN}(v_3^{4k+2})=v_3^{4k}h_{20}^3$ for $k>0$. q.e.d.

References

[1] J.F. Adams: Stable homotopy and generalised homology, University of Chicago Press, Chicago, 1974.

[2] A.K. Bousfield: *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[3] M. Hikida and K. Shimomura: *An exact sequence related to Adams-Novikov E_2-terms of a cofibering*, J. Math. Soc. Japan **46** (1994), 645–661.

[4] P.S. Landweber: *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. **98** (1976), 591–610.

[5] M. Mahowald: *Ring spectra which are Thom complexes*, Duke Math. J. **46** (1979), 549–559.

[6] H.R. Miller and D.C. Ravenel: *Morava Stabilizer Algebras and the localization of Novikov's E_2-term*. Duke Math. J. **44** (1977), 433–447.

[7] M. Mahowald and K. Shimomura: *The Adams-Novikov spectral sequence for the L_2-localization of a v_2-spectrum*, the Proceedings of the International Congress in Algebraic Topology, Edited by M. Tangora, 1991, Contemporary. Math. **146**, A.M.S. (1993), 237-250.

[8] H.R. Miller, D.C. Ravenel, and W.S. Wilson: *Periodic phenomena in Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469-516.

[9] M.R.F. Moreira: *Primitives of BP_*BP modulo an invariant prime ideal*, Amer. J. Math. **100** (1978), 1247-1273.

[10] D.G. Quillen: *On the formal group laws of unoriented and complex cobordism theory*, Bull. A. M.S. **75** (1969), 1293-1298.

[11] D.C. Ravenel: *The structure of BP_*BP modulo an invariant prime ideal*, Topology **15** (1976), 149-153.

[12] D.C. Ravenel: *The cohomology of the Morava stabilizer algebras*, Math. Z. **152** (1977), 287-297.

[13] D.C. Ravenel: *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351-414.

[14] D.C. Ravenel: Complex cobordism and stable homotopy groups of spheres, Academic Press, 1986.

[15] D.C. Ravenel: *The geometric realization of the chromatic resolution*, In W. Browder, editor, Algebraic topology and algebraic K-theory, 1987, 168-179.

[16] D.C. Ravenel: Nilpotence and periodicity in stable homotopy theory, Ann. of Math. Studies 128, Princeton Univ. Press, 1992.

[17] K. Shimomura: *The homotopy groups of the L_2-localized Mahowald spectrum $X\langle 1\rangle$*, Forum Math. **7** (1995), 685-707.

[18] D-Y. Yan: Brown-Peterson homology of Mahowald's X_k spectra, Trans. Amer. Math. Soc. **344** (1994), 261-289.

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