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# THE HOMOTOPY GROUPS OF A SPECTRUM WHOSE *BP*<sub>\*</sub>-HOMOLOGY IS $v_2^{-1} BP_* / (2, v_1^{\infty}) [t_1] \otimes \Lambda (t_2)$

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# 1. Introduction

In [5], Mahowald gave some examples of ring spectra obtained as Thom spectra. One of them is  $X_2$  in [5], which is a Thom spectrum associated to  $\omega$ :  $\Omega S^2 \rightarrow BO$ , where  $\omega$  is a mapping corresponding to the generator of  $\pi_1(BO)$ . Let BP denote the Brown-Peterson spectrum at the prime 2. Then the spectrum  $X_2$  is also characterized by the  $BP_*$ -homology  $BP_*(X_2) = BP_*/(2)[t_1]$  as a subcomodule algera of  $BP_*(BP)/(2) = BP_*/(2)[t_1, t_2, \cdots]$ , where  $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \cdots]$  over Hazewinkel's generators  $v_i$  (cf. [14]).

Relating to  $X_2$ , consider a spectrum X constructed as follows: Let C be a cofiber of the Bousfield localization map  $X_2 \rightarrow L_1 X_2$  with respect to the Johnson-Wilson spectrum E(1) with  $\pi_*(E(1)) = \mathbb{Z}_{(2)}[v_1, v_1^{-1}]$ . Then C is an  $X_2$ -module spectrum since  $X_2$  is a ring spectrum. Consider the element  $h_{20} \in \pi_5(X_2)$ . Now the spectrum X is a cofiber of a map  $h_{20}: \Sigma^5 C \to C$ . By this definition, the  $BP_*$ -homology of X is  $BP_*(X) = BP/(2, v_1^{\infty})[t_1] \otimes A(t_2)$ . Once we determined the homotopy groups  $\pi_*(L_2X_2)$  in [17], the homotopy groups  $\pi_*(L_2X)$  can be obtained from it. Here  $L_2$  denotes the Bousfield localization functor with respect to the Johnson-Wilson spectrum E(2) with  $\pi_*(E(2)) = \mathbb{Z}_{(2)}[v_1, v_2, v_2^{-1}]$  as a subalgebra of  $v_2^{-1}BP_*$ . But, in this paper, we compute, independently of [17], the homotopy groups  $\pi_*(L_2X)$  of the  $E(2)_*$ -localized spectrum of X by using the Adams-Novikov spectral sequence. The computation of the  $E_2$ -term is done in the same manner as that of [17], using the  $v_1$ -Bockstein spectral sequence. Different from the odd prime case, there may involve non-trivial differentials of the Adams-Novikov spectral sequence. On the other hand, different from the case for  $X_2$ , this case may support at most one family of non-trivial differentials. In this sense, it is a little easier to determine the homotopy groups of  $L_2X$  than those of  $L_2X_2$ . By using the results of [7], we show here that the differentials are all trivial, in a different fashion from that of [17], and have the  $E_{\infty}$ -term of the spectral sequence. In order to state the result, consider the integers  $A_n$  defined by

 $A_0=1, A_{2n+1}=1+2A_{2n} \text{ and } A_{2n+2}=2A_{2n+1}$ 

for  $n \ge 0$ , and use the notations :

 $\begin{array}{l} C_{\infty}\langle x \rangle \text{ is a } \mathbb{Z}/2[v_{1}, v_{2}, v_{2}^{-1}]\text{-module isomorphic to} \\ \mathbb{Z}/2[v_{1}, v_{1}^{-1}, v_{2}, v_{2}^{-1}]/\mathbb{Z}/2[v_{1}, v_{2}, v_{2}^{-1}] \\ \text{generated by elements } \{x/v_{1}^{i}\}_{j>0} \text{ such that } v_{1}(x/v_{1}^{j}) = x/v_{1}^{j-1}. \\ C_{j}\langle x \rangle \text{ is a cyclic } \mathbb{Z}/2[v_{1}, v_{2}, v_{2}^{-1}]\text{-module isomorphic to} \\ \mathbb{Z}/2[v_{1}, v_{2}, v_{2}^{-1}]/(v_{1}^{j}) \\ \text{generated by an element } x/v_{1}^{j}. \end{array}$ 

**Theorem.** The  $E_{\infty}$ -term of the Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$  is a  $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ -module

 $M_* \otimes \Lambda(\rho).$ 

Here, the graded  $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ -module  $M_*$  is given by :

$$\begin{split} M_{0} &= C_{\infty} \langle 1 \rangle \oplus \oplus_{n,t \geq 0} C_{An} \langle v_{3}^{2(2t+1)} \rangle, \\ M_{1} &= \bigoplus_{t \geq 0} (C_{1} \langle v_{3}^{2t+1} h_{30} \rangle \oplus C_{1} \langle v_{3}^{2t+1} h_{31} \rangle \oplus C_{3} \langle v_{3}^{4t+2} h_{30} \rangle) \\ &\oplus \bigoplus_{n \geq 0, t \geq 0} C_{An} \langle v_{3}^{2^{2}(2t+1)+1} h_{21} \rangle \\ &\oplus \bigoplus_{t,k \geq 0} (C_{A_{2k+1}} \langle v_{3}^{4^{k}(4t+2)+b_{k+1}} h_{30} \rangle \oplus C_{A_{2k}} \langle x_{3}^{4^{k}(2t+1)+b_{k+1}/2} h_{31} \rangle), \\ M_{2} &= \bigoplus_{t \geq 0} C_{1} \langle v_{3}^{2^{2}t+1} h_{30} h_{31} \rangle \\ &\oplus \bigoplus_{t,k \geq 0} (C_{A_{2k+1}} \langle v_{3}^{4^{k}(4t+2)+b_{k+1}+1} h_{21} h_{30} \rangle \\ &\oplus C_{A_{2k}} \langle v_{3}^{4^{k}(2t+1)+(b_{k+1}/2)+1} h_{21} h_{31} \rangle) \text{ and } \\ M_{n} &= 0 \text{ for } n > 2. \end{split}$$

Furthermore, the generators have the following degrees :

 $|v_3|=14, |h_{20}|=5, |h_{21}|=11, |h_{30}|=13, and |h_{31}|=27.$ 

In the theorem, an element x has a degree r if  $x \in \pi_r(L_2X)$ .

This paper is organized as follows: In the next section, we recall some facts known about the  $v_1$ -Bockstein spectral sequence. In §3, we define elements  $x_n$ , which will play the main role in the computation of the Bockstein spectral sequence. We compute  $E_2$ -terms of the Adams-Novikov spectral sequence computing the homotopy groups  $\pi_*(L_2X)$  in §4, by using the tools given in the previous sections. In section 5, we prepare some lemmas to compute the Adams-Novikov differentials in the last section using the results of [7].

### 2. The Bockstein spectral sequence

Let  $(A, \Gamma)$  denote a Hopf algebroid with  $\Gamma$  A-flat. Then it is known (cf. [14, Ch. A1]) that the category of  $\Gamma$ -comodules has enough injectives and so we can define the Ext groups as a cohomology of an injective resolution. Furthermore it

is given by a cohomology of the cobar resolution. So we can define  $\operatorname{Ext}_{\Gamma}^{n}(A, M) = H^{n}(\Omega_{\Gamma}^{*}M)$  for a  $\Gamma$ -comodule M, where  $\Omega_{\Gamma}^{*}M$  is a cobar complex (cf. [14]). The cobar complex  $\Omega_{\Gamma}^{*}M$  is a defineratial graded module with

$$\Omega^{s}_{\Gamma}M = M \otimes_{A} \Gamma \otimes_{A} \cdots \otimes_{A} \Gamma \quad (s \text{ copies of } \Gamma),$$

and the differentials  $d_r: \Omega_r^r M \to \Omega_r^{r+1} M$  defined inductively by

$$d_0(m) = \psi(m) - m \otimes 1$$
 and  $d_r(x \otimes y) = d_s(x) \otimes y + (-1)^s x \otimes d_t(y)$ 

for  $x \in \Omega_r^s M$  and  $y \in \Omega_r^t A$ . Here  $\psi : M \to M \otimes_A \Gamma$  denotes the comodule structure map of M. In the following, every comodule is induced from A and so we use  $\eta_R$  for  $\psi$ .

Suppose that  $A = \mathbb{Z}_{(2)}[v_1, v_2, \cdots]$  and  $\Gamma = A[t_1, t_2, \cdots]$ . Consider a Hopf algebroid  $\mathcal{P} = A[t_1] \otimes A(t_2)$  and a coalgebroid  $\Sigma = \Gamma \Box \Phi A$  over A. Then  $\Sigma = A[t_2^2, t_3, \cdots]$  and we have the change of rings theorem :

Lemma 2.1. For a comodule A, there is an isomorphism

$$\operatorname{Ext}_{\Gamma}^{*}(A, M \otimes_{A} \Phi) \cong \operatorname{Ext}_{\Sigma}^{*}(A, M).$$

Proof. Consider a relative injective  $\Gamma$ -resolution of  $M \otimes_A \varphi$ :

$$M \otimes_A \varPhi \longrightarrow I_0 \otimes_A \Gamma \longrightarrow I_1 \otimes_A \Gamma \longrightarrow \cdots,$$

which is split as A-modules. Then apply the cotensor product  $-\Box \varphi A$  and we obtain a relative injective  $\Sigma$ -resolution of M:

 $M \longrightarrow I_0 \otimes_A \Sigma \longrightarrow I_1 \otimes_A \Sigma \longrightarrow \cdots,$ 

since  $\Sigma = \Gamma \Box \phi A$ . Thus the both Ext groups are obtained from the same complex  $I_0 \rightarrow I_1 \rightarrow \cdots$ . q.e.d.

In this paper, we will compute  $\operatorname{Ext}_{F}^{*}(A, v_{2}^{-1}A/(2, v_{1}^{\infty})\otimes_{A} \boldsymbol{\Phi})$ . By virtue of this lemma, we will work in the category of  $\Sigma$ -comodules. In order to compute the Ext groups  $\operatorname{Ext}_{\Sigma}^{*}(A, v_{2}^{-1}A/(2, v_{1}^{\infty}))$ , we adopt the  $v_{1}$ -Bockstein spectral sequence with  $E_{1}$ -term

Ext<sup>\*</sup><sub>$$\Sigma$$</sub>(A,  $v_2^{-1}A/(2, v_1)$ ).

To compute the  $E_1$ -term we recall [7] the structure

(2.2) 
$$\operatorname{Ext}_{F}^{*}(A, v_{2}^{-1}A/(2, v_{1})[t_{1}]) = K(2)_{*}[v_{3}, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_{2}).$$

This is shown by using the change of rings theorems

$$\operatorname{Ext}_{F}^{*}(A, v_{2}^{-1}A/(2, v_{1})[t_{1}]) = \operatorname{Ext}_{K(2)*K(2)}^{*}(K(2)*, K(2)*[t_{1}])$$
  
= 
$$\operatorname{Ext}_{S(2,2)}^{*}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes_{K(2)*}K(2)*[v_{3}],$$

in which  $K(2)_* = \mathbb{Z}/2[v_2, v_2^{-1}], K(2)_*K(2) = K(2)_* \otimes_A \Gamma \otimes_A K(2)_*$  and S(2,2) =

 $\mathbb{Z}/2[t_2, t_3, \cdots]/(t_i^4 - t_i: i > 1)$ . Note here that the action of A on  $K(2)_*$  is given by sending  $v_i$  to 0 for  $i \neq 2$  and  $v_2$  to  $v_2$ , and  $(K(2)_*, K(2)_*K(2))$  becomes a Hopf algebroid induced from  $(A, \Gamma)$ . The second equation follows from the  $K(2)_*K(2)$ -comodule structure  $K(2)_*[t_1] = K(2)_*[t_1]/(v_2t_1^4 + v_2^2t_1) \otimes_{K(2)_*}K(2)_*[v_3]$ which is obtained from Landweber's formula  $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \mod (2, v_1)$ .

**Lemma 2.3.** The  $E_1$ -term is given by

Ext<sup>\*</sup><sub>2</sub>(A,  $v_2^{-1}A/(2, v_1)) = K(2)_*[v_3] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho),$ 

where  $K(2)_* = \mathbb{Z}/(2)[v_2, v_2^{-1}]$  and  $h_{21}$ ,  $h_{30}$ ,  $h_{31}$  and  $\rho$  are the homology classes represented by  $t_2^2$ ,  $t_3$ ,  $t_3^2$  and  $v_2^5 t_4 + t_4^2$  in the cobar complex, respectively.

Proof. Let  $H^*M$  for a  $\Gamma$ -comodule M denote the Ext group  $\operatorname{Ext}_{\Gamma}^*(A, M)$ , and  $E_*$  and  $D_*$  be  $\Gamma$ -comodules

$$E_* = v_2^{-1} A/(2, v_1)[t_1] \otimes A(t_2)$$
 and  $D_* = v_2^{-1} A/(2, v_1)[t_1].$ 

Then the short exact sequence  $0 \to D_* \subset E_* \to \Sigma^{-6}D_* \to 0$  of  $\Gamma$ -comodules yields the long exact sequence

$$\cdots \longrightarrow H^{s,t}D_* \longrightarrow H^{s,t}E_* \longrightarrow H^{s,t-6}D_* \stackrel{\delta}{\longrightarrow} H^{s+1,t}D_* \longrightarrow \cdots$$

with  $\delta(x) = h_{20}x$ . By (2.2),

$$H^*D_* = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This shows that  $h_{20}: H^s D_* \rightarrow H^{s+1} D_*$  is a monomorphism and we have the lemma. q.e.d.

#### 3. The elements $x_n$

In this section we will define elements  $x_n$  such that

$$x_n \equiv v_3^{2^n} \mod(2, v_1) \text{ and } d_0(x_n) \equiv v_1^{e_n} g_n,$$

in which  $g_n$  repesents a generator of  $\operatorname{Ext}_{\Sigma}^{1}(A, v_2^{-1}A/(2, v_1))$  and  $e_n$  to be taken as greate as possible. These elements play a central role in the Bockstein spectral sequence.

Hereafter we use the following abbreviation :

Ext\*(N)=Ext
$$\sharp(A, N)$$
 for a comodule N,  
 $M(j)=v_2^{-1}A/(2, v_1^j)$  and  $M=\lim_{i \to j} M(j)=v_2^{-1}A/(2, v_1^\infty).$ 

Then note that

$$BP_*(L_2X) = M \bigotimes_A \Phi$$
 and  $Ext^*(M) = Ext^*_{\Gamma}(A, BP_*(L_2X)).$ 

In  $v_2^{-1}BP_*/(2)$ , we define elements  $x_n$ , which will be used to define elements of Ext\*(*M*). From here on, we compute everything with setting  $v_2=1$  for the sake of simplicity. We also write

$$x \equiv y \mod(v_1^j)$$

for x,  $y \in \Omega_{\Sigma}^* M$  if x = y in the cobar complex  $\Omega_{\Sigma}^* M(j)$ .

We first introduce elements  $c_{3i}$  (i=0, 1) and  $\tilde{c}_{31}$  in  $\Sigma = A[t_2^2, t_3, \cdots]$  defined by

(3.1)  
$$v_{1}^{2}c_{30} = d_{0}(v_{4}^{2} + v_{1}^{2}v_{5}) + t_{2}^{8} + t_{2}^{2}, v_{1}c_{31} = d_{0}(v_{4}) + t_{4}^{4} \text{ and} \widetilde{c}_{31} = c_{31} + v_{1}(v_{3}^{2}c_{31} + v_{3}t_{2}^{2}).$$

**Lemma 3.2.** The cochains  $c_{30}$  and  $c_{31}$  are cocycles of the cobar complex  $\Omega_{\Sigma}^{1}M(j)$  for any j > 0. Furthermore,

$$c_{30} \equiv t_3 + v_3 t_2^8 \mod(v_1) \text{ and } c_{31} \equiv t_3^2 + v_1 v_3 t_2^2 \mod(v_1^4).$$

Proof. Since  $d_1d_0=0$ ,  $d_1(t_2)=0$  and  $d_0(v_1)=0$ , the first part of the lemma follows immediately from the definition, since the multiplication by  $v_1$  on  $\Omega_z^1 M(j)$  is monomorphic. The latter half is shown by the direct computation using

(3.3) 
$$\begin{aligned} \eta_R(v_1^2) &= v_1^2, \ \eta_R(v_4) \equiv v_4 + v_2 t_2^4 + v_1 t_3^2 + v_1^2 v_3 t_2^2 \mod(v_1^5), \\ \eta_R(v_4^2) \equiv v_4^2 + v_2^2 t_2^8 + v_2^8 t_2^2 + v_1^2 t_3^4 + v_1^4 v_3^2 t_2^4 \mod(v_1^{10}), \text{ and} \\ \eta_R(v_5) \equiv v_5 + v_3 t_2^8 + v_2 t_3^4 + v_2^8 t_3 \mod(v_1) \end{aligned}$$

in  $\Sigma$ , noticing that  $d_0(x) = \eta_R(x) - x$ . In fact,  $d_0(v_4^2 + v_1^2 v_5) \equiv t_2^8 + t_2^2 + v_1^2 t_3 + v_1^2 v_3 t_2^8 \mod(v_1^3)$ , by setting  $v_2 = 1$ , which gives  $c_{30}$ . For  $c_{31}$ , follows from  $\eta_R(v_4)$ . q.e.d.

**Lemma 3.4.** Put  $\varphi_1 = v_1 v_3^2 (v_4 + v_4^4)$ , and we have

$$d_0(\varphi_1) \equiv v_1(c_{30}^2 + \tilde{c}_{31}) \mod(v_1^3)$$

in  $v_2^{-1}\Sigma = v_2^{-1}A[t_2^2, t_3, \cdots].$ 

Proof. Since  $d_0(x) = \eta_R(x) - x$  and  $\eta_R$  is a map of algebras, this is verified by Lemma 3.2 and the following facts on  $\eta_R$ :

$$\eta_R(v_1) = v_1, \ \eta_R(v_2) = v_2, \\ \eta_R(v_3^2) \equiv v_3^2 \ \operatorname{mod}(v_1^2), \\ \eta_R(v_4) = v_4 + t_2^4 + v_1 c_{31} \ \operatorname{and} \ \eta_R(v_4^4) \equiv v_4^4 + t_2^{16} + t_2^4 \ \operatorname{mod}(v_1^4)$$

in  $v_2^{-1}\Sigma$ . In fact, by Lemma 3.2, we see that

$$c_{30}^2 + \tilde{c}_{31} \equiv v_3^2 t_2^{16} + v_1 v_3^2 c_{31}.$$

On the other hand, we compute

$$d_0(\varphi_1) \equiv v_1 v_3^2 d_0(v_4 + v_4^4) \equiv v_1 v_3^2(v_1 c_{31} + t_2^{16}).$$
 q.e.d.

Note that  $v_2^{-1}\Sigma$  is not a Hopf algebroid and so (3.1) does not imply the above lemma. In fact,  $d_0(v_4^2) = d_0(v_4)^2 + t_2^2$ . This with (3.1) yields the following

**Lemma 3.5.** In  $v_2^{-1}\Sigma$ ,

$$d_0(v_1^6v_5) = v_1^6(c_{31}^2 + c_{30}).$$

**Lemma 3.6.** There exist elements  $x_i$  of  $v_2^{-1}A$  with  $x_i \equiv v_3^{2i} \mod(2, v_1)$  such that

$$d_{0}(x_{0}) = v_{1}t_{2}^{2},$$
  

$$d_{0}(x_{1}) = v_{1}^{3}c_{31},$$
  

$$d_{0}(x_{2}) = v_{1}^{6}c_{30},$$
  

$$d_{0}(x_{2n+1}) \equiv v_{1}^{1+2a_{n}}v_{3}^{2b_{n}}(v_{3}^{2}c_{31}+v_{3}t_{2}^{2}) \mod(v_{1}^{2+2a_{n}}) \text{ and }$$
  

$$d_{0}(x_{2n+2}) \equiv v_{1}^{a_{n+1}}v_{3}^{b_{n+1}}c_{30} \mod(v_{1}^{1+a_{n+1}})$$

for n > 0. Here the integers  $a_n$  and  $b_n$  are given by

$$a_0=1$$
 and  $a_n=4a_{n-1}+2$   $(n>0)$   
 $b_0=0, b_1=0$  and  $b_n=4b_{n-1}+4$   $(n>1)$ .

**Proof.** Define the elements  $x_i$  inductively as follows :

(3.7)  

$$\begin{array}{rcl}
x_{0} = v_{3}, \\
x_{1} = v_{3}^{2} + v_{1}^{2} v_{4}, \\
x_{2} = x_{1}^{2} + v_{1}^{6} v_{5}, \\
x_{2n} = x_{2n-1}^{2} + v_{1}^{an} v_{3}^{bn} v_{5} \text{ and} \\
x_{2n+1} = x_{2n}^{2} + v_{1}^{2an-1} v_{3}^{2bn} \varphi_{1} + v_{1}^{2an-3} v_{3}^{2bn} x_{1}.
\end{array}$$

Then the lemma will be proved by induction. The first equation follows immediately from the Landweber formula:  $\eta_R(v_3) = v_3 + v_1 t_2^2$ . The second and the third are verified by (3.1). The others are inductively shown by Lemmas 3.4 and 3.5.

q.e.d.

## 4. The $E_2$ -term

Put  $L = v_2^{-1}BP_*/(2, v_1)$  and  $M = v_2^{-1}BP_*/(2, v_1^{\infty})$ . Then we have the short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{v_1} M \longrightarrow 0,$$

which yields the long exact sequence

(4.1) 
$$0 \longrightarrow \operatorname{Ext}^{0}(\underline{L}) \xrightarrow{f_{*}} \operatorname{Ext}^{0}(\underline{M}) \xrightarrow{v_{1}} \operatorname{Ext}^{0}(\underline{M}) \xrightarrow{\delta_{0}} \cdots \cdots \xrightarrow{\delta_{n-1}} \operatorname{Ext}^{n}(\underline{L}) \xrightarrow{f_{*}} \operatorname{Ext}^{n}(\underline{M}) \xrightarrow{v_{1}} \operatorname{Ext}^{n}(\underline{M}) \longrightarrow \cdots$$

Here f is a  $\Sigma$ -comodule map given by  $f(x) = x/v_1$ ,

$$\operatorname{Ext}^{n}(N) = \operatorname{Ext}^{n}_{\Sigma}(A, N)$$

for a  $\Sigma$ -comodule N, and note that the Ext group  $\text{Ext}^*(L)$  is determined in Lemma 2.3.

We here introduce some notations :

$$K(2)_* = \mathbb{Z}/2[v_2, v_2^{-1}], K = K(2)_*[v_1] = \mathbb{Z}/2[v_1, v_2, v_2^{-1}].$$

For an element  $x \in Ext^*(L)$ ,

- $C_n\langle x \rangle$  denotes a cyclic K-module isomorphic to  $K/(v_1^n)$ generated by  $\{x/v_1^n + z/v_1^{n-1}\} \in \operatorname{Ext}^*(M)$  for some  $z \in \mathcal{Q}_x^* v_2^{-1} BP_*/(2).$
- $C_{\infty}\langle x \rangle$  denotes a K-module isomorphic to  $v_1^{-1}K/K$  with basis  $\{x/v_1^i + z/v_1^{j-1}\}_{j>0} \subset \operatorname{Ext}^*(M)$  for some  $z \in \Omega_{\Sigma}^{\infty} v_2^{-1}BP_*/(2)$ .

Note that these  $C_*\langle x \rangle$  are sub-K-module of  $\text{Ext}^*(M)$ .

We compute  $\operatorname{Ext}^*(M) = \operatorname{Ext}^*_{\Sigma}(A, v_2^{-1}A/(2, v_1^{\infty}))$  from  $\operatorname{Ext}^*(L) = \operatorname{Ext}^*_{\Sigma}(A, v_2^{-1}A/(2, v_1))$  by using the following

**Lemma 4.2.** ([8, Remark 3.11]) Let  $\{x_{\lambda}\}_{\lambda \in \Lambda}$  be a set of generators of  $K(2)_*$ -module  $\operatorname{Ext}^i(L)$ , and  $\{\xi_{\lambda}\}_{\lambda \in \Lambda_0}$  and  $\{\xi_{\lambda,j}\}_{\lambda \in \Lambda_1}$  subsets of  $\operatorname{Ext}^i(M)$  such that  $\Lambda = \Lambda_0 \prod \Lambda_1$ ,

1) there exists a positive integer  $a(\lambda)$  for each  $\lambda \in \Lambda_0$  such that

$$v_1^{a(\lambda)-1}\xi_{\lambda} = f_*(x_{\lambda})$$
 and  
 $\delta_i(\xi_{\lambda}) \neq 0,$ 

2)  $\xi_{\lambda,1} = f_*(x_\lambda)$ ,  $v_1 \xi_{\lambda,j} = \xi_{\lambda,j-1}$  and  $\delta_i(\xi_{\lambda,j}) = 0$  for  $\lambda \in \Lambda_1$ .

Suppose that the set  $\{\delta_i(\xi_\lambda)\}_{\lambda \in \Lambda_0}$  is linearly independent over  $K(2)_*$ . Then Ext<sup>i</sup> $(M) = \bigoplus_{\lambda \in \Lambda_0} C_{a(\lambda)}\langle x_\lambda \rangle \oplus \bigoplus_{\lambda \in \Lambda_1} C_{\infty}\langle x_\lambda \rangle$ .

In this section, we will use Lemma 4.2 to compute  $\text{Ext}^*(M)$ , which is the  $E_2$ -term of the Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$ . Let  $\rho$  denote the homology class of  $\text{Ext}^1(L)$  given in Lemma 2.3.

**Lemma 4.3.** There exist elements  $\rho_i \in \Omega_{\Sigma}^1 v_2^{-1} A/(2)$  such that

 $\rho_i \equiv \rho \mod(2, v_1)$ 

up to homology and

$$d_1(\rho_i) \equiv 0 \mod(2, v_1^{2^i}).$$

Proof. In [9], Moreira constructed an element  $u \in \Omega_{\Sigma}^{1}L$  such that

$$d_0(u) \!=\! (\, \widetilde{
ho} \!+\! \zeta) \!+\! (\, \widetilde{
ho} \!+\! \zeta)^2 \ =\! (\, \widetilde{
ho} \!+\! t_2^2) \!+\! \widetilde{
ho}^2 \!+\! t_2^2 \!+\! t_2^4$$

in the cobar complex  $\Omega_{\Sigma}^{2}L$ . Here  $\zeta$  is represented by a cochain  $t_{2} + t_{2}^{2}$  in  $\Omega_{\Gamma}^{1}L$ , and  $\tilde{\rho}$  denotes a cocycle which represents the cohomology class  $\rho$ . Since  $t_{2}^{4}$  is homologous to 0, so is  $\tilde{\rho}$  to  $\tilde{\rho}^{2}$ . Hence define  $\rho_{i} = \tilde{\rho}^{2i}$  and we have the lemma. q.e.d.

For each j, there is an integer i such that  $\rho_i/v_1^j$  is a cocycle. In this case, we write

 $x\rho/v_1^j = x\rho_i/v_1^j$ .

Such an abbreviation would not cause any confusion.

The main lemma of the last section implies

**Lemma 4.4.** For the connecting homomorphism  $\delta_0$  in (4.1),

$$\begin{array}{c} \delta_0(v_3^{2t+1}/v_1) = v_3^{2t}h_{21},\\ \delta_0(v_3^{4t+2}/v_1^3) = v_3^{4t}h_{31},\\ \delta_0(v_3^{8t+4}/v_1^6) = v_3^{8t}h_{30},\\ \delta_0(v_3^{4n(4t+2)}/v_1^{1+2a_n}) = v_3^{4n+1t+2bn}(v_3^2h_{31}+v_3h_{21}) \text{ and }\\ \delta_0(v_3^{4n+1(2t+1)}/v_1^{an+1}) = v_3^{2\cdot4^{n+1}t+bn+1}h_{30} \end{array}$$

for  $t \ge 0, n > 0$ .

Here  $v_3^s/v_1^j$  denotes a cocycle of the cobar complex whose leading term is  $v_3^s/v_1^j$ . Therefore, we obtain the lemma by setting  $v_3^{2^*s}/v_1^j = x_n^s/v_1^j$  from Lemma 3.6. Now apply Lemma 4.2 to obtain

**Proposition 4.5.** The Ext group  $\text{Ext}^{0}(M)$  is a direct sum of  $C_{\infty}\langle 1 \rangle$  and  $C_{An}\langle v_{3}^{2^{n}(2t+1)} \rangle$  for  $n \geq 0$  and  $t \geq 0$ . Here  $A_{2n} = a_{n}$  and  $A_{2n+1} = 1 + 2a_{n}$ .

These give us the cokernel of  $\delta_0$ :

**Corollary 4.6.** The cokernel of  $\delta_0$ : Ext<sup>0</sup>(M)  $\rightarrow$  Ext<sup>1</sup>(L) is a  $K(2)_*$ -free module generated by

$$v_3^{2t+1}h_{21}, v_3^{u'}h_{30}, v_3^{u}h_{31} and v_3^{t}\rho$$

for  $t \ge 0$ ,  $u \notin T$  and  $u' \notin 2T$ . Here T is a subset of the natural numbers N:

$$T = \{n: 4 | n \text{ or } 4^{i+1} | (n-2b_i-2) \text{ for some } i > 0\},\$$

for  $b_i = 4(4^{i-1}-1)/3$ .

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**Lemma 4.7.** The complement U=N-T is given as

$$U = \{n: 2 \nmid n \text{ or } n = 2 \cdot 4^{k} t + 6 \cdot 4^{k-1} + 2(4^{k-1} - 1)/3$$
  
for some  $k > 0$  and  $t \ge 0\}$ 

For the computation of  $\delta_1$ , we introduce other elements :

**Lemma 4.8.** Consider an element  $\varphi = v_5 + v_3 v_4^2$ . Then there exist elements  $H_{21}$  and  $H_{32}$  in  $\Sigma$  such that

$$d_0(\varphi) = H_{32} + t_3 + H_{21}, \ d_1(H_{21}) = 0 = d_1(H_{32}), H_{21} \equiv t_2^2 \quad and \quad H_{32} \equiv t_3^4 \mod(v_1)$$

in the cobar complex  $\Omega_{\Sigma}^1 v_2^{-1} A/(2)$ .

Proof. For an element  $\psi = v_3^2 + v_1^7 v_3$ , we compute  $d_0(\psi) = v_1^2 t_2^4$  by  $\eta_R(v_3) = v_3 + v_1 t_2^2 + v_1^4 t_2$  in  $BP_*[t_2, t_3, \cdots]$ . Now put

$$H_{32} = t_3^4 + v_1^2 \psi t_2^4$$

Then, the formula  $\Delta(t_3^4) = t_3^4 \otimes 1 + 1 \otimes t_3^4 + v_1^4 t_2^4 \otimes t_2^4$  yields

$$d_1(H_{32}) = 0$$
 and  $H_{32} \equiv t_3^4 \mod(v_1)$ 

Furthermore, we compute

$$d_0(\varphi) \equiv t_3^4 + t_3 + v_3 t_2^2 \mod(v_1),$$

and so

$$d_0(\varphi) \equiv H_{32} + t_3 + v_3 t_2^2 \mod(v_1).$$

Put, then,

$$H_{21} = d_0(\varphi) + H_{32} + t_3$$

and we have

$$d_1(H_{21})=0$$
 and  $H_{21}\equiv v_3t_2^2 \mod(v_1)$ .

q.e.d.

**Lemma 4.9.** For the connecting homomorphism  $\delta_1 : \operatorname{Ext}^1(M) \longrightarrow \operatorname{Ext}^2(L)$ , we have

$$\begin{split} \delta_1(v_3^{4^{t+3}}h_{21}/v_1^3) &= v_3^{4^{t+1}}h_{21}h_{31},\\ \delta_1(v_3^{8^{t+5}}h_{21}/v_1^6) &= v_3^{8^{t+1}}h_{21}h_{30},\\ \delta_1(v_3^{4^{n(4t+2)+1}}h_{21}/v_1^{1+2a_n}) &= v_3^{4^{n+1}t+2b_n+1}h_{21}(v_3^2h_{31}+v_3h_{21}),\\ \delta_1(v_3^{4^{n+1}(2t+1)+1}h_{21}/v_1^{a_{n+1}}) &= v_3^{2\cdot 4^{n+1}t+b_{n+1}+1}h_{21}h_{30}\\ \delta_1(v_3^{2^{t+1}}h_{30}/v_1) &= v_3^{2^{t}}h_{21}h_{30}, \end{split}$$

$$\begin{split} \delta_1(v_3^{4^{t+2}}h_{30}/v_1^3) &= v_3^{4^t}h_{30}h_{31},\\ \delta_1(v_3^{4^{k}(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k}) &= v_3^{4^{k}(4t+2)-2}h_{30}(h_{31}+v_3^{-1}h_{21}),\\ \delta_1(v_3^{2^{t+1}}h_{31}/v_1) &= v_3^{2^t}h_{21}h_{31} \quad and\\ \delta_1(v_3^{4^{k}(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k}) &= v_3^{4^{k}(2t+1)-2}h_{30}(h_{31}+v_3^{-1}h_{21}). \end{split}$$

Proof. The first four equations follow immediately from Lemmas 4.4 and 4.8 with replacing  $v_3h_{21}$  by  $H_{21}$ . The fifth, sixth and eighth equations follow immediately from Lemmas 3.2 and 3.6. For the other equations, just put

$$v_{3}^{4^{k}(4t+2)+b_{k+1}}h_{30}/v_{1}^{1+2a_{k}} = v_{3}^{4^{k}(4t+2)}d_{0}(x_{2k+2})/v_{1}^{1+2a_{k}+a_{k+1}} \text{ and } \\ v_{3}^{4^{k}(2t+1)+b_{k+1}/2}h_{31}/v_{1}^{a_{k}} = v_{3}^{4^{k}(2t+1)}d_{0}(x_{2k+1})/v_{1}^{a_{k}+1+2a_{k}},$$

and we have the result by Lemma 3.6.

Now use Lemma 4.2, and we obtain

**Proposition 4.10.** Ext<sup>1</sup>(M) is a direct sum of 
$$\rho$$
Ext<sup>0</sup>(M) and

$$e^{1}(M) = \bigoplus_{t \ge 0} (C_{1} \langle v_{3}^{2t+1} h_{30} \rangle \oplus C_{1} \langle v_{3}^{2t+1} h_{31} \rangle \oplus C_{3} \langle v_{3}^{4t+2} h_{30} \rangle) \\ \oplus \bigoplus_{n > 0, t \ge 0} C_{A_{n}} \langle v_{3}^{2^{n}(2t+1)+1} h_{21} \rangle \\ \oplus \bigoplus_{t,k \ge 0} (C_{1+2a_{k}} \langle v_{3}^{4^{k}(4t+2)+b_{k+1}} h_{30} \rangle \oplus C_{a_{k}} \langle v_{3}^{4^{k}(2t+1)+b_{k+1}/2} h_{31} \rangle).$$

**Corollary 4.11.** The cokernel of  $\delta_1$ :  $\operatorname{Ext}^1(M) \to \operatorname{Ext}^2(L)$  is a direct sum of  $\rho$ Coker  $\delta_0$  and a  $K(2)_*$ -module generated by

$$v_3^{2t+1}h_{30}h_{31}, v_3^{2u+1}h_{21}h_{31} and v_3^{2u'+1}h_{21}h_{30}$$

for  $t \ge 0$ ,  $2u \notin T$  and  $u' \notin 2T$ .

**Lemma 4.12.** For the connecting homomorphism  $\delta_2 : \operatorname{Ext}^1(M) \to \operatorname{Ext}^2(L)$ , we have

$$\begin{split} \delta_2(v_3^{2t+1}h_{30}h_{31}/v_1) &= v_3^{2t}h_{21}h_{30}h_{31}, \\ \delta_2(v_3^{4t+3}h_{21}h_{30}/v_1^3) &= v_3^{4t+1}h_{21}h_{30}h_{31}, \\ \delta_2(v_3^{4k(4t+2)+b_{k+1}+1}h_{21}h_{30}/v_1^{1+2a_k}) &= v_3^{4k(4t+2)-1}h_{21}h_{30}h_{31}, \\ \delta_2(v_3^{4k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}/v_1^{a_k}) &= v_3^{4k(2t+1)-1}h_{21}h_{30}h_{31}. \end{split}$$

Proof. Note that  $\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1) = \delta_0(v_3^{2t+1}/v_1)h_{30}h_{31}$  since  $h_{3i} = c_{3i}$ 's are cocycles by Lemma 3.2. Now the first equation follows from Lemmas 4.4 and 4.9. For the other equations, use Lemmas 4.8 and 4.9 since  $\delta_2(v_3^{2t+1}h_{21}h_{3i}/v_1^i) = \delta_1(v_3^{2t}h_{3i}/v_1^i)v_3h_{21}$  if we use the representative  $H_{21}$  for the cohomology class  $v_3h_{21}$ .

Again by Lemma 4.2, we obtain

q.e.d.

**Proposition 4.13.** Ext<sup>2</sup>(M) is a direct sum of  $\rho e^{1}(M)$  and

$$e^{2}(M) = \bigoplus_{t,k\geq 0} (C_{1+2a_{k}} \langle v_{3}^{4^{k}(4t+2)+b_{k+1}+1}h_{21}h_{30} \rangle \\ \oplus C_{a_{k}} \langle v_{3}^{4^{k}(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31} \rangle) \oplus C_{1} \langle v_{3}^{2t+1}h_{30}h_{31} \rangle)$$

q.e.d.

**Corollary 4.14.** The cokernel of  $\delta_2 : \operatorname{Ext}^2(M) \to \operatorname{Ext}^3(L)$  is a  $K(2)_*$ -module  $\rho$ Coker  $\delta_1$ .

Now the following proposition follows immediately, by the same argument as above.

**Proposition 4.15.** For n > 3,  $Ext^n(M) = 0$ , and

$$\operatorname{Ext}^{3}(M) = \rho e^{2}(M).$$

5. On the map  $j_*: E_2(X) \rightarrow E_2(C)$ 

As is stated in the introduction, C denotes the cofiber of  $X_2 \rightarrow L_2 X_2$ . Then it is an  $X_2$ -module spectrum and  $h_{20} \in \pi_5(X_2)$  induces a map  $h_{20}: C \rightarrow C$ . In fact, it is the composition

$$C = S^0 \wedge C \xrightarrow{h_{20} \wedge C} X_2 \wedge C \xrightarrow{\nu} C,$$

in which  $\nu$  denotes the X<sub>2</sub>-module structure. Then we have a cofiber sequence

$$\Sigma^5 C \xrightarrow{h_{20}} C \xrightarrow{i} X \xrightarrow{j} \Sigma^6 C.$$

Let  $E_r^*(Y)$  denote the  $E_r$ -term of the Adams-Novikov spectral sequence converging to  $\pi_*(L_2Y)$  for a spectrum Y, and  $d_r^{AN}$ , its differentials. Then this gives rise to the exact sequence

$$0 \longrightarrow E_2^{0,t}(C) \xrightarrow{i_*} E_2^{0,t}(X) \xrightarrow{j_*} E_2^{0,t-6}(C) \xrightarrow{\delta} E_2^{1,t}(C) \longrightarrow \cdots.$$

Here  $E_2^{s,t}(X) = \operatorname{Ext}^{s,t}(M)$ , whose structure is given in the previous section. We further consider a cofiber E of  $h_{20}: C \to C$ . Then we have a commutative diagram

in which rows and columns are cofibrations.

**Lemma 5.2.** Let  $v_3^t/v_1^A$  denote a generator of  $E_2(X)$  as a  $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ -module. Then

$$j_*(v_3^t/v_1^{A-1})=0.$$

Proof. If  $t=2^n(2s+1)$  for some  $n, s\geq 0$ , then  $v_3^t/v_1^A$  is a homology class represented by  $x_n^{2s+1}/v_1^{An}$ . For n=0, the lemma is trivial. Now suppose that  $j_*(x_n^{2s+1}/v_1^{An})=0$  for even n=2m. Then squaring this, we obtain

$$j_*(x_{n+1}^{2s+1}/v_1^{A_{n+1}})=v_3^w/v_1$$

for some  $w \ge 0$ . Consider the diagram

$$\begin{array}{cccc} E_2^0(X) & \stackrel{j_*}{\longrightarrow} & E_2^0(C) \\ & \downarrow \delta & & \downarrow \delta \\ E_2^1(D) & \stackrel{i_*}{\longrightarrow} & E_2^1(E) & \stackrel{j_*}{\longrightarrow} & E_2^1(D) \end{array}$$

induced from (5.1). Since  $\delta(x_{n+1}^{2s+1}/v_1^{4n+1})$  is in the image of  $i_*$  by Lemma 4.4,  $\delta(v_3^w/v_1)=0$  in  $E_2^1(D)$  by the above diagram, and so 2|w since  $\delta(v_3^w/v_1)=wv_3^{w-1}h_{21}$  by Landweber's formula  $d_0(v_3)=v_1t_2^2+v_1^4t_2$  in  $BP_*[t_2, t_3, \cdots]$ . Thus we have

$$j_*(x_{n+1}^{2s+1}/v_1^{A_{n+1}}) = v_3^{2u}/v_1.$$

Square this, and we have

$$j_*(x_{n+2}^{2s+1}/v_1^{A_{n+2}}) = v_3^{4u}/v_1^2$$

Notice that  $j_*(x) = y$  if  $d_0(x) = yt_2$ , where  $d_0(x) = \eta_R(x) - x$ . A direct computation shows us  $d_0(v_3^{4u}x_1/v_1^4) = v_3^{4u}t_2/v_1^2$  in the cobar complex  $\Omega_2^2 M$ . Thus we have shown inductively that  $j_*(v_3^{2^{n}(2s+1)}/v_1^{4n})$  equals to 0 if *n* is even, and to  $v_3^{2^s}/v_1$  for some *u* if *n* is odd. q.e.d.

# 6. The Adams-Novikov differential

Consider the cofiber E of  $h_{20}$ :  $\Sigma^5 D \rightarrow D$ . Then by [7, Th. 7.1], we immediately obtain the following

**Proposition 6.1.** The Adams-Novikov spectral sequence for computing  $\pi_*(L_2E)$  collapses from the  $E_2$ -term.

Note that the  $E_2$ -term for our X is

$$E_2^*(X) = \operatorname{Ext}_{\Gamma}^*(A, v_2^{-1}BP_*(X)) = \operatorname{Ext}^*(M).$$

**Lemma 6.2.** For the Adams-Novikov differential  $d_3^{AN}: E_2^0(X) \rightarrow E_2^3(X)$ ,  $d_3^{AN}(v_3^t/v_1^A)$  is a sum of the elements of the form  $v_3^{2u+1}h_{21}h_{3i}\rho/v_1^k$  for i=0, 1 and k>1. Here  $v_3^t/v_1^A$  is a generator of the  $Z/2[v_1, v_2, v_2^{-1}]$ -module  $M_0$ .

Proof. Consider the diagram (5.1). The third column induces the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{3}(M) \xrightarrow{\nu_{1}} \operatorname{Ext}^{3}(M) \xrightarrow{\delta_{3}} \operatorname{Ext}^{4}(L) \longrightarrow \cdots$$

of the  $E_2$ -terms. If the  $\delta_0$  image of  $v_3^t/v_1^A$  is  $x \neq 0$ , then  $\delta_3(d_3^{AN}(v_3^t/v_1^A)) = d_3^{AN}(x) = 0$  by Proposition 6.1. Thus  $d_3^{AN}(v_3^t/v_1^A)$  is divisible by  $v_1$ . Furthermore it implies that  $v_3^{2t+1}h_{30}h_{31}\rho/v_1$  cannot be a target of  $d_3^{AN}$ . In fact, it is not divisible by  $v_1$  by Proposition 4.15. Now the lemma follows from Lemma 4.15. q.e.d.

**Theorem 6.3.** The Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$  collapses from the  $E_2$ -term.

Proof. By proposition 4.15, the Adams-Novikov differentials are all trivial except for  $d_3^{AN}$ :  $E_2^0(X) \to E_2^3(X)$ . So it is sufficient to show that  $d_3^{AN}(v_3^t/v_1^i) = 0$  for each  $v_3^t/v_1^i \in E_2^0(X)$ . By Lemma 6.2,

(6.4) 
$$d_3^{AN}(v_3^t/v_1^{A-k}) = \sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho/v_1^2$$

for some  $k \ge 0$ , where  $\lambda_{u,i} \in \mathbb{Z}/2$ . Since

$$d_3(v_3^{2u+1}h_{21}h_{3i}\rho/v_1^2) = v_3^{2u}h_{20}^2h_{3i}\rho/v_1 \neq 0$$

in the cobar complex  $\Omega^4_{\Gamma}BP_*(C)$ , we see that

(6.5) 
$$j_*(\sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho / v_1 \neq 0.$$

Now send (6.4) by  $j_*$  and we have a contradiction to Lemma 5.2, which says  $j_*(v_3^t/v_1^{A-k})=0$  if k>0. If k=0 and  $j_*(v_3^t/v_1^A)\neq 0$ , then

$$j_*(v_3^t/v_1^A) = v_3^{2u}/v_1$$

for some  $u \ge 0$  as is seen in the proof of Lemma 5.2. Therefore, (6.4) and (6.5) yield

$$d_3^{AN}(v_3^{2u}/v_1) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho/v_1 \neq 0$$

in  $E_2^*(C)$  for some  $\lambda_{u,i} \in \mathbb{Z}/2$ . Now pull this back to  $E_2^*(D)$  under the map  $i_*$ :  $E_2^*(D) \to E_2^*(C)$  to obtain the non-trivial differential

$$d_3^{AN}(v_3^{2u}) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho \neq 0$$

in  $E_2^*(D)$ , which again contradicts to a result of [7] which says  $d_3^{AN}(v_3^{4k})=0$  and  $d_3^{AN}(v_3^{4k+2})=v_3^{4k}h_{20}^3$  for k>0. q.e.d.

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