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Osaka University
THE HOMOTOPY GROUPS OF A SPECTRUM
WHOSE $BP_*$-HOMOLOGY IS
$v_2^1 BP_*/(2, v_1^\infty) [t_1] \otimes (t_2)$

Keiko Masamoto, Tsuyoshi Matsuhisa and Katsumi Shimomura

(Received May 12, 1994)

1. Introduction

In [5], Mahowald gave some examples of ring spectra obtained as Thom spectra. One of them is $X_2$ in [5], which is a Thom spectrum associated to $\omega : QS^2 \to BO$, where $\omega$ is a mapping corresponding to the generator of $\pi_1(BO)$. Let $BP$ denote the Brown-Peterson spectrum at the prime 2. Then the spectrum $X_2$ is also characterized by the $BP_*$-homology $BP_*(X_2) = BP_*/(2)[t]$ as a sub-comodule algebra of $BP_*(BP)/(2) = BP_*/(2)[t_1, t_2, \cdots]$, where $BP_* = \mathbb{Z}[
u_1, \nu_2, \cdots]$ over Hazewinkel’s generators $v_i$ (cf. [14]).

Relating to $X_2$, consider a spectrum $X$ constructed as follows: Let $C$ be a cofiber of the Bousfield localization map $X_2 \to L_1 X_2$ with respect to the Johnson-Wilson spectrum $E(1)$ with $\pi_*(E(1)) = \mathbb{Z}[v_1, v_1^{-1}]$. Then $C$ is an $X_2$-module spectrum since $X_2$ is a ring spectrum. Consider the element $h_20 \in \pi_0(X_2)$. Now the spectrum $X$ is a cofiber of a map $h_20 : \Sigma^5 C \to C$. By this definition, the $BP_*$-homology of $X$ is $BP_*(X) = BP_d/(2, v^0_0)[t_1] \otimes A(t_2)$. Once we determined the homotopy groups $\pi_*(L_2 X_2)$ in [17], the homotopy groups $\pi_*(L_2 X)$ can be obtained from it. Here $L_2$ denotes the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$ with $\pi_*(E(2)) = \mathbb{Z}[v_1, v_2, v_2^{-1}]$ as a subalgebra of $v_2^{-1}BP_*$. But, in this paper, we compute, independently of [17], the homotopy groups $\pi_*(L_2 X)$ of the $E(2)_d$-localized spectrum of $X$ by using the Adams-Novikov spectral sequence. The computation of the $E_2$-term is done in the same manner as that of [17], using the $v_1$-Bockstein spectral sequence. Different from the odd prime case, there may involve non-trivial differentials of the Adams-Novikov spectral sequence. On the other hand, different from the case for $X_2$, this case may support at most one family of non-trivial differentials. In this sense, it is a little easier to determine the homotopy groups of $L_2 X$ than those of $L_2 X_2$. By using the results of [7], we show here that the differentials are all trivial, in a different fashion from that of [17], and have the $E_\infty$-term of the spectral sequence. In order to state the result, consider the integers $A_n$ defined by
for $n \geq 0$, and use the notations:

$C_\omega(x)$ is a $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to

$\mathbb{Z}/2[v_1, v_1^{-1}, v_2, v_2^{-1}]/\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$

generated by elements $\{x/v_i\}_{i \geq 0}$ such that $v_i(x/v_i) = x_v/v_i^{-1}$.

$C_{f}(x)$ is a cyclic $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to

$\mathbb{Z}/2[v_1, v_2, v_2^{-1}]/(v_i)$

generated by an element $x/v_i$.

**Theorem.** The $E_\infty$-term of the Adams-Novikov spectral sequence for computing $\pi_*(\mathbb{L}_X)$ is a $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$-module

$$M_\infty \otimes A(\rho).$$

Here, the graded $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$-module $M_\infty$ is given by:

$$M_0 = C_\omega \langle \chi \rangle \otimes \bigoplus_{i,j \geq 0} C_{A_0} \langle v_0^{2(i+1)} \rangle,$$

$$M_1 = \bigoplus_{i,j \geq 0} \left( C_1 \langle v_0^{2(i+1)} \rangle \otimes C_1 \langle v_0^{2(i+1)} \rangle \otimes C_3 \langle v_0^{4(i+1)} \rangle \right),$$

$$M_2 = \bigoplus_{i,j \geq 0} \left( C_{A_2} \langle v_0^{4(i+1)} \rangle \right),$$

$$M_n = 0 \text{ for } n > 2.$$

Furthermore, the generators have the following degrees:

$$|v_0| = 14, \quad |h_0| = 5, \quad |h_2| = 11, \quad |h_3| = 13, \text{ and } |h_4| = 27.$$

In the theorem, an element $x$ has a degree $r$ if $\pi_r(\mathbb{L}_X)$.

This paper is organized as follows: In the next section, we recall some facts known about the $v_1$-Bockstein spectral sequence. In §3, we define elements $x_n$, which will play the main role in the computation of the Bockstein spectral sequence. We compute $E_\infty$-terms of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(\mathbb{L}_X)$ in §4, by using the tools given in the previous sections. In section 5, we prepare some lemmas to compute the Adams-Novikov differentials in the last section using the results of [7].

### 2. The Bockstein spectral sequence

Let $(A, \Gamma)$ denote a Hopf algebroid with $\Gamma$ A-flat. Then it is known (cf. [14, Ch. A1]) that the category of $\Gamma$-comodules has enough injectives and so we can define the Ext groups as a cohomology of an injective resolution. Furthermore it
is given by a cohomology of the cobar resolution. So we can define \( \text{Ext}^\mathbb{F}(A, M) = H^\mathbb{F}(\Omega^\mathbb{F}M) \) for a \( \Gamma \)-comodule \( M \), where \( \Omega^\mathbb{F}M \) is a cobar complex (cf. \([14]\)). The cobar complex \( \Omega^\mathbb{F}M \) is a differential graded module with

\[
\Omega^\mathbb{F}M = M \otimes \Lambda \Gamma \otimes \cdots \otimes \Lambda \Gamma \quad (s \text{ copies of } \Gamma),
\]

and the differentials \( d_r : \Omega^\mathbb{F}M \to \Omega^{r+1}\mathbb{F}M \) defined inductively by

\[
d_0(m) = \phi(m) - m \otimes 1 \quad \text{and} \quad d_r(x \otimes y) = d_s(x) \otimes y + (-1)^s x \otimes d_r(y)
\]

for \( x \in \Omega^\mathbb{F}M \) and \( y \in \Omega^\mathbb{F}A \). Here \( \phi : M \to M \otimes \Lambda \Gamma \) denotes the comodule structure map of \( M \). In the following, every comodule is induced from \( A \) and so we use \( \eta^\mathbb{F}_r \) for \( \phi \).

Suppose that \( A = \mathbb{Z}[(\nu_1, \nu_2, \cdots) \quad \text{and} \quad \Gamma = A[(t_1, t_2, \cdots) \quad \text{Consider a Hopf algebroid } \Phi = A[\{t_2, t_3, \cdots \} \quad \text{and a coalgebroid } \Sigma = \Gamma \square \phi \Lambda \text{ over } A. \quad \text{Then } \Sigma = A[[t_2, t_3, \cdots] \quad \text{and we have the change of rings theorem :}

**Lemma 2.1.** For a comodule \( A \), there is an isomorphism

\[
\text{Ext}^\mathbb{F}(A, M \otimes \Lambda \Phi) \cong \text{Ext}^\mathbb{F}(A, M).
\]

**Proof.** Consider a relative injective \( \Gamma \)-resolution of \( M \otimes \Lambda \Phi : \)

\[
M \otimes \Lambda \Phi \longrightarrow I_0 \otimes \Lambda \Gamma \longrightarrow I_1 \otimes \Lambda \Gamma \longrightarrow \cdots,
\]

which is split as \( A \)-modules. Then apply the cotensor product \( - \square \phi \Lambda \) and we obtain a relative injective \( \Sigma \)-resolution of \( M : \)

\[
M \longrightarrow I_0 \otimes \Lambda \Sigma \longrightarrow I_1 \otimes \Lambda \Sigma \longrightarrow \cdots,
\]

since \( \Sigma = \Gamma \square \phi \Lambda \). Thus the both Ext groups are obtained from the same complex \( I_0 \to I_1 \to \cdots \).

In this paper, we will compute \( \text{Ext}^\mathbb{F}(A, v^{-1}_2 A/(2, \nu^\mathbb{F}_1) \otimes \Lambda \Phi) \). By virtue of this lemma, we will work in the category of \( \Sigma \)-comodules. In order to compute the Ext groups \( \text{Ext}^\mathbb{F}(A, v^{-1}_2 A/(2, \nu^\mathbb{F}_1)) \), we adopt the \( \nu_1 \)-Bockstein spectral sequence with \( E_1 \)-term

\[
\text{Ext}^\mathbb{F}(A, v^{-1}_2 A/(2, \nu_1)).
\]

To compute the \( E_1 \)-term we recall \([7]\) the structure

\[
\text{Ext}^\mathbb{F}(A, v^{-1}_2 A/(2, \nu_1)[t_1]) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).
\]

This is shown by using the change of rings theorems

\[
\text{Ext}^\mathbb{F}(A, v^{-1}_2 A/(2, \nu_1)[t_1]) = \text{Ext}^\mathbb{F}_{(2), K(2)_*}(K(2)_*, K(2)_*[t_1]) = \text{Ext}^\mathbb{F}_{(2), K(2)_*}(Z/2, Z/2 \otimes K(2)_* K(2)_*[v_3]),
\]

in which \( K(2)_* = Z/2[v_2, v^{-1}_2], \quad K(2)_* K(2) = K(2)_* \otimes \Lambda \Gamma \otimes K(2)_* \text{ and } S(2,2) = \cdots \).
Note here that the action of $A$ on $K(2)_\ast$ is given by sending $v_i$ to 0 for $i \neq 2$ and $v_3$ to $v_3$, and $(K(2)_\ast, K(2)_\ast K(2))$ becomes a Hopf algebroid induced from $(A, \Gamma)$. The second equation follows from the $K(2)_\ast K(2)$-comodule structure $K(2)_\ast[t_i] = K(2)_\ast[t_i]/(v_2t_i^4 + v_3t_i) \otimes_{K(2), K(2)_\ast} v_3$ which is obtained from Landweber's formula $\eta_r(v_3) = v_3 + v_2t_i^4 + v_3t_1 \mod (2, v_1)$.

**Lemma 2.3.** The $E_1$-term is given by

$$\text{Ext}^*(A, v_3^{-1}A/(2, v_1)) = K(2)_\ast[v_3] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho),$$

where $K(2)_\ast = \mathbb{Z}/(2)[v_2, v_3^{-1}]$ and $h_{21}, h_{30}, h_{31}$ and $\rho$ are the homology classes represented by $t_2^3, t_3, t_2^3$ and $v_2t_i^4 + t_2^4$ in the cobar complex, respectively.

**Proof.** Let $H^*M$ for a $\Gamma$-comodule $M$ denote the Ext group $\text{Ext}^*(A, M)$, and $E_\ast$ and $D_\ast$ be $\Gamma$-comodules

$$E_\ast = v_3^{-1}A/(2, v_1)[t_i] \otimes \Lambda(t_2) \quad \text{and} \quad D_\ast = v_3^{-1}A/(2, v_1)[t_i].$$

Then the short exact sequence $0 \to D_\ast \subset E_\ast \to \Sigma^{-6}D_\ast \to 0$ of $\Gamma$-comodules yields the long exact sequence

$$\cdots \to H^{s+t}D_\ast \to H^{s+t}E_\ast \to H^{s+t-6}D_\ast \to \delta \to H^{s+1+t}D_\ast \to \cdots$$

with $\delta(x) = h_{20}x$. By (2.2),

$$H^*D_\ast = K(2)_\ast[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This shows that $h_{20}: H^*D_\ast \to H^{s+1}D_\ast$ is a monomorphism and we have the lemma.

**q.e.d.**

3. *The elements $x_n$*

In this section we will define elements $x_n$ such that

$$x_n = v_3^\alpha \mod (2, v_1) \quad \text{and} \quad d_0(x_n) = v_3^\alpha g_n,$$

in which $g_n$ represents a generator of $\text{Ext}^i_k(A, v_3^{-1}A/(2, v_1))$ and $e_n$ to be taken as great as possible. These elements play a central role in the Bockstein spectral sequence.

Hereafter we use the following abbreviation :

$$\text{Ext}^*(N) = \text{Ext}^*(A, N) \quad \text{for a comodule} \ N,$$

$$M(j) = v_3^{-1}A/(2, v_1) \quad \text{and} \quad M = \lim_{\longrightarrow i} M(j) = v_3^{-1}A/(2, v_1).$$

Then note that

$$BP_\ast(L_2X) = M \otimes_A \Phi \quad \text{and} \quad \text{Ext}^*(M) = \text{Ext}^*(A, BP_\ast(L_2X)).$$
In $\nu^{-1}BP_{n}/(2)$, we define elements $x_{n}$, which will be used to define elements of $\text{Ext}^{*}(M)$. From here on, we compute everything with setting $v_{2}=1$ for the sake of simplicity. We also write

$$x \equiv y \mod(v_{1})$$

for $x, y \in \Omega^{*}M$ if $x=y$ in the cobar complex $\Omega^{*}M(j)$.

We first introduce elements $c_{3i}$ ($i=0, 1$) and $\overline{c_{31}}$ in $\Sigma = A[t_{1}, t_{3}, \cdots]$ defined by

$$v_{1}^{i}c_{30} = d_{0}(v_{1}^{i} + v_{1}^{i}v_{3}) + t_{2}^{i} + t_{3}^{i},$$

$$v_{1}c_{31} = d_{0}(v_{4}) + t_{4}^{i}$$

and

$$\overline{c_{31}} = c_{31} + v_{1}(v_{3}^{i}c_{31} + v_{3}t_{3}^{i}).$$

**Lemma 3.2.** The cochains $c_{30}$ and $c_{31}$ are cocycles of the cobar complex $\Omega^{*}M(j)$ for any $j>0$. Furthermore,

$$c_{30} = t_{3} + v_{3}t_{3}^{i} \mod(v_{1}) \quad \text{and} \quad c_{31} = t_{3} + v_{1}v_{3}t_{3}^{i} \mod(v_{1}).$$

**Proof.** Since $d_{1}d_{0}=0$, $d_{1}(t_{3})=0$ and $d_{0}(v_{1})=0$, the first part of the lemma follows immediately from the definition, since the multiplication by $v_{1}$ on $\Omega^{*}M(j)$ is monomorphic. The latter half is shown by the direct computation using

$$\eta_{R}(v_{1}^{i}) = v_{1}^{i}, \quad \eta_{R}(v_{3}) = v_{3} + v_{1}t_{3}^{i} + v_{3}t_{3}^{i} + v_{1}v_{3}t_{3}^{i} \mod(v_{1}),$$

$$\eta_{R}(v_{4}) = v_{4} + v_{1}t_{4}^{i} + v_{3}t_{4}^{i} + v_{1}v_{3}t_{4}^{i} \mod(v_{1}),$$

and

$$\eta_{R}(v_{5}) = v_{5} + v_{1}t_{5}^{i} + v_{3}t_{5}^{i} + v_{1}v_{3}t_{5}^{i} \mod(v_{1})$$

in $\Sigma$, noticing that $d_{0}(x) = \eta_{R}(x) - x$. In fact, $d_{0}(v_{4} + v_{3}t_{3}^{i}) = t_{4}^{i} + t_{4}^{i}t_{3}^{i} + v_{1}v_{3}t_{3}^{i} \mod(v_{1})$, by setting $v_{2}=1$, which gives $c_{30}$. For $c_{31}$, follows from $\eta_{R}(v_{4})$.

**q.e.d.**

**Lemma 3.4.** Put $\varphi_{1} = v_{1}v_{3}^{i}(v_{4} + v_{3}^{i})$, and we have

$$d_{0}(\varphi_{1}) = v_{1}(c_{30} + \overline{c_{31}}) \mod(v_{1})$$

in $\nu^{-1}M = \nu^{-1}A[t_{3}, t_{3}, \cdots]$.

**Proof.** Since $d_{0}(x) = \eta_{R}(x) - x$ and $\eta_{R}$ is a map of algebras, this is verified by Lemma 3.2 and the following facts on $\eta_{R}$:

$$\eta_{R}(v_{1}) = v_{1}, \quad \eta_{R}(v_{2}) = v_{2},$$

$$\quad \eta_{R}(v_{3}) = v_{3} \mod(v_{1}),$$

$$\quad \eta_{R}(v_{4}) = v_{4} + t_{4}^{i} + v_{1}c_{31} \quad \text{and} \quad \eta_{R}(v_{4}) = v_{4} + t_{4}^{16} + t_{4}^{14} \mod(v_{1})$$

in $\nu^{-1}M$. In fact, by Lemma 3.2, we see that

$$c_{30} + \overline{c_{31}} = v_{3}^{i}t_{3}^{16} + v_{1}v_{3}c_{31}.$$
Note that \( v_2^{-1} \Sigma \) is not a Hopf algebroid and so (3.1) does not imply the above lemma. In fact, \( d_0(v_2^2) = d_0(v_4^2 + t_2^2) \). This with (3.1) yields the following

**Lemma 3.5.** In \( v_2^{-1} \Sigma \),

\[
d_0(v_i^k v_6) = v_i^k (c_{31} + c_{30}).
\]

**Lemma 3.6.** There exist elements \( x_i \) of \( v_2^{-1} \Lambda \) with \( x_i \equiv v_3^i \text{ mod}(2, v_1) \) such that

\[
d_0(x_0) = v_1 t_2^2,
\]
\[
d_0(x_1) = v_1^3 c_{31},
\]
\[
d_0(x_2) = v_1^5 c_{30},
\]
\[
d_0(x_{2n+1}) = v_1^{3+2a_n} v_3^{2b_n}(v_3^2 c_{31} + v_3 t_2^2) \text{ mod}(v_1^{2+2a_n}) \quad \text{and}
\]
\[
d_0(x_{2n+2}) = v_1^{5a_n} v_3^{5b_n-1} c_{30} \text{ mod}(v_1^{1+2a_n+1})
\]

for \( n > 0 \). Here the integers \( a_n \) and \( b_n \) are given by

\[
a_0 = 1 \quad \text{and} \quad a_n = 4a_{n-1} + 2 \quad \text{for } n > 0,
\]
\[
b_0 = 0, \quad b_1 = 0 \quad \text{and} \quad b_n = 4b_{n-1} + 4 \quad \text{for } n > 1.
\]

**Proof.** Define the elements \( x_i \) inductively as follows:

\[
egin{align*}
x_0 &= v_3, \\
x_1 &= v_2^3 + v_1^7 v_4, \\
x_2 &= x_1^2 + v_1^5 v_5, \\
x_{2n} &= x_{2n-1}^2 + v_1^{2^{n-1}} v_3^5 v_5 \quad \text{and} \\
x_{2n+1} &= x_{2n}^2 + v_1^{2^{n-1}} v_3^{2b_n} x_1 + v_1^{2a_n-3} v_3^{2b_n} x_1.
\end{align*}
\]

Then the lemma will be proved by induction. The first equation follows immediately from the Landweber formula: \( \eta_*(v_3) = v_3 + v_1 t_2^2 \). The second and the third are verified by (3.1). The others are inductively shown by Lemmas 3.4 and 3.5.

q.e.d.

4. The \( E_2 \)-term

Put \( L = v_2^{-1} \Lambda BP_*/(2, v_1) \) and \( M = v_2^{-1} \Lambda BP_*/(2, v_1) \). Then we have the short exact sequence

\[
0 \rightarrow L \rightarrow M \rightarrow M \rightarrow 0,
\]

which yields the long exact sequence

\[
0 \rightarrow \text{Ext}^0(L) \rightarrow \text{Ext}^0(M) \rightarrow \text{Ext}^0(M) \rightarrow \text{Ext}^0(M) \rightarrow \cdots
\]
Here $f$ is a $\Sigma$-comodule map given by $f(x) = x/v_1$,

$$\text{Ext}^\ast(N) = \text{Ext}^\ast_\Sigma(A, N)$$

for a $\Sigma$-comodule $N$, and note that the Ext group $\text{Ext}^\ast(L)$ is determined in Lemma 2.3.

We here introduce some notations:

$$K(2)\ast = \mathbb{Z}/2[v_2, v_2^{-1}], \ K = K(2)_\ast[v_1] = \mathbb{Z}/2[v_1, v_2, v_2^{-1}].$$

For an element $x \in \text{Ext}^\ast(L)$,

$$C_n\langle x \rangle$$

denotes a cyclic $K$-module isomorphic to $K/(v_1)$ generated by \{x/v_1 + z/v_1^{-1}\} $\in \text{Ext}^\ast(M)$ for some $z \in \Omega_{v_2}^1 v_2^{-1}BP_\ast/(2)$.

$$C_\omega\langle x \rangle$$

denotes a $K$-module isomorphic to $v_1^{-1}K/K$ with basis \{x/v_1 + z/v_1^{-1}\} $\in \text{Ext}^\ast(M)$ for some $z \in \Omega_{v_2}^1 v_2^{-1}BP_\ast/(2)$.

Note that these $C_n\langle x \rangle$ are sub-$K$-module of $\text{Ext}^\ast(M)$.

We compute $\text{Ext}^\ast(M) = \text{Ext}^\ast_\Sigma(A, v_2^{-1}A/(2, v_2^\ast))$ from $\text{Ext}^\ast(L) = \text{Ext}^\ast_\Sigma(A, v_2^{-1}A/(2, v_1))$ by using the following

**Lemma 4.2.** ([8, Remark 3.1]) Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a set of generators of $K(2)_\ast$-module $\text{Ext}^i(L)$, and $\{\xi_\lambda\}_{\lambda \in \Lambda_0}$ and $\{\xi_{\lambda,j}\}_{\lambda \in \Lambda_1}$ subsets of $\text{Ext}^i(M)$ such that $\Lambda = \Lambda_0 \sqcup \Lambda_1$,

1) there exists a positive integer $a(\lambda)$ for each $\lambda \in \Lambda_0$ such that

$$v_1^{a(\lambda)-1} \xi_\lambda = f_\ast(x_\lambda) \text{ and } \delta_i(\xi_\lambda) \neq 0,$$

2) $\xi_{\lambda,1} = f_\ast(x_\lambda), \ v_1 \xi_{\lambda,j} = \xi_{\lambda,j-1} \text{ and } \delta_i(\xi_{\lambda,j}) = 0 \text{ for } \lambda \in \Lambda_1.$$

Suppose that the set $\{\delta_i(\xi_\lambda)\}_{\lambda \in \Lambda_0}$ is linearly independent over $K(2)_\ast$. Then $\text{Ext}^i(M) = \bigoplus_{\lambda \in \Lambda_0} C_{a(\lambda)}\langle x_\lambda \rangle \bigoplus \bigoplus_{\lambda \in \Lambda_1, C_{\omega}\langle x_\lambda \rangle}.$

In this section, we will use Lemma 4.2 to compute $\text{Ext}^\ast(M)$, which is the $E_2$-term of the Adams-Novikov spectral sequence for computing $\pi_\ast(L_2X)$. Let $\rho$ denote the homology class of $\text{Ext}^i(L)$ given in Lemma 2.3.

**Lemma 4.3.** There exist elements $\rho_1 \in \Omega^1_{v_2} v_2^{-1}A/(2)$ such that $\rho_i \equiv \rho \mod(2, v_1)$ up to homology and

$$d_i(\rho_i) \equiv 0 \mod(2, v_1).$$
Proof. In [9], Moreira constructed an element \( u \in \Omega^k L \) such that
\[
d_0(u) = (\tilde{\rho} + \zeta) + (\tilde{\rho} + \zeta)^2
\]
in the cobar complex \( \Omega^k L \). Here \( \zeta \) is represented by a cochain \( t_2 + tl \) in \( \Omega^k L \), and \( \tilde{\rho} \) denotes a cocycle which represents the cohomology class \( \rho \). Since \( t_2 \) is homologous to 0, so is \( \tilde{\rho} \) to \( \tilde{\rho}^2 \). Hence define \( \rho_i = \tilde{\rho}^{2i} \) and we have the lemma.
\[\text{q.e.d.}\]

For each \( j \), there is an integer \( i \) such that \( \rho_i/v_i \) is a cocycle. In this case, we write
\[
x^i \rho_i/v_i = x^i \rho / v_i.
\]
Such an abbreviation would not cause any confusion.

The main lemma of the last section implies

**Lemma 4.4.** For the connecting homomorphism \( \delta_0 \) in (4.1),
\[
\begin{align*}
\delta_0(v_3^{3t+1}/v_1) &= v_3^{3t} h_{21}, \\
\delta_0(v_3^{4t+2}/v_1) &= v_3^{4t} h_{31}, \\
\delta_0(v_3^{6t+4}/v_1) &= v_3^{6t} h_{30}, \\
\delta_0(v_3^{6t+4}/v_1) &= v_3^{6t} h_{30}, \\
\delta_0(v_3^{5(t+2)/v_1}) &= v_3^{5(t+2)/v_1} h_{21} + v_3 h_{21} \quad \text{and} \\
\delta_0(v_3^{5(t+2)/v_1}) &= v_3^{5(t+2)/v_1} h_{21} + v_3 h_{21}
\end{align*}
\]
for \( t \geq 0, n > 0 \).

Here \( v_3/v_1 \) denotes a cocycle of the cobar complex whose leading term is \( v_3/v_1 \).

Therefore, we obtain the lemma by setting \( v_3^n/v_1 = x^n v_1 \) from Lemma 3.6. Now apply Lemma 4.2 to obtain

**Proposition 4.5.** The Ext group \( \text{Ext}^0(M) \) is a direct sum of \( C_{m}\langle 1 \rangle \) and \( C_{kn}\langle v_3^{2(n+1)} \rangle \) for \( n \geq 0 \) and \( t \geq 0 \). Here \( A_{2n} = a_n \) and \( A_{2n+1} = 1 + 2a_n \).

These give us the cokernel of \( \delta_0 \):

**Corollary 4.6.** The cokernel of \( \delta_0 : \text{Ext}^0(M) \to \text{Ext}^1(L) \) is a \( K(2)^* \)-free module generated by \( v_3^{t+1} h_{21}, v_3^{t} h_{30}, v_3^{t} h_{31} \) and \( v_3^{t+1} \rho \)
for \( t \geq 0, u \not\in T \) and \( u' \not\in 2 T \). Here \( T \) is a subset of the natural numbers \( N \):
\[
T = \{ n : 4 \mid n \text{ or } 4^{i+1} \mid (n - 2b_i - 2) \text{ for some } i > 0 \}
\]
for \( b_i = 4(4^{i-1} - 1)/3 \).
Lemma 4.7. The complement \( U = N - T \) is given as
\[
U = \{ n : 2 \times n \text{ or } n = 2 \cdot 4^k t + 6 \cdot 4^{k-1} + 2(4^{k-1} - 1)/3 \\
\text{for some } k > 0 \text{ and } t \geq 0 \}
\]

For the computation of \( \delta_1 \), we introduce other elements:

Lemma 4.8. Consider an element \( \varphi = v_5 + v_3 v_2^2 \). Then there exist elements \( H_{21} \) and \( H_{32} \) in \( \Sigma \) such that
\[
\begin{align*}
d_0(\varphi) &= H_{32} + t_3 + H_{21}, \\
d_1(H_{21}) &= 0 = d_1(H_{32}), \\
H_{21} &= t_2^2 \text{ and } H_{32} = t_3^4 \mod(v_1)
\end{align*}
\]
in the cobar complex \( \Omega^1 \mathcal{V}^{-1} \Lambda(2) \).

Proof. For an element \( \psi = v_3^2 + v_1 v_3 \), we compute \( d_0(\psi) = v_1 t_2^2 \) by \( \eta_R(v_3) = v_3 + v_1 t_2^2 + v_1^2 t_2 \) in \( BP_*[t_2, t_3, \ldots] \). Now put
\[
H_{32} = t_3^4 + v_1^2 \phi t_3^4.
\]
Then, the formula \( \Delta(t_3^2) = t_3^2 \otimes 1 + 1 \otimes t_3^2 + v_1 t_3^2 \otimes t_3^2 \) yields
\[
d_1(H_{32}) = 0 \text{ and } H_{32} = t_3^4 \mod(v_1).
\]
Furthermore, we compute
\[
d_0(\varphi) = t_3^4 + t_3 + v_3 t_2^2 \mod(v_1),
\]
and so
\[
d_0(\varphi) = H_{32} + t_3 + v_3 t_2^2 \mod(v_1).
\]
Put, then,
\[
H_{21} = d_0(\varphi) + H_{32} + t_3
\]
and we have
\[
d_1(H_{21}) = 0 \text{ and } H_{21} = v_3 t_2^2 \mod(v_1).
\]
q.e.d.

Lemma 4.9. For the connecting homomorphism \( \delta_1 : \text{Ext}^1(M) \rightarrow \text{Ext}^2(L) \), we have
\[
\begin{align*}
\delta_1(v_3^{4t+3} h_{21}/v_3^3) &= v_3^{4t+1} h_{21} h_{31}, \\
\delta_1(v_3^{4t+5} h_{21}/v_3^5) &= v_3^{4t+1} h_{21} h_{30}, \\
\delta_1(v_3^{4k+2+1} h_{21}/v_1^{1+2n}) &= v_3^{4k+1+2bn+1} h_{21}(v_3 h_{31} + v_3 h_{21}), \\
\delta_1(v_3^{4k+1+2t+1} h_{21}/v_3^{4n+1}) &= v_3^{4k+1+t+bn+1} h_{21} h_{30} \\
\delta_1(v_3^{2t+1} h_{30}/v_1) &= v_3^{2t} h_{21} h_{30},
\end{align*}
\]
\[
\delta_1(v_3^{4,t+2}h_{30}/v_1) = v_3^{4,t}h_{30}h_{31}, \\
\delta_1(v_3^{4(t+2)+b_{a+1}}h_{30}/v_1^{1+2a}) = v_3^{4(t+2)−2}h_{30}(h_{31} + v_3^{−1}h_{21}), \\
\delta_1(v_3^{4,t+1}h_{31}/v_1) = v_3^{4,t}h_{21}h_{31} \quad \text{and} \\
\delta_1(v_3^{4(2t+1)+b_{a+1}/2}h_{31}/v_1) = v_3^{4(2t+1)−2}h_{30}(h_{31} + v_3^{−1}h_{21}).
\]

Proof. The first four equations follow immediately from Lemmas 4.4 and 4.8 with replacing \(v_3h_{21}\) by \(H_{21}\). The fifth, sixth and eighth equations follow immediately from Lemmas 3.2 and 3.6. For the other equations, just put
\[
v_3^{4(t+2)+b_{a+1}}h_{30}/v_1^{1+2a} = v_3^{4(t+2)}d_0(x_{2a+2})/v_1^{1+2a+2a+1} \quad \text{and} \\
v_3^{4(2t+1)+b_{a+1}/2}h_{31}/v_1 = v_3^{4(2t+1)}d_0(x_{2a+1})/v_1^{a+1+2a},
\]
and we have the result by Lemma 3.6. \(\text{q.e.d.}\)

Now use Lemma 4.2, and we obtain

**Proposition 4.10.** Ext\(^1\)(\(M\)) is a direct sum of \(\rho\)Ext\(^0\)(\(M\)) and
\[
e^1(M) = \bigoplus_{i > 0}(C_i<v_3^{4i+1}h_{30}>) + C_C<v_3^{4t+2}h_{30}>) \\
\bigoplus_{i > 0}(C_i<v_3^{4i+t+1}h_{31}>) + C_C<v_3^{4t+2}h_{30}>)
\]

**Corollary 4.11.** The cokernel of \(\delta_1 : \text{Ext}^1(M) \to \text{Ext}^2(L)\) is a direct sum of \(\rho\)Coker \(\delta_0\) and a \(K(2)\)-module generated by
\[
v_3^{3t+1}h_{30}h_{31}, \quad v_3^{2a+1}h_{21}h_{31} \quad \text{and} \quad v_3^{2u+1}h_{21}h_{30}
\]
for \(t \geq 0, 2u \notin T\) and \(u' \notin 2T\).

**Lemma 4.12.** For the connecting homomorphism \(\delta_2 : \text{Ext}^1(M) \to \text{Ext}^2(L)\), we have
\[
\delta_2(v_3^{4i+1}h_{30}h_{31}/v_1) = v_3^{4,i}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4i+3}h_{31}/v_1) = v_3^{4i+1}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4(4t+2)+b_{a+1}}h_{21}h_{30}/v_1^{1+2a}) = v_3^{4(4t+2)−2}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4(2t+1)+b_{a+1}/2}h_{31}/v_1) = v_3^{4(2t+1)−1}h_{21}h_{30}h_{31}.
\]

Proof. Note that \(\delta_2(v_3^{4i+1}h_{30}h_{31}/v_1) = \delta_0(v_3^{4i+1}/v_1)h_{30}h_{31}\) since \(h_{3i} = c_3\)'s are cocycles by Lemma 3.2. Now the first equation follows from Lemmas 4.4 and 4.9. For the other equations, use Lemmas 4.8 and 4.9 since \(\delta_2(v_3^{4i+1}h_{21}h_{30}/v_1) = \delta_1(v_3^{4i+1}h_{31}/v_1)v_3h_{21}\) if we use the representative \(H_{21}\) for the cohomology class \(v_3h_{21}\).

Again by Lemma 4.2, we obtain \(\text{q.e.d.}\)

**Proposition 4.13.** Ext\(^2\)(\(M\)) is a direct sum of \(\rho e^1(M)\) and
\[
e^2(M) = \bigoplus_{i > 0}(C_i<v_3^{4i(t+2)+b_{a+1}+1}h_{21}h_{30}>) \\
\bigoplus_{i > 0}(C_C<v_3^{4i(2t+1)+b_{a+1}/2}h_{21}h_{31}>) + C_C<v_3^{4t+1}h_{30}h_{31}>)
\]
Corollary 4.14. The cokernel of $\delta_2 : \text{Ext}^2(M) \to \text{Ext}^3(L)$ is a $K(2)_*$-module $\rho \text{Coker} \delta_1$.

Now the following proposition follows immediately, by the same argument as above.

Proposition 4.15. For $n > 3$, $\text{Ext}^n(M) = 0$, and $\text{Ext}^3(M) = \rho e^3(M)$.

5. On the map $j_\#: E_2(X) \to E_2(C)$

As is stated in the introduction, $C$ denotes the cofiber of $X_2 \to L_2 X_2$. Then it is an $X_2$-module spectrum and $h_{20} \in \pi_0(X_2)$ induces a map $h_{20} : C \to C$. In fact, it is the composition

$$C = S^0 \vee C \xrightarrow{h_{20} \vee C} X_2 \vee C \xrightarrow{\nu} C,$$

in which $\nu$ denotes the $X_2$-module structure. Then we have a cofiber sequence

$$\Sigma^g C \xrightarrow{h_{20}} C \xrightarrow{i} X \xrightarrow{j} \Sigma^0 C.$$

Let $E_r^s(Y)$ denote the $E_r$-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_2 Y)$ for a spectrum $Y$, and $d^{AN}_r$, its differentials. Then this gives rise to the exact sequence

$$0 \to E_2^{0,t}(C) \xrightarrow{j_*} E_2^{0,t}(X) \xrightarrow{j_*} E_2^{0,t-b}(C) \xrightarrow{\delta} E_2^{1,t}(C) \to \cdots.$$

Here $E_2^{0,t}(X) = \text{Ext}^{s+t}(M)$, whose structure is given in the previous section. We further consider a cofiber $E$ of $h_{20} : C \to C$. Then we have a commutative diagram

$$\begin{array}{cccccc}
C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C \\
\downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 & & \\
C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\Sigma D & \xrightarrow{h_{20}} & \Sigma D & \xrightarrow{i} & \Sigma E & \xrightarrow{j} & \Sigma^2 D \\
\end{array}$$

(5.1)

in which rows and columns are cofibrations.

Lemma 5.2. Let $v_3^4/v_1^A$ denote a generator of $E_2(X)$ as a $\mathbb{Z}/2[v_1, v_2, v_3^2]$-module. Then

$$j_*(v_3^4/v_1^A) = 0.$$

Proof. If $t = 2^n(2s + 1)$ for some $n, s \geq 0$, then $v_3^4/v_1^A$ is a homology class represented by $x_n^{2s+1}/v_1^{2n}$. For $n = 0$, the lemma is trivial. Now suppose that $j_*(x_n^{2s+1}/v_1^{2n}) = 0$ for even $n = 2m$. Then squaring this, we obtain

$$j_*(x_n^{2s+1}/v_1^{2n+1}) = v_3^w/v_1.$$
for some \( w \geq 0 \). Consider the diagram

\[
\begin{array}{ccc}
E_2^2(X) & \xrightarrow{j_*} & E_2^2(C) \\
\downarrow{i_*} & & \downarrow{j_*}
\end{array}
\]

\[
E_2^3(D) \xrightarrow{i_*} E_2^3(E) \xrightarrow{j_*} E_2^3(D)
\]

induced from (5.1). Since \( \delta(x_{n+1}/v_1) \) is in the image of \( i_* \) by Lemma 4.4, \( \delta(v_3/v_1) = 0 \) in \( E_2^3(D) \) by the above diagram, and so \( 2|w \) since \( \delta(v_3/v_1) = uv_3v_1^{w-1}h_2 \) by Landweber’s formula \( d_0(v_3) = v_1t_2 + v_4t_2 \) in \( BP_*(t_2, t_3, \cdots) \). Thus we have

\[
j_*(x_{n+1}/v_1) = v_3^u/v_1.
\]

Square this, and we have

\[
j_*(x_{n+2}/v_1) = v_3^u/v_1.
\]

Notice that \( j_*(x) = y \) if \( d_0(x) = \eta(x) - x \). A direct computation shows us \( d_0(v_3^u x/v_1) = v_3^u t_2/v_1^2 \) in the cobar complex \( \Omega_2^2 M \). Thus we have shown inductively that \( j_*(v_3^{2n+1}/v_1^{4n+2}) \) equals to 0 if \( n \) is even, and to \( v_3^2/v_1 \) for some \( u \) if \( n \) is odd. q.e.d.

### 6. The Adams-Novikov differential

Consider the cofiber \( E = h_{30} : \Sigma^5 D \to D \). Then by [7, Th. 7.1], we immediately obtain the following

**Proposition 6.1.** The Adams-Novikov spectral sequence for computing \( \pi_*(L_2E) \) collapses from the \( E_2 \)-term.

Note that the \( E_2 \)-term for our \( X \) is

\[
E_2^*(X) = \text{Ext}^*(A, v_3^{-1}BP_*(X)) = \text{Ext}^*(M).
\]

**Lemma 6.2.** For the Adams-Novikov differential \( d^{AN}_k : E_2^k(X) \to E_3^k(X) \), \( d^{AN}_k(v_3/v_1^k) \) is a sum of the elements of the form \( v_3^{2u+1}h_2h_3^k/v_1^i \) for \( i = 0, 1 \) and \( k > 1 \). Here \( v_3/v_1 \) is a generator of the \( \mathbb{Z}/2[v_1, v_2, v_3^{-1}] \)-module \( M_0 \).

Proof. Consider the diagram (5.1). The third column induces the long exact sequence

\[
\cdots \to \text{Ext}^3(M) \xrightarrow{a} \text{Ext}^4(M) \xrightarrow{b} \text{Ext}^4(L) \to \cdots
\]

of the \( E_2 \)-terms. If the \( \delta_0 \) image of \( v_3/v_1 \) is \( x \neq 0 \), then \( \delta_0(d^{AN}_3(v_3/v_1^i)) = d^{AN}_4(x) = 0 \) by Proposition 6.1. Thus \( d^{AN}_3(v_3/v_1^i) \) is divisible by \( v_1 \). Furthermore it implies that \( v_3^{2u+1}h_2h_3^k/v_1 \) cannot be a target of \( d^{AN}_3 \). In fact, it is not divisible by \( v_1 \) by Proposition 4.15. Now the lemma follows from Lemma 4.15. q.e.d.
Theorem 6.3. The Adams-Novikov spectral sequence for computing $$\pi_*(L_2X)$$ collapses from the $$E_2$$-term.

Proof. By proposition 4.15, the Adams-Novikov differentials are all trivial except for $$d_3^{AN}: E_3^2(X) \to E_3^3(X)$$. So it is sufficient to show that $$d_3^{AN}(v^i/v_1) = 0$$ for each $$v^i/v_1 \in E_2^0(X)$$. By Lemma 6.2,

$$d_3^{AN}(v^i/v_1) = \sum_{u,t} \lambda_{u,t} v_3^{2u+1} h_{3t} h_{3i} / v_1^2$$

for some $$k \geq 0$$, where $$\lambda_{u,t} \in \mathbb{Z}/2$$. Since

$$d_3(v_3^{2u+1} h_{3t} h_{3i} / v_1^2) = v_3^{2u} h_2 h_{3t} h_{3i} / v_1 \neq 0$$

in the cobar complex $$\Omega^*BP_*(C)$$, we see that

$$j_*(v_3^{2u+1} h_{3t} h_{3i} / v_1^2) = \sum_{u,t} \lambda_{u,t} v_3^{2u} h_2 h_{3t} h_{3i} / v_1 \neq 0.$$ (6.5)

Now send (6.4) by $$j_*$$ and we have a contradiction to Lemma 5.2, which says $$j_*(v^i/v_1^{i+k}) = 0$$ if $$k > 0$$. If $$k = 0$$ and $$j_*(v^i/v_1^i) \neq 0$$, then

$$j_*(v^i/v_1^i) = v_3^u / v_1$$

for some $$u \geq 0$$ as is seen in the proof of Lemma 5.2. Therefore, (6.4) and (6.5) yield

$$d_3^{AN}(v_3^u / v_1) = \sum_{u,t} \lambda_{u,t} v_3^{2u} h_2 h_{3t} h_{3i} / v_1 \neq 0$$

in $$E_3^u(C)$$ for some $$\lambda_{u,t} \in \mathbb{Z}/2$$. Now pull this back to $$E_3^u(D)$$ under the map $$i_*: E_3^u(D) \to E_3^u(C)$$ to obtain the non-trivial differential

$$d_3^{AN}(v_3^{2u}) = \sum_{u,t} \lambda_{u,t} v_3^{2u} h_2 h_{3t} h_{3i} \neq 0$$

in $$E_3^u(D)$$, which again contradicts to a result of [7] which says $$d_3^{AN}(v_3^k) = 0$$ and $$d_3^{AN}(v_3^{k+2}) = v_3^{k} h_2^3$$ for $$k > 0$$. q.e.d.

References

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