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ON A GENERALIZATION OF THE RING THEORY

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1. Introduction. A ring of endomorphisms of a module plays a very important role in many parts of mathematics; the property of a ring itself is also clarified when we consider it as a ring of endomorphisms of a module. As a generalization of this idea, we can consider a set of homomorphisms of a module to another module which is closed under the addition and subtraction defined naturally but has no more a structure of a ring since we can not define the product. However, suppose that we have an additive group M consisting of homomorphisms of a module A to a module B and that we have also an additive group N consisting of homomorphisms of B to A . In this case we can define the product of three elements f_1 , g and f_2 where f_1 and f_2 are elements of M and g is an element of N . If this product $f_1 g f_2$ is also an element of M for every f_1 , g and f_2 , we say that M is closed under the multiplication using N between. Similarly we can define that N is closed under the multiplication using M between. Take f_1 , f_2 and f_3 in M and g_1 and g_2 in N in the above case. Then we have

$$(f_1 g_1 f_2) g_2 f_3 = f_1 g_1 (f_2 g_2 f_3) = f_1 (g_1 f_2 g_2) f_3.$$

When we define this situation abstractly, we can get a new algebraic system.

DEFINITION. Let M be an additive group whose elements are denoted by a, b, c, \dots , and Γ another additive group whose elements are $\gamma, \beta, \alpha, \dots$. Suppose that $a\gamma b$ is defined to be an element of M and that $\gamma a \beta$ is defined to be an element of Γ for every a, b, γ and β . If the products satisfy the following three conditions:

$$\begin{aligned} & (a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b, \\ 1) \quad & a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b, \\ & a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2, \end{aligned}$$

$$2) \quad (a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c,$$

3) if $a\gamma b=0$ for any a and b in M , then $\gamma=0$,
then M is called a Γ -ring

The purpose of this note is to determine the structure of Γ -rings under the following conditions which are called semi-simple and simple according to the usual ring theory.

DEFINITION. Let M be a Γ -ring as above. If for any non-zero element a of M there exists such an element γ (depending on a) in Γ that $a\gamma a \neq 0$, we say that M is *semi-simple*. If for any non-zero elements a and b of M there exists γ (depending on a and b) in Γ such that $a\gamma b \neq 0$, we say that M is *simple*.

The main result obtained in this note is that a simple Γ -ring which satisfies the chain condition for left and right ideals (defined in § 3) is the set $D_{n,m}$ of all rectangular matrices of type $n \times m$ over some division ring D and Γ is $D_{m,n}$ of type $m \times n$. The product $a\gamma b$ is the same as the usual matrix product of elements a , γ and b of $D_{n,m}$, $D_{m,n}$ and $D_{n,m}$. This is a generalization of the theorem of Wedderburn on simple rings. Subsequently, a semi-simple Γ -ring satisfying the chain condition for left and right ideals will be shown to be a direct sum of simple Γ_i -rings, where $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$ (direct).

2. Examples. Suppose we have a right R -module M with an operator ring R . Take a submodule Γ of $\text{Hom}_R(M, R)$. Then M is a Γ -ring as follows: If a and b are elements of M and if γ is an element of Γ , then we define

$$a\gamma b = a \cdot \gamma(b),$$

where $\gamma(b)$ is an image of b by γ and is an element of R . It is easy to verify that

$$(a\gamma b)\beta c = (a \cdot \gamma(b)) \cdot \beta(c) = a(\gamma(b)\beta(c)) = a \cdot \gamma(b \cdot \beta(c)) = a\gamma(b\beta c).$$

We also define that

$$\gamma b\beta = \beta \cdot \gamma(b)_l \quad (\beta \text{ operating first}),$$

where $\gamma(b)_l$ means the left multiplication of $\gamma(b)$. Then

$$(a\gamma b)\beta c = a(\gamma(b)\beta(c)) = a(\gamma b\beta)c.$$

The conditions 1) and 3) hold naturally and M is a Γ -ring. But it will be shown in § 3 that every Γ -ring is given in this way.

To illustrate further this new algebraic system, we introduce the

definition and examples of cubic rings.

DEFINITION. We call that M is a *cubic ring* when we can define the product of three elements of M which is an additive group such that it satisfies

- $$\begin{aligned} & (a_1 + a_2)bc = a_1bc + a_2bc, \\ 4) & \quad a(b_1 + b_2)c = ab_1c + ab_2c, \\ & \quad ab(c_1 + c_2) = abc_1 + abc_2, \\ 5) & \quad ab(cde) = (abc)de, \\ 6) & \quad \text{if } abc=0 \text{ for all } a \text{ and } c, \text{ then } b=0. \end{aligned}$$

If we take the product in a cubic ring M as the product of two elements of M using one element of $\Gamma=M$ between, then conditions 1) and 3) for a Γ -ring are satisfied. Also the first part of 2) is satisfied. Hence, in order that M is a Γ -ring, we must be able to define the product $\Gamma \times M \times \Gamma$ such that the latter part of 2) holds. In the following examples, we can find it easily.

EXAMPLE 1. Let $V_n(F)$ be a vector space of dim n over a field F . If a, b and c are vectors in it, we define $abc=(a \cdot b)c$, where $(a \cdot b)$ is the inner product of a and b . It is easy to see that $V_n(F)$ is a cubic ring. Now we define $(bcd)'=b(c \cdot d)$. Then $ab(cde)=(a \cdot b)(c \cdot d)e=a(bcd)'e$, i.e., $V_n(F)$ is a Γ -ring with $\Gamma=V_n(F)$.

EXAMPLE 2. Let $D_{n,m}$ be the set of all rectangular matrices of type $n \times m$ over a division ring D . If a, b and c are elements in it, we define $abc=ab^t c$, where b^t is the transpose of a matrix b and the above product is well-defined. Then $D_{n,m}$ is clearly a cubic ring. Now we define $(bcd)'=dc^t b$. Then $ab(cde)=ab^t cd^t e=a(bcd)'e$, i.e., $D_{n,m}$ is a Γ -ring with $\Gamma=D_{n,m}$.

EXAMPLE 3. Let I be the set of all purely imaginary complex numbers. Then it is a cubic ring with the usual multiplication. Also it is a Γ -ring with $\Gamma=I$. However, even with the same I , we can define another cubic ring. For example, if a, b and c are elements in I , we define the product of a, b and c as $a\bar{b}c$ where \bar{b} is the conjugate of b , i.e., $-b$. This product also satisfies 4), 5) and 6) of the definition of cubic rings. In this case, we put $(bcd)'=-bcd$.

3. The operator rings and ideals. Let M be a Γ -ring. Consider the additive group generated by pairs (γ, a) , where $\gamma \in \Gamma$ and $a \in M$ with defining relations $(\gamma_1 + \gamma_2, a) = (\gamma_1, a) + (\gamma_2, a)$ and $(\gamma, a_1 + a_2) = (\gamma, a_1) + (\gamma, a_2)$. We define the multiplication of the elements of this additive group such that

$$(\gamma, a)(\beta, b) = (\gamma, a\beta b).$$

Using the condition 2), we can verify that

$$((\gamma, a)(\beta, b))(\alpha, c) = (\gamma, a)((\beta, b)(\alpha, c)).$$

Thus we get a ring which we denote by F . Now we can see that F is a right operator ring of M by the following definition:

$$a(\gamma, b) = a\gamma b,$$

for, we have

$$(a(\gamma, b))(\beta, c) = (a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma, b\beta c) = a((\gamma, b)(\beta, c)).$$

The set of all elements of F that annihilate M forms an ideal which we denote by A , and we denote F/A by R and call it *the right operator ring of M* . We use γa for an element of R which is gained from (γ, a) . Thus $a\gamma b = a(\gamma b)$. Then, take an element γ of Γ . It induces an R -homomorphism of M to R such that $\gamma(a) = \gamma a$. The condition 3) implies that Γ induces the zero homomorphism if and only if $\gamma = 0$. Thus Γ is considered to be a subset of the total set of R -homomorphisms of M to R ; $\Gamma \subset \text{Hom}_R(M, R)$.

Similarly we can define *the left operator ring L of M* . We start with (a, γ) and define the product such that $(a, \gamma)(b, \beta) = (a\gamma b, \beta)$. Also we define the left operation such that $(a, \gamma)b = a\gamma b$, and so on. $a\gamma$ is an element of L given from (a, γ) and $a\gamma b = (a\gamma)b$. And we can say that $\Gamma \subset \text{Hom}_L(M, L)$.

DEFINITION. R -submodules of M are called *right ideals* of M , and L -submodules of M are *left ideals*.

A right ideal \mathfrak{r} is nothing but a submodule of M such that $\mathfrak{r}\Gamma M \subset \mathfrak{r}$. A left ideal \mathfrak{l} is a submodule of M such that $M\Gamma \mathfrak{l} \subset \mathfrak{l}$.

4. Peirce decomposition in semi-simple Γ -rings. Assume that M is semi-simple, and let \mathfrak{r} be a minimal right ideal. Then by semi-simplicity there exists an element ε in Γ such that $a\varepsilon a \neq 0$ for a non-zero element a in \mathfrak{r} . Then $0 \neq a\varepsilon \mathfrak{r} \subset \mathfrak{r}$ and hence $\mathfrak{r} = a\varepsilon \mathfrak{r}$, for \mathfrak{r} is minimal. Therefore $a = a\varepsilon e$ with some element e of \mathfrak{r} . Then $e = e\varepsilon e$, since from $a = a\varepsilon e = (a\varepsilon e)\varepsilon e$

we have $a\varepsilon(e - e\varepsilon e) = 0$ which means $e - e\varepsilon e = 0$, for a set $\{c | a\varepsilon c = 0, c \in \mathfrak{r}\}$ is a right ideal contained in a minimal ideal \mathfrak{r} and is $\{0\}$. Since $e \in \mathfrak{r}$, $eR \subset \mathfrak{r}$, i.e., $eR = \mathfrak{r}$. εM being a right ideal of R , $e\varepsilon M$ is a right ideal of M contained in \mathfrak{r} , and hence $e\varepsilon M = \mathfrak{r}$. Thus we get

Lemma 1. *If M is semi-simple and \mathfrak{r} is a minimal right ideal, then $\mathfrak{r} = eR = e\varepsilon M$ with $e \in \mathfrak{r}$ and $\varepsilon \in \Gamma$, where $e\varepsilon e = e$.*

Now we use the idea of Peirce decomposition of the ring theory. Suppose that we have a right ideal $\mathfrak{r} = e\varepsilon M$ such that $e\varepsilon e = e$. Then

$$M = e\varepsilon M + M_1 \quad (\text{direct}),$$

where $M_1 = \{b | e\varepsilon b = 0\}$, since any element a of M is written

$$a = e\varepsilon a + (a - e\varepsilon a),$$

and $e\varepsilon(a - e\varepsilon a) = 0$. M_1 is clearly a right ideal of M . Now we can get a decomposition theorem.

Theorem 1. *If M is semi-simple and satisfies the minimum condition for right ideals, then*

$$M = e_1R + e_2R + \dots + e_nR \quad (\text{direct}),$$

where e_iR are minimal right ideals and $e_iR = e_i\varepsilon_iM$, and $e_i\varepsilon_i e_i = e_i$ and $e_i\varepsilon_i e_j = 0$ if $i \neq j$.

Proof. Suppose that we have

$$M = e_1\varepsilon_1M + \dots + e_{k-1}\varepsilon_{k-1}M + M_{k-1} \quad (\text{direct})$$

such that $e_i\varepsilon_iM$ are minimal right ideals and

$$e_i\varepsilon_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and that $e_i\varepsilon_i a = 0$ if $a \in M_{k-1}$ for $i = 1, 2, \dots, k-1$. This is true for $k=2$ as above. Apply the above discussion on M_{k-1} , and we get

$$M_{k-1} = e_k\varepsilon'_kM + M_k \quad (\text{direct})$$

as in the above. Here $e_i\varepsilon_i e_k = 0$ if $i < k$, but we can not say that $e_k\varepsilon'_k e_i = 0$. So, we change ε'_k suitably. Put

$$\varepsilon_k = \varepsilon'_k - \varepsilon'_k(e_1\varepsilon_1 + \dots + e_{k-1}\varepsilon_{k-1}).$$

Then we can see that $e_k\varepsilon_k e_k = e_k$ and $e_k\varepsilon_k e_i = 0$. Thus we have a decomposition for k . Since M satisfies the minimum condition for right ideals, we

can get the decomposition in Theorem 1.

Similarly we can get

Theorem 1'. *If M is semi-simple and satisfies the minimum condition for left ideals, then*

$$M = Ld_1 + Ld_2 + \cdots + Ld_m \quad (\text{direct}),$$

where Ld_i are minimal left ideals and $Ld_i = M\delta_i d_i$, and $d_i \delta_i d_i = d_i$ and $d_j \delta_i d_i = 0$ if $i \neq j$.

5. Simple Γ -rings. Assume M is simple and satisfies the minimum condition for right and left ideals in this section. First we want to show that $e_i R$ and $e_j R$ are isomorphic as R -modules. M being simple, we can find an element γ in Γ such that $e_i \gamma e_j \neq 0$. Then $e_i \gamma r_j = r_i$ where r_i and r_j are $e_i R$ and $e_j R$. By a correspondence:

$$(r_j \ni) x \longrightarrow e_i \gamma x (\in r_i)$$

we have a one-one mapping of r_j onto r_i . If $x \neq 0$, $e_i \gamma x \neq 0$, because $\{c | e_i \gamma c = 0, c \in r_j\}$ is a right ideal contained in r_j and is $\{0\}$ as r_j is minimal. This mapping is "onto" because r_i is minimal. Since $x(\beta c) = x\beta c$ corresponds to $e_i \gamma (x\beta c) = (e_i \gamma x)(\beta c)$, this mapping is an R -homomorphism, i.e., an R -isomorphism. Similarly $Ld_i \cong Ld_j$ (L -isomorphic). Next, we want to show that all L -endomorphisms of M are given by the right multiplication of R . Let ϕ be an L -endomorphism of M and put $\phi(d_i) = u_i$. Since $d_i = d_i \delta_i d_i$, $u_i = d_i \delta_i u_i$. Therefore, $u_i = d_i (\sum_j \delta_j d_j \delta_j u_j)$ where $\sum_j \delta_j d_j \delta_j u_j$ is an element of R . On the other hand, by the definition of the right operator ring, R is considered to be the set of all L -endomorphisms of M . Then the ring theory shows us that the latter ring is a matrix ring D_m over a division ring D , where D_m is $D_{m,m}$. Matrix units $E_{r,s}$ of D_m map d_r to d_s and d_t to 0 if $t \neq r$.

Now we can determine M with respect to R which is identified with D_m as above. Since minimal right ideals of D_m are $E_{r,r} D_m$, $e_i D_m (= e_i R$ in Theorem 2) $= e_i E_{r,r} D_m$ with some r . Then put $e_i E_{r,s} = e_{i,s}$. We get $e_{i,s}$ ($i=1, 2, \dots, n$; $s=1, 2, \dots, m$) such that

$$e_{i,s} E_{r,t} = \begin{cases} e_{i,t} & s = r, \\ 0 & s \neq r. \end{cases}$$

Thus we can say that $M = \sum_{i,s} e_{i,s} D$, i.e., $e_{i,s}$ are matrix units of $D_{n,m}$ and M is (isomorphic to) $D_{n,m}$ as a right D_m -module.

Next we must determine Γ . An element γ of Γ is considered to

induce a mapping from M to R as in § 3, and Γ is considered to be a subset of the set of all R -homomorphisms of $M=D_{n,m}$ to $R=D_m$. On the other hand, D_m -homomorphisms of $D_{n,m}$ to D_m are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose ϕ is a D_m -homomorphism of $D_{n,m}$ to D_m such that

$$\phi(e_{i,s}) = \sum_{p,q} E_{p,q} T_{p,q}(i, s)$$

with $T_{p,q}(i, s)$ in D . Multiply $E_{s,s}$, and we can see $T_{p,q}(i, s)=0$ if $q \neq s$. Multiply $E_{s,t}$, and we can see $T_{p,s}(i, s)=T_{p,t}(i, t)$. Putting $T_{p,s}(i, s)=T_p(i)$, we have

$$\phi(e_{i,s}) = \sum_p E_{p,s} T_p(i) = (\sum_{p,j} e'_{p,j} T_p(j)) e_{i,s},$$

where $e'_{p,i}$ are matrix units of $D_{m,n}$ such that

$$e'_{p,j} e_{i,s} = \begin{cases} E_{p,s} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Hence ϕ is induced by the left multiplication of an element $A = \sum e'_{p,j} T_p(j)$ of $D_{m,n}$. Identifying γ which induces ϕ and A which corresponds to ϕ , we can say that $\Gamma \subset D_{m,n}$. What we want to show is that $\Gamma = D_{m,n}$. But Γ is a two sided D_m - D_n module and must be identical with $D_{m,n}$. Summarizing all the discussions, we get the main theorem.

Theorem 2. *If M is a simple Γ -ring satisfying the minimum condition for left and right ideals, then M is $D_{n,m}$ and Γ is $D_{m,n}$. The product ayb is the usual matrix product of three elements a , γ and b of $D_{n,m}$, $D_{m,n}$ and $D_{n,m}$.*

6. Semi-simple Γ -rings. Let M be a semi-simple Γ -ring which satisfies the minimum condition for left and right ideals in this section. Arranging suitably, we can see that M is expressed as follows:

$$M = Ld_1^{(1)} + \dots + Ld_{m(1)}^{(1)} + Ld_1^{(2)} + \dots + Ld_{m(2)}^{(2)} + \dots + Ld_1^{(q)} + \dots + Ld_{m(q)}^{(q)},$$

where $d_i^{(j)}$ are some d_k of Theorem 1'. Moreover we take the order such that in the above $Ld_i^{(j)} \cong Ld_k^{(j)}$ (L -isomorphic) and $Ld_i^{(j)} \not\cong Ld_k^{(j')}$ if $j \neq j'$. Then, R is, as the right multiplication ring of the L -module M , equal to a direct ring sum $\sum_j D_{m(j)}^{(j)}$, where $D_{m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $m(j) \times m(j)$. Furthermore $D_{m(j)}^{(j)}$ operate on $Ld_i^{(j)}$ as usual and are zero on $Ld_i^{(j')}$ if $j \neq j'$. On the other hand, we have in Theorem 1

$$M = e_1R + e_2R + \dots + e_nR.$$

e_iR being minimal, $e_iR = e_i D_{m(j)}^{(j)}$ with some j . Rearranging the order suitably, we have

$$M = e_1^{(1)}R + \dots + e_{n(1)}^{(1)}R + e_1^{(2)}R + \dots + e_{n(2)}^{(2)}R + \dots + e_1^{(q)}R + \dots + e_{n(q)}^{(q)}R,$$

where $e_i^{(j)}R = e_i^{(j)} D_{m(j)}^{(j)}$ and $e_i^{(j)}$ are some e_n . Hence $n = \sum_j n(j)$. With the same discussion as in §5, we can say that

$$\begin{aligned} e_1^{(1)}R + \dots + e_{n(1)}^{(1)}R &= D_{n(1), m(1)}^{(1)}, \\ e_1^{(2)}R + \dots + e_{n(2)}^{(2)}R &= D_{n(2), m(2)}, \\ &\dots\dots\dots \\ e_1^{(q)}R + \dots + e_{n(q)}^{(q)}R &= D_{n(q), m(q)}, \end{aligned}$$

i.e.,

$$M = D_{n(1), m(1)}^{(1)} + \dots + D_{n(q), m(q)}^{(q)},$$

where $D_{n(j), m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $n(j) \times m(j)$. Naturally $D_{m(j)}^{(j)}$ operate on $D_{n(j), m(j)}^{(j)}$ as usual and are zero on $D_{n(j'), m(j')}^{(j')}$ if $j \neq j'$. Γ is then a set of R -homomorphisms of M to R and is contained in $\sum_j D_{m(j), n(j)}^{(j)}$. Here the product of elements of $D_{m(j), n(j)}^{(j)}$ and of $D_{n(j'), m(j')}^{(j')}$ is performed as usual if $j = j'$ and is 0 if $j \neq j'$. On the other hand, the condition of semi-simplicity means that for any non-zero element a of $D_{n(j), m(j)}^{(j)}$ there exists γ in Γ such that $a\gamma a \neq 0$. Now we want to show that each $D_{n(j), m(j)}^{(j)}$ is a simple Γ_j -ring. Let V and V' be left $D^{(j)}$ -modules of dim $n(j)$ and of dim $m(j)$. $D_{n(j), m(j)}^{(j)}$ and $D_{m(j), n(j)}^{(j)}$ are considered to be the sets of all $D^{(j)}$ -homomorphisms of V to V' and of V' to V . When we notice that $D_{m(j'), n(j')}^{(j')}$ induce zero mapping on V' if $j \neq j'$, we can say that elements of Γ induce mappings of V' to V . In this case we can show that $X\Gamma = V$ for any subspace X of dim 1 of V' . For, suppose that $X\Gamma \subsetneq V$. Then we can find an element a in $D_{n(j), m(j)}^{(j)}$ such that $Va = X$ and $(X\Gamma)a = 0$. Then $a\gamma a = 0$ for every γ in Γ , which is a contradiction. Now this fact implies the existence of γ such that $a\gamma b \neq 0$ for any non-zero a and b , for we can take a subspace of Va as X and take γ such that $(X\gamma)b \neq 0$. Thus we can conclude that $\Gamma = \sum_j D_{m(j), n(j)}^{(j)}$. Now put $\Gamma_j = D_{m(j), n(j)}^{(j)}$.

Theorem 3. *If M is a semi-simple Γ -ring satisfying the minimum condition for left and right ideals, then M is a direct sum of simple Γ_i -rings where $\Gamma = \Gamma_1 + \dots + \Gamma_q$ (direct):*

$$M = M_1 + M_2 + \cdots + M_q \quad (\text{direct}),$$

where M_i are simple Γ_i -rings and $M_i\Gamma_j = 0$ if $i \neq j$, and $M_i\Gamma_j M_i = 0$ if $i \neq j$.

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