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Osaka University
ON A GENERALIZATION OF THE RING THEORY

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(Received May 20, 1964)

1. Introduction. A ring of endomorphisms of a module plays a
very important role in many parts of mathematics; the property of a
ring itself is also clarified when we consider it as a ring of endomor-
phisms of a module. As a generalization of this idea, we can consider
a set of homomorphisms of a module to another module which is closed
under the addition and subtraction defined naturally but has no more a
structure of a ring since we can not define the product. However, sup-
pose that we have an additive group $M$ consisting of homomorphisms of
a module $A$ to a module $B$ and that we have also an additive group $N$
consisting of homomorphisms of $B$ to $A$. In this case we can define the
product of three elements $f_1$, $g$ and $f_2$ where $f_1$ and $f_2$ are elements of
$M$ and $g$ is an element of $N$. If this product $f_1gf_2$ is also an element
of $M$ for every $f_1$, $g$ and $f_2$, we say that $M$ is closed under the mul-
tiplication using $N$ between. Similarly we can define that $N$ is closed
under the multiplication using $M$ between. Take $f_1$, $f_2$ and $f_3$ in $M$ and
g_1$ and $g_2$ in $N$ in the above case. Then we have

$$(f_1g_1f_1)g_2f_3 = f_1g_1(f_2g_2f_3) = f_1(g_1f_2g_2)f_3.$$

When we define this situation abstractly, we can get a new algebraic
system.

DEFINITION. Let $M$ be an additive group whose elements are denoted
by $a, b, c, \ldots$, and $\Gamma$ another additive group whose elements are $\gamma, \beta, \alpha, \ldots$.
Suppose that $a\gamma b$ is defined to be an element of $M$ and that $\gamma a\beta$ is de-
defined to be an element of $\Gamma$ for every $a, b, \gamma$ and $\beta$. If the products
satisfy the following three conditions:

1 $$(a_i + a_2)\gamma b = a_i\gamma b + a_2\gamma b,$$

2 $a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b,$$

3 $a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2,$$
2) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$,

3) if $a\gamma b = 0$ for any $a$ and $b$ in $M$, then $\gamma = 0$,
then $M$ is called a Γ-ring

The purpose of this note is to determine the structure of Γ-rings under the following conditions which are called semi-simple and simple according to the usual ring theory.

**Definition.** Let $M$ be a Γ-ring as above. If for any non-zero element $a$ of $M$ there exists such an element $\gamma$ (depending on $a$) in Γ that $a\gamma a = 0$, we say that $M$ is semi-simple. If for any non-zero elements $a$ and $b$ of $M$ there exists $\gamma$ (depending on $a$ and $b$) in Γ such that $a\gamma b = 0$, we say that $M$ is simple.

The main result obtained in this note is that a simple Γ-ring which satisfies the chain condition for left and right ideals (defined in §3) is the set $D_{nm}$ of all rectangular matrices of type $n \times m$ over some division ring $D$ and Γ is $D_{m,n}$ of type $m \times n$. The product $a\gamma b$ is the same as the usual matrix product of elements $a$, $\gamma$ and $b$ of $D_{n,m}$, $D_{m,n}$ and $D_{s,m}$. This is a generalization of the theorem of Wedderburn on simple rings. Subsequently, a semi-simple Γ-ring satisfying the chain condition for left and right ideals will be shown to be a direct sum of simple Γ-rings, where $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ (direct).

2. **Examples.** Suppose we have a right $R$-module $M$ with an operator ring $R$. Take a submodule $\Gamma$ of $\text{Hom}_R(M, R)$. Then $M$ is a Γ-ring as follows: If $a$ and $b$ are elements of $M$ and if $\gamma$ is an element of $\Gamma$, then we define

$$a\gamma b = a \cdot \gamma(b),$$

where $\gamma(b)$ is an image of $b$ by $\gamma$ and is an element of $R$. It is easy to verify that

$$(a\gamma b)\beta c = (a \cdot \gamma(b)) \cdot \beta(c) = a(\gamma(b)\beta(c)) = a \cdot \gamma(b \cdot \beta(c)) = a\gamma(b\beta)c.$$

We also define that

$$\gamma b\beta = \beta \cdot \gamma(b),$$

(\beta \text{ operating first}),

where $\gamma(b)$ means the left multiplication of $\gamma(b)$. Then

$$(a\gamma b)\beta c = a(\gamma(b)\beta(c)) = a(\gamma b\beta)c.$$

The conditions 1) and 3) hold naturally and $M$ is a Γ-ring. But it will be shown in §3 that every Γ-ring is given in this way.

To illustrate further this new algebraic system, we introduce the
DEFINITION. We call that $M$ is a cubic ring when we can define the product of three elements of $M$ which is an additive group such that it satisfies

\[(a_1 + a_2)b = a_1b + a_2b,
4)\]
\[a(b_1 + b_2)c = ab_1c + ab_2c,
5)\]
\[ab(c_1 + c_2) = abc_1 + abc_2,
6)\]
\[ab(cde) = (abc)de,
\]
\[\text{if } abc = 0 \text{ for all } a \text{ and } c, \text{ then } b = 0.\]

If we take the product in a cubic ring $M$ as the product of two elements of $M$ using one element of $\Gamma = M$ between, then conditions 1) and 3) for a $\Gamma$-ring are satisfied. Also the first part of 2) is satisfied. Hence, in order that $M$ is a $\Gamma$-ring, we must be able to define the product $\Gamma \times M \times \Gamma$ such that the latter part of 2) holds. In the following examples, we can find it easily.

**Example 1.** Let $V_n(F)$ be a vector space of dim $n$ over a field $F$. If $a$, $b$, and $c$ are vectors in it, we define $abc = (a \cdot b)c$, where $(a \cdot b)$ is the inner product of $a$ and $b$. It is easy to see that $V_n(F)$ is a cubic ring. Now we define $(bcd)' = b(c \cdot d)$. Then $ab(cde) = (a \cdot b)(c \cdot d)e = a(bcd)'e$, i.e., $V_n(F)$ is a $\Gamma$-ring with $\Gamma = V_n(F)$.

**Example 2.** Let $D_{n,m}$ be the set of all rectangular matrices of type $n \times m$ over a division ring $D$. If $a$, $b$, and $c$ are elements in it, we define $abc = ab'tc$, where $b'$ is the transpose of a matrix $b$ and the above product is well-defined. Then $D_{n,m}$ is clearly a cubic ring. Now we define $(bcd)' = dc'b$. Then $ab(cde) = ab'cd'e = a(bcd)'e$, i.e., $D_{n,m}$ is a $\Gamma$-ring with $\Gamma = D_{n,m}$.

**Example 3.** Let $I$ be the set of all purely imaginary complex numbers. Then it is a cubic ring with the usual multiplication. Also it is a $\Gamma$-ring with $\Gamma = I$. However, even with the same $I$, we can define another cubic ring. For example, if $a$, $b$, and $c$ are elements in $I$, we define the product of $a$, $b$, and $c$ as $\overline{abc}$ where $\overline{b}$ is the conjugate of $b$, i.e., $-b$. This product also satisfies 4), 5) and 6) of the definition of cubic rings. In this case, we put $(bcd)' = -bcd$. 


3. The operator rings and ideals. Let $M$ be a $\Gamma$-ring. Consider the additive group generated by pairs $(\gamma, a)$, where $\gamma \in \Gamma$ and $a \in M$ with defining relations $(\gamma_1 + \gamma_2, a) = (\gamma_1, a) + (\gamma_2, a)$ and $(\gamma, a_1 + a_2) = (\gamma, a_1) + (\gamma, a_2)$. We define the multiplication of the elements of this additive group such that

$$(\gamma, a)(\beta, b) = (\gamma, a\beta b).$$

Using the condition 2), we can verify that

$$((\gamma, a)(\beta, b))(\alpha, c) = (\gamma, a)((\beta, b)(\alpha, c)).$$

Thus we get a ring which we denote by $F$. Now we can see that $F$ is a right operator ring of $M$ by the following definition:

$$a(\gamma, b) = a\gamma b,$$

for, we have

$$(a(\gamma, b))(\beta, c) = (a\gamma b)\beta c = a(\gamma, b\beta c) = a((\gamma, b)(\beta, c)).$$

The set of all elements of $F$ that annihilate $M$ forms an ideal which we denote by $A$, and we denote $F/A$ by $R$ and call it the right operator ring of $M$. We use $\gamma a$ for an element of $R$ which is gained from $(\gamma, a)$. Thus $a\gamma b = a(\gamma b)$. Then, take an element $\gamma$ of $\Gamma$. It induces an $R$-homomorphisms of $M$ to $R$ such that $\gamma(a) = \gamma a$. The condition 3) implies that $\Gamma$ induces the zero homomorphism if and only if $\gamma = 0$. Thus $\Gamma$ is considered to be a subset of the total set of $R$-homomorphisms of $M$ to $R$; $\Gamma \subseteq \text{Hom}_R(M, R)$.

Similarly we can define the left operator ring $L$ of $M$. We start with $(a, \gamma)$ and define the product such that $(a, \gamma)(b, \beta) = (a\gamma b, \beta)$. Also we define the left operation such that $(a, \gamma)b = a\gamma b$, and so on. $a\gamma b$ is an element of $L$ given from $(\gamma, a)$ and $a\gamma b = (a\gamma)b$. And we can say that $\Gamma \subseteq \text{Hom}_L(M, L)$.

**Definition.** $R$-submodules of $M$ are called right ideals of $M$, and $L$-submodules of $M$ are left ideals.

A right ideal $\tau$ is nothing but a submodule of $M$ such that $\tau M \subseteq \tau$. A left ideal $\imath$ is a submodule of $M$ such that $M\imath \subseteq \imath$.

4. Peirce decomposition in semi-simple $\Gamma$-rings. Assume that $M$ is semi-simple, and let $\tau$ be a minimal right ideal. Then by semi-simplicity there exists an element $e$ in $\Gamma$ such that $a\varepsilon a = 0$ for a non-zero element $a$ in $\tau$. Then $0 = a\varepsilon \tau \subseteq \tau$ and hence $\tau = a\varepsilon \tau$, for $\tau$ is minimal. Therefore $a = a\varepsilon e$ with some element $e$ of $\tau$. Then $e = e\varepsilon e$, since from $a = a\varepsilon e = (a\varepsilon e)e$
We have \( a \varepsilon (e - e \varepsilon e) = 0 \) which means \( e - e \varepsilon e = 0 \), for a set \( \{ c | a \varepsilon c = 0, c \in \mathfrak{r} \} \) is a right ideal contained in a minimal ideal \( \mathfrak{r} \) and is \{0\}. Since \( e \in \mathfrak{r} \), \( e \mathfrak{r} \subset \mathfrak{r} \), i.e., \( e \mathfrak{r} = \mathfrak{r} \). \( \varepsilon \mathfrak{M} \) being a right ideal of \( \mathfrak{R} \), \( e \varepsilon \mathfrak{M} \) is a right ideal of \( \mathfrak{M} \) contained in \( \mathfrak{r} \), and hence \( e \varepsilon \mathfrak{M} = \mathfrak{r} \). Thus we get

**Lemma 1.** If \( \mathfrak{M} \) is semi-simple and \( \mathfrak{r} \) is a minimal right ideal, then \( \mathfrak{r} = e \mathfrak{R} = e \varepsilon \mathfrak{M} \) with \( e \in \mathfrak{r} \) and \( \varepsilon \in \Gamma \), where \( e \varepsilon e = e \).

Now we use the idea of Peirce decomposition of the ring theory. Suppose that we have a right ideal \( \mathfrak{r} = e \varepsilon \mathfrak{M} \) such that \( e \varepsilon e = e \). Then

\[ M = e \varepsilon \mathfrak{M} + M_i \] (direct),

where \( M_i = \{ b | b e b = 0 \} \), since any element \( a \) of \( \mathfrak{M} \) is written

\[ a = e \varepsilon a + (a - e \varepsilon a) , \]

and \( e \varepsilon (a - e \varepsilon a) = 0 \). \( \mathfrak{M} \) is clearly a right ideal of \( \mathfrak{M} \). Now we can get a decomposition theorem.

**Theorem 1.** If \( \mathfrak{M} \) is semi-simple and satisfies the minimum condition for right ideals, then

\[ M = e_i \mathfrak{R} + e_2 \mathfrak{R} + \cdots + e_n \mathfrak{R} \] (direct),

where \( e_i \mathfrak{R} \) are minimal right ideals and \( e_i \mathfrak{R} = e_i \varepsilon_i \mathfrak{M} \), and \( e_i \varepsilon_i e_i = e_i \) and \( e_i \varepsilon_i e_j = 0 \) if \( i \neq j \).

**Proof.** Suppose that we have

\[ M = e_i \varepsilon_i \mathfrak{M} + \cdots + e_k \varepsilon_{k-1} \mathfrak{M} + M_{k-1} \] (direct)

such that \( e_i \varepsilon_i \mathfrak{M} \) are minimal right ideals and

\[ e_i \varepsilon_i e_j = \begin{cases} e_i & \text{if } i = j , \\ 0 & \text{if } i \neq j , \end{cases} \]

and that \( e_i \varepsilon_i a = 0 \) if \( a \in M_{k-1} \) for \( i = 1, 2, \ldots, k-1 \). This is true for \( k = 2 \) as above. Apply the above discussion on \( M_{k-1} \), and we get

\[ M_{k-1} = e_k \varepsilon_k \mathfrak{M} + M_k \] (direct)

as in the above. Here \( e_i \varepsilon_k e_k = 0 \) if \( i < k \), but we can not say that \( e_k \varepsilon_k e_i = 0 \). So, we change \( \varepsilon_k \) suitably. Put

\[ \varepsilon_k = \varepsilon_k - \varepsilon_k (e_1 \varepsilon_1 + \cdots + e_{k-1} \varepsilon_{k-1}) . \]

Then we can see that \( e_k \varepsilon_k e_k = e_k \) and \( e_k \varepsilon_k e_i = 0 \). Thus we have a decomposition for \( k \). Since \( \mathfrak{M} \) satisfies the minimum condition for right ideals, we
can get the decomposition in Theorem 1. Similarly we can get

**Theorem 1'**. If \( M \) is semi-simple and satisfies the minimum condition for left ideals, then

\[
M = Ld_1 + Ld_2 + \cdots + Ld_m \quad \text{(direct),}
\]

where \( Ld_i \) are minimal left ideals and \( Ld_i = M \delta_i d_i \), and \( d_i \delta_i d_i = d_i \) and \( d_j \delta_i d_i = 0 \) if \( i \neq j \).

5. **Simple \( \Gamma \)-rings.** Assume \( M \) is simple and satisfies the minimum condition for right and left ideals in this section. First we want to show that \( e_i R \) and \( e_j R \) are isomorphic as \( \Gamma \)-modules. \( M \) being simple, we can find an element \( \gamma \) in \( \Gamma \) such that \( e_i \gamma e_j = 0 \). Then \( e_i \gamma \tau_j = \tau_i \) where \( \tau_i \) and \( \tau_j \) are \( e_i R \) and \( e_j R \). By a correspondence:

\[
(\tau_j \ni x) \rightarrow e_i \gamma x (\in \tau_i)
\]

we have a one-one mapping of \( \tau_j \) onto \( \tau_i \). If \( x = 0 \), \( e_i \gamma x = 0 \), because \( \{c | e_i \gamma c = 0, c \in \tau_j\} \) is a right ideal contained in \( \tau_j \) and is \( \{0\} \) as \( \tau_j \) is minimal. This mapping is “onto” because \( \tau_i \) is minimal. Since \( x(\beta c) = x\beta c \) corresponds to \( e_i \gamma (x\beta c) = (e_i \gamma x)(\beta c) \), this mapping is an \( R \)-homomorphism, i.e., an \( R \)-isomorphism. Similarly \( Ld_i \approx Ld_j \) (\( L \)-isomorphic). Next, we want to show that all \( L \)-endomorphisms of \( M \) are given by the right multiplication of \( R \). Let \( \phi \) be an \( L \)-endomorphism of \( M \) and put \( \phi(d_i) = u_i \). Since \( d_i = d_i \delta_i d_i, u_i = d_i \delta_i u_i \). Therefore, \( u_i = d_i(\sum_j \delta_j d_j \delta_j u_j) \) where \( \sum_j \delta_j d_j \delta_j u_j \)

is an element of \( R \). On the other hand, by the definition of the right operator ring, \( R \) is considered to be the set of all \( L \)-endomorphisms of \( M \). Then the ring theory shows us that the latter ring is a matrix ring \( D_m \) over a division ring \( D \), where \( D_m \) is \( D_{mm} \). Matrix units \( E_{r,s} \) of \( D_m \) map \( d_r \) to \( d_s \) and \( d_t \) to 0 if \( t \neq r \).

Now we can determine \( M \) with respect to \( R \) which is identified with \( D_m \) as above. Since minimal right ideals of \( D_m \) are \( E_{r,s}D_m, e_i D_m \) in Theorem 2) = \( e_i E_{r,s}D_m \) with some \( r \). Then put \( e_i E_{r,s} = e_{i,s} \). We get \( e_{i,s} \) \( (i = 1, 2, \ldots, n; s = 1, 2, \ldots, m) \) such that

\[
e_{i,s}E_{r,s} = \begin{cases} e_{i,t} & s = r, \\ 0 & s \neq r. \end{cases}
\]

Thus we can say that \( M = \sum_{i,s} e_{i,s} D \), i.e., \( e_{i,s} \) are matrix units of \( D_{n,m} \) and \( M \) is (isomorphic to) \( D_{n,m} \) as a right \( D_m \)-module.

Next we must determine \( \Gamma \). An element \( \gamma \) of \( \Gamma \) is considered to
induce a mapping from $M$ to $R$ as in §3, and $\Gamma$ is considered to be a subset of the set of all $R$-homomorphisms of $M=D_{n,m}$ to $R=D_m$. On the other hand, $D_m$-homomorphisms of $M=D_{n,m}$ to $D_m$ are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose $\phi$ is a $D_m$-homomorphism of $M$ to $R$ as in §3, and $\Gamma$ is considered to be a subset of the set of all $R$-homomorphisms of $M=D_{n,m}$ to $R=D_m$.

On the other hand, $D_m$-homomorphisms of $D_{n,m}$ to $D_m$ are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose $\phi$ is a $D_m$-homomorphism of $D_{n,m}$ to $D_m$ such that

$$\phi(e_{i,s}) = \sum_{p,q} E_{p,q} T_{p,q}(i, s)$$

with $T_{p,q}(i, s)$ in $D$. Multiply $E_{s,s}$, and we can see $T_{p,q}(i, s)=0$ if $q \neq s$. Multiply $E_{s,t}$, and we can see $T_{p,s}(i, s)=T_{p,t}(i, t)$. Putting $T_{p,s}(i, s)=T_{p}(i)$, we have

$$\phi(e_{i,s}) = \sum_{p} E_{p,s} T_{p}(i) = \left(\sum_{j} e'_{p,j} T_{p}(j)\right) e_{i,s},$$

where $e'_{p,i}$ are matrix units of $D_{m,n}$ such that

$$e'_{p,j} e_{i,s} = \begin{cases} E_{p,s} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Hence $\phi$ is induced by the left multiplication of an element $A=\sum e'_{p,j} T_{p}(j)$ of $D_{m,n}$. Identifying $\gamma$ which induces $\phi$ and $A$ which corresponds to $\phi$, we can say that $\Gamma \subset D_{m,n}$. What we want to show is that $\Gamma = D_{m,n}$. But $\Gamma$ is a two sided $D_m - D_n$ module and must be identical with $D_{m,n}$.

Summarizing all the discussions, we get the main theorem.

**Theorem 2.** If $M$ is a simple $\Gamma$-ring satisfying the minimum condition for left and right ideals, then $M$ is $D_{n,m}$ and $\Gamma$ is $D_{m,n}$. The product $ab\gamma$ is the usual matrix product of three elements $a$, $\gamma$ and $b$ of $D_{n,m}$, $D_{m,n}$ and $D_{m,n}$.

6. Semi-simple $\Gamma$-rings. Let $M$ be a semi-simple $\Gamma$-ring which satisfies the minimum condition for left and right ideals in this section. Arranging suitably, we can see that $M$ is expressed as follows:

$$M = Ld_{1}^{(1)} + \cdots + Ld_{m(1)}^{(1)} + Ld_{1}^{(2)} + \cdots + Ld_{m(2)}^{(2)} + \cdots + Ld_{1}^{(p)} + \cdots + Ld_{m(p)}^{(p)};$$

where $d_{j}^{(p)}$ are some $d_b$ of Theorem 1. Moreover we take the order such that in the above $Ld_{i}^{(j)} \cong Ld_{k}^{(j)}$ (L-isomorphic) and $Ld_{i}^{(j)} \cong Ld_{j}^{(j')}$ if $j \neq j'$. Then, $R$ is, as the right multiplication ring of the $L$-module $M$, equal to a direct ring sum $\sum_{j} D_{m(j)}^{(j)}$, where $D_{m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $m(j) \times m(j)$. Furthermore $D_{m(j)}^{(j)}$ operate on $Ld_{i}^{(j)}$ as usual and are zero on $Ld_{i}^{(j')}$ if $j \neq j'$. On the other hand, we have in Theorem 1
$M = e_1R + e_2R + \cdots + e_nR.$
e_iR$ being minimal, $e_iR = e_iD^{(j)}_{m(j)}$ with some $j$. Rearranging the order suitably, we have

$$M = e_1^{(1)}R + \cdots + e_{n(1)}^{(1)}R + e_1^{(2)}R + \cdots + e_{n(2)}^{(2)}R + \cdots + e_1^{(q)}R + \cdots + e_{n(q)}^{(q)}R,$$

where $e_i^{(j)}R = e_i D^{(j)}_{m(j)}$ and $e_i^{(j)}$ are some $e_k$. Hence $n = \sum_j n(j)$. With the same discussion as in §5, we can say that

$$e_1^{(1)}R + \cdots + e_{n(1)}^{(1)}R = D^{(1)}_{m(1)},$$
$$e_1^{(2)}R + \cdots + e_{n(2)}^{(2)}R = D^{(2)}_{m(2)},$$
\[\vdots\]
$$e_1^{(q)}R + \cdots + e_{n(q)}^{(q)}R = D^{(q)}_{m(q)},$$
i.e.,

$$M = D^{(1)}_{m(1), m(1)} + \cdots + D^{(q)}_{m(q), m(q)},$$

where $D^{(j)}_{m(j), m(j)}$ are matrix rings over division rings $D^{(j)}$ of type $n(j) \times m(j)$. Naturally $D^{(j)}_{m(j), m(j)}$ operate on $D^{(j)}_{m(j), m(j)}$ as usual and are zero on $D^{(j)}_{m(j'), m(j')}$ if $j \neq j'$. $\Gamma$ is then a set of $R$-homomorphisms of $M$ to $R$ and is contained in $\sum_j D^{(j)}_{m(j), m(j)}$. Here the product of elements of $D^{(j)}_{m(j), m(j)}$ and of $D^{(j')}_{m(j), m(j')}$ is performed as usual if $j = j'$ and is 0 if $j \neq j'$. On the other hand, the condition of semi-simplicity means that for any non-zero element $a$ of $D^{(j)}_{m(j), m(j)}$ there exists $\gamma$ in $\Gamma$ such that $a\gamma a = 0$. Now we want to show that each $D^{(j)}_{m(j), m(j)}$ is a simple $\Gamma_j$-ring. Let $V$ and $V'$ be left $D^{(j)}$-modules of dim $n(j)$ and of dim $m(j)$. $D^{(j)}_{m(j), m(j)}$ and $D^{(j')}_{m(j), m(j)}$ are considered to be the sets of all $D^{(j)}$-homomorphisms of $V$ to $V'$ and of $V'$ to $V$. When we notice that $D^{(j)}_{m(j), m(j)}$ induce zero mapping on $V'$ if $j \neq j'$, we can say that elements of $\Gamma$ induce mappings of $V'$ to $V$. In this case we can show that $X\Gamma = V$ for any subspace $X$ of dim 1 of $V$. For, suppose that $X\Gamma \subseteq V$. Then we can find an element $a$ in $D^{(j)}_{m(j), m(j)}$ such that $Va = X$ and $(X\Gamma)a = 0$. Then $a\gamma a = 0$ for every $\gamma$ in $\Gamma$, which is a contradiction. Now this fact implies the existance of $\gamma$ such that $a\gamma b = 0$ for any non-zero $a$ and $b$, for we can take a subspace of $Va$ as $X$ and take $\gamma$ such that $(X\gamma)b = 0$. Thus we can conclude that $\Gamma = \sum_j D^{(j)}_{m(j), m(j)}$. Now put $\Gamma_j = D^{(j)}_{m(j), m(j)}$.

**Theorem 3.** If $M$ is a semi-simple $\Gamma$-ring satisfying the minimum condition for left and right ideals, then $M$ is a direct sum of simple $\Gamma_i$-rings where $\Gamma = \Gamma_1 + \cdots + \Gamma_q$ (direct):
\[ M = M_1 + M_2 + \cdots + M_n \quad (direct), \]

where \( M_i \) are simple \( \Gamma_i \)-rings and \( M_i \Gamma_j = 0 \) if \( i \neq j \), and \( M_i \Gamma_j M_i = 0 \) if \( i = j \).

Osaka University and University of Alberta