ON A GENERALIZATION OF THE RING THEORY

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(Received May 20, 1964)

1. Introduction. A ring of endomorphisms of a module plays a very important role in many parts of mathematics; the property of a ring itself is also clarified when we consider it as a ring of endomorphisms of a module. As a generalization of this idea, we can consider a set of homomorphisms of a module to another module which is closed under the addition and subtraction defined naturally but has no more a structure of a ring since we can not define the product. However, suppose that we have an additive group $M$ consisting of homomorphisms of a module $A$ to a module $B$ and that we have also an additive group $N$ consisting of homomorphisms of $B$ to $A$. In this case we can define the product of three elements $f_1, g$ and $f_2$ where $f_1$ and $f_2$ are elements of $M$ and $g$ is an element of $N$. If this product $f_1gf_2$ is also an element of $M$ for every $f_1, g$ and $f_2$, we say that $M$ is closed under the multiplication using $N$ between. Similarly we can define that $N$ is closed under the multiplication using $M$ between. Take $f_1, f_2$ and $f_3$ in $M$ and $g_1$ and $g_2$ in $N$ in the above case. Then we have

$$(f_1g_1f_3)g_2f_3 = f_1(g_1g_2f_3) = f_1(g_1f_3g_2)f_3.$$  

When we define this situation abstractly, we can get a new algebraic system.

DEFINITION. Let $M$ be an additive group whose elements are denoted by $a, b, c, \ldots$, and $Γ$ another additive group whose elements are $γ, β, α, \ldots$. Suppose that $aγb$ is defined to be an element of $M$ and that $γαβ$ is defined to be an element of $Γ$ for every $a, b, γ$ and $β$. If the products satisfy the following three conditions:

1) $$(a_1 + a_2)γb = a_1γb + a_2γb,$$
2)  

$$(γ_1 + γ_2)b = aγb + aγb,$$
3)  

$$(aγ(b_1 + b_2) = aγb_1 + aγb_2.$$
2) \((a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta) c\),

3) if \(a\gamma b = 0\) for any \(a\) and \(b\) in \(M\), then \(\gamma = 0\),
then \(M\) is called a \(\Gamma\)-ring

The purpose of this note is to determine the structure of \(\Gamma\)-rings under the following conditions which are called semi-simple and simple according to the usual ring theory.

**Definition.** Let \(M\) be a \(\Gamma\)-ring as above. If for any non-zero element \(a\) of \(M\) there exists such an element \(\gamma\) (depending on \(a\)) in \(\Gamma\) that \(a\gamma a \neq 0\), we say that \(M\) is semi-simple. If for any non-zero elements \(a\) and \(b\) of \(M\) there exists \(\gamma\) (depending on \(a\) and \(b\)) in \(\Gamma\) such that \(a\gamma b \neq 0\), we say that \(M\) is simple.

The main result obtained in this note is that a simple \(\Gamma\)-ring which satisfies the chain condition for left and right ideals (defined in §3) is the set \(D_{nm}\) of all rectangular matrices of type \(n \times m\) over some division ring \(D\) and \(\Gamma\) is \(D_{mn}\) of type \(m \times n\). The product \(a\gamma b\) is the same as the usual matrix product of elements \(a\), \(\gamma\) and \(b\) of \(D_{nm}\), \(D_{mn}\) and \(D_{nm}\).

This is a generalization of the theorem of Wedderburn on simple rings. Subsequently, a semi-simple \(\Gamma\)-ring satisfying the chain condition for left and right ideals will be shown to be a direct sum of simple \(\Gamma\)-rings, where \(\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n\) (direct).

**2. Examples.** Suppose we have a right \(R\)-module \(M\) with an operator ring \(R\). Take a submodule \(\Gamma\) of \(\text{Hom}_R(M, R)\). Then \(M\) is a \(\Gamma\)-ring as follows: If \(a\) and \(b\) are elements of \(M\) and if \(\gamma\) is an element of \(\Gamma\), then we define

\[a\gamma b = a \cdot \gamma(b),\]

where \(\gamma(b)\) is an image of \(b\) by \(\gamma\) and is an element of \(R\). It is easy to verify that

\[(a\gamma b)\beta c = (a \cdot \gamma(b)) \cdot \beta(c) = a\gamma(b\beta(c)) = a \cdot \gamma(b \cdot \beta(c)) = a\gamma(b\beta) c\].

We also define that

\[\gamma b\beta = \beta \cdot \gamma(b), \quad (\beta \text{ operating first}),\]

where \(\gamma(b)\) means the left multiplication of \(\gamma(b)\). Then

\[(a\gamma b)\beta c = a\gamma(b\beta(c)) = a(\gamma b\beta)c\].

The conditions 1) and 3) hold naturally and \(M\) is a \(\Gamma\)-ring. But it will be shown in §3 that every \(\Gamma\)-ring is given in this way.

To illustrate further this new algebraic system, we introduce the
definition and examples of cubic rings.

**Definition.** We call that \( M \) is a cubic ring when we can define the product of three elements of \( M \) which is an additive group such that it satisfies

\[
\begin{align*}
(a_1+a_2)bc &= a_1bc+a_2bc, \\
ab(c_1+c_2) &= abc_1+abc_2, \\
ab(cde) &= (abc)de, \\
\text{if } abc = 0 \text{ for all } a \text{ and } c, \text{ then } b = 0.
\end{align*}
\]

If we take the product in a cubic ring \( M \) as the product of two elements of \( M \) using one element of \( \Gamma = M \) between, then conditions 1) and 3) for a \( \Gamma \)-ring are satisfied. Also the first part of 2) is satisfied. Hence, in order that \( M \) is a \( \Gamma \)-ring, we must be able to define the product \( \Gamma \times M \times \Gamma \) such that the latter part of 2) holds. In the following examples, we can find it easily.

**Example 1.** Let \( V_n(F) \) be a vector space of dim \( n \) over a field \( F \). If \( a, b \) and \( c \) are vectors in it, we define \( abc = (a \cdot b)c \), where \( (a \cdot b) \) is the inner product of \( a \) and \( b \). It is easy to see that \( V_n(F) \) is a cubic ring. Now we define \((bcd)' = b(c \cdot d)\). Then \( ab(cde) = ab(c \cdot d)e = a(bcd)'e \), i.e., \( V_n(F) \) is a \( \Gamma \)-ring with \( \Gamma = V_n(F) \).

**Example 2.** Let \( D_{n,m} \) be the set of all rectangular matrices of type \( n \times m \) over a division ring \( D \). If \( a, b \) and \( c \) are elements in it, we define \( abc = ab'c \), where \( b' \) is the transpose of a matrix \( b \) and the above product is well-defined. Then \( D_{n,m} \) is clearly a cubic ring. Now we define \((bcd)' = dc'b \). Then \( ab(cde) = ab'cd'e = a(bcd)'e \), i.e., \( D_{n,m} \) is a \( \Gamma \)-ring with \( \Gamma = D_{n,m} \).

**Example 3.** Let \( I \) be the set of all purely imaginary complex numbers. Then it is a cubic ring with the usual multiplication. Also it is a \( \Gamma \)-ring with \( \Gamma = I \). However, even with the same \( I \), we can define another cubic ring. For example, if \( a, b \) and \( c \) are elements in \( I \), we define the product of \( a, b \) and \( c \) as \( \bar{a}bc \) where \( \bar{b} \) is the conjugate of \( b \), i.e., \( -b \). This product also satisfies 4), 5) and 6) of the definition of cubic rings. In this case, we put \((bcd)' = -bcd\).
3. The operator rings and ideals. Let $M$ be a $\Gamma$-ring. Consider the additive group generated by pairs $(\gamma, a)$, where $\gamma \in \Gamma$ and $a \in M$ with defining relations $(\gamma_1 + \gamma_2, a) = (\gamma_1, a) + (\gamma_2, a)$ and $(\gamma, a_1 + a_2) = (\gamma, a_1) + (\gamma, a_2)$. We define the multiplication of the elements of this additive group such that

$$(\gamma, a)(\beta, b) = (\gamma, a\beta b).$$

Using the condition 2), we can verify that

$$(\gamma, a)(\beta, b)(\alpha, c) = (\gamma, a)((\beta, b)(\alpha, c)).$$

Thus we get a ring which we denote by $F$. Now we can see that $F$ is a right operator ring of $M$ by the following definition:

$$a(\gamma, b) = a\gamma b,$$

for, we have

$$(a(\gamma, b))(\beta, c) = (a\gamma b)\beta c = a\gamma(b\beta c) = a((\gamma, b)(\beta, c)).$$

The set of all elements of $F$ that annihilate $M$ forms an ideal which we denote by $A$, and we denote $F/A$ by $R$ and call it the right operator ring of $M$. We use $\gamma a$ for an element of $R$ which is gained from $(\gamma, a)$. Thus $a\gamma b = a(\gamma b)$. Then, take an element $\gamma$ of $\Gamma$. It induces an $R$-homomorphisms of $M$ to $R$ such that $\gamma(a) = a\gamma$. The condition 3) implies that $\Gamma$ induces the zero homomorphism if and only if $\gamma = 0$. Thus $\Gamma$ is considered to be a subset of the total set of $R$-homomorphisms of $M$ to $R$; $\Gamma \subseteq \text{Hom}_R(M, R)$.

Similarly we can define the left operator ring $L$ of $M$. We start with $(a, \gamma)$ and define the product such that $(a, \gamma)(b, \beta) = (a\gamma b, \beta)$. Also we define the left operation such that $(a, \gamma)b = a\gamma b$, and so on. $a\gamma$ is an element of $L$ given from $(a, \gamma)$ and $a\gamma b = (a\gamma)b$. And we can say that $\Gamma \subseteq \text{Hom}_L(M, L)$.

**Definition.** $R$-submodules of $M$ are called right ideals of $M$, and $L$-submodules of $M$ are left ideals.

A right ideal $\tau$ is nothing but a submodule of $M$ such that $\tau M \subseteq \tau$. A left ideal $\iota$ is a submodule of $M$ such that $M\iota \subseteq \iota$.

4. Peirce decomposition in semi-simple $\Gamma$-rings. Assume that $M$ is semi-simple, and let $\tau$ be a minimal right ideal. Then by semi-simplicity there exists an element $\varepsilon$ in $\Gamma$ such that $a\varepsilon a \neq 0$ for a non-zero element $a$ in $\tau$. Then $0 \neq a\varepsilon \tau \subseteq \tau$ and hence $\tau = a\varepsilon \tau$, for $\tau$ is minimal. Therefore $a = a\varepsilon e$ with some element $e$ of $\tau$. Then $e = e\varepsilon e$, since from $a = a\varepsilon e = (a\varepsilon e)e$
we have $ae(e-e\varepsilon e)=0$ which means $e-e\varepsilon e=0$, for a set $\{c|ae\varepsilon e=0, c \in r\}$ is a right ideal contained in a minimal ideal $r$ and is $\{0\}$. Since $e \in r$, $eR \subset r$, i.e., $eR=r$. $\varepsilon M$ being a right ideal of $R$, $e\varepsilon M$ is a right ideal of $M$ contained in $r$, and hence $e\varepsilon M=r$. Thus we get

**Lemma 1.** If $M$ is semi-simple and $r$ is a minimal right ideal, then $r=eR=e\varepsilon M$ with $e \in r$ and $e \in \Gamma$, where $e\varepsilon e=e$.

Now we use the idea of Peirce decomposition of the ring theory. Suppose that we have a right ideal $r=e\varepsilon M$ such that $e\varepsilon e=e$. Then

$$M = e\varepsilon M + M_1 \quad (\text{direct}),$$

where $M_1 = \{b|e\varepsilon b=0\}$, since any element $a$ of $M$ is written

$$a = e\varepsilon a + (a-e\varepsilon a),$$

and $e\varepsilon (a-e\varepsilon a)=0$. $M_1$ is clearly a right ideal of $M$. Now we can get a decomposition theorem.

**Theorem 1.** If $M$ is semi-simple and satisfies the minimum condition for right ideals, then

$$M = e_1 R + e_2 R + \cdots + e_n R \quad (\text{direct}),$$

where $e_i R$ are minimal right ideals and $e_i R = e_i \varepsilon_i M$, and $e_i e_i e_i = e_i$ and $e_i e_i e_j = 0$ if $i \neq j$.

**Proof.** Suppose that we have

$$M = e_1 \varepsilon_1 M + \cdots + e_{k-1} \varepsilon_{k-1} M + M_{k-1} \quad (\text{direct})$$

such that $e_i \varepsilon_i M$ are minimal right ideals and

$$e_i \varepsilon_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and that $e_i \varepsilon_i a = 0$ if $a \in M_{k-1}$ for $i=1, 2, \cdots, k-1$. This is true for $k=2$ as above. Apply the above discussion on $M_{k-1}$, and we get

$$M_{k-1} = e_k \varepsilon_k M + M_k \quad (\text{direct})$$

as in the above. Here $e_i \varepsilon_i e_k = 0$ if $i < k$, but we can not say that $e_k \varepsilon_k e_i = 0$. So, we change $\varepsilon_k$ suitably. Put

$$\varepsilon_k = \varepsilon_k - \varepsilon_k (e_1 \varepsilon_1 + \cdots + e_{k-1} \varepsilon_{k-1}).$$

Then we can see that $e_k \varepsilon_k \varepsilon_k = e_k$ and $e_k \varepsilon_k e_i = 0$. Thus we have a decomposition for $k$. Since $M$ satisfies the minimum condition for right ideals, we
can get the decomposition in Theorem 1.
Similarly we can get

Theorem 1'. If M is semi-simple and satisfies the minimum condition for left ideals, then

\[ M = Ld_1 + Ld_2 + \cdots + Ld_m \quad \text{(direct)}, \]

where \( Ld_i \) are minimal left ideals and \( Ld_i = M\delta_i d_i \), and \( d_i \delta_i d_i = d_i \) and \( d_i \delta_i d_i = 0 \) if \( i \neq j \).

5. Simple \( \Gamma \)-rings. Assume \( M \) is simple and satisfies the minimum condition for right and left ideals in this section. First we want to show that \( e_i R \) and \( e_j R \) are isomorphic as \( \Gamma \)-modules. \( M \) being simple, we can find an element \( \gamma \) in \( \Gamma \) such that \( e_i \gamma e_j = 0 \). Then \( e_i \gamma \tau_j = \tau_i \) where \( \tau_i \) and \( \tau_j \) are \( e_i R \) and \( e_j R \). By a correspondence:

\[(\tau_j \ni x \mapsto e_i \gamma x (\in \tau_i)\]

we have a one-one mapping of \( \tau_j \) onto \( \tau_i \). If \( x = 0 \), \( e_i \gamma x = 0 \), because \( \{ c \mid e_i \gamma c = 0, c \in \tau_j \} \) is a right ideal contained in \( \tau_j \) and is \{0\} as \( \tau_j \) is minimal. This mapping is "onto" because \( \tau_i \) is minimal. Since \( x(\beta c) = x\beta c \) corresponds to \( e_i \gamma (x\beta c) = (e_i \gamma x)(\beta c) \), this mapping is an \( R \)-homomorphism, i.e., an \( R \)-isomorphism. Similarly \( Ld_i = Ld_j \) (\( L \)-isomorphic). Next, we want to show that all \( L \)-endomorphisms of \( M \) are given by the right multiplication of \( R \). Let \( \phi \) be an \( L \)-endomorphism of \( M \) and put \( \phi(d_i) = u_i \). Since \( d_i = d_i \delta_i d_i, u_i = d_i \delta_i u_i \). Therefore, \( u_i = d_i (\sum \delta_j d_j \delta_j u_j) \) where \( \sum \delta_j d_j \delta_j u_j \) is an element of \( R \). On the other hand, by the definition of the right operator ring, \( R \) is considered to be the set of all \( L \)-endomorphisms of \( M \). Then the ring theory shows us that the latter ring is a matrix ring \( D_m \) over a division ring \( D \), where \( D_m \) is \( D_{mm} \). Matrix units \( E_{r,s} \) of \( D_m \) map \( d_r \) to \( d_s \) and \( d_t \) to 0 if \( t \neq r \).

Now we can determine \( M \) with respect to \( R \) which is identified with \( D_m \) as above. Since minimal right ideals of \( D_m \) are \( E_{r,s} D_m = e_i E_{r,s} D_m \) with some \( r \). Then put \( e_i E_{r,s} = e_{i,s} \). We get \( e_{i,s} \) \((i = 1, 2, \ldots, n; s = 1, 2, \ldots, m) \) such that

\[ e_{i,s} E_{r,t} = \begin{cases} e_{i,t} & s = r, \\ 0 & s \neq r. \end{cases} \]

Thus we can say that \( M = \sum_{i,t} e_{i,s} D \), i.e., \( e_{i,s} \) are matrix units of \( D_{n,m} \) and \( M \) is (isomorphic to) \( D_{n,m} \) as a right \( D_m \)-module.

Next we must determine \( \Gamma \). An element \( \gamma \) of \( \Gamma \) is considered to
induce a mapping from $M$ to $R$ as in § 3, and $\Gamma$ is considered to be a subset of the set of all $R$–homomorphisms of $M=D_{n,m}$ to $R=D_m$. On the other hand, $D_{m}$–homomorphisms of $D_{n,m}$ to $D_m$ are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose $\phi$ is a $D_m$–homomorphism of $D_{n,m}$ to $D_m$ such that

$$\phi(e_{i,s}) = \sum_{j} E_{p,q} T_{p,q}(i, s)$$

with $T_{p,q}(i, s)$ in $D$. Multiply $E_{s,s}$, and we can see $T_{p,q}(i, s)=0$ if $q \neq s$. Multiply $E_{s,t}$, and we can see $T_{p,s}(i, s)=T_{p,i}(i, t)$. Putting $T_{p,q}(i, s)=T_p(i)$, we have

$$\phi(e_{i,s}) = \sum_p E_{p,s} T_p(i) = \left(\sum_{p,j} e'_{p,i} T_p(j)\right)e_{i,s},$$

where $e'_{p,i}$ are matrix units of $D_{m,n}$ such that

$$e'_{p,i}e_{i,s} = \begin{cases} E_{p,s} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Hence $\phi$ is induced by the left multiplication of an element $A=\sum e'_{p,i} T_p(j)$ of $D_{m,n}$. Identifying $\gamma$ which induces $\phi$ and $A$ which corresponds to $\phi$, we can say that $\Gamma \subseteq D_{m,n}$. What we want to show is that $\Gamma = D_{m,n}$. But $\Gamma$ is a two sided $D_m$ module and must be identical with $D_{m,n}$. Summarizing all the discussions, we get the main theorem.

**Theorem 2.** If $M$ is a simple $\Gamma$–ring satisfying the minimum condition for left and right ideals, then $M$ is $D_{n,m}$ and $\Gamma$ is $D_{m,n}$. The product $ab\gamma$ is the usual matrix product of three elements $a, \gamma$ and $b$ of $D_{n,m}, D_{m,n}$ and $D_{m,n}$.

6. Semi-simple $\Gamma$–rings. Let $M$ be a semi-simple $\Gamma$–ring which satisfies the minimum condition for left and right ideals in this section. Arranging suitably, we can see that $M$ is expressed as follows:

$$M = LD_{1}^{(1)} + \cdots + LD_{m(1)}^{(1)} + LD_{1}^{(2)} + \cdots + LD_{m(2)}^{(2)} + \cdots + LD_{1}^{(n)} + \cdots + LD_{m(n)}^{(n)};$$

where $d_{i}^{(j)}$ are some $d_k$ of Theorem 1. Moreover we take the order such that in the above $LD_{1}^{(j)} \cong LD_{k}^{(j)} (L$–isomorphic) and $LD_{i}^{(j)} \cong LD_{i}^{(j')} if j \neq j'$. Then, $R$ is, as the right multiplication ring of the $L$–module $M$, equal to a direct ring sum $\sum_j D_{m(j)}^{(j)}$, where $D_{m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $m(j) \times m(j)$. Furthermore $D_{m(j)}^{(j)}$ operate on $LD_{i}^{(j)}$ as usual and are zero on $LD_{i}^{(j')} if j \neq j'$. On the other hand, we have in Theorem 1
$M = e_1 R + e_2 R + \cdots + e_m R$.

e_i R being minimal, $e_i R = e_i D^{(j)}_{m(j)}$ with some $j$. Rearranging the order suitably, we have

$M = e^{(1)}_1 R + \cdots + e^{(1)}_{m(1)} R + e^{(2)}_1 R + \cdots + e^{(2)}_{m(2)} R + \cdots + e^{(q)}_1 R + \cdots + e^{(q)}_{m(q)} R$,

where $e^{(j)}_i R = e^{(j)}_i D^{(j)}_{m(j)}$ and $e^{(j)}_i$ are some $e_i$. Hence $n = \sum_j n(j)$. With the same discussion as in §5, we can say that

\[
e^{(1)}_1 R + \cdots + e^{(1)}_{m(1)} R = D^{(1)}_{n(1), m(1)},
\]

\[
e^{(2)}_1 R + \cdots + e^{(2)}_{m(2)} R = D^{(2)}_{n(2), m(2)},
\]

\[
\ldots
\]

\[
e^{(q)}_1 R + \cdots + e^{(q)}_{m(q)} R = D^{(q)}_{n(q), m(q)},
\]

i.e.,

$M = D^{(1)}_{n(1), m(1)} + \cdots + D^{(q)}_{n(q), m(q)}$,

where $D^{(j)}_{m(j), m(j)}$ are matrix rings over division rings $D^{(j)}$ of type $n(j) \times m(j)$. Naturally $D^{(j)}_{m(j)}$ operate on $D^{(j)}_{m(j), m(j)}$ as usual and are zero on $D^{(j')}_{m(j'), m(j')}$, if $j \neq j'$. $\Gamma$ is then a set of $R$-homomorphisms of $M$ to $R$ and is contained in $\sum_j D^{(j)}_{m(j), m(j)}$. Here the product of elements of $D^{(j)}_{m(j), m(j)}$ and of $D^{(j')}_{m(j'), m(j')}$ is performed as usual if $j = j'$ and is 0 if $j \neq j'$. On the other hand, the condition of semi-simplicity means that for any non-zero element $a$ of $D^{(j)}_{m(j), m(j)}$, there exists $\gamma$ in $\Gamma$ such that $a \gamma a = 0$. Now we want to show that each $D^{(j)}_{m(j), m(j)}$ is a simple $\Gamma_j$-ring. Let $V$ and $V'$ be left $D^{(j)}$-modules of dim $n(j)$ and of dim $m(j)$. $D^{(j)}_{m(j), m(j)}$ and $D^{(j')}_{m(j'), m(j')}$ are considered to be the sets of all $D^{(j)}$-homomorphisms of $V$ to $V$ and of $V'$ to $V$. When we notice that $D^{(j')}_{m(j'), m(j')}$ induce zero mapping on $V'$ if $j \neq j'$, we can say that elements of $\Gamma$ induce mappings of $V'$ to $V$. In this case we can show that $X \Gamma = V$ for any subspace $X$ of dim 1 of $V$. For, suppose that $X \Gamma \subseteq V$. Then we can find an element $a$ in $D^{(j)}_{m(j), m(j)}$ such that $V a = X$ and $(X \Gamma) a = 0$. Then $a \gamma a = 0$ for every $\gamma$ in $\Gamma$, which is a contradiction. Now this fact implies the existence of $\gamma$ such that $a \gamma b = 0$ for any non-zero $a$ and $b$, for we can take a subspace of $V a$ as $X$ and take $\gamma$ such that $(X \gamma) b = 0$. Thus we can conclude that $\Gamma = \sum_j D^{(j)}_{m(j), m(j)}$.

Now put $\Gamma_j = D^{(j)}_{m(j), m(j)}$.

**Theorem 3.** If $M$ is a semi-simple $\Gamma$-ring satisfying the minimum condition for left and right ideals, then $M$ is a direct sum of simple $\Gamma_i$-rings where $\Gamma = \Gamma_1 + \cdots + \Gamma_q$ (direct):
$M = M_1 + M_2 + \cdots + M_r$ \quad (direct),

where $M_i$ are simple $\Gamma_i$-rings and $M_i\Gamma_j = 0$ if $i \neq j$, and $M_i\Gamma_i M_i = 0$ if $i \neq j$.

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