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ON A GENERALIZATION OF THE RING THEORY

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1. Introduction. A ring of endomorphisms of a module plays a very important role in many parts of mathematics; the property of a ring itself is also clarified when we consider it as a ring of endomorphisms of a module. As a generalization of this idea, we can consider a set of homomorphisms of a module to another module which is closed under the addition and subtraction defined naturally but has no more a structure of a ring since we can not define the product. However, suppose that we have an additive group M consisting of homomorphisms of a module A to a module B and that we have also an additive group Nconsisting of homomorphisms of B to A. In this case we can define the product of three elements f_1 , g and f_2 where f_1 and f_2 are elements of M and g is an element of N. If this product f_1gf_2 is also an element of M for every f_1 , g and f_2 , we say that M is closed under the multiplication using N between. Similarly we can define that N is closed under the multiplication using M between. Take f_1 , f_2 and f_3 in M and g_1 and g_2 in N in the above case. Then we have

$$(f_1g_1f_2)g_2f_3 = f_1g_1(f_2g_2f_3) = f_1(g_1f_2g_2)f_3.$$

When we define this situation abstractly, we can get a new algebraic system.

DEFINITION. Let M be an additive group whose elements are denoted by a, b, c, \dots , and Γ another additive group whose elements are $\gamma, \beta, \alpha, \dots$. Suppose that $a\gamma b$ is defined to be an element of M and that $\gamma a\beta$ is defined to be an element of Γ for every a, b, γ and β . If the products satisfy the following three conditions:

$$\begin{array}{l} (a_1+a_2)\gamma b = a_1\gamma b + a_2\gamma b \ , \\ a(\gamma_1+\gamma_2)b = a\gamma_1 b + a\gamma_2 b \ , \\ a\gamma(b_1+b_2) = a\gamma b_1 + a\gamma b_2 \ , \end{array}$$

2) $(a\gamma b)\beta c = a\gamma (b\beta c) = a(\gamma b\beta)c$,

3) if $a\gamma b=0$ for any a and b in M, then $\gamma=0$,

then M is called a Γ -ring

The purpose of this note is to determine the structure of Γ -rings under the following conditions which are called semi-simple and simple according to the usual ring theory.

DEFINITION. Let M be a Γ -ring as above. If for any non-zero element a of M there exists such an element γ (depending on a) in Γ that $a\gamma a \neq 0$, we say that M is *semi-simple*. If for any non-zero elements a and b of M there exists γ (depending on a and b) in Γ such that $a\gamma b \neq 0$, we say that M is *simple*.

The main result obtained in this note is that a simple Γ -ring which satisfies the chain condition for left and right ideals (defined in § 3) is the set $D_{n,m}$ of all rectangular matrices of type $n \times m$ over some division ring D and Γ is $D_{m,n}$ of type $m \times n$. The product $a\gamma b$ is the same as the usual matrix product of elements a, γ and b of $D_{n,m}$, $D_{m,n}$ and $D_{n,m}$. This is a generalization of the theorem of Wedderburn on simple rings. Subsequently, a semi-simple Γ -ring satisfying the chain condition for left and right ideals will be shown to be a direct sum of simple Γ_i rings, where $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ (direct).

2. Examples. Suppose we have a right *R*-module *M* with an operator ring *R*. Take a submodule 1' of $\operatorname{Hom}_{R}(M, R)$. Then *M* is a 1'-ring as follows: If *a* and *b* are elements of *M* and if γ is an element of 1', then we define

$$a\gamma b = a \cdot \gamma(b)$$
,

where $\gamma(b)$ is an image of b by γ and is an element of R. It is easy to verify that

$$(a\gamma b)eta c = (a \cdot \gamma(b)) \cdot eta(c) = a(\gamma(b)eta(c)) = a \cdot \gamma(b \cdot eta(c)) = a\gamma(beta c)$$
.

We also define that

 $\gamma b\beta = \beta \cdot \gamma(b)_l$ (β operating first),

where $\gamma(b)_l$ means the left multiplication of $\gamma(b)$. Then

$$(a\gamma b)\beta c = a(\gamma(b)\beta(c)) = a(\gamma b\beta)c$$
.

The conditions 1) and 3) hold naturally and M is a Γ -ring. But it will be shown in §3 that every Γ -ring is given in this way.

To illustrate further this new algebraic system, we introduce the

definition and examples of cubic rings.

DEFINITION. We call that M is a *cubic ring* when we can define the product of three elements of M which is an additive group such that it satisfies

(4)

$$(a_1+a_2)bc = a_1bc+a_2bc$$
,
 $a(b_1+b_2)c = ab_1c+ab_2c$,
 $ab(c_1+c_2) = abc_1+abc_2$,

ab(cde) = (abc)de,

6) if abc=0 for all a and c, then b=0.

If we take the product in a cubic ring M as the product of two elements of M using one element of $\Gamma = M$ between, then conditions 1) and 3) for a Γ -ring are satisfied. Also the first part of 2) is satisfied. Hence, in order that M is a Γ -ring, we must be able to define the product $\Gamma \times M \times \Gamma$ such that the latter part of 2) holds. In the following examples, we can find it easily.

EXAMPLE 1. Let $V_n(F)$ be a vector space of dim *n* over a field *F*. If *a*, *b* and *c* are vectors in it, we define $abc = (a \cdot b)c$, where $(a \cdot b)$ is the inner product of *a* and *b*. It is easy to see that $V_n(F)$ is a cubic ring. Now we define $(bcd)' = b(c \cdot d)$. Then $ab(cde) = (a \cdot b)(c \cdot d)e = a(bcd)'e$, i.e., $V_n(F)$ is a Γ -ring with $\Gamma = V_n(F)$.

EXAMPLE 2. Let $D_{n,m}$ be the set of all rectangular matrices of type $n \times m$ over a division ring D. If a, b and c are elements in it, we define $abc=ab^{t}c$, where b^{t} is the transpose of a matrix b and the above product is well-defined. Then $D_{n,m}$ is clearly a cubic ring. Now we define $(bcd)'=dc^{t}b$. Then $ab(cde)=ab^{t}cd^{t}e=a(bcd)'e$, i.e., $D_{n,m}$ is a Γ -ring with $\Gamma=D_{n,m}$.

EXAMPLE 3. Let I be the set of all purely imaginary complex numbers. Then it is a cubic ring with the usual multiplication. Also it is a Γ -ring with $\Gamma = I$. However, even with the same I, we can define another cubic ring. For example, if a, b and c are elements in I, we define the product of a, b and c as $a\bar{b}c$ where \bar{b} is the conjugate of b, i.e., -b. This product also satisfies 4), 5) and 6) of the definition of cubic rings. In this case, we put (bcd)' = -bcd.

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3. The operator rings and ideals. Let M be a Γ -ring. Consider the additive group generated by pairs (γ, a) , where $\gamma \in \Gamma$ and $a \in M$ with defining relations $(\gamma_1 + \gamma_2, a) = (\gamma_1, a) + (\gamma_2, a)$ and $(\gamma, a_1 + a_2) = (\gamma, a_1) + (\gamma, a_2)$. We define the multiplication of the elements of this additive group such that

$$(\gamma, a)(\beta, b) = (\gamma, a\beta b).$$

Using the condition 2), we can verify that

$$((\gamma, a)(\beta, b))(\alpha, c) = (\gamma, a)((\beta, b)(\alpha, c)).$$

Thus we get a ring which we denote by F. Now we can see that F is a right operator ring of M by the following definition:

$$a(\gamma, b) = a\gamma b$$
,

for, we have

$$(a(\gamma, b))(\beta, c) = (a\gamma b)\beta c = a\gamma (b\beta c) = a(\gamma, b\beta c) = a((\gamma, b)(\beta, c))$$

The set of all elements of F that annihilate M forms an ideal which we denote by A, and we denote F/A by R and call it *the right operator ring of* M. We use γa for an element of R which is gained from (γ, a) . Thus $a\gamma b = a(\gamma b)$. Then, take an element γ of Γ . It induces an Rhomomorphisms of M to R such that $\gamma(a) = \gamma a$. The condition 3) implies that Γ induces the zero homomorphism if and only if $\gamma = 0$. Thus Γ is considered to be a subset of the total set of R-homomorphisms of M to R; $\Gamma \subset \operatorname{Hom}_{R}(M, R)$.

Similary we can define the left operator ring L of M. We start with (a, γ) and define the product such that $(a, \gamma)(b, \beta) = (a\gamma b, \beta)$. Also we define the left operation such that $(a, \gamma)b = a\gamma b$, and so on. $a\gamma$ is an element of L given from (a, γ) and $a\gamma b = (a\gamma)b$. And we can say that $\Gamma \subset \operatorname{Hom}_L(M, L)$.

DEFINITION. R-submodules of M are called *right ideals* of M, and L-submodules of M are *left ideals*.

A right ideal \mathfrak{r} is nothing but a submodule of M such that $\mathfrak{r}\Gamma M \subset \mathfrak{r}$. A left ideal \mathfrak{l} is a submodule of M such that $M\Gamma\mathfrak{l} \subset \mathfrak{l}$.

4. Peirce decomposition in semi-simple Γ -rings. Assume that M is semi-simple, and let \mathfrak{r} be a minimal right ideal. Then by semi-simplicity there exists an element \mathfrak{E} in Γ such that $a\mathfrak{E}a \neq 0$ for a non-zero element a in \mathfrak{r} . Then $0 \neq a\mathfrak{E}\mathfrak{r} \subset \mathfrak{r}$ and hence $\mathfrak{r} = a\mathfrak{E}\mathfrak{r}$, for \mathfrak{r} is minimal. Therefore $a = a\mathfrak{E}e$ with some element e of \mathfrak{r} . Then $e = e\mathfrak{E}e$, since from $a = a\mathfrak{E}e = (a\mathfrak{E}e)\mathfrak{E}e$

we have $a\mathcal{E}(e-e\mathcal{E}e)=0$ which means $e-e\mathcal{E}e=0$, for a set $\{c|a\mathcal{E}c=0, c\in r\}$ is a right ideal contained in a minimal ideal r and is $\{0\}$. Since $e\in r$, $eR \subset r$, i.e., eR=r. $\mathcal{E}M$ being a right ideal of R, $e\mathcal{E}M$ is a right ideal of M contained in r, and hence $e\mathcal{E}M=r$. Thus we get

Lemma 1. If M is semi-simple and \mathfrak{r} is a minimal right ideal, then $\mathfrak{r}=eR=e\mathcal{E}M$ with $e\in\mathfrak{r}$ and $\mathcal{E}\in\Gamma$, where $e\mathcal{E}e=e$.

Now we use the idea of Peirce decomposition of the ring theory. Suppose that we have a right ideal $r = e \in M$ such that $e \in e = e$. Then

$$M = e \varepsilon M + M_1$$
 (direct),

where $M_1 = \{b | e \in b = 0\}$, since any element *a* of *M* is written

$$a = e \varepsilon a + (a - e \varepsilon a)$$
,

and $e\varepsilon(a-e\varepsilon a)=0$. M_1 is clearly a right ideal of M. Now we can get a decomposition theorem.

Theorem 1. If M is semi-simple and satisfies the minimum condition for right ideals, then

$$M = e_1 R + e_2 R + \dots + e_n R \qquad (direct),$$

where e_iR are minimal right ideals and $e_iR=e_i\varepsilon_iM$, and $e_i\varepsilon_ie_i=e_i$ and $e_i\varepsilon_ie_j=0$ if $i \neq j$.

Proof. Suppose that we have

$$M = e_1 \varepsilon_1 M + \dots + e_{k-1} \varepsilon_{k-1} M + M_{k-1} \qquad \text{(direct)}$$

such that $e_i \varepsilon_i M$ are minimal right ideals and

$$e_i \mathcal{E}_i e_j = egin{cases} e_i & ext{if} \quad i=j \ 0 & ext{if} \quad i\neq j \ , \ \end{cases}$$

and that $e_i \varepsilon_i a = 0$ if $a \in M_{k-1}$ for $i=1, 2, \dots, k-1$. This is true for k=2 as above. Apply the above discussion on M_{k-1} , and we get

$$M_{k-1} = e_k \varepsilon'_k M + M_k$$
 (direct)

as in the above. Here $e_i \varepsilon_i e_k = 0$ if i < k, but we can not say that $e_k \varepsilon'_k e_i = 0$. So, we change ε'_k suitably. Put

$$\mathcal{E}_{\boldsymbol{k}} = \mathcal{E}_{\boldsymbol{k}}' - \mathcal{E}_{\boldsymbol{k}}' (e_1 \mathcal{E}_1 + \dots + e_{\boldsymbol{k}-1} \mathcal{E}_{\boldsymbol{k}-1}).$$

Then we can see that $e_k \varepsilon_k e_k = e_k$ and $e_k \varepsilon_k e_i = 0$. Thus we have a decomposition for k. Since M satisfies the minimum condition for right ideals, we

can get the decomposition in Theorem 1. Similarly we can get

Theorem 1'. If M is semi-simple and satisfies the minimum condition for left ideals, then

$$M = Ld_1 + Ld_2 + \dots + Ld_m \qquad (direct),$$

where Ld_i are minimal left ideals and $Ld_i = M\delta_i d_i$, and $d_i\delta_i d_i = d_i$ and $d_j\delta_i d_i = 0$ if $i \neq j$.

5. Simple Γ -rings. Assume M is simple and satisfies the minimum condition for right and left ideals in this section. First we want to show that e_iR and e_jR are isomorphic as R-modules. M being simple, we can find an element γ in Γ such that $e_i\gamma e_j \pm 0$. Then $e_i\gamma r_j = r_i$ where r_i and r_j are e_iR and e_jR . By a correspondence:

$$(\mathfrak{r}_{j} \ni) x \longrightarrow e_{i} \gamma x (\in \mathfrak{r}_{i})$$

we have a one-one mapping of r_j onto r_i . If $x \neq 0$, $e_i\gamma x \neq 0$, because $\{c | e_i\gamma c = 0, c \in r_j\}$ is a right ideal contained in r_j and is $\{0\}$ as r_j is minimal. This mapping is "onto" because r_i is minimal. Since $x(\beta c) = x\beta c$ corresponds to $e_i\gamma(x\beta c) = (e_i\gamma x)(\beta c)$, this mapping is an *R*-homomorphism, i.e., an *R*-isomorphism. Similarly $Ld_i \simeq Ld_j$ (*L*-isomorphic). Next, we want to show that all *L*-endomorphisms of *M* are given by the right multiplication of *R*. Let ϕ be an *L*-endomorphism of *M* and put $\phi(d_i) = u_i$. Since $d_i = d_i\delta_id_i$, $u_i = d_i\delta_iu_i$. Therefore, $u_i = d_i(\sum_j \delta_j d_j\delta_j u_j)$ where $\sum_j \delta_j d_j\delta_j u_j$ is an element of *R*. On the other hand, by the definition of the right operator ring, *R* is considered to be the set of all *L*-endomorphisms of *M*. Then the ring theory shows us that the latter ring is a matrix ring D_m over a division ring *D*, where D_m is $D_{m,m}$. Matrix units $E_{r,s}$ of D_m map d_r to d_s and d_i to 0 if $t \neq r$.

Now we can determine M with respect to R which is identified with D_m as above. Since minimal right ideals of D_m are $E_{r,r}D_m$, e_iD_m $(=e_iR)$ in Theorem 2)= $e_iE_{r,r}D_m$ with some r. Then put $e_iE_{r,s}=e_{i,s}$. We get $e_{i,s}$ $(i=1, 2, \dots, n; s=1, 2, \dots, m)$ such that

$$e_{i,s}E_{r,t} = \begin{cases} e_{i,t} & s=r, \\ 0 & s \neq r. \end{cases}$$

Thus we can say that $M = \sum_{i,s} e_{i,s}D$, i.e., $e_{i,s}$ are matrix units of $D_{n,m}$ and M is (isomorphic to) $D_{n,m}$ as a right D_m -module.

Next we must determine Γ . An element γ of Γ is considered to

induce a mapping from M to R as in §3, and Γ is considered to be a subset of the set of all R-homomorphisms of $M=D_{n,m}$ to $R=D_m$. On the other hand, D_m -homomorphisms of $D_{n,m}$ to D_m are induced by the left multiplications of elements of $D_{m,n}$. In fact, suppose ϕ is a D_m -homomorphism of $D_{n,m}$ to D_m such that

$$\phi(e_{i,s}) = \sum_{p,q} E_{p,q} T_{p,q}(i,s)$$

with $T_{p,q}(i, s)$ in *D*. Multiply $E_{s,s}$, and we can see $T_{p,q}(i, s)=0$ if $q \neq s$. Multiply $E_{s,t}$, and we can see $T_{p,s}(i, s)=T_{p,t}(i, t)$. Putting $T_{p,s}(i, s)=T_{p}(i)$, we have

$$\phi(e_{i,s}) = \sum_{p} E_{p,s} T_{p}(i) = (\sum_{p,j} e'_{p,j} T_{p}(j)) e_{i,s},$$

where $e'_{p,i}$ are matrix units of $D_{m,n}$ such that

$$e'_{p,j}e_{i,s} = \begin{cases} E_{p,s} & \text{ if } j = i, \\ 0 & \text{ if } j \neq i. \end{cases}$$

Hence ϕ is induced by the left multiplication of an element $A = \sum e'_{p,j} T_p(j)$ of $D_{m,n}$. Identifying γ which induces ϕ and A which corresponds to ϕ , we can say that $\Gamma \subset D_{m,n}$. What we want to show is that $\Gamma = D_{m,n}$. But Γ is a two sided $D_m - D_n$ module and must be identical with $D_{m,n}$. Summarizing all the dicussions, we get the main theorem.

Theorem 2. If M is a simple Γ -ring satisfying the minimum condition for left and right ideals, then M is $D_{n,m}$ and Γ is $D_{m,n}$. The product ayb is the usual matrix product of three elements a, γ and b of $D_{n,m}, D_{m,n}$ and $D_{n,m}$.

6. Semi-simple Γ -rings. Let M be a semi-simple Γ -ring which satisfies the minimum condition for left and right ideals in this section. Arranging suitably, we can see that M is expressed as follows:

$$M = Ld_1^{(1)} + \dots + Ld_{m(1)}^{(1)} + Ld_1^{(2)} + \dots + Ld_{m(2)}^{(2)} + \dots + Ld_1^{(q)} + \dots + Ld_{m(q)}^{(q)},$$

where $d_i^{(j)}$ are some d_k of Theorem 1'. Moreover we take the order such that in the above $Ld_i^{(j)} \cong Ld_k^{(j)}$ (*L*-isomorphic) and $Ld_i^{(j)} \cong Ld_k^{(j')}$ if $j \neq j'$. Then, *R* is, as the right multiplication ring of the *L*-module *M*, equal to a direct ring sum $\sum_j D_{m(j)}^{(j)}$, where $D_{m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $m(j) \times m(j)$. Furthermore $D_{m(j)}^{(j)}$ operate on $Ld_i^{(j)}$ as usual and are zero on $Ld_i^{(j')}$ if $j \neq j'$. On the other hand, we have in Theorem 1 N. NOBUSAWA

$$M = e_1 R + e_2 R + \cdots + e_n R.$$

 $e_i R$ being minimal, $e_i R = e_i D_{m(j)}^{(j)}$ with some *j*. Rearranging the order suitably, we have

$$M = e_1^{(1)}R + \dots + e_{n(1)}^{(1)}R + e_1^{(2)}R + \dots + e_{n(2)}^{(2)}R + \dots + e_1^{(q)}R + \dots + e_{n(q)}^{(q)}R$$
 ,

where $e_i^{(j)}R = e_i^{(j)}D_{m(i)}^{(j)}$ and $e_i^{(j)}$ are some e_k . Hence $n = \sum_j n(j)$. With the same discussion as in §5, we can say that

i.e.,

$$M = D_{n(1),m(1)}^{(1)} + \cdots + D_{n(q),m(q)}^{(q)}$$

where $D_{n(j),m(j)}^{(j)}$ are matrix rings over division rings $D^{(j)}$ of type $n(j) \times m(j)$. Naturally $D_{m(j)}^{(j)}$ operate on $D_{n(j),m(j)}^{(j)}$ as usual and are zero on $D_{n(j'),m(j')}^{(j')}$ if $j \neq j'$. Γ is then a set of *R*-homomorphisms of *M* to *R* and is contained in $\sum_{j} D_{m(j),n(j)}^{(j)}$. Here the product of elements of $D_{m(j),n(j)}^{(j)}$ and of $D_{n(j'),m(j')}^{(j')}$ is performed as usual if j=j' and is 0 if j=j'. On the other hand, the condition of semi-simplicity means that for any non-zero element a of $D_{n(j),m(j)}^{(j)}$ there exists γ in Γ such that $a\gamma a \neq 0$. Now we want to show that each $D_{n(j),m(j)}^{(j)}$ is a simple Γ_j -ring. Let V and V' be left $D^{(j)}$ modules of dim n(j) and of dim m(j). $D_{n(j),m(j)}^{(j)}$ and $D_{m(j),n(j)}^{(j)}$ are considered to be the sets of all $D^{(j)}$ -homomorphisms of V to V' and of V' to V. When we notice that $D_{m(j'),n(j')}^{(j')}$ induce zero mapping on V' if $j \neq j'$, we can say that elements of Γ induce mappings of V' to V. In this case we can show that $X\Gamma = V$ for any subspace X of dim 1 of V'. For, suppose that $X\Gamma \subseteq V$. Then we can find an element a in $D_{n(j),m(j)}^{(j)}$ such that Va=X and $(X\Gamma)a=0$. Then $a\gamma a=0$ for every γ in Γ , which is a contradiction. Now this fact implies the existance of γ such that $a\gamma b \neq 0$ for any non-zero a and b, for we can take a subspace of Va as X and take γ such that $(X\gamma)b \neq 0$. Thus we can conclude that $\Gamma = \sum_{j} D_{m(j),n(j)}^{(j)}$. Now put $\Gamma_{j} = D_{m(j), n(j)}^{(j)}$.

Theorem 3. If M is a semi-simple Γ -ring satisfying the minimum condition for left and right ideals, then M is a direct sum of simple Γ_i -rings where $\Gamma = \Gamma_1 + \cdots + \Gamma_q$ (direct):

$$M=M_{\scriptscriptstyle 1}\!+\!M_{\scriptscriptstyle 2}\!+\!\cdots\!+\!M_{\scriptscriptstyle q} \qquad (direct)$$
 ,

where M_i are simple Γ_i -rings and $M_i\Gamma M_j=0$ if $i \neq j$, and $M_i\Gamma_j M_i=0$ if $i \neq j$.

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