<table>
<thead>
<tr>
<th>Title</th>
<th>Relative projectivity and extendibility of Auslander-Reiten sequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Uno, Katsuhiro</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 25(3) P.499-P.518</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1988</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/12357">https://doi.org/10.18910/12357</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/12357</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
RELATIVE PROJECTIVITY AND EXTENDIBILITY OF AUSLANDER-REITEN SEQUENCES

KATSUHIRO UNO

(Received February 23, 1987)

Introduction

Let $kG$ be the group algebra of a finite group $G$ over a field $k$ of characteristic $p$. For any non-projective indecomposable right $kG$-module $W$, there is a so called Auslander-Reiten sequence $SW: 0 \to \Omega^2 W \to X \to W \to 0$ (exact) terminating at $W$, where $\Omega$ denotes the Heller operator. (See [2, 2.17.6] for the definition of Auslander-Reiten sequences.) From this sequence, we get the exact sequence $0 \to \text{Hom}_{kG}(\cdot, \Omega^2 W) \to \text{Hom}_{kG}(\cdot, X) \to \text{Hom}_{kG}(\cdot, W) \to \text{Ext}^1_{kG}(\cdot, \Omega^2 W)$ of contravariant functors from the category of $kG$-modules into that of $k$-spaces. Those functors and natural transformations among them form a category. This functor category possesses properties similar to those of the category of $kG$-modules. For instance, we can give notions of simplicity, indecomposability and so on for its objects. It is known that the image of the above $\sigma$ is a simple object. Moreover, each simple object of the functor category gives rise to a simple object of the module category or an Auslander-Reiten sequence, and this gives a one-to-one correspondence between the set of isomorphism classes of simple objects of the functor category and the union of the set of isomorphism classes of simple $kG$-modules with the set of equivalence classes of Auslander-Reiten sequences. In this way, Auslander-Reiten sequences are often identified with simple objects of the functor category. In this paper, we consider $SW$ as an Auslander-Reiten sequence and as a simple object simultaneously.

Up to this point, these facts hold for any finite dimensional $k$-algebras if we replace $\Omega^2$ by a certain operator. One can see a brief review of these facts (Auslander-Reiten theory) in [9, §1].

Recently in [9] Green studied Auslander-Reiten theory for group algebras and gave several notions for the functor category, which have analogues in the $kG$-module category. "Restrictions", "inductions" and "trace maps" are examples of them.

In this paper, we consider "relative projectivity" and "extendibility" of Auslander-Reiten sequences for group modules, which can be defined as soon as the above notions are given.
Concerning relative projectivity, Green showed that each Auslander-Reiten sequence $SW$ has "a vertex", which is a $p$-subgroup of $G$ determined uniquely up to $G$-conjugate and that some conjugate of a vertex of $SW$ contains a vertex of $W$ and is contained in the normalizer of a vertex of $W$. Moreover, an analogue of Green correspondence exists. Thus, in order to find a vertex of an Auslander-Reiten sequence $SW$, we may assume that a vertex of $W$ is normal in $G$. (See the first paragraph of [9, §8].)

Concerning extendibility, we are given a normal subgroup $N$ of $G$ and a simple object $SV$ of the functor category corresponding to an indecomposable $kN$-module $V$. We consider when we can extend $SV$ to $G$.

In view of the above, one might notice that we will have to study modules over $kG$ and over $kN$ for a normal subgroup $N$ of $G$. So it seems that Clifford theory is useful. As a matter of fact, using Clifford theory, it can be shown that, if $k$ is sufficiently large, then a simple object corresponding to an $N$-projective indecomposable $kG$-module having a $G$-invariant $N$-source gives a simple module over some twisted group algebra of $G/N$ over $k$.

We can prove that, for any subgroup $H$ of $G$ with $N \subseteq H \subseteq G$, $SW$ is $H$-projective if and only if the simple module given by $SW$ is $H/N$-projective (Theorem 5.4). In certain cases, one can apply this fact to determine a vertex of $SW$.

On the other hand, for any indecomposable $kN$-module $V$, we can see that if $V$ extends to $G$, then so does $SV$. Also, if $V$ is $G$-invariant, then for any indecomposable direct summand $W$ of the induced module $V^G$, $SW$ is an extension of $SV$ if and only if the simple module given by $SW$ is 1-dimensional. (See Theorem 6.1).

We note that the same idea is already used to give a sufficient condition which guarantees that $W$ and $SW$ have vertices in common. ([13, Theorem 2.5].)

This paper is organized as follows. After introducing terminologies and notations in Section 1, we will briefly review the Auslander-Reiten theory (Section 2) and Clifford theory (Section 3). Some results concerning trace maps are proved in Section 4. Relative projectivity and extendibility of simple objects of the functor category are studied in Sections 5 and 6, respectively.

1. Notations and conventions

Throughout this paper, we see the following notations and conventions. $G$ is a fixed finite group and $k$ is a field of characteristic $p, p \neq 0$. All modules considered here are finitely generated and, unless otherwise noted, every module is a right module. $\text{Mod}kG$ denotes the category whose objects are all the (finite dimensional) modules over the group algebra $kG$ and whose morphisms are all the $kG$-homomorphisms among them. For any $kG$-modules $W$ and $W'$,
we use \( \sigma(W, W') \) to denote \( \text{Hom}_{kG}(W, W') \) for notational convenience. For any finite dimensional \( k \)-algebra \( R \), we write \( JR \) to denote its radical, and for any \( R \)-module \( W \), \( \delta_g(W) \) means the dimension of \( \text{End}_R(W)/J \text{End}_R(W) \) over \( k \). Other notations and terminologies in representation theory are standard. (See for example [8].)

In addition to the above, the letters \( N \) and \( V \) are reserved to mean a normal subgroup of \( G \) and a \( kN \)-module, respectively. Usually \( V \) is assumed to be non-projective, indecomposable or \( G \)-invariant. Here we say that \( V \) is \( G \)-invariant if \( V \otimes_{kN} g=V \) as \( kN \)-modules for all \( g \in G \).

Whenever \( V \) is given, \( E \) denotes the \( \Lambda G \)-endomorphism ring of the induced module \( V^G=V \otimes_{kN} kG \). Fix representatives \( G/N \) of cosets of \( N \) in \( G \), and for any subgroup \( H \) of \( G \) with \( N \subset H \subset G \), choose representatives \( G/H \) of left cosets of \( H \) in \( G \). We can and will regard \( V^H \) as a \( kH \)-direct summand of \( V^G \). In fact, by Mackey's theorem, we have the decomposition

\[
V^G = \bigoplus_{x \in G/H} (V \otimes_{kN} x)^H
\]

of \( V^G \) into the direct sum of \( kH \)-submodules \( (V \otimes_{kN} x)^H \). We write, for instance, \( Vx \) instead of \( V \otimes_{kN} x \). Letting \( E_H = \text{End}_{kH}(V^H) \), we can consider \( E_H \) as a subalgebra of \( E \) via the injective \( \Lambda \)-algebra homomorphism \( i_H: E_H \rightarrow E \) given by \( i_H(f) = f \otimes_{kH} \text{Id}_{kG} \). Note that for any \( N \subset H \subset G \), we have \( i_H|_{E_H} = i_H \). Furthermore, if \( V \) is \( G \)-invariant, then for each \( x \in G/N \), there exists a unit \( u_x \) of \( E \) such that \( u_x(V) = Vx \), and \( E \) has a decomposition \( E = \bigoplus_{x \in G/N} u_x E_N \) into the direct sum of \( k \)-subspaces \( E_N = u_x E_N \). Also, there hold \( E_H = \bigoplus_{x \in G/H} E_H u_x = \bigoplus_{x \in G/H} u_x E_H \) and \( E = \bigoplus_{x \in G/H} E_H u_x = \bigoplus_{x \in G/H} u_x E_H \). For these facts see for example [4] and [5].

For a \( kG \)-module \( W \), let \( a = a_H(W) \) be the isomorphism

\[
a_H(W): \sigma(V^G, W) \rightarrow \sigma(V^H, W)
\]
in the Frobenius reciprocity law. Note that \( a \) is an isomorphism of \( \text{End}_{kG}(W)\)-\( E_H \)-bimodules. For each \( f \in \sigma(V^G, W) \), the image of \( f \) by \( a \) is denoted by \( f^a \).

Also for any \( kG \)-modules \( W \) and \( W' \), we use \( t_{W,W'}^H(V) \) to denote the usual trace map from \( \sigma(W, W') \) into \( \sigma(W, W') \). Note that, since we are taking representatives of left cosets of \( H \) in \( G \), for any \( f \in \sigma(W, W') \), \( t_{W,W'}^H(f) \) sends any \( \omega \in W \) into \( \sum_{x \in G/H} f(\omega x) x^{-1} \).

Regarding \( (Vx)^H \) as a \( kH \)-direct summand of \( V^G \), we may apply \( t_{W,W'}^H(V^G, W) \) to elements of \( \sigma(H((Vx)^H, W)) \). In doing so, of course, each element of \( \sigma(H((Vx)^H, W)) \) is considered to vanish on \( (Vy)^H \) for all \( y \in G/H \) with \( y \neq x \). When \( V \) is \( G \)-invariant, we let \( s_x (x \in G/H) \) be the element of \( \sigma(H(V^G, V^H)) \) defined by
Lemma 1.2. Suppose that $V$ is $G$-invariant. Let $W$ be a $kH$-module. Then we have $H(V^G, W) = \bigoplus_{y \in G/H} H(V^H, W) s_x$.

Proof. By (1.1) it suffices to see $H(V^H, W) s_x = H((Vx)^H, W)$. But this is clear because the restriction of $s_x$ to $(Vx)^H$ gives an isomorphism from $(Vx)^H$ onto $V^H$.

Lemma 1.3. Suppose that $V$ is $G$-invariant. Let $W$ be a $kG$-module. Then for any $f \in H(V^G, W) \subseteq H(V^G, W)$, we have

$$t_H^G(V^G, W)(fs_x) = f^{-1}u_x^{-1}$$

for all $x \in G/H$.

In particular, we get

$$t_H^G(V^G, W)(f) = f^{-1}.$$

Proof. For any $v \in Vx$, we have

$$t_H^G(fs_x)(v) = \sum_{y \in G/H} fs_x(vy) y^{-1} = fs_x(v),$$

since $s_x$ vanishes on $Vxy$ if $y \notin H$. It follows by the definition of $s_x$ that $t_H^G(fs_x)$ agrees with $fu_x^{-1}$ on $Vx$, and hence it is equal to $f^{-1}u_x^{-1}$ on $Vx$ by the definition of $a$. Since both $t_H^G(fs_x)$ and $f^{-1}u_x^{-1}$ are $kG$-homomorphisms, they must coincide with each other. The last statement holds since we can take the identity element of $E$ for $u$. Now the proof is complete.

2. Preliminaries for the Auslander-Reiten and Green theories

In this section, we review a part of the Auslander-Reiten and Green theories following [9], while some of the results will be stated in a way convenient for our use.

Let $\text{MMOD}_k G$ be the category whose objects are all the $k$-linear contravariant functors from $\text{MOD}_k G$ into $\text{MOD}_k k$ and whose morphisms are all the natural transformations between those functors. It is known from the Auslander-Reiten theory that for each indecomposable $kG$-module $W$, there is a simple object $SW$ of $\text{MMOD}_k G$, which is unique up to isomorphisms, such that $\iota(\cdot, W)$ is the projective cover of $SW$. Moreover, each simple object appears in this way. Furthermore, $SW$ corresponds to a simple $kG$-module, if $W$ is projective, or to an Auslander-Reiten sequence terminating at $W$, otherwise. Recall also that any $SW$ is known to be finitely presented.

For any non-projective indecomposable module $W$, we can obtain $SW$ as
the image of some $\alpha: G(\cdot, W) \to D_G(W, \cdot)$, where $D_G(W, \cdot)$ is the space dual to $G(\cdot, W)$ ([9, Theorem 1.7]). However, when $W$ is a direct summand of $V^G$, it seems better to use $D_G(V^G, \cdot)$ instead of $D_G(W, \cdot)$. The following should be compared with [9, Theorem 1.8].

**Proposition 2.1.** Let $e$ be a primitive idempotent of $E$ and let $W = eV^G$. Then a functor $\gamma = \gamma_e: G(\cdot, W) \to D_G(V^G, \cdot)$ satisfies $\text{Im} \gamma = SW$ if and only if

(i) $\gamma(W)(\text{Id}_W) \neq 0$, and

(ii) $\gamma(W)(\text{Id}_W)(eJ) = 0$.

**Proof.** By an argument similar to the one in the proof of [9, Theorem 1.8], $\text{Im} \gamma = SW$ if and only if

$$f \circ \alpha(W, X) \subseteq J(eJ) \Leftrightarrow \gamma(W)(\text{Id}_W)(f \circ \alpha(V^G, X)) = 0,$$

for any $kG$-module $X$ and $f \in \alpha(X, W)$. Thus it suffices to show that

$$f \circ \alpha(W, X) \subseteq J(eJ) \Leftrightarrow f \circ \alpha(V^G, X) \subseteq eJ.$$

Now, notice that $f \circ \alpha(V^G, X)$ is an $E$-submodule of $eE$. Since $eE$ has the unique maximal submodule $eJ$, the above is easy to see.

**Remark.** If $\gamma: G(\cdot, W) \to D_G(V^G, \cdot)$ satisfies (i) and (ii) of Proposition 2.1, we have the following exact sequence in $\text{MMod } kG$.

$$0 \to \text{rad}_G(\cdot, W) \to \alpha(\cdot, W) \xrightarrow{\gamma} SW \to 0$$

Here, for each $kG$-module $X$, $\text{rad}_G(X, W) = \{f \in \alpha(X, W): fg \in J \text{End}_G(W) \text{ for all } g \in \alpha(W, X)\}$. See [9, Theorem 1.4].

Let $W$ and $W'$ be $kG$-modules. Then each $f \in \alpha(W, W')$ gives a morphism $f_\#: \alpha(\cdot, W) \to \alpha(\cdot, W')$, which is defined as follows. For any $kG$-module $X$ and any $\phi \in \alpha(X, W)$, $f_\#(\phi) = f \circ \phi$. Yoneda's lemma says that the above map $f \to f_\#$ is bijective, namely;

**Lemma 2.2 ([9, 1.1]).** For any $kG$-modules $W$ and $W'$, the map

$$Y(W, W') : \alpha(W, W') \to \text{Hom}(\alpha(\cdot, W), \alpha(\cdot, W'))$$

given by $Y(W, W')(f) = f_\#$ for all $f \in \alpha(W, W')$ is an isomorphism. Moreover, if $W = W'$, then $Y(W, W')$ is an isomorphism of $k$-algebras.

Let $H$ be a subgroup of $G$. Then at any $kH$-module $X$, the restriction $(\alpha(\cdot, W))_H$ of $\alpha(\cdot, W)$ to $H$ has the value $\alpha(X^G, W)$ by its definition ([9, §2]). Moreover, the isomorphism $\alpha(X, W): \alpha(X^G, W) \to \mu(X, W)$ in the Frobenius reciprocity law yields $\alpha(\cdot, W)_H \cong (\alpha(\cdot, W))_H$ ([9, Prop. 2.12]). Hence, for any
$f \in \mathcal{H}(W, W')$, the composite $a(\cdot, W')^{-1} Y(W_H, W'_H)(f) a(\cdot, W)$ gives a morphism from $(\mathcal{G}(\cdot, W))_H$ into $(\mathcal{G}(\cdot, W'))_H$. Furthermore, the map sending any such $f$ into $a(\cdot, W')^{-1} Y(W_H, W'_H)(f) a(\cdot, W)$ is clearly an isomorphism from $\mathcal{H}(W, W')$ onto $\text{Hom}((\mathcal{G}(\cdot, W))_H, (\mathcal{G}(\cdot, W'))_H)$. We denote this isomorphism by $Y(W_H, W'_H)$. Note that for any $kH$-module $X$, $Y_H(W_H, W'_H)(f)(X)$ is a homomorphism from $\mathcal{G}(X^G, W)$ into $\mathcal{G}(X^G, W')$.

Next, we recall some properties of trace maps and relative projectivity for $\text{MMod } kG$. In [9, §5] Green gave a definition of trace maps. For any objects $F$ and $F'$ of $\text{MMod } kG$ and any subgroup $H$ of $G$, the trace map $T^G_H = T^G_H(F, F')$ is a $k$-linear map from $\text{Hom}(F_H, F'_H)$ into $\text{Hom}(F, F')$.

One of his interesting results is as follows.

**Lemma 2.3** ([9, Prop. 6.4]). For any $kG$-modules $W$ and $W'$ and any $f \in \mathcal{H}(W, W')$, we have $T^G_H(Y_H(W, W')(f)) = Y(W, W')(t^G_H(f))$.

We shall study more about the trace maps in Section 4.

Using the notion of induction, we can give a definition of relative projectivity for $\text{MMod } kG$. Here we remark that relative projectivity is defined only among finitely presented objects of $\text{MMod } kG$. Moreover, it can be shown that "Higman's criterion" exists [9, Th. 5.11]. Hence a (finitely presented) object of $\text{MMod } kG$ is $H$-projective if and only if its identity automorphism lies in the image of the trace map $T^G_H$.

If $F$ is a finitely presented indecomposable object of $\text{MMod } kG$, then there is a unique (up to $G$-conjugate) $p$-subgroup $P$ of $G$ such that $F$ is $H$-projective if and only if $H \subseteq G$. ([9, Theorem 4.7]) We call this $P$ a vertex of $F$ and denote it by $vtx(F)$.

Let $W$ be an indecomposable $kG$-module. Then $SW$ is finitely presented and simple. Now we have the following result which follows from [9, 5.12 and 7.7].

**Proposition 2.4.** (i) There holds $vtx(W) \subseteq G \subseteq vtx(SW) \subseteq G I$, where $I$ is the inertial subgroup of a $vtx(SW)$-source of $W$ in $N_G(vtx(W))$.

(ii) Let $W''$ be the $kN_G(vtx(W))$-module that corresponds to $W$ via the Green correspondence with respect to $(G, vtx(W), N_G(vtx(W)))$, and let $W''$ be a $kI$-module such that $W''^N = W'$ and that $W'$ and $W''$ have vertices in common. (Note: Such $W''$ always exists.) Then we have $vtx(SW) = G vtsx(SW') = vtsx(SW'')$.

By the above proposition, some problems concerning $vtx(SW)$ can be reduced to the case where $vtx(W)$ is normal and a $vtx(W)$-source of $W$ is $G$-invariant. If it is the case, letting $V$ be a source of $W$, we should consider direct summands of $V^G$. For this, Clifford theory (see Section 3) is useful. Using the above technique, we shall investigate vertices of simple objects of $\text{MMod } kG$. 


in Section 5.

The final result in this section is the following, which will be used in Sections 5 and 6. This is a special case of [9, Prop. 7.9].

**Lemma 2.5.** Let $N$ be a normal subgroup of $G$ and fix a $G$-invariant indecomposable $kN$-module $V$. Suppose that an indecomposable $kG$-module $W$ has multiplicity $r(r>0)$ as a direct summand of $V^G$. Then we have $(SW)_N=\delta r(SV)$, where $\delta=\delta_{kG}(W)/\delta_{kN}(V)$.

3. Clifford theory

In this section we review Clifford theory. Following [5], we state the results in terms of group-graded rings and modules. The main theorem in Clifford theory is Theorem 3.4 below. We also give a criterion on extendibility of modules.

Let $H$ be a finite group. If a ring $R$ (with 1) has a direct sum decomposition $R=\bigoplus_{h\in H} R_h$ into additive subgroups $R_h$, $h\in H$, such that $R_h R_{h'}=R_{h h'}$ for all $h, h'\in H$, we say that $R$ is a (fully) $H$-graded ring. For those $R$, it is clear that $R_1$ is a subring of $R$ and $R_h$ is an $R_1$-$R_1$-bimodule.

Note that $kG$ is a fully $G/N$-graded ring with the decomposition $kG=\bigoplus_{x \in G/N} kx$.

For any ring $R$, let $R^*$ denote the unit group of $R$.

In the rest of this section, $R$ is always assumed to be a fully $H$-graded ring.

We set $GU(R)=\bigcup_{h\in H}(R_h \cap R^*)$ and define a map $d_\cdot: GU(R)\to H$ by $d(r)=h$ if $r\in R_h$, $h\in H$. The elements of $GU(R)$ form a subgroup of $R^*$. It is not difficult to see that

$$\mathcal{X}(R)^*: 1 \to R^*_1 \to GU(R) \to H \to 1$$

is exact except possibly at $H$. ([5, Prop. 5.2].)

Let $X$ be an $R_1$-module. The induced $R$-module $X^R=X\otimes_{R_1} R$ has a decomposition $X^R=\bigoplus_{h\in H} X\otimes_{R_1} R_h$ into a direct sum of $R_1$-submodules $X\otimes_{R_1} R_h$. Now as in [5, §3, 4, 5] we have the following. (See also the first half of Section 1.)

(3.1) $\text{End}_R(X^R)=\bigoplus_{h\in H} E_h$, 

where $E_h=\{\phi \in \text{End}_R(X^R)| \phi(X\otimes_{R_1} R_h) \subset X\otimes_{R_1} R_{h'}$ for all $h'\in H\}$. As in Section 1, we put $E=\text{End}_R(X^R)$ for convenience. It is easy to see that the map from $\text{End}_{R_1}(X)$ into $E_1$ sending any $\Phi \in \text{End}_{R_1}(X)$ into $\Phi \otimes_{R_1} \text{Id}_R$ gives a ring
isomorphism. Moreover, if $X$ is $R$-invariant, i.e., $X$ is isomorphic to each $X \otimes_{R_h} R_h$, $h \in H$, as $R_i$-modules, then $E$ is fully graded with the decomposition (3.1). Furthermore, if this is the case, the sequence

$$\mathcal{X}(E) : 1 \rightarrow E^* \rightarrow GU(E) \rightarrow H \rightarrow 1.$$ 

is exact ([5, §4, 5]).

On the extendibility of $X$, we have;

**Theorem 3.2** ([6, Theorem 2.8]). $X$ extends to $R$ if and only if $\mathcal{X}(E)$ splits.

**Remark.** Suppose that $\mathcal{X}(E)$ is a split exact sequence. Let $\gamma : H \rightarrow GU(E)$ be a splitting homomorphism. Then for each $h, h' \in H$, the restriction of $\gamma(h)$ to $X \otimes_{R_h} R_h$, gives an $R_i$-isomorphism from $X \otimes_{R_h} R_{h'}$ onto $X \otimes_{R_h} R_{h''}$.

As a corollary to the above theorem, we obtain;

**Corollary 3.3.** Assume further that $R$ is a finite dimensional $k$-algebra and each $R_h$ is a $k$-subspace of $R$ and is free as a left $R_i$-module. Then, if $X$ extends to $R$, then $X=X/JX$ extends to $R$. Here $JX$ denotes the radical of $X$.

**Proof.** By Theorem 3.2, we have a splitting homomorphism $\gamma : H \rightarrow GU(E)$. Since $R$ is fully graded, it is easy to see that the radical of the $R_i$-module $X \otimes_{R_i} R_h$ is precisely $JX \otimes_{R_i} R_h$ for all $h$ in $H$. Hence by the remark following Theorem 3.2, each $\gamma(h)$ sends $JX \otimes_{R_i} R_{h'}$ into $JX \otimes_{R_i} R_{h''}$ for all $h' \in H$. The sum $\bigoplus_{h \in H} JX \otimes_{R_i} R_h = (JX)^R$ is an $R$-submodule of $X^R$ and it follows that

$$X^R/(JX)^R = \bigoplus_{h \in H}(X/JX) \otimes_{R_i} R_h = (X/JX)^R = X^R.$$

Since $(JX)^R$ is fixed by all $\gamma(h)$, each $\gamma(h)$ gives an $R$-automorphism $\gamma(h)$ of $X^R$. Therefore, by the choice of $\{\gamma(h)\}$, the map $\tilde{\gamma} : H \rightarrow GU(End_R(X^R))$ defined by $\tilde{\gamma}(h) = \gamma(h), h \in H$, gives a splitting homomorphism for the sequence $\mathcal{X}(End_R(X^R))$. Now the proof is completed by Theorem 3.2.

In the case where $X$ is $R$-invariant so that $E(=End_R(X^R))$ is fully $H$-graded, we have a nice correspondence theorem originally due to Clifford. Before we state it, we introduce some notations.

Let $\text{Mod}(R | X)$ be the additive full subcategory of $\text{Mod} R$ whose objects are those $R$-modules such that their restrictions to $R_i$ are isomorphic to direct summands of direct sums of some copies of $X$. Also $\text{Mod}(E | E_i)$ denotes the additive full subcategory of $\text{Mod} E$ whose objects are those $E$-modules such that their restrictions to $E_i$ are projective $E_i$-modules.

Regarding $X^R$ as an $E$-$R$-bimodule naturally, for any $R$-module $W$, $\text{Hom}_R(X^R, W)$ can be considered as an object of $\text{Mod} E$. And for any $E$-module $Y$,
the tensor product $Y \otimes^\mathbb{E} X^R$ is an object of $\text{ModR}$. In fact, $\text{Hom}_R(X^R, \cdot)$ (resp. $\cdot \otimes^\mathbb{E} X^R$) is an additive functor from $\text{Mod R}$ (resp. $\text{Mod E}$) into $\text{Mod E}$ (resp. $\text{Mod R}$). (See [5, §7] for detail.)

The following is the main theorem of Clifford theory.

**Theorem 3.4.** Suppose that an $R_1$-module $X$ is $R$-invariant. Then:

1. The restrictions of $\text{Hom}_R(X^R, \cdot)$ and $\cdot \otimes^\mathbb{E} X^R$ give an equivalence between $\text{Mod}(R \mid X)$ and $\text{Mod}(E \mid E_1)$.
2. Assume that an object $W$ of $\text{Mod}(R \mid X)$ corresponds to an object $Y$ of $\text{Mod}(E \mid E_1)$ under the equivalence. Then we have a ring isomorphism $\text{End}_R(W) \cong \text{End}_E(Y)$.

The definition of equivalence used here can be found on page 65 of [7].

Proof. (i) This is [5, Theorem 7.4].

(ii) This is an immediate consequence of (i).

**Corollary 3.5.** Suppose that $X$ is $R$-invariant. Then $X$ extends to $R$ if and only if the regular $E_1$-module $E_1$ extends to $E$.

Proof. It is obvious that extensions of $X$ are objects of $\text{Mod}(R \mid X)$. One can show easily that extensions of $X$ correspond to extensions of $E_1$ under the equivalence. (See also [12, Cor. 3.16].)

**Remark 3.6.** If $X$ is $R$-invariant, then $X^R$ is an object of $\text{Mod}(R \mid X)$. Since $\text{Hom}_R(X^R, X^R)=E$, the regular $E$-module $E$ is the object of $\text{Mod}(E \mid E_1)$ that corresponds to $X^R$ under the equivalence. Since both $\text{Hom}_R(X^R, \cdot)$ and $\cdot \otimes^\mathbb{E} X^R$ are additive, every direct summand of $X^R$ (resp. $E$) is an object of $\text{Mod}(R \mid X)$ (resp. $\text{Mod}(E \mid E_1)$). Moreover, if an indecomposable $R$-module $W$ is a direct summand of $X^R$ and corresponds to an indecomposable direct summand $Y$ of $E$ under the equivalence, then $W$ and $Y$ have the same multiplicity in $X^R$ and $E$.

Finally we consider so called Nakayama relations. Now we further assume that our ring $R$ is a (finite dimensional) $k$-algebra and that each $R_1$ is a $k$-subspace of $R$. Let $\{P_i\}$ and $\{Q_i\}$ be basic sets of non-isomorphic projective indecomposable $R$- and $R_1$-modules, respectively. And let $P_i=P_i JP_i$ and $Q_i=Q_i JP_i$. Then an argument similar to the one in the proof of [8, III, Theorem 2.6] can be applied to obtain the following.

**Theorem 3.7.** Suppose that $Q_i^R=\oplus a_i P_i$, and let $a_i$ be the multiplicity of $Q_i$ in a composition series of $(P_i)_{E_1}$. Then we have $a_i \delta_R(P_i)=a_i \delta_{E_1}(Q_i)$.

**4. Trace maps**

In this section we study trace maps in various categories. The main
purposes of this section are to give a definition of a trace map in the category Mod\(E\) and to show that an assertion similar to Lemma 2.3 holds. After this, we consider the trace maps for quotient objects.

Throughout this section, we assume that \(V\) is \(G\)-invariant and fix a subgroup \(H\) of \(G\) with \(N \subset H \subset G\). As in Section 1, we put \(E_H = \text{End}_k(H(V^G))\) which is regarded as a subalgebra of \(E\), and choose units \(u_x, x \in G/\right\rangle N\), of \(E\) with \(E = \bigoplus_{x \in G/\right\rangle N} E_{N} u_x = \bigoplus_{x \in G/\right\rangle N} u_x E_{N}\).

We now define a trace map for \(E\)-modules. Let \(Y\) and \(Y'\) be \(E\)-modules. Then \(\tau^0_Y = \tau^0_Y(Y, Y'): \text{Hom}_{E_H}(Y, Y') \rightarrow \text{Hom}_{E}(Y, Y')\) is defined as follows.

\[
\tau^0_H(\xi)(y) = \sum_{x \in G/\right\rangle N} \xi(y u_x) u_x^{-1}
\]

for all \(\xi \in \text{Hom}_{E_H}(Y, Y')\) and \(y \in Y\).

Remark. (i) It is an easy exercise to check that the above \(\tau^0_H\) does not depend on the choice of \(\{u_x\}_{x \in G/\right\rangle H}\).

(ii) Once we obtain a notion of a trace map, we can give a definition of relative projectivity for Mod\(E\). An \(E\)-module \(Y\) is \(H/\right\rangle N\)-projective if \(Id_Y\) lies in \(\tau^0_H(\text{End}_{E_H}(Y))\).

Let \(W\) and \(W'\) be objects of Mod\((kG|V)\) and let \(Y = \rho(V^G, W)\) and \(Y' = \rho(V^G, W')\). Then by Theorem 3.4, \(\rho(W, W') \sim \text{Hom}_E(Y, Y')\) as additive groups. We denote this isomorphism by \(z = z(W, W')\). This \(z\) is described as follows. For any \(f \in \rho(W, W')\) and \(y \in Y = \rho(V^G, W)\), we have \(z(f)(y) = fy\).

Notice that \(W_H\) is an object of Mod\((kH|V)\), and that the Frobenius reciprocity law yields \(a_H(W): H(V^H, W) \sim H(V^G, W) = Y\) as \(E_H\)-modules. Thus again by Theorem 3.4, \(W_H\) corresponds to the \(E_H\)-module \(Y_{EH}\) and we have \(z_H = z_H(W, W'): H(W_H, W') \sim \text{Hom}_{E_H}(Y, Y')\). To describe \(z_H\) explicitly, recall that the action of \(E_H\) on \(Y\) is given via the isomorphism \(a_H(W)\). Thus for any \(f \in \rho(H(W, W')\) and any \(y \in Y\), we have \(z_H(f)(y) = (fy^H)^{a_H^{-1}}\).

Now we have the following diagram.

\[
uo(W, W') \xrightarrow{z} \text{Hom}_{E}(Y, Y') \xrightarrow{\tau^0_H} \text{Hom}_{E_H}(Y, Y') \xrightarrow{z_H} \text{Hom}_{E_H}(Y, Y')
\]

Lemma 4.1. The above diagram commutes.

Proof. Let \(f \in \rho(H(W, W'))\) and \(y \in Y\). By the definition of \(\tau\) we have

\[
(\tau^0_H z_H(f))(y)(v) = \sum_{x \in G/\right\rangle H} (f(y u_x)^{x^{-1}}) u_x^{-1}(v)
\]

for all \(v \in V^G\). Suppose that \(v \in V\). Then we have \(u_x^{-1}(v) = v' x^{-1}\) for some \(v' \in V\). Hence we have
Therefore we get

\[(\tau_H^g z_H(f))(y) = \sum_{v \in G/H} (fy(vx)) x^{-1}\]

for all \(v \in V\). Since both \((\tau_H^g z_H(f))(y)\) and \(t_H^g(f) y\) are \(kG\)-homomorphisms from \(V^G\) into \(W'\), they must agree on \(V^G\). Hence we obtain \(\tau_H^g z_H(f)(y) = t_H^g(f) y = z(t_H^g(f))(y)\) for all \(y \in Y\). The proof is now complete.

Next, we study the trace maps for quotient objects. The following lemma is easy to show and we omit the proof.

**Lemma 4.2.** Let \(W'\) be a \(kG\)-submodule of \(W\). Suppose that \(f \in H(W, W)\) satisfies \(f(W') \subseteq W'\). Then \(f\) induces \(f \in H(W/W', W/W')\), \(t_H^g(f)(W') \subseteq W'\), and \(t_H^g(f)\) naturally induces \(t_H^g(f)\).

**Remark.** Replacing \(kG\) and \(kH\) by \(E\) and \(E_H\) respectively, we can prove a similar statement for \(T_H^g\).

It is nontrivial to show a similar assertion for \(T_H^g\), while this is essentially discussed in the first half of [9, §6]. Before proving it, we must give a notion that an endomorphism of an object \(\sigma(\cdot, W)\) of \(\operatorname{MMod} kG\) “preserves” a subobject of it.

Let \(K\) be a subobject of \(\sigma(\cdot, W)\). Then an element \(f \in \operatorname{End}_{kH}(W)\) (or the corresponding element \(Y_H(W, W)(f)\) of \(\operatorname{End}((\sigma(\cdot, W))_H)\)) is said to preserve \(K\) if the following holds.

\[f(K_H(X)) \subseteq K_H(X) \quad \text{for all } kH\text{-module } X.\]

More precisely, by the definition of the restriction in \(\operatorname{MMod} kG\), the above is equivalent to:

\[(f g_{a(X, W)}) a(X, W)^{-1} \subseteq K_H(X) = K(X^G)\]

for all \(g \in K_H(X) = K(X^G) \subseteq \sigma(X^G, W)\), where \(a(X, W)\) is the isomorphism from \(\sigma(X^G, W)\) onto \(H(X, W)\) in the Frobenius reciprocity law. In other words, \(f\) preserves \(K\) if and only if
Lemma 4.3. In the above situation, suppose that \( f \in \text{End}_H(W) \) preserves \( K \). Then \( f \) naturally induces \( f^* \in \text{End}(\langle \cdot, W \rangle/K)_H \), \( t^H(f) \) preserves \( K \), and \( t^G_H(f) \) naturally induces \( T^G_H(f^*) \).

Proof. Since \( f \) preserves \( K \), it clearly induces some \( f^* \in \text{End}(\langle \cdot, W \rangle/K)_H \), and by the above argument, \( f^* \) must come from \( Y_H(W, W)(f) \). To see the other statements, taking any \( kG \)-module \( X \), consider the following sequence.

\[
(4.3)' \quad \phi(X, W) \xrightarrow{n} \phi(X_H^G, W) \xrightarrow{Y_H(W, W)(f)(X_H)} \phi(X_H^G, W) \xrightarrow{m} \phi(X, W).
\]

Here the maps \( n \) and \( m \) are defined as follows.

\[
n(g)(\sum_{s \in G/H} \alpha_s \otimes x) = g(\sum_{s \in G/H} \alpha_s \otimes x) \quad \text{and} \quad m(h)(\alpha) = h(\sum_{s \in G/H} \alpha_s \otimes x^{-1})
\]

for all \( g \in \phi(X, W) \), \( h \in \phi(X_H^G, W) \) and \( \alpha, \alpha_s \in X \). An easy calculation shows that \( m \) \( Y_H(W, W)(f)(X_H) \cdot n(g) = t^H(f) g \) for all \( g \in \phi(W, X) \). On the other hand, since \( m \) and \( n \) are natural and since \( f \) preserves \( K \), \( (4.3)' \) induces

\[
(4.3)'' \quad K(X) \xrightarrow{n} K(X_H^G) \xrightarrow{Y_H(W, W)(f)(X_H)} K(X_H^G) \xrightarrow{m} K(X).
\]

Hence \( t^H(f) \) preserves \( K \). Now let \( Q \) be the quotient object \( \phi(\cdot, W)/K \). Noticing \( Q(X_H^G) = Q_H(X_H) \), \( (4.3)' \) and \( (4.3)'' \) give

\[
Q(X) \xrightarrow{n'} Q_H(X_H) \xrightarrow{f^*(X_H)} Q_H(X_H) \xrightarrow{m'} Q(X).
\]

Since \( Q_H(X_H) = Q_H(X) \), it follows from the definition of the trace map [9, 5.6] that the composite of the above sequence is precisely \( T^G_H(f^*)(X) \). On the other hand, the early computation yields that the composite of the above takes any \( g \in Q(X) \) into \( t^H_H(f)(g) \in Q(X) \), where \( f \) means the image of \( f \) under the natural epimorphism from \( \phi(X, W) \) onto \( Q(X) \). Therefore, \( t^H_H(f) \) induces \( T^G_H(f^*) \in \text{End}(\phi(X_H^G)/K) \). Now the proof is complete.

5. Relative projectivity of simple objects

In this section, we study relative projectivity of \( SW \) for a fixed non-projective indecomposable \( kG \)-module \( W \). As remarked in Section 2, to determine a vertex of \( SW \), we may consider the case where a vertex of \( W \) is normal in \( G \) and a source of \( W \) is \( G \)-invariant. Thus in this section, we assume that \( W \) is a direct summand of \( V^G \) and that \( V \) is \( G \)-invariant.

Let \( e \) be the primitive idempotent of \( E = \text{End}_{kG}(V^G) \) corresponding to \( W \) so
that \( eV^G = W \). As before we fix a subgroup \( H \) of \( G \) with \( N \subset H \subset G \). The results in the previous section and [9, Prop. 6.4] give the following commutative diagram.

\[
\begin{array}{cccc}
\text{End}(G(\cdot, W)) & \xrightarrow{Y(W, W)} & \text{End}_{k}(W) & \xrightarrow{\tau} \text{End}_{k}(eE) \\
T^G_H & \xrightarrow{\theta} & \text{End}_{k}(W) & \xrightarrow{\tau} \text{End}_{k}(eE) \\
\text{End}((G(\cdot, W))_H) & \xrightarrow{Y^H(W, W)} & \text{End}_{k}(W) & \xrightarrow{\tau} \text{End}_{k}(eE)
\end{array}
\]

For simplicity we write \( Y_H \) to denote \( Y_H(W, W) \). The following is a key result.

**Theorem 5.2.** For any subgroup \( H \) of \( G \) with \( N \subset H \subset G \), the composite \( z_H Y_H \) induces an isomorphism

\[ \theta_H : \text{End}((SW)_H) \rightarrow \text{End}_{k}(eE/eJE) \]

of \( k \)-algebras such that the following diagram commutes.

\[
\begin{array}{cccc}
\text{End}(SW) & \xrightarrow{\theta} & \text{End}_{k}(eE/eJE) \\
T^G_H & \xrightarrow{\theta} & \text{End}_{k}(W) & \xrightarrow{\tau} \text{End}_{k}(eE/eJE) \\
\text{End}((SW)_H) & \xrightarrow{\theta} & \text{End}_{k}(eE/eJE)
\end{array}
\]

We first prove the following.

**Proposition 5.3.** Let \( f \) be an element of \( \text{End}_{kH}(W) \). Then the following (i) (resp. (iii)) is equivalent to (ii) (resp. (iv)).

(i) \( z_H(f) \) preserves \( eJE \).

(ii) \( Y_H(f) \) preserves \( \text{rad}G(\cdot, W) \).

(iii) \( z_H(f)(eE) \subset eJE \).

(iv) \( Y_H(f)(G(\cdot, W))_H \subset (\text{rad}G(\cdot, W))_H \).

Proof. We rewrite the above conditions as follows. By the explicit description of \( z_H \) in the paragraph preceding Lemma 4.1, (i) is equivalent to

(i)' \( f(eJE)^s e^{-1} \subset eJE \).

Moreover, since \( f(eJE)^s \subset H(V^H, W) \), it follows by Lemma 1.3 that (i)' is equivalent to

(i)'' \( I^H(f(eJE)^s \subset eJE \).

Similarly, (iii) is equivalent to

(iii)' \( t^0_H(f(eE)^s) \subset eJE \).

Next, we claim that (ii) is equivalent to

(ii)' For any \( kH \)-module \( X \) and any \( h \in H(X, W_H) \),

\[ t^0_H(h_H(V^H, X)) \subset eJE \text{ implies } t^0_H(fh_H(V^H, X)) \subset eJE. \]
Let $K = \text{rad}_G(\cdot, W)$. Then by Proposition 2.1, we have

$$0 \to K_H(X) \to H(X, W) \xrightarrow{\gamma_H(X)} (D(V^g, X))_H \quad \text{(exact)}.$$

Also let $T_\gamma = \gamma(W)(\text{Id}_W) \in D(V^g, W)$. Now (ii) holds if and only if $\gamma_H(X)(fh) = 0$ in $(D(V^g, X))_H$ for those $h \in H(X, W)$ with $\gamma_H(X)(h) = 0$. Now recall that $\gamma_H(X)(h)(g) = \gamma_H(W_H)(\text{Id}_W)(hg) - T_\gamma(hg)$ for all $g \in H(V^g, X)$ and that $T_\gamma = T_\gamma t_\theta^g[9, \text{Prop. 6.7}]$. On the other hand, since $t_\theta^g(h_n(V^g_H, X))$ and $t_\theta^g(fh_n(V^g_H, X))$ are right $E$-submodules of $eE$, Proposition 2.1 yields that $\text{Ker} T_\gamma$ includes $t_\theta^g(h_n(V^g_H, X))$ (resp. $t_\theta^g(fh_n(V^g_H, X))$) if and only if $eE$ includes $t_\theta^g(h_n(V^g_H, X))$ (resp. $t_\theta^g(fh_n(V^g_H, X))$). Thus (ii) is equivalent to (ii)'.

A similar argument shows that (iv) is equivalent to (iv)' For any $kH$-module $X$ and any $h \in H(X, W_H)$,

$$t_\theta^g(fh_n(V^g_H, X)) \subset eE.$$

We will prove that (i)'' (resp. (iii)'') is equivalent to (ii)' (resp. (iv)'').

Next we claim that for any $kH$-module $X$ and any $h^g \in H(X, W_H)$, we have

$$(5.3)' \quad t_\theta^g(fh_n(V^g_H, X)) = \sum_{s \in G/H} t_\theta^g(fh_n(V^g_H, X)) u^{-1}.$$

In fact, the left hand side of (5.3)' is equal to

$$\sum_{s \in G/H} t_\theta^g(fh_n(V^g_H, X)) s, \quad \text{(by Lemma 1.2.)}$$

$$= \sum_{s \in G/H} (fh_n(V^g_H, X))^s u^{-1}, \quad \text{(by Lemma 1.3),}$$

which is equal to the right hand side of (5.3)' again by Lemma 1.3. Since (5.3)' holds for any choice of $f \in \text{End}_G(W)$, we also have

$$t_\theta^g(h_n(V^g_H, X)) = \sum_{s \in G/H} t_\theta^g(h_n(V^g_H, X)) u^{-1}.$$

Furthermore, taking $X = V^H$, we obtain

$$(5.3)'' \quad t_\theta^g(h_n(V^g_H, V^H)) = \sum_{s \in G/H} (hE_H)^s e^{-1} u^{-1} = \sum_{s \in G/H} h^{s-1} E_H u^{-1}.$$

Here the last equality holds since $a$ is an isomorphism of $E_H$-modules.

We first show that (ii)' is equivalent to (i)'''. Assume (ii)'. Let $X = V^H$ in (ii)', $g$ an element of $eJE$, and let $h = g \in H(V^H, W_H)$. Then since $h^{s-1} = g \in eJE$, (5.3)'' implies that $t^g_\theta(h_n(V^g_H, V^H)) \subset eE$, and hence (ii) yields that $t^g_\theta(fh_n(V^g_H, V^H)) \subset eE$. In particular, taking the element $s$ of $h_n(V^g_H, V^H)$ defined by

$$s(v) = \begin{cases} v & \text{if } v \in V^H \\ 0 & \text{if } v \in \oplus_{s \in G/H} (V^H)^H \end{cases},$$

we can conclude that $t^g_\theta(fh) = t^g_\theta(fg)$ lies in $eE$. Thus (i)''' holds.

Conversely, suppose that (i)''' holds. For a $kH$-module $X$ and $h \in H(X, W_H)$,
assume that \( t^E(h(V^H, X)) \subseteq eJE \). Then we in particular get \( t^E(h(V^H, X)) \subseteq eJE \). Hence Lemma 1.3 yields that \( (h(V^H, X))^{-1} \subseteq \) eJE, i.e., that \( (h(V^H, X)) \subseteq (eJE)^e \). Now using (5.3)', (i)' implies that \( tE(fh(V^H, X)) = \sum_{s \in G/H} tE(fh(V^H, X)) u_s^{-1} \subseteq \sum_{s \in G/H} (eJE) u_s^{-1} = eJE \). Hence (ii)' holds.

Let us now show that (iii)' is equivalent to (iv)'. Assume (iii)' . Notice that for any \( kH \)-module \( X \) and any \( h \in h(V^H, X) \) is included in \( eJE \). Thus by (5.3)' we have \( tE(fh(V^H, X)) = \sum_{s \in G/H} tE(fh(V^H, X)) u_s^{-1} \subseteq \sum_{s \in G/H} (eJE) u_s^{-1} = eJE \). Hence (iv)' holds. Conversely assume that (iv)' holds. Then we have \( tE(fh(V^H, X)) \subseteq tE(fh(V^H, X)) \subseteq \) eJE. Since \( (V^H, W_H) = (eE)^e \), we get (iii)' . Now the proof is complete.

Proof of Theorem 5.2. For any \( H \) with \( N \subset H \subset G \), define \( \theta_H : End((\mathcal{G}(\cdot, W)_H) \rightarrow End_{kH}(eE) \) by \( \theta_H = z_H Y_H^{-1} \). Then \( \theta_H \) is an isomorphism of \( k \)-algebras. Note that \( eE_{kH} \) (resp. \( (\mathcal{G}(\cdot, W)_H) \) is a projective object of \( \text{Mod} E_H \) (resp. \( \text{MMod} kH \)). Hence by Proposition 5.3, \( \theta_H \) induces an isomorphism of \( k \)-algebras from \( \text{End}((SW)_H) \) onto \( \text{End}_{kH}(eE/eJE) \). Namely, for any \( \xi \in \text{End}((SW)_H) \), there is an element \( \eta \) of \( \text{End}((\mathcal{G}(\cdot, W)_H)) \) such that \( \eta \) preserves \( \text{rad} \mathcal{G}(\cdot, W) \) and induces \( \xi \). Thus by Proposition 5.3, \( \theta_H(\eta) \) preserves \( eJE \) and \( \theta_H(\xi) \) is defined to be the \( E_N \)-endomorphism of \( eE/eJE \) induced by \( \theta_H(\eta) \). By Proposition 5.3 again, this does not depend on the choice of those \( \eta \) that induce \( \xi \).

Now we prove that (5.2)' commutes. For any \( \xi \in \text{End}((SW)_H) \), Lemma 4.3 implies that \( T^H_H(\xi) \) is induced by \( T^H_H(\eta) \), where \( \eta \) is an element of \( \text{End}((\mathcal{G}(\cdot, W)_H)) \) which induces \( \xi \). It follows by the definition of the \( \theta_H \) and commutativity of the diagram (5.1) that \( \theta_G T^H_H(\xi) \) is induced from \( \theta_G \theta_H(\eta) \). Now by Lemma 4.2 \( \theta_G \theta_H(\eta) \) induces \( \tau^H_H(\eta) \). Therefore, we have \( \theta_G T^H_H(\xi) = \tau^H_H(\eta) \) as desired. This completes the proof.

The following theorem, which is an easy consequence of Theorem 5.2, is the main result of this section.

**Theorem 5.4.** Let \( H \) be a subgroup of \( G \) with \( N \subset H \subset G \). Then \( SW \) is \( H \)-projective if and only if \( eE/eJE \) is \( H/N \)-projective.

Proof. Using Higman’s criterion, the result holds immediately from Theorem 5.2.

Using the above theorem, to study relative projectivity of \( SW \), we may consider that of a simple \( E \)-module \( eE/eJE \).

In the rest of this section, we assume further that \( V \) is indecomposable and that \( k \) is algebraically closed.

Then we have the following, which is well known. (See, for example, [4. Propositions 2.4 and 5.2].)
Lemma 5.5. Let \( I = J(E_N) E \). Then:

(i) \( I \) is a two sided ideal of \( E \).

(ii) \( I \) is included in \( JE \), and hence annihilates each simple \( E \)-module.

(iii) \( E/I \) is isomorphic to a twisted group algebra of \( G/N \) over \( k \) with a basis \( \{ u_x \} \), where \( u_x \) is the image of \( u_x \) in \( E/I \).

By the above lemma, each simple \( E \)-module can be considered as a simple module over a twisted group algebra \( E/I \). Thus knowledge of modules over those algebras (see [3] and [10] for example) will help to determine vertices of a simple object \( SW \).

Remark. Relative projectivity for twisted group algebras is defined in a way similar to that for group algebras ([3, §4]), and we can use Higman’s criterion, as well. It follows at once from the definition of the relative projectivity for \( \text{Mod} E \) that \( eE/eJE \) is \( H/N \)-projective as an \( E/I \)-module if and only if it is \( H/N \)-projective as an \( E/I \)-module.

As an application of Theorem 5.4, we give the following example.

Example. Let \( G = GL(3, q) \), where \( q \) is a power of \( p \). Put
\[
D = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : c \in F_q^* , \quad P = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in F_q^* ,
\]
and let \( H \) be the subgroup of \( G \) consisting of all the upper triangular matrices. Then an easy calculation shows that \( H = N_G(D) \), \( H/D \) has a normal \( p \)-Sylow subgroup \( P/D \), and \( C_G(D) \supseteq P \). Hence every \( kD \)-module \( S \) is \( P \)-invariant, i.e., the inertial subgroup \( I_H(S) \) of \( S \) in \( H \) contains \( P \), and every simple module over any twisted group algebra of \( I_H(S)/D \) has \( P/D \) as its vertex. Therefore, Proposition 2.4 and Theorem 5.4 imply that if \( D \) is a vertex of an indecomposable \( kG \)-module \( W \), then \( \text{vtx}(SW) = _eP \).

As another application of Theorem 5.4, we prove;

Theorem 5.6. Suppose that \( U \) is an indecomposable \( kG \)-module. Let \( P \) be a vertex of \( U \) with \( N_G(P) \supseteq \text{vtx}(SU) \supseteq P \), (see Proposition 2.4 (i)), and \( S \) a \( P \)-source of \( U \). If \( U \) has \( p' \)-multiplicity as a direct summand of \( S \), then \( \text{vtx}(SU) \) is a \( p' \)-Sylow subgroup of the inertial subgroup \( I_G(S) \) of \( S \) in \( N_G(P) \).

Proof. Let \( U' \) be the Green correspondent of \( U \) with respect to \( (G, P, N_G(P)) \). Let \( U'' \) be an indecomposable direct summand of \( S^{(5)} \) such that \( U''N_G(P) = U' \). Then \( U'' \) has \( p' \)-multiplicity as a direct summand of \( S^{(5)} \). Thus by Proposition 2.4 (ii), we may assume that \( P \) is normal in \( G \) and \( S \) is \( G \)-invariant. Using the letters \( W, V \) and \( N \) instead of \( U, S \) and \( P \), respectively,
we are in the same situation as in Theorem 5.4. So use the same notation as there. Consider the simple $E/I$-module $eE/eJE$, which is isomorphic to $SW(V^o)$ as $E$-modules. Since $SW(V^o)=(SW)_N(V)$, Lemma 2.5 and our assumption imply that $\dim_e eE/eJE$ is relatively prime to $p$. Hence by [10, Chap. 5 Theorem 9.8], $vtx(eE/eJE)$ is a $p$-Sylow subgroup of $G/N$. Therefore the result follows from Theorem 5.4. This completes the proof.

As a corollary to the above, we can prove the first half of [9, Theorem 8.2] as follows.

**Corollary 5.7.** Assume that $G$ is a $p$-group. Let $W$ be an indecomposable $kG$-module, $P$ a vertex of $W$ and $S$ a $P$-source of $W$. Then $vtx(SW)=G/N$.

**Proof.** By Green's theorem, $SW(P)$ is indecomposable and hence it is the Green correspondent of $W$ with respect to $(G,P,N_G(P))$. Thus $W$ has multiplicity 1 as a direct summand of $S^G$. Therefore the above theorem yields the results.

**6. Extendibility of simple objects**

As before, $N$ is a normal subgroup of $G$ and $V$ is an indecomposable $kN$-module.

We say that $SV$ extends to $G$ if there exists a finitely presented object $F$ of $\mathcal{MMod}_{kG}$ such that $F_N=SV$. When $V$ is non-projective, by a standard argument [1, Prop. 4.9], $SV$ extends to $G$ if and only if there exists a short exact sequence of $kG$-modules such that upon the restriction to $N$ it is isomorphic to the direct sum of $SV$ with a split short exact sequence. The above $F$ (or short exact sequence) is called an extension of $SV$ to $G$.

The main result of this section is as follows.

**Theorem 6.1.** (i) If $V$ extends to $G$, then so does $SV$.

(ii) Suppose that $E_N/JE_N=k$. Then, if there is an indecomposable $kG$-module $W$ such that $(SW)_N=SV$, the number of isomorphism classes of those modules is equal to that of 1-dimensional representations of $G/N$ over $k$.

**Proof.** We first claim that if $SW$ is an extension of $SV$, then $W$ is isomorphic to a direct summand of $V^G$. Assume that it does not hold. Then there holds $SW(V^o)=\{0\}$ for some $W$ with $(SW)_N=SV$. Thus it follows by the definition of the restriction for $\mathcal{MMod}_{kG}$ that $(SW)_N(V)=\{0\}$. This contradicts the fact that $(SW)_N=SV$.

For any indecomposable direct summand $W$ of $V^G$, let $r_W$ denote the multiplicity of $W$.

If $V$ extends to $G$, then it is $G$-invariant. Thus, if $W$ is isomorphic to a direct summand of $V^G$, then $W_N$ is isomorphic to a direct sum of some copies
of $V$. It follows by Lemma 2.5 that for any indecomposable direct summand $W$ of $V^G$ the simple object $SW$ is an extension of $SV$ if and only if $r_w \delta_{kN}(W) = \delta_{kN}(V)$. We now apply Theorem 3.4. By the equivalence between $\text{Mod}(kG | V)$ and $\text{Mod}(E|E_N)$, each (isomorphism class of) indecomposable direct summand $W$ of $V^G$ corresponds to an (isomorphism class of) indecomposable direct summand $Y$ of $E$ with the same multiplicity. (Remark 3.6.) Combining this with the claim in the first paragraph, it follows that the number of isomorphism classes of indecomposable $kG$-modules $W$ such that $SW$ are extensions of $SV$ is equal to the number of isomorphism classes of indecomposable direct summands of $E$ with the multiplicity $b_Y$ satisfying

$$(6.1)' \quad b_Y \delta_n(Y) = \delta_{E_N}(E_N).$$

(See Theorem 3.4 (ii).) Now recall that $E$ is a fully $G/N$-graded ring and that $E_N = \text{End}_{kN}(V)$ is a local ring. So, in particular, $E_N$ (resp. $E_N = E_N|JE_N$) is the unique projective indecomposable (resp. simple) $E_N$-module. Hence applying Theorem 3.7 to $E$, the number of isomorphism classes of those $Y$ satisfying $(6.1)'$ is equal to the number of isomorphism classes of extensions of $E_N$ to $E$. On the other hand, since $V$ extends to $G$, it follows by Corollary 3.5 that $E_N$ extends to $E$, and hence $E_N$ extends to $E$ by Corollary 3.3. Therefore, especially, an argument given above implies that $SV$ extends to $G$.

Now assume that $E_N = k$. First note that if an indecomposable $kG$-module $W$ satisfies $(SW)_N = SV$, then $W$ is $N$-projective by the argument in the first paragraph. It is easily seen from [9, Prop. 7.9] that our assumption implies that $V$ is $G$-invariant. Thus to prove the second statement, it suffices to show that the number of isomorphism classes of extensions of $E_N = k$ to $E$ coincides with the number of 1-dimensional representations of $G/N$ over $k$. Now our previous argument yields that there is an extension of $E_N$ to $E$. Let $I = (JE_N)E$. Since $I$ annihilates any extension of $E_N$ (Lemma 5.5 (ii)), $E/I$ has a 1-dimensional representation. Thus $E/I$ is isomorphic to the group algebra of $G/N$ over $k$. (See Lemma 5.5 (iii).) Hence each extension of $E_N$ can be considered as a 1-dimensional representation of $G/N$. Since $I \subset JE_N$, any two extensions of $E_N$ are isomorphic to each other as $E$-modules if and only if they are so as $E/I$-modules. Therefore the second statement has been proved.

Remark. (i) Suppose that $G/N$ is a $p$-group and that $k$ is sufficiently large. Then $V^G$ is indecomposable by Green's theorem. Thus, if $V$ is $G$-invariant, then the proof of Theorem 6.1 implies that $S(V^G)$ is a unique simple extension of $SV$.

(ii) There might be an extension of $SV$ which is not simple. For example, let $N$ be a cyclic group of order $p$, $G$ the direct product of $N$ with another cyclic group $M$ of order $p$. Take a non-projective indecomposable $kN$-module $V$ and
the trivial \( kM \)-module \( kM \). Then the tensor product \( (SV) \otimes_k kM \) is naturally a short exact sequence of \( kG \)-modules and it is clearly an extension of \( SV \). However, this sequence is not an Auslander-Reiten sequence. In particular, this gives an object of \( \text{MMod}_kG \) different from \( S(VG) \).

Assume that \( V \) is \( G \)-invariant and \( E\pi k \). Then \( E/(JE\pi) E \) is isomorphic to a twisted group algebra (Lemma 5.5 (iii)), and hence it determines an element \( \varphi \) of \( H^2(G/N, k^*) \). See also [11, §1]. Now we have;

**Corollary 6.2.** In the above situation, there is an indecomposable \( kG \)-module \( W \) such that \( (SW)_N \cong SV \) if and only if \( \varphi = 0 \) in \( H^2(G/N, k^*) \).

Proof. This is clear by the proof of Theorem 6.1 since a twisted group algebra has a 1-dimensional representation if and only if \( \varphi = 0 \).

**Remark.** If \( G/N \) is a \( p' \)-group, then by [11, Cor. 1.12] \( \varphi = 0 \) if and only if \( V \) extends to \( G \). Therefore, Theorem 6.1 and Corollary 6.2 yield that there is an indecomposable \( kG \)-module \( W \) such that \( (SW)_N \cong SV \) if and only if \( V \) extends to \( G \).

**References**

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan