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Author(s)	Uno, Katsuhiro
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Uno, K.
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RELATIVE PROJECTIVITY AND EXTENDIBILITY OF AUSLANDER-REITEN SEQUENCES

KATSUHIRO UNO

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Introduction

Let kG be the group algebra of a finite group G over a field k of characteristic p. For any non-projective indecomposable right kG-module W, there is a so called Auslander-Reiten sequence $SW\colon 0\to\Omega^2W\to X\to W\to 0$ (exact) terminating at W, where Ω denotes the Heller operator. (See [2, 2.17.6] for the definition of Auslander-Reiten sequences.) From this sequence, we get the exact sequence $0\to\mathrm{Hom}_{kG}(\cdot,\Omega^2W)\to\mathrm{Hom}_{kG}(\cdot,X)\to\mathrm{Hom}_{kG}(\cdot,W)\overset{\sigma}{\to}\mathrm{Ext}^1_{kG}(\cdot,\Omega^2W)$ of contravariant functors from the category of kG-modules into that of k-spaces. Those functors and natural transformations among them form a category. This functor category possesses properties similar to those of the category of kG-modules. For instance, we can give notions of simplicity, indecomposability and so on for its objects. It is known that the image of the above σ is a simple object. Moreover, each simple object of the functor category gives rise to a simple object of the module category or an Auslander-Reiten sequence, and this gives a one-to-one correspondence between the set of isomorphism classes of simple objects of the functor category and the union of the set of isomorphism classes of simple kG-modules with the set of equivalence classes of Auslander-Reiten sequences. In this way, Auslander-Reiten sequences are often identified with simple objects of the functor category. In this paper, we consider SW as an Auslander-Reiten sequence and as a simple object simultaneously.

Up to this point, these facts hold for any finite dimensional k-algebras if we replace Ω^2 by a certain operator. One can see a brief review of these facts (Auslander-Reiten theory) in [9, §1].

Recently in [9] Green studied Auslander-Reiten theory for group algebras and gave several notions for the functor category, which have analogues in the kG-module category. "Restrictions", "inductions" and "trace maps" are examples of them.

In this paper, we consider "relative projectivity" and "extendibility" of Auslander-Reiten sequences for group modules, which can be defined as soon as the above notions are given.

Concerning relative projectivity, Green showed that each Auslander-Reiten sequence SW has "a vertex", which is a p-subgroup of G determined uniquely up to G-conjugate and that some conjugate of a vertex of SW contains a vertex of W and is contained in the normalizer of a vertex of W. Moreover, an analogue of Green correspondence exists. Thus, in order to find a vertex of an Auslander-Reiten sequence SW, we may assume that a vertex of W is normal in G. (See the first paragraph of [9, §8].)

Concerning extendibility, we are given a normal subgroup N of G and a simple object SV of the functor category corresponding to an indecomposable kN-module V. We consider when we can extend SV to G.

In view of the above, one might notice that we will have to study modules over kG and over kN for a normal subgroup N of G. So it seems that Clifford theory is useful. As a matter of fact, using Clifford theory, it can be shown that, if k is sufficiently large, then a simple object corresponding to an N-projective indecomposable kG-module having a G-invariant N-source gives a simple module over some twisted group algebra of G/N over k.

We can prove that, for any subgroup H of G with $N \subset H \subset G$, SW is H-projective if and only if the simple module given by SW is H/N-projective (Theorem 5.4). In certain cases, one can apply this fact to determine a vertex of SW.

On the other hand, for any indecomposable kN-module V, we can see that if V extends to G, then so does SV. Also, if V is G-invariant, then for any indecomposable direct summand W of the induced module V^G, SW is an extension of SV if and only if the simple module given by SW is 1-dimensional. (See Theorem 6.1).

We note that the same idea is already used to give a sufficient condition which guarantees that W and SW have vertices in common. ([13, Theorem 2.5].)

This paper is organized as follows. After introducing terminologies and notations in Section 1, we will briefly review the Auslander-Reiten theory (Section 2) and Clifford theory (Section 3). Some results concerning trace maps are proved in Section 4. Relative projectivity and extendibility of simple objects of the functor category are studied in Sections 5 and 6, respectively.

1. Notations and conventions

Throughout this paper, we see the following notations and conventions.

G is a fixed finite group and k is a field of characteristic $p, p \neq 0$. All modules considered here are finitely generated and, unless otherwise noted, every module is a right module. Mod kG denotes the category whose objects are all the (finite dimensional) modules over the group algebra kG and whose morphisms are all the kG-homomorphisms among them. For any kG-modules W and W', we use ${}_G(W, W')$ to denote $\mathrm{Hom}_{kG}(W, W')$ for notational convenience. For any finite dimensional k-algebra R, we write JR to denote its radical, and for any R-module W, $\delta_R(W)$ means the dimension of $\mathrm{End}_R(W)/J\,\mathrm{End}_R(W)$ over k. Other notations and terminologies in representation theory are standard. (See for example [8].)

In addition to the above, the letters N and V are reserved to mean a normal subgroup of G and a kN-module, respectively. Usually V is assumed to be non-projective, indecomposable or G-invariant. Here we say that V is G-invariant if $V\otimes_{kN} g\simeq V$ as kN-modules for all $g\in G$.

Whenever V is given, E denotes the kG-endomorphism ring of the induced module $V^G=V\otimes_{kN} kG$. Fix representatives G/N of cosets of N in G, and for any subgroup H of G wtih $N\subset H\subset G$, choose representatives G/H of left cosets of H in G from G/N. We can and will regard V^H as a kH-direct summand of V^G. In fact, by Mackey's theorem, we have the decomposition

$$V^G = \oplus_{x\in G/H}(V\otimes_{kN} x)^H \tag{1.1}$$

of V^G into the direct sum of kH-submodules $(V\otimes_{kN} x)^H$. We write, for instance, Vx instead of $V\otimes_{kN} x$. Letting E_H be $\mathrm{End}_{kH}(V^H)$, we can consider E_H as a subalgebra of E via the injective k-algebra homomorphism $i_H\colon E_H\to E$ given by $i_H(f)=f\otimes_{kH}\mathrm{Id}_{kG}$. Note that for any $N\subset H\subset H'\subset G$, we have $i_{H'}|_{E_H}=i_H$. Furthermore, if V is G-invariant, then for each $x\in G/N$, there exists a unit u_x of E such that $u_x(V)=Vx$, and E has a decomposition $E=\oplus_{x\in G/N} E_N u_x=\oplus_{x\in G/N} u_x E_N$ into the direct sum of k-subspaces $E_N u_x=u_x E_N$. Also, there hold $E_H=\oplus_{x\in H/N} E_N u_x=\oplus_{x\in H/N} u_x E_N$ and $E=\oplus_{x\in G/H} E_H u_x=\oplus_{x\in G/H} u_x E_H$. For these facts see for example [4] and [5].

For a kG-module W, let $a=a_H(W)$ be the isomorphism

$$a_H(W)\colon {}_G(V^G, W)\to {}_H(V^H, W)$$

in the Frobenius reciprocity law. Note that a is an isomorphism of $\mathrm{End}_{kG}(W)$-E_H-bimodules. For each $f\in{}_G(V^G, W)$, the image of f by a is denoted by f^a.

Also for any kG-modules W and W', we use $t_H^G=t_H^G(W, W')$ to denote the usual trace map from ${}_H(W, W')$ into ${}_G(W, W')$. Note that, since we are taking representatives of left cosets of H in G, for any $f\in{}_H(W, W')$, $t_H^G(f)$ sends any $w\in W$ into $\sum_{x\in G/H} f(wx)x^{-1}$.

Regarding $(Vx)^H$ as a kH-direct summand of V^G, we may apply $t_H^G(V^G, W)$ to elements of ${}_H((Vx)^H, W)$. In doing so, of course, each element of ${}_H((Vx)^H, W)$ is considered to vanish on $(Vy)^H$ for all $y\in G/H$ with $y\neq x$. When V is G-invariant, we let s_x $(x\in G/H)$ be the element of ${}_H(V^G, V^H)$ defined by

$$s_x(v)=\begin{cases} u_x^{-1}(v) & \text{if } v\in(Vx)^H,\\ 0 & \text{if } v\in\oplus_{y\in G/H, y\neq x}(Vy)^H.\end{cases}$$

The following are of later use. (See Section 5.)

Lemma 1.2. *Suppose that V is G-invariant. Let W be a kH-module. Then we have ${}_H(V^G, W)=\oplus_{x\in G/H}\,{}_H(V^H, W)\,s_x$.*

Proof. By (1.1) it suffices to see ${}_H(V^H, W)\,s_x={}_H((Vx)^H, W)$. But this is clear because the restriction of s_x to $(Vx)^H$ gives an isomorphism from $(Vx)^H$ onto V^H.

Lemma 1.3. *Suppose that V is G-invariant. Let W be a kG-module. Then for any $f\in{}_H(V^H, W)\subset{}_H(V^G, W)$, we have*

$$t_H^G(V^G, W)\,(fs_x)=f^{a^{-1}}u_x^{-1} \qquad \textit{for all} \quad x\in G/H\,.$$

In particular, we get

$$t_H^G(V^G, W)\,(f)=f^{a^{-1}}\,.$$

Proof. For any $v\in Vx$, we have

$$t_H^G(fs_x)\,(v)=\textstyle\sum_{y\in G/H} fs_x(vy)\,y^{-1}=fs_x(v)\,,$$

since s_x vanishes on Vxy if $y\notin H$. It follows by the definition of s_x that $t_H^G(fs_x)$ agrees with fu_x^{-1} on Vx, and hence it is equal to $f^{a^{-1}}u_x^{-1}$ on Vx by the definition of a. Since both $t_H^G(fs_x)$ and $f^{a^{-1}}u_x^{-1}$ are kG-homomorphisms, they must coincide with each other. The last statement holds since we can take the identity element of E for u_1. Now the proof is complete.

2. Preliminaries for the Auslander-Reiten and Green theories

In this section, we review a part of the Auslander-Reiten and Green theories following [9], while some of the results will be stated in a way convenient for our use.

Let MMod kG be the category whose objects are all the k-linear contravariant functors from Mod kG into Mod k and whose morphisms are all the natural transformations between those functors. It is known from the Auslander-Reiten theory that for each indecomposable kG-module W, there is a simple object SW of MMod kG, which is unique up to isomorphisms, such that ${}_G(\cdot, W)$ is the projective cover of SW. Moreover, each simple object appears in this way. Furthermore, SW corresponds to a simple kG-module, if W is projective, or to an Auslander-Reiten sequence terminating at W, otherwise. Recall also that any SW is known to be finitely presented.

For any non-projective indecomposable module W, we can obtain SW as the image of some $\alpha: {}_G(\cdot, W) \to D_G(W, \cdot)$, where $D_G(W, \cdot)$ is the space dual to ${}_G(\cdot, W)$ ([9, Theorem 1.7]). However, when W is a direct summand of V^G, it seems better to use $D_G(V^G, \cdot)$ instead of $D_G(W, \cdot)$. The following should be compared with [9, Theorem 1.8].

Proposition 2.1. *Let e be a primitive idempotent of E and let $W=eV^G$. Then a functor $\gamma=\gamma_e: {}_G(\cdot, W) \to D_G(V^G, \cdot)$ satisfies* $\mathrm{Im}\,\gamma \simeq SW$ *if and only if*

(i) $\gamma(W)(\mathrm{Id}_W) \neq 0$, *and*

(ii) $\gamma(W)(\mathrm{Id}_W)(eJE)=0$.

Proof. By an argument similar to the one in the proof of [9, Theorem 1.8], $\mathrm{Im}\,\gamma \simeq SW$ if and only if

$$f\cdot {}_G(W, X) \subset J(eEe) \Leftrightarrow \gamma(W)(\mathrm{Id}_W)(f\cdot {}_G(V^G, X)) = 0\,,$$

for any kG-module X and $f \in {}_G(X, W)$. Thus it suffices to show that

$$f\cdot {}_G(W, X) \subset J(eEe) \Leftrightarrow f\cdot {}_G(V^G, X) \subset eJE\,.$$

Now, notice that $f\cdot {}_G(V^G, X)$ is an E-submodule of eE. Since eE has the unique maximal submodule eJE, the above is easy to see.

REMARK. If $\gamma: {}_G(\cdot, W) \to D_G(V^G, \cdot)$ satisfies (i) and (ii) of Proposition 2.1, we have the following exact sequence in MMod kG.

$$0 \to \mathrm{rad}_G(\cdot, W) \to {}_G(\cdot, W) \xrightarrow{\gamma} SW \to 0$$

Here, for each kG-module X, $\mathrm{rad}_G(X, W)=\{f\in {}_G(X, W): fg \in J\,\mathrm{End}_{kG}(W)$ for all $g\in {}_G(W, X)\}$. See [9, Theorem 1.4].)

Let W and W' be kG-modules. Then each $f\in {}_G(W, W')$ gives a morphism $f_*: {}_G(\cdot, W) \to {}_G(\cdot, W')$, which is defined as follows. For any kG-module X and any $\varphi \in {}_G(X, W)$, $f_*(\varphi)=f\cdot\varphi$. Yoneda's lemma says that the above map $f \to f_*$ is bijective, namely;

Lemma 2.2 ([9, 1.1]). *For any kG-modules W and W', the map*

$$Y(W, W'): {}_G(W, W') \to \mathrm{Hom}({}_G(\cdot, W), {}_G(\cdot, W'))$$

given by $Y(W, W')(f)=f_*$ for all $f\in {}_G(W, W')$ is an isomorphism. Moreover, if $W=W'$, then $Y(W, W')$ is an isomorphism of k-algebras.

Let H be a subgroup of G. Then at any kH-module X, the restriction $({}_G(\cdot, W))_H$ of ${}_G(\cdot, W)$ to H has the value ${}_G(X^G, W)$ by its definition ([9, §2]). Moreover, the isomorphism $a(X, W)$: ${}_G(X^G, W) \to {}_H(X, W)$ in the Frobenius reciprocity law yields $({}_G(\cdot, W))_H \xrightarrow{\sim} {}_H(\cdot, W_H)$ ([9, Prop. 2.12]). Hence, for any

$f \in {}_H(W, W')$, the composite $a(\cdot, W')^{-1}\, Y(W_H, W'_H)\,(f)\, a(\cdot, W)$ gives a morphism from $({}_G(\cdot, W))_H$ into $({}_G(\cdot, W'))_H$. Furthermore, the map sending any such f into $a(\cdot, W')^{-1}\, Y(W_H, W'_H)\,(f)\, a(\cdot, W)$ is clearly an isomorphism from ${}_H(W, W')$ onto $\mathrm{Hom}(({}_G(\cdot, W))_H, ({}_G(\cdot, W'))_H)$. We denote this isomorphism by $Y_H(W, W')$. Note that for any kH-module X, $Y_H(W, W')\,(f)\,(X)$ is a homomorphism from ${}_G(X^G, W)$ into ${}_G(X^G, W')$.

Next, we recall some properties of trace maps and relative projectivity for MMod kG. In [9, §5] Green gave a definition of trace maps. For any objects F and F' of MMod kG and any subgroup H of G, the trace map $T_H^G = T_H^G(F, F')$ is a k-linear map from $\mathrm{Hom}(F_H, F'_H)$ into $\mathrm{Hom}(F, F')$.

One of his interesting results is as follows.

Lemma 2.3 ([9, Prop. 6.4]). *For any kG-modules W and W' and any $f \in {}_H(W, W')$, we have $T_H^G(Y_H(W, W')\,(f)) = Y(W, W')\,(t_H^G(f))$.*

We shall study more about the trace maps in Section 4.

Using the notion of induction, we can give a definition of relative projectivity for MMod kG. Here we remark that relative projectivity is defined only among finitely presented objects of MMod kG. Moreover, it can be shown that "Higman's criterion" exists [9, Th. 5.11]. Hence a (finitely presented) object of MMod kG is H-projective if and only if its identity automorphism lies in the image of the trace map T_H^G.

If F is a finitely presented indecomposable object of MMod kG, then there is a unique (up to G-conjugate) p-subgroup P of G such that F is H-projective if and only if $H \supset_G P$. ([9, Theorem 4.7]). We call this P a vertex of F and denote it by $vtx(F)$.

Let W be an indecomposable kG-module. Then SW is finitely presented and simple. Now we have the following result which follows from [9, 5.12 and 7.7].

Proposition 2.4. (i) *There holds $vtx(W) \subset_G vtx(SW) \subset_G I$, where I is the inertial subgroup of a $vtx(W)$-source of W in $N_G(vtx(W))$.*

(ii) *Let W' be the $kN_G(vtx(W))$-module that corresponds to W via the Green correspondence with respect to $(G, vtx(W), N_G(vtx(W)))$, and let W'' be a kI-module such that $W''^{N_G(vtx(W))} \simeq W'$ and that W' and W'' have vertices in common. (Note: Such W'' always exists.) Then we have $vtx(SW) =_G vtx(SW') = vtx(SW'')$.*

By the above proposition, some problems concerning $vtx(SW)$ can be reduced to the case where $vtx(W)$ is normal and a $vtx(W)$-source of W is G-invariant. If it is the case, letting V be a source of W, we should consider direct summands of V^G. For this, Clifford theory (see Section 3) is useful. Using the above technique, we shall investigate vertices of simple objects of MMod kG

in Section 5.

The final result in this section is the following, which will be used in Sections 5 and 6. This is a special case of [9, Prop. 7.9].

Lemma 2.5. *Let N be a normal subgroup of G and fix a G-invariant indecomposable kN-module V. Suppose that an indecomposable kG-module W has multiplicity $r (r>0)$ as a direct summand of V^G. Then we have $(SW)_N \simeq \delta r(SV)$, where $\delta = \delta_{kG}(W)/\delta_{kN}(V)$.*

3. Clifford theory

In this section we review Clifford theory. Following [5], we state the results in terms of group-graded rings and modules. The main theorem in Clifford theory is Theorem 3.4 below. We also give a criterion on extendibility of modules.

Let H be a finite group. If a ring R (with 1) has a direct sum decomposition $R = \bigoplus_{h\in H} R_h$ into additive subgroups R_h, $h \in H$, such that $R_h R_{h'} = R_{hh'}$ for all $h, h' \in H$, we say that R is a (fully) H-graded ring. For those R, it is clear that R_1 is a subring of R and R_h is an R_1-R_1-bimodule.

Note that kG is a fully G/N-graded ring with the decomposition $kG \simeq \bigoplus_{x\in G/N} kNx$.

For any ring R, let R^* denote the unit group of R.

In the rest of this section, R is always assumed to be a fully H-graded ring.

We set $GU(R) = \bigcup_{h\in H}(R_h \cap R^*)$ and define a map d:. $GU(R) \to H$ by $d(r) = h$ if $r \in R_h$, $h \in H$. The elements of $GU(R)$ form a subgroup of R^*. It is not difficult to see that

$$\mathscr{X}\langle R\rangle\colon 1 \to R_1^* \to GU(R) \overset{d}{\to} H \to 1$$

is exact except possibly at H. ([5, Prop. 5.2].)

Let X be an R_1-module. The induced R-module $X^R = X \otimes_{R_1} R$ has a decomposition $X^R = \bigoplus_{h\in H} X \otimes_{R_1} R_h$ into a direct sum of R_1-submodules $X \otimes_{R_1} R_h$. Now as in [5, §3, 4, 5] we have the following. (See also the first half of Section 1.)

$$\mathrm{End}_R(X^R) = \bigoplus_{h\in H} E_h\,, \tag{3.1}$$

where $E_h = \{\varphi \in \mathrm{End}_R(X^R) \mid \varphi(X \otimes_{R_1} R_h) \subset X \otimes_{R_1} R_{hh'}$ for all $h' \in H\}$. As in Section 1, we put $E = \mathrm{End}_R(X^R)$ for convenience. It is easy to see that the map from $\mathrm{End}_{R_1}(X)$ into E_1 sending any $\Phi \in \mathrm{End}_{R_1}(X)$ into $\Phi \otimes_{R_1} \mathrm{Id}_R$ gives a ring isomorphism. Moreover, if X is R-invariant, i.e., X is isomorphic to each $X\otimes_{R_1} R_h$, $h\in H$, as R_1-modules, then E is fully graded with the decomposition (3.1). Furthermore, if this is the case, the sequence

$$\mathcal{X}\langle E\rangle : 1 \to E_1^* \to GU(E) \to H \to 1 .$$

is exact ([5, §4, 5].)

On the extendibility of X, we have;

Theorem 3.2 ([6, Theorem 2.8]). *X extends to R if and only if $\mathcal{X}\langle E\rangle$ splits.*

REMARK. Suppose that $\mathcal{X}\langle E\rangle$ is a split exact sequence. Let $\gamma\colon H\to GU(E)$ be a splitting homomorphism. Then for each $h, h'\in H$, the restriction of $\gamma(h)$ to $X\otimes_{R_1} R_{h'}$ gives an R_1-isomorphism from $X\otimes_{R_1} R_{h'}$ onto $X\otimes_{R_1} R_{hh'}$.

As a corollary to the above theorem, we obtain;

Corollary 3.3. *Assume further that R is a finite dimensional k-algebra and each R_h is a k-subspace of R and is free as a left R_1-module. Then, if X extends to R, then $\bar{X}=X/JX$ extends to R. Here JX denotes the radical of X.*

Proof. By Theorem 3.2, we have a splitting homomorphism $\gamma\colon H\to GU(E)$. Since R is fully graded, it is easy to see that the radical of the R_1-module $X\otimes_{R_1} R_h$ is precisely $JX\otimes_{R_1} R_h$ for all h in H. Hence by the remark following Theorem 3.2, each $\gamma(h)$ sends $JX\otimes_{R_1} R_{h'}$ into $JX\otimes_{R_1} R_{hh'}$ for all $h'\in H$. The sum $\bigoplus_{h\in H} JX\otimes_{R_1} R_h=(JX)^R$ is an R-submodule of X^R and it follows that

$$X^R/(JX)^R \simeq \bigoplus_{h\in H}(X/JX)\otimes_{R_1} R_h = (X/JX)^R = \bar{X}^R .$$

Since $(JX)^R$ is fixed by all $\gamma(h)$, each $\gamma(h)$ gives an R-automorphism $\overline{\gamma(h)}$ of $\bar{X}^R$. Therefore, by the choice of $\{\gamma(h)\}$, the map $\bar{\gamma}\colon H\to GU(\mathrm{End}_R(\bar{X}^R))$ defined by $\bar{\gamma}(h)=\overline{\gamma(h)}$, $h\in H$, gives a splitting homomorphism for the sequence $\mathcal{X}\langle \mathrm{End}_R(\bar{X}^R)\rangle$. Now the proof is completed by Theorem 3.2.

In the case where X is R-invariant so that $E(=\mathrm{End}_R(X^R))$ is fully H-graded, we have a nice correspondence theorem originally due to Clifford. Before we state it, we introduce some notations.

Let $\mathrm{Mod}(R\,|\,X)$ be the additive full subcategory of $\mathrm{Mod}\,R$ whose objects are those R-modules such that their restrictions to R_1 are isomorphic to direct summands of direct sums of some copies of X. Also $\mathrm{Mod}(E\,|\,E_1)$ denotes the additive full subcategory of $\mathrm{Mod}\,E$ whose objects are those E-modules such that their restrictions to E_1 are projective E_1-modules.

Regarding X^R as an E-R-bimodule naturally, for any R-module W, $\mathrm{Hom}_R(X^R, W)$ can be considered as an object of $\mathrm{Mod}\,E$. And for any E-module Y, the tensor product $Y \otimes_E X^R$ is an object of Mod$R$. In fact, $\mathrm{Hom}_R(X^R, \cdot)$ (resp. $\cdot \otimes_E X^R$) is an additive functor from Mod R (resp. Mod E) into Mod E (resp. Mod R). (See [5, §7] for detail.)

The following is the main theorem of Clifford theory.

Theorem 3.4. *Suppose that an R_1-module X is R-invariant. Then:*

(1) *The restrictions of* $\mathrm{Hom}_R(X^R, \cdot)$ *and* $\cdot \otimes_E X^R$ *give an equivalence between* $\mathrm{Mod}(R|X)$ *and* $\mathrm{Mod}(E|E_1)$.

(ii) *Assume that an object W of* $\mathrm{Mod}(R|X)$ *corresponds to an object Y of* $\mathrm{Mod}(E|E_1)$ *under the equivalence. Then we have a ring isomorphism* $\mathrm{End}_R(W) \simeq \mathrm{End}_E(Y)$.

The definition of equivalence used here can be found on page 65 of [7].

Proof. (i) This is [5, Theorem 7.4].

(ii) This is an immediate consequence of (i).

Corollary 3.5. *Suppose that X is R-invariant. Then X extends to R if and only if the regular E_1-module E_1 extends to E.*

Proof. It is obvious that extensions of X are objects of $\mathrm{Mod}(R|X)$. One can show easily that extensions of X correspond to extensions of E_1 under the equivalence. (See also [12, Cor. 3.16].)

REMARK 3.6. If X is R-invariant, then X^R is an object of $\mathrm{Mod}(R|X)$. Since $\mathrm{Hom}_R(X^R, X^R)=E$, the regular E-module E is the object of $\mathrm{Mod}(E|E_1)$ that corresponds to X^R under the equivalence. Since both $\mathrm{Hom}_R(X^R, \cdot)$ and $\cdot \otimes_E X^R$ are additive, every direct summand of X^R (resp. E) is an object of $\mathrm{Mod}(R|X)$ (resp. $\mathrm{Mod}(E|E_1)$). Moreover, if an indecomposable R-module W is a direct summand of X^R and corresponds to an indecomposable direct summand Y of E under the equivalence, then W and Y have the same multiplicity in X^R and E.

Finally we consider so called Nakayama relations. Now we further assume that our ring R is a (finite dimensional) k-algebra and that each R_h is a k-subspace of R. Let $\{P_s\}$ and $\{Q_t\}$ be basic sets of non-isomorphic projective indecomposable R- and R_1-modules, respectively. And let $\bar{P}_s=P_s/JP_s$ and $\bar{Q}_t=Q_t/JQ_t$. Then an argument similar to the one in the proof of [8, III, Theorem 2.6] can be applied to obtain the following.

Theorem 3.7. *Suppose that $Q_t^R \simeq \oplus_s a_{st} P_s$, and let a'_{st} be the multiplicity of $\bar{Q}_t$ in a composition series of $(\bar{P}_s)_{R_1}$. Then we have $a_{st}\, \delta_R(P_s)=a'_{st}\, \delta_{R_1}(Q_t)$.*

4. Trace maps

In this section we study trace maps in various categories. The main purposes of this section are to give a definition of a trace map in the category Mod E and to show that an assertion similar to Lemma 2.3 holds. After this, we consider the trace maps for quotient objects.

Throughout this section, we assume that V is G-invariant and fix a subgroup H of G with $N \subset H \subset G$. As in Section 1, we put $E_H = \mathrm{End}_{kH}(V^H)$ which is regarded as a subalgebra of E, and choose units u_x, $x \in G/N$, of E with $E = \bigoplus_{x \in G/N} E_N u_x = \bigoplus_{x \in G/N} u_x E_N$.

We now define a trace map for E-modules. Let Y and Y' be E-modules. Then $\tau_H^G = \tau_H^G(Y, Y')$: $\mathrm{Hom}_{E_H}(Y, Y') \to \mathrm{Hom}_E(Y, Y')$ is defined as follows.

$$\tau_H^G(\xi)(y) = \sum_{x \in G/H} \xi(y u_x) u_x^{-1} \qquad \text{for all} \quad \xi \in \mathrm{Hom}_{E_H}(Y, Y') \quad \text{and} \quad y \in Y .$$

REMARK. (i) It is an easy exercise to check that the above τ_H^G does not depend on the choice of $\{u_x\}_{x \in G/H}$.

(ii) Once we obtain a notion of a trace map, we can give a definition of relative projectivity for Mod E. An E-module Y is H/N-projective if Id_Y lies in $\tau_H^G(\mathrm{End}_{E_H}(Y))$.

Let W and W' be objects of Mod$(kG \mid V)$ and let $Y = {}_G(V^G, W)$ and $Y' = {}_G(V^G, W')$. Then by Theorem 3.4, ${}_G(W, W') \widetilde{\to} \mathrm{Hom}_E(Y, Y')$ as additive groups. We denote this isomorphism by $z = z(W, W')$. This z is described as follows. For any $f \in {}_G(W, W')$ and $y \in Y = {}_G(V^G, W)$, we have $z(f)(y) = fy$.

Notice that W_H is an object of Mod$(kH \mid V)$, and that the Frobenius reciprocity law yields $a_H(W)$: ${}_H(V^H, W) \widetilde{\to} {}_H(V^G, W) = Y$ as E_H-modules. Thus again by Theorem 3.4, W_H corresponds to the E_H-module Y_{E_H} and we have $z_H = z_H(W, W')$: ${}_H(W, W') \widetilde{\to} \mathrm{Hom}_{E_H}(Y, Y')$. To describe z_H explicitly, recall that the action of E_H on Y is given via the isomorphism $a_H(W)$. Thus for any $f \in {}_H(W, W')$ and any $y \in Y$, we have $z_H(f)(y) = (fy^a)^{a^{-1}}$.

Now we have the following diagram.

$$\begin{array}{ccc} {}_G(W, W') & \xrightarrow{\;z\;} & \mathrm{Hom}_E(Y, Y') \\ t_H^G \uparrow & & \uparrow \tau_H^G \\ {}_H(W, W') & \xrightarrow{\;z_H\;} & \mathrm{Hom}_{E_H}(Y, Y') \end{array}$$

Lemma 4.1. *The above diagram commutes.*

Proof. Let $f \in {}_H(W, W')$ and $y \in Y$. By the definition of τ we have

$$(\tau_H^G z_H(f))(y)(v) = \sum_{x \in G/H} (f(y u_x)^a)^{a^{-1}} u_x^{-1}(v)$$

for all $v \in V^G$. Suppose that $v \in V$. Then we have $u_x^{-1}(v) = v' x^{-1}$ for some $v' \in V$. Hence we have

$$\begin{aligned}(f(yu_x)^a)^{a^{-1}}u_x^{-1}(v) &= (f(yu_x)^a)^{a^{-1}}(v'x^{-1}) \\ &= ((f(yu_x)^a)^{a^{-1}}(v'))\,x^{-1} \\ &= ((f(yu_x)\,(v'))\,x^{-1} \\ &= (fyu_x(v'))\,x^{-1} \\ &= (fy(vx))\,x^{-1}\,.\end{aligned}$$

Therefore we get

$$\begin{aligned}(\tau_H^G\, z_H(f))\,(y)\,(v) &= \textstyle\sum_{x\in G/H}(fy(vx))\,x^{-1} \\ &= t_H^G(fy)\,(v) \\ &= t_H^G(f)\,y(v)\end{aligned}$$

for all $v\in V$. Since both $(\tau_H^G\, z_H(f))\,(y)$ and $t_H^G(f)\,y$ are kG-homomorphisms from V^G into W', they must agree on V^G. Hence we obtain $\tau_H^G\, z_H(f)\,(y)=t_H^G(f)\,y=z(t_H^G(f))\,(y)$ for all $y\in Y$. The proof is now complete.

Next, we study the trace maps for quotient objects. The following lemma is easy to show and we omit the proof.

Lemma 4.2. *Let W' be a kG-submodule of W. Suppose that $f\in{}_H(W, W)$ satisfies $f(W')\subset W'$. Then f induces $\bar{f}\in{}_H(W/W', W/W')$, $t_H^G(f)\,(W')\subset W'$, and $t_H^G(f)$ naturally induces $t_H^G(\bar{f})$.*

REMARK. Replacing kG and kH by E and E_H respectively, we can prove a similar statement for τ_H^G.

It is nontrivial to show a similar assertion for T_H^G, while this is essentially discussed in the first half of [9, §6]. Before proving it, we must give a notion that an endomorphism of an object ${}_G(\cdot, W)$ of MMod kG "preserves" a subobject of it.

Let K be a subobject of ${}_G(\cdot, W)$. Then an element $f\in\mathrm{End}_{kH}(W)$ (or the corresponding element $Y_H(W, W)\,(f)$ of $\mathrm{End}(({}_G(\cdot, W))_H)$ is said to preserve K if the following holds.

$$f(K_H(X))\subset K_H(X) \qquad \text{for all } kH\text{-module } X.$$

More precisely, by the definition of the restriction in MMod kG, the above is equivalent to:

$$(fg^{a(X,W)})^{a(X,W)^{-1}}\in K_H(X) = K(X^G)$$

for all $g\in K_H(X)=K(X^G)\in{}_G(X^G, W)$, where $a(X, W)$ is the isomorphism from ${}_G(X^G, W)$ onto ${}_H(X, W)$ in the Frobenius reciprocity law. In other words, f preserves K if and only if

$$Y_H(W, W)(f)(X)(K(X^G)) \subset K(X^G).$$

(See the argument following Lemma 2.2.)

Lemma 4.3. *In the above situation, suppose that* $f \in \mathrm{End}_{kH}(W)$ *preserves* K. *Then* f *naturally induces* $f^* \in \mathrm{End}((_G(\cdot, W)/K)_H)$, $t_H^G(f)$ *preserves* K, *and* $t_H^G(f)$ *naturally induces* $T_H^G(f^*)$.

Proof. Since f preserves K, it clearly induces some $f^* \in \mathrm{End}((_G(\cdot, W)/K)_H)$, and by the above argument, f^* must come from $Y_H(W, W)(f)$. To see the other statements, taking any kG-module X, consider the following sequence.

$$(4.3)' \quad {}_G(X, W) \xrightarrow{n} {}_G(X_H{}^G, W) \xrightarrow{Y_H(W, W)(f)(X_H)} {}_G(X_H{}^G, W) \xrightarrow{m} {}_G(X, W).$$

Here the maps n and m are defined as follows.

$$n(g)(\textstyle\sum_{x\in G/H}\alpha_x \otimes x) = g(\sum_{x\in G/H}\alpha_x x) \quad \text{and} \quad m(h)(\alpha) = h(\sum_{x\in G/H}\alpha x \otimes x^{-1})$$

for all $g \in {}_G(X, W)$, $h \in {}_G(X_H{}^G, W)$ and $\alpha, \alpha_x \in X$. An easy calculation shows that $m\ Y_H(W, W)(f)(X_H)\ n(g) = t_H^G(f)\, g$ for all $g \in {}_G(W, X)$. On the other hand, since m and n are natural and since f preserves K, (4.3)$'$ induces

$$(4.3)'' \quad K(X) \xrightarrow{n} K(X_H{}^G) \xrightarrow{Y_H(W, W)(f)(X_H)} K(X_H{}^G) \xrightarrow{m} K(X)$$

Hence $t_H^G(f)$ preserves K. Now let Q be the quotient object ${}_G(\cdot, W)/K$. Noticing $Q(X_H{}^G) = Q_H(X_H)$, (4.3)$'$ and (4.3)$''$ give

$$Q(X) \xrightarrow{n'} Q_H(X_H) \xrightarrow{f^*(X_H)} Q_H(X_H) \xrightarrow{m'} Q(X).$$

Since $Q_H(X_H) = Q_H{}^G(X)$, it follows from the definition of the trace map [9, 5.6] that the composite of the above sequence is precisely $T_H^G(f^*)(X)$. On the other hand, the early computation yields that the composite of the above takes any $\bar{g} \in Q(X)$ into $\overline{t_H^G(f)\, g} \in Q(X)$, where $\bar{f}$ means the image of f under the natural epimorphism from ${}_G(X, W)$ onto $Q(X)$. Therefore, $t_H^G(f)$ induces $T_H^G(f^*) \in \mathrm{End}(_G(\cdot, W)/K)$. Now the proof is complete.

5. Relative projectinity of simple objects

In this section, we study relative projectivity of SW for a fixed non-projective indecomposable kG-module W. As remarked in Section 2, to determine a vertex of SW, we may consider the case where a vertex of W is normal in G and a source of W is G-invariant. Thus in this section, we assume that W is a direct summand of V^G and that V is G-invariant.

Let e be the primitive idempotent of $E = \mathrm{End}_{kG}(V^G)$ corresponding to W so that $eV^G=W$. As before we fix a subgroup H of G with $N\subset H\subset G$. The results in the previous section and [9, Prop. 6.4] give the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{End}({}_G(\cdot, W)) & \xleftarrow{Y(W,W)} & \mathrm{End}_{kG}(W) & \xrightarrow{z} & \mathrm{End}_E(eE) \\ T_H^G \uparrow & & t_H^G \uparrow & & \tau_H^G \uparrow \\ \mathrm{End}(({}_G(\cdot, W))_H) & \xleftarrow{Y_H(W,W)} & \mathrm{End}_{kH}(W) & \xrightarrow{z_H} & \mathrm{End}_{E_H}(eE) \end{array} \tag{5.1}$$

For simplicity we write Y_H to denote $Y_H(W, W)$. The following is a key result.

Theorem 5.2. *For any subgroup H of G with $N\subset H\subset G$, the composite $z_H Y_H^{-1}$ induces an isomorphism*

$$\theta_H\colon \mathrm{End}((SW)_H) \to \mathrm{End}_{E_H}(eE/eJE)$$

of k-algebras such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{End}(SW) & \xrightarrow{\theta_G} & \mathrm{End}_E(eE/eJE) \\ T_H^G \uparrow & & \tau_H^G \uparrow \\ \mathrm{End}((SW)_H) & \xrightarrow{\theta_H} & \mathrm{End}_{E_H}(eE/eJE) \end{array} \tag{5.2$'$}$$

We first prove the following.

Proposition 5.3. *Let f be an element of* $\mathrm{End}_{kH}(W)$. *Then the following* (i) (*resp.* (iii)) *is equivalent to* (ii) (*resp.* (iv)).

(i) $z_H(f)$ *preserves* eJE.

(ii) $Y_H(f)$ *preserves* $rad_G(\cdot, W)$.

(iii) $z_H(f)\,(eE)\subset eJE$.

(iv) $Y_H(f)\,({}_G(\cdot, W))_H\subset(rad_G(\cdot, W))_H$.

Proof. We rewrite the above conditions as follows. By the explicit description of z_H in the paragraph preceding Lemma 4.1, (i) is equivalent to

(i)′ $(f(eJE)^a)^{a^{-1}}\subset eJE$.

Moreover, since $f(eJE)^a\subset{}_H(V^H, W)$, it follows by Lemma 1.3 that (i)′ is equivalent to

(i)″ $t_H^G(f(eJE)^a)\subset eJE$.

Similarly, (iii) is equivalent to

(iii)′ $t_H^G(f(eE)^a)\subset eJE$.

Next, we claim that (ii) is equivalent to

(ii)′ For any kH-module X and any $h\in{}_H(X, W_H)$,

$$t_H^G(h\,{}_H(V^G{}_H, X))\subset eJE \quad \text{implies} \quad t_H^G(fh\,{}_H(V^G{}_H, X))\subset eJE\,.$$

Let $K=\mathrm{rad}_G(\cdot, W)$. Then by Propositipon 2.1, we have

$$0 \to K_H(X) \to {}_H(X, W) \xrightarrow{\gamma_H(X)} (D(V^G, X))_H \quad \text{(exact)}.$$

Also let $T_\gamma=\gamma(W)(\mathrm{Id}_W)\in D(V^G, W)$. Now (ii) holds if and only if $\gamma_H(X)(fh)=0$ in $(D(V^G, X))_H$ for those $h\in{}_H(X, W)$ with $\gamma_H(X)(h)=0$. Now recall that $\gamma_H(X)(h)(g)=\gamma_H(W_H)(\mathrm{Id}_W)(hg)=T_{\gamma_H}(hg)$ for all $g\in{}_H(V^G{}_H, X)$ and that $T_{\gamma_H}=T_\gamma t_H^G$([9, Prop. 6.7]). On the other hand, since $t_H^G(h\,{}_H(V^G{}_H, X))$ and $t_H^G(fh\,{}_H(V^G{}_H, X))$ are right E-submodules of eE, Proposition 2.1 yields that Ker T_γ includes $t_H^G(h\,{}_H(V^G{}_H, X))$ (resp. $t_H^G(fh\,{}_H(V^G{}_H, X))$) if and only if eJE includes $t_H^G(h\,{}_H(V^G{}_H, X))$ (resp. $t_H^G(fh\,{}_H(V^G{}_H, X))$). Thus (ii) is equivalent to (ii)$'$.

A similar argument shows that (iv) is equivalent to

(iv)$'$ For any kH-module X and any $h\in{}_H(X, W_H)$,

$$t_H^G(fh\,{}_H(V^G{}_H, X))\subset eJE\,.$$

We will prove that (i)$''$ (resp. (iii)$'$) is equivalent to (ii)$'$ (resp. (iv)$'$).

Next we claim that for any kH-module X and any $h\in{}_H(X, W_H)$ we have

$$(5.3)' \qquad t_H^G(fh\,{}_H(V^G{}_H, X)) = \textstyle\sum_{x\in G/H} t_H^G(fh\,{}_H(V^H, X))\,u_x^{-1}\,.$$

In fact, the left hand side of (5.3)$'$ is equal to

$$\begin{aligned}&\textstyle\sum_{x\in G/H} t_H^G(fh\,{}_H(V^H, X)\,s_x) \quad \text{(by Lemma 1.2.)}\\ &= \textstyle\sum_{x\in G/H}(fh\,{}_H(V^H, X))^{a^{-1}}\,u_x^{-1} \quad \text{(by Lemma 1.3)},\end{aligned}$$

which is equal to the right hand side of (5.3)$'$ again by Lemma 1.3. Since (5.3)$'$ holds for any choice of $f\in\mathrm{End}_{kH}(W)$, we also have

$$t_H^G(h\,{}_H(V^G{}_H, X)) = \textstyle\sum_{x\in G/H} t_H^G(h\,{}_H(V^H, X))\,u_x^{-1}\,.$$

Furthermore, taking $X=V^H$, we obtain

$$(5.3)'' \qquad t_H^G(h\,{}_H(V^G{}_H, V^H)) = \textstyle\sum_{x\in G/H}(hE_H)^{a^{-1}}\,u_x^{-1} = \sum_{x\in G/H} h^{a^{-1}}\,E_H\,u_x^{-1}\,.$$

Here the last equality holds since a is an isomorphism of E_H-modules.

We first show that (ii)$'$ is equivalent to (i)$''$. Assume (ii)$'$. Let $X=V^H$ in (ii)$'$, g an element of eJE, and let $h=g^a\in{}_H(V^H, W)$. Then since $h^{a^{-1}}=g\in eJE$, (5.3)$''$ implies that $t_H^G(h\,{}_H(V^G{}_H, V^H))\subset eJE$, and hence (ii)$'$ yields that $t_H^G(fh\,{}_H(V^G{}_H, V^H))\subset eJE$. In particular, taking the element s of ${}_H(V^G{}_H, V^H)$ defined by

$$s(v) = \begin{cases} v & \text{if } \ v\in V^H \\ 0 & \text{if } \ v\in \bigoplus_{x\neq 1}(Vx)^H\,, \end{cases}$$

we can conclude that $t_H^G(fh)=t_H^G(fg^a)$ lies in eJE. Thus (i)$''$ holds.

Conversely, suppose that (i)$''$ holds. For a kH-module X and $h\in{}_H(X, W_H)$, assume that $t_H^G(h_H(V^G{}_H, X)) \subset eJE$. Then we in particular get $t_H^G(h_H(V^H, X)) \subset eJE$. Hence Lemma 1.3 yields that $(h_H(V^H, X))^{a-1} \subset eJE$, i.e., that $(h_H(V^H, X)) \subset (eJE)^a$. Now using (5.3)′, (i)″ implies that $t_H^G(fh_H(V^G{}_H, X)) = \sum_{x \in G/H} t_H^G(fh_H(V^H, X))\, u_x^{-1} \subset \sum_{x \in G/H}(eJE)\, u_x^{-1} = eJE$. Hence (ii)′ holds.

Let us now show that (iii)′ is equivalent to (iv)′. Assume (iii)′. Notice that for any kH-module X and any $h \in {}_H(X, W_H)$, $h_H(V^H, X)$ is included in $(eE)^a$. Thus by (5.3)′ we get $t_H^G(fh_H(V^G{}_H, X)) = \sum_{x \in G/H} t_H^G(fh_H(V^H, X))\, u_x^{-1} \subset \sum_{x \in G/H}(eJE)\, u_x^{-1} = eJE$. Hence (iv)′ holds. Conversely assume that (iv)′ holds. Then we have $t_H^G(f_H(V^H, W_H)) \subset t_H^G(f_H(V^G{}_H, W_H)) \subset eJE$. Since ${}_H(V^H, W_H) = (eE)^a$, we get (iii)′. Now the proof is complete.

Proof of Theorem 5.2. For any H with $N \subset H \subset G$, define θ'_H: End $(({}_G(\cdot, W))_H) \to \mathrm{End}_{E_H}(eE)$ by $\theta'_H = z_H Y_H^{-1}$. Then θ'_H is an isomorphism of k-algebras. Note that eE_{E_H}(resp. $({}_G(\cdot, W))_H$) is a projective object of $\mathrm{Mod}\, E_H$ (resp. $\mathrm{MMod}\, kH$). Hence by Proposition 5.3, θ'_H induces an isomorphism of k-algebras from $\mathrm{End}((SW)_H)$ onto $\mathrm{End}_{E_H}(eE/eJE)$. Namely, for any $\xi \in \mathrm{End}((SW)_H)$, there is an element η of $\mathrm{End}(({}_G(\cdot, W))_H)$ such that η preserves $\mathrm{rad}_G(\cdot, W)$ and induces ξ. Thus by Proposition 5.3, $\theta'_H(\eta)$ preserves eJE and $\theta_H(\xi)$ is defined to be the E_H-endomorphism of eE/eJE induced by $\theta'_H(\eta)$. By Proposition 5.3 again, this does not depend on the choice of those η that induce ξ.

Now we prove that (5.2)′ commutes. For any $\xi \in \mathrm{End}((SW)_H)$, Lemma 4.3 implies that $T_H^G(\xi)$ is induced by $T_H^G(\eta)$, where η is an element of $\mathrm{End}(({}_G(\cdot, W))_H)$ which induces ξ. It follows by the definition of the θ_H and commutativity of the diagram (5.1) that $\theta_G\, T_H^G(\xi)$ is induced from $\tau_H^G\, \theta'_H(\eta)$. Now by Lemma 4.2 $\tau_H^G\, \theta'_H(\eta)$ induces $\tau_H^G\, \theta_H(\xi)$. Therefore, we have $\theta_G\, T_H^G(\xi) = \tau_H^G\, \theta_H(\xi)$ as desired. This completes the proof.

The following theorem, which is an easy consequence of Theorem 5.2, is the main result of this section.

Theorem 5.4. *Let H be a subgroup of G with $N \subset H \subset G$. Then SW is H-projective if and only if eE/eJE is H/N-projective.*

Proof. Using Higman's criterion, the result holds immediately from Theorem 5.2.

Using the above theorem, to study relative projectivity of SW, we may consider that of a simple E-module eE/eJE.

In the rest of this section, we assume further that V is indecomposable and that k is algebraically closed.

Then we have the following, which is well known. (See, for example, [4. Propositions 2.4 and 5.2].)

Lemma 5.5. *Let $I=J(E_N)E$. Then;*

(i) *I is a two sided ideal of E.*

(ii) *I is included in JE, and hence annihilates each simple E-module.*

(iii) *E/I is isomorphic to a twisted group algebra of G/N over k with a basis $\{\bar{u}_x\}$, where $\bar{u}_x$ is the image of u_x in E/I.*

By the above lemma, each simple E-module can be considered as a simple module over a twisted group algebra E/I. Thus knowledge of modules over those algebras (see [3] and [10] for example) will help to determine vertices of a simple object SW.

Remark. Relative projectivity for twisted group algebras is defined in a way similar to that for group algebras ([3, §4]), and we can use Higman's criterion, as well. It follows at once from the definition of the relative projectivity for Mod$E$ that eE/eJE is H/N-projective as an E-module if and only if it is H/N-projective as an E/I-module.

As an application of Theorem 5.4, we give the following example.

Example. Let $G=GL(3, q)$, where q is a power of p. Put

$$D=\left\{\begin{bmatrix}1&0&c\\0&1&0\\0&0&1\end{bmatrix} : c\in \boldsymbol{F}_p\right\}, \quad P=\left\{\begin{bmatrix}1&a&c\\0&1&b\\0&0&1\end{bmatrix} : a, b, c\in \boldsymbol{F}_q\right\},$$

and let H be the subgroup of G consisting of all the upper triangular matrices. Then an easy calculation shows that $H=N_G(D)$, H/D has a normal p-Sylow subgroup P/D, and $C_G(D)\supset P$. Hence every kD-module S is P-invariant, i.e., the inertial subgroup $I_H(S)$ of S in H contains P, and every simple module over any twisted group algebra of $I_H(S)/D$ has P/D as its vertex. Therefore, Proposition 2.4 and Theorem 5.4 imply that if D is a vertex of an indecomposable kG-module W, then $vtx(SW)=_G P$.

As another application of Theorem 5.4, we prove;

Theorem 5.6. *Suppose that U is an indecomposable kG-module. Let P be a vertex of U with $N_G(P)\supset vtx(SU)\supset P$, (see Proposition 2.4 (i)), and S a P-source of U. If U has p'-multiplicity as a direct summand of S^G, then $vtx(SU)$ is a p-Sylow subgroup of the inertial subgroup $I_G(S)$ of S in $N_G(P)$.*

Proof. Let U' be the Green correspondent of U with respect to $(G, P, N_G(P))$. Let U'' be an indecomposable direct summand of $S^{I_G(S)}$ such that $U''^{N_G(P)}=U'$. Then U'' has p'-multiplicity as a direct summand of $S^{I_G(S)}$. Thus by Proposition 2.4 (ii), we may assume that P is normal in G and S is G-invariant. Using the letters W, V and N instead of U, S and P, respectively, we are in the same situation as in Theorem 5.4. So use the same notation as there. Consider the simple E/I-module eE/eJE, which is isomorphic to $SW(V^G)$ as E-modules. Since $SW(V^G)=(SW)_N(V)$, Lemma 2.5 and our assumption imply that $\dim_k eE/eJE$ is relatively prime to p. Hence by [10, Chap. 5 Theorem 9.8], $vtx(eE/eJE)$ is a p-Sylow subgroup of G/N. Therefore the result follows from Theorem 5.4. This completes the proof.

As a corollary to the above, we can prove the first half of [9, Theorem 8.2] as follows.

Corollary 5.7. *Assume that G is a p-group. Let W be an indecomposable kG-module, P a vertex of W and S a P-source of W. Then $vtx(SW)=_G I_G(S)$.*

Proof. By Green's theorem, $S^{N_G(P)}$ is indecomposable and hence it is the Green correspondent of W with respect to $(G, P, N_G(P))$. Thus W has multiplicity 1 as a direct summand of S^G. Therefore the above theorem yields the results.

6. Extendibility of simple objects

As before, N is a normal subgroup of G and V is an indecomposable kN-module.

We say that SV extends to G if there exists a finitely presented object F of MModkG such that $F_H=SV$. When V is non-projective, by a standard argument [1, Prop. 4.9], SV extends to G if and only if there exists a short exact sequence of kG-modules such that upon the restriction to N it is isomorphic to the direct sum of SV with a split short exact sequence. The above F (or short exact sequence) is called an extension of SV to G.

The main result of this section is as follows.

Theorem 6.1. (i) *If V extends to G, then so does SV.*

(ii) *Suppose that $E_N/JE_N\simeq k$. Then, if there is an indecomposable kG-module W such that $(SW)_N\simeq SV$, the number of isomorphism classes of those modules is equal to that of 1-dimensional representations of G/N over k.*

Proof. We first claim that if SW is an extension of SV, then W is isomorphic to a direct summand of V^G. Assume that it does not hold. Then there holds $SW(V^G)=\{0\}$ for some W with $(SW)_N\simeq SV$. Thus it follows by the definition of the restriction for MMod kG that $(SW)_N(V)=\{0\}$. This contradicts the fact that $(SW)_N\simeq SV$.

For any indecomposable direct summand W of V^G, let r_W denote the multiplicity of W.

If V extends to G, then it is G-invariant. Thus, if W is isomorphic to a direct summand of V^G, then W_N is isomorphic to a direct sum of some copies of V. It follows by Lemma 2.5 that for any indecomposable direct summand W of V^G the simple object SW is an extension of SV if and only if $r_W\, \delta_{kG}(W) = \delta_{kN}(V)$. We now apply Theorem 3.4. By the equivalence between $\mathrm{Mod}(kG \mid V)$ and $\mathrm{Mod}(E \mid E_N)$, each (isomorphism class of) indecomposable direct summand W of V^G corresponds to an (isomorphism class of) indecomposable direct summand Y of E with the same multiplicity. (Remark 3.6.) Combining this with the claim in the first paragraph, it follows that the number of isomorphism classes of indecomposable kG-modules W such that SW are extensions of SV is equal to the number of isomorphism classes of indecomposable direct summands of E with the multiplicity b_Y satisfying

$$b_Y\, \delta_E(Y) = \delta_{E_N}(E_N)\,. \tag{6.1)'$$

(See Theorem 3.4 (ii).) Now recall that E is a fully G/N-graded ring and that $E_N \simeq \mathrm{End}_{kN}(V)$ is a local ring. So, in particular, E_N (resp. $\bar{E}_N = E_N/JE_N$) is the unique projective indecomposable (resp. simple) E_N-module. Hence applying Theorem 3.7 to E, the number of isomorphism classes of those Y satisfying (6.1)′ is equal to the number of isomorphism classes of extensions of $\bar{E}_N$ to E. On the other hand, since V extends to G, it follows by Corollary 3.5 that E_N extends to E, and hence $\bar{E}_N$ extends to E by Corollary 3.3. Therefore, especially, an argument given above implies that SV extends to G.

Now assume that $\bar{E}_N \simeq k$. First note that if an indecomposable kG-module W satisfies $(SW)_N \simeq SV$, then W is N-projective by the argument in the first paragraph. It is easily seen from [9, Prop. 7.9] that our assumption implies that V is G-invariant. Thus to prove the second statement, it suffices to show that the number of isomorphism classes of extensions of $\bar{E}_N \simeq k$ to E coincides with the number of 1-dimensional representations of G/N over k. Now our previous argument yields that there is an extension of $\bar{E}_N$ to E. Let $I = (JE_N)\, E$. Since I annihilates any extension of $\bar{E}_N$ (Lemma 5.5 (ii)), E/I has a 1-dimensional representation. Thus E/I is isomorphic to the group algebra of G/N over k. (See Lemma 5.5 (iii).) Hence each extension of $\bar{E}_N$ can be considered as a 1-dimensional representation of G/N. Since $I \subset JE$, any two extensions of $\bar{E}_N$ are isomorphic to each other as E-modules if and only if they are so as E/I-modules. Therefore the second statement has been proved.

REMARK. (i) Suppose that G/N is a p-group and that k is sufficiently large. Then V^G is indecomposable by Green's theorem. Thus, if V is G-invariant, then the proof of Theorem 6.1 implies that $S(V^G)$ is a unique simple extension of SV.

(ii) There might be an extension of SV which is not simple. For example, let N be a cyclic group of order p, G the direct product of N with another cyclic group M of order p. Take a non-projective indecomposable kN-module V and the trivial kM-module k_M. Then the tensor product $(SV)\otimes_k k_M$ is naturally a short exact sequence of kG-modules and it is clearly an extension of SV. However, this sequence is not an Auslander-Reiten sequence. In particular, this gives an object of MModkG different from $S(V^G)$.

Assume that V is G-invariant and $\bar{E}_N \simeq k$. Then $E/(JE_N)\,E$ is isomorphic to a twisted group algebra (Lemma 5.5 (iii)), and hence it determines an element φ of $\mathrm{H}^2(G/N, k^*)$. See also [11, §1]. Now we have;

Corollary 6.2. *In the above situation, there is an indecomposable kG-module W such that* $(SW)_N \simeq SV$ *if and only if* $\varphi=0$ *in* $\mathrm{H}^2(G/N, k^*)$.

Proof. This is clear by the proof of Theorem 6.1 since a twisted group algebra has a 1-dimensional representation if and only if $\varphi=0$.

REMARK. If G/N is a p'-group, then by [11, Cor. 1.12] $\varphi=0$ if and only if V extends to G. Therefore, Theorem 6.1 and Corollary 6.2 yield that there is an indecomposable kG-module W such that $(SW)_N \simeq SV$ if and only if V extends to G.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan