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Osaka University
ON THE REGULARITY OF THE WEAK SOLUTIONS
OF ABSTRACT DIFFERENTIAL EQUATIONS

VIOREL BARBU

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1. Preliminaries

Let $H$ be a Hilbert space; $(\cdot, \cdot)$, $||\cdot||$ and are the notations for the scalar product and for the norm. Denote by $\mathbb{R}$ the real axis, $-\infty < t < \infty$, and $\mathcal{D}(\mathbb{R})$, $\mathcal{D}(\mathbb{R}, H)$, $\mathcal{D}'(\mathbb{R}, H)$ the spaces of infinitely differentiable scalar functions with compact support, infinitely differentiable $H$-valued functions with compact support and $H$-valued distribution, respectively, on $\mathbb{R}$ with their usual topologies (see L. Schwartz [7]). The space of $H$-valued distributions with compact support will be denoted by $\mathcal{E}'(\mathbb{R}, H)$. If $u \in \mathcal{D}'(\mathbb{R}, H)$ we may define $D^k u \in \mathcal{D}'(\mathbb{R}, H)$ by the formula: $D^k u(\phi) = (-1)^k u(D^k \phi)$, $\forall \phi \in \mathcal{D}(\mathbb{R}, H)$. If $\phi \in \mathcal{D}(\mathbb{R}, H)$ then for each complex $\lambda$, $\hat{\phi}(\lambda)$ denotes its Fourier-Laplace transform. (Here $D^k \phi = \frac{d^k}{dt^k}\phi$).

Let $A : D_A \subset H \to H$ be a closed linear operator with the domain $D_A$ dense in $H$ and let $A^*$ be its adjoint. The domain $D_A^*$ of the operator $A^*$ is Banach space in the norm $||x|| = ||x|| + ||A^*x||$. We denote by $\mathcal{D}(\mathbb{R}, D_A^*)$ the space of infinitely differentiable $D_A^*$-valued functions with compact support on $\mathbb{R}$ and by $\mathcal{D}'(\mathbb{R}, D_A^*)$ its dual. Since $\mathcal{D}(\mathbb{R}, D_A^*) = \mathcal{D}(\mathbb{R}) \overline{\otimes} D_A^*$ it is easy to see that the space $\mathcal{D}(\mathbb{R}, D_A^*)$ is dense in $\mathcal{D}(\mathbb{R}, H)$. In an analogous manner, we may define the spaces $\mathcal{D}(a, b; H)$, $\mathcal{D}'(a, b; H)$ and $\mathcal{D}(a, b; D_A^*)$. Let $L^* : \mathcal{D}'(\mathbb{R}, D_A^*) \to \mathcal{D}(\mathbb{R}, H)$ be the linear operator

\begin{equation}
L^* \phi = -\left(\frac{1}{i} \frac{d\phi}{dt} + A^* \phi\right)
\end{equation}

and let $L : \mathcal{D}'(\mathbb{R}, H) \to \mathcal{D}'(\mathbb{R}, D_A^*)$ be its adjoint defined by

\begin{equation}
L u(\phi) = u(L^* \phi), \quad \phi \in \mathcal{D}(\mathbb{R}, D_A^*).
\end{equation}

Let $\mathcal{L}(\mathcal{D}, H)$ be the space of all vector $H$-valued distributions $\mathcal{L}(\mathcal{D}(\mathbb{R}, H), H)$. For $E \in \mathcal{L}(\mathcal{D}, H)$ we define $LE$ by the formula

\begin{equation}
LE(\phi) = E(L^* \phi), \quad \phi \in \mathcal{D}(\mathbb{R}, D_A^*),
\end{equation}

where

\begin{equation}
E(t) = \int_{\mathbb{R}} e^{-i t \lambda} \hat{E}(\lambda) \, d\lambda,
\end{equation}

and $\hat{E}(\lambda)$ is the Fourier transform of $E(t)$. Following Barbu, let $E \in \mathcal{L}(\mathcal{D}, H)$, with $E(t) = \mathcal{E}_t E(0)$, $\forall t \geq 0$, and $\mathcal{E}_t$ is the semigroup of operators generated by $E(t)$. Then $LE(\phi) = \mathcal{E}_t \mathcal{L}E(\phi)$. Let $E(\cdot) = E_{\cdot}$. If $x$ is a function, $x(t) = \mathcal{E}_t x(0)$, $\forall t \geq 0$. For $E \in \mathcal{L}(\mathcal{D}, H)$ and $x \in H$ we define $E x$ by the formula

\begin{equation}
E x = E(\cdot) x(\cdot),
\end{equation}

where $E(\cdot) x(\cdot)$ is the function generated by $E(\cdot)$ and $x(\cdot)$.
and denote, for every $\varphi \in \mathcal{D}(R, H)$

$$(E*\varphi)(t) = E_s(\varphi(t-s)).$$

It is easy to see that $E*\varphi \in C^\infty(R, H)$. If $u \in \mathcal{E}'(R, H)$ and $E \in \mathcal{L}(\mathcal{D}, H)$, $E*u$ denotes the distribution defined by

$$(E*u)(\varphi) = u(E_s(\varphi(t+s)), \varphi^3)(R, H).$$

If $\varphi(t) \in \mathcal{D}(R)$, then as in the scalar case it follows immediately

$$\text{supp } (\rho E*u) \subseteq \text{supp } \rho + \text{supp } u.$$

**Definition.** We say that the distribution $u \in \mathcal{D}'(a, b; H)$ is a weak solution on $(a, b)$ of the equation

$$(E) \quad \frac{1}{i} \frac{du}{dt}Au = f,$$

where $f \in \mathcal{D}'(a, b; D_A*)$, if the following relation

$$(1.4) \quad u(L^*\varphi) = f(\varphi)$$

holds for any $\varphi \in \mathcal{D}(a, b; D_A*)$.

The existence theorems for the weak solutions of the equation (E) have been obtained by T. Kato and H. Tanabe [3], S. Zaidman [8] and M. A. Malik [6]. We give in this paper some results concerning the regularity of the weak solutions of (E). For the strict solutions of (E) a similar result has been proved by S. Agmon and L. Nirenberg [1].

2. **Differentiability of solutions**

In the following we denote by $R(\lambda, A^*)$ the resolvent $(\lambda I - A^*)^{-1}$ of the operator $A^*$.

**Theorem 1.** Suppose that for every $m \geq 0$ there exists a number $C_m > 0$ such that the resolvent $R(\lambda, A^*)$ exists in the domain

$$(2.1) \quad \Lambda_m = \{ \lambda; |\text{Im } \lambda| \leq m \log |\text{Re } \lambda|; |\text{Re } \lambda| \geq C_m \},$$

and

$$(2.2) \quad ||R(\lambda, A^*)|| \leq C_m^1 |\lambda|^M \exp (N |\text{Im } \lambda|), \quad \text{in } \Lambda_m,$$

where $M > 0$, $N > 0$ are constants independent of $m$ and $C_m^1 > 0$. Then every weak solution $u \in \mathcal{D}'(-a, a; H)$ of (E) with $f \in C^\infty(-a, a; H)$ is infinitely differentiable on the interval $|t| < a - N$. 
Proof. Let $E \in L(D, D_{A^*})$ be defined by the equality

$$
E(\varphi) = -(2\pi)^{-1} \int_{|\sigma| \geq C_m} R(-\sigma, A^*) \hat{\varphi}(\sigma) d\sigma; \quad \varphi \in D(R, H)
$$

Obviously

$$
E(L^* \varphi) = \varphi(0) - \int_{|\sigma| \leq C_m} \hat{\varphi}(\sigma) d\sigma; \quad \varphi \in D(R, D_{A^*}).
$$

We denote by $\Delta$ the interval $(-a', a')$ where $a' < a - N_1$, $N < N_1 < a$, and consider $\varphi(t) \in D(R)$ such that $\varphi(t) = 1$ for $|t| \leq a' + \delta$ and $\varphi(t) = 0$ in $|t| \geq a' + \delta'$. Assume that $N < \delta < \delta' < N_1$. If $u \in D'(-a, a; H)$ is a weak solution of (E) then we have

$$
L(u \varphi)(\psi) = (f \varphi)(\psi) + (D^t \varphi u)(\psi)
$$

for every $\varphi \in D(-a, a; H)$. On the other hand since $A^*$ is closed, from (2.2) it follows

$$
(L(u \varphi)^*) \varphi = (L \varphi u)(\psi).
$$

Let us denote by $g$ the distribution $D^t \varphi \cdot u$. Then from (2.4) and (2.5) we get

$$
u \varphi(\psi) = (E \varphi + (E^t \varphi) \psi + u \varphi(\hat{\varphi}^t))
$$

for every $\varphi \in D(-a, a; H)$, where $h_\psi(t) = \int_{|\sigma| \leq C_m} e^{it\sigma} \hat{\varphi}(\sigma) d\sigma$.

Obviously

$$
||D^k h_\varphi(t)|| \leq M_k ||\psi||_{L^2}, \quad t \in R
$$

for any $\varphi \in D(-a, a; H)$, where $|| \ ||_{L^2}$ denotes the norm in the space $L^2(R, H)$.

Since $f \varphi \in C^\infty(R, H)$ it follows that $E \varphi \varphi \in C^\infty(R, H)$ which implies that

$$
|D^k (E \varphi \varphi)(\psi)| \leq M_k^2 ||\psi||_{L^2}; \quad \psi \in D(-a, a; H).
$$

Let $\rho(t)$ be a scalar $C^\infty$ function on the real line such that $\rho(t) = 1$ for $|t| \leq \varepsilon$ and $\rho(t) = 0$ for $|t| \geq \varepsilon'$; $0 < \varepsilon < \varepsilon'$. Since supp $g \subset \{t; a' + \delta < |t| \leq a' + \delta'\}$, taking $\varepsilon$ so small such that $\varepsilon' < \delta$, from an above remark we deduce that $(\rho E^t g)(\varphi) = 0$ for any $\varphi \in D(-a', a'; H)$.

Hence

$$
(E^t g)(\varphi) = ((1 - \rho)E^t g)(\varphi), \quad \varphi \in D(-a', a'; H).
$$

Now we introduce the function $\phi^t(s) = (1 - \rho(s))D^k \varphi(t + s)$ and denote by $\hat{\phi}^t(s)(\lambda)$ its Laplace transform. Let $m$ be an arbitrary non-negative integer. We may write $\hat{\phi}^t(s)(\lambda)$ in the form...
where
\[ \hat{\psi}_{t,1}^{(k)}(\lambda) = \int_{s \geq s} e^{-is\lambda} (1 - \rho(s)) D^k \phi(t + s) ds \]
and
\[ \hat{\psi}_{t,2}^{(k)}(\lambda) = \int_{s < -s} e^{-is\lambda} (1 - \rho(s)) D^k \phi(t + s) ds. \]

A simple computation shows that with another constant \( M_k \), one must have the estimates,

\begin{equation}
(2.11) \quad \|\hat{\psi}_{t,1}^{(k)}(\sigma - i m \log |\sigma|)\| \leq M_k |\sigma|^{-k} \|\phi\|_{L^2}, \quad \sigma \in \mathbb{R}
\end{equation}

and

\begin{equation}
(2.12) \quad \|\hat{\psi}_{t,2}^{(k)}(\sigma + i m \log |\sigma|)\| \leq M_k |\sigma|^{-k} \|\phi\|_{L^2}, \quad \sigma \in \mathbb{R}
\end{equation}

Let \( f_i^{(k)}(t) \) be the functions

\begin{equation}
(2.13) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{|r| \geq C_m} R(-\sigma, A^*) \hat{\psi}_{t,i}^{(k)}(\sigma) d\sigma; \quad i = 1, 2; \quad t \in \mathbb{R}.
\end{equation}

After a suitable deformation of contours in the complex plane, the functions \( f_i^{(k)}(t) \) can be expressed in the following form

\begin{equation}
(2.14) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{\Gamma_m} R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda) d\lambda; \quad i = 1, 2;
\end{equation}

where \( \Gamma_m \) is the frontier of the domain \( \{ \lambda; \ Im \lambda \geq -m \log |Re \lambda|; |Re \lambda| \geq C_m \} \) and \( \Gamma_m^* \) the frontier of \( \{ \lambda; \ Im \lambda \leq m \log |Re \lambda|; |Re \lambda| \geq C_m \} \). It is easy to see that the shift of the integration contour is legitimate. Now we have on \( \Gamma_m^* \),

\[ \|R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda)\| \leq M_k |\lambda|^{-m+k-\varepsilon-N} \|\phi\|_{L^2}; \quad \sigma = Re \lambda. \]

Choosing \( \varepsilon \) so that \( \varepsilon > N \) and \( m \) so large such that \( M + k - m(\varepsilon - N) < -1 \), one obtains

\begin{equation}
(2.15) \quad \|f_i^{(k)}(t)\| \leq M_k \|\phi\|_{L^2}; \quad t \in \mathbb{R}; \quad \phi \in \mathcal{D}(-a', a'; H).
\end{equation}

We remark that

\[ D^h((1 - \rho)E^*g)(\psi) = (-1)^h (g(f^{(k)}_1) + g(f^{(k)}_2)). \]

From (2.15) this implies that

\begin{equation}
(2.16) \quad \|D^h((1 - \rho)E^*g)(\psi)\| \leq M_k \|\phi\|_{L^2}; \quad \phi \in \mathcal{D}(-a', a'; D_A^*).
\end{equation}

Using (2.7), (2.8), (2.9) and (2.16) we obtain

\begin{equation}
(2.17) \quad |D^h(\omega \varphi)(\psi)| \leq M_k \|\psi\|_{L^2}; \quad \varphi \in \mathcal{D}(-a', a'; D_A^*).
\end{equation}
Since the space $\mathcal{D}(-a', a'; D_A^*)$ is dense in $\mathcal{D}(-a', a'; H)$ from the Hahn-Banach theorem it follows that $D^k(u\varphi) \in L^2(-a', a'; H)$ for any $k=0, 1, \ldots$. Hence $u\varphi \in C^\infty(-a', a'; H)$. Because the number $N_i > N$ is arbitrary, the proof is complete.

**Corollary 1.** Suppose that there exist some non-negative numbers $N$, $C$, $N_0$ such that $R(\lambda, A^*)$ exists in the domain

$$(2.18) \quad \Lambda = \{\lambda; \ |Im\ \lambda| \leq C \log |Re\ \lambda|; \ |Re\ \lambda| \geq N_0\}$$

and

$$(2.19) \quad ||R(\lambda, A^*)|| \leq \text{pol} (|\lambda|) \exp (N |Im\ \lambda|).$$

Then every solution $u\in \mathcal{D}(R, H)$ of (E), with $f \in C^\infty(R, H)$, is infinitely differentiable on $R$.

**Corollary 2.** Suppose that $f \in \mathcal{D}'(-a, a; D_A^*)$ such that

$$(2.20) \quad |D^k f(\varphi)| \leq M_k ||\varphi + A^* \varphi||_{L^2}; \quad \forall \varphi \in \mathcal{D}(-a, a; D_A^*).$$

If the hypotheses of theorem 1 are satisfied, then every solution $u \in \mathcal{D}'(-a, a; H)$ of (E) is infinitely differentiable on the subinterval $|t| < a - N$.

Proof. The proof in this case is very much the same, except the inequality (2.9). To estimate $|D^k(E*f\varphi)(\varphi)|$ we remark that

$$||A^*(E*f\varphi)(t)|| \leq M ||D^l \varphi||_{L^2} \quad \forall \varphi \in \mathcal{D}(-a, a; H)$$

where $l$ is a non-negative integer. From (2.6), (2.7) and (2.16) this implies that

$$(2.21) \quad |D^k(u\varphi)(\varphi)| \leq M_k ||D^l \varphi||_{L^2}; \quad \forall \varphi \in \mathcal{D}(-a', a'; D_A^*).$$

As in the proof of theorem 1 this implies that $u \in C^\infty(-a', a'; H)$.

**Remark.** If $u \in C^\infty(\Delta, H)$ is a weak solution of (E) with $f \in C^\infty(\Delta, H)$, then $u(t)$ is a strict solution of (E). To prove this it is enough to choose in the equality (1.2), $\varphi = \varphi_0 \otimes x$ where $\varphi_0 \in \mathcal{D}(\Delta)$ and $x \in H$. Hence the necessity results for differentiability, proved by Agmon-Nirenberg [1], are true in our case.

### 3. Hypoanalyticity of solutions

**Definitions.** A $C^\infty H$-valued function $u(t)$ is said to be $d$-hypoanalytic on $\Delta \subset R$ if for any compact subset $K \subset \Delta$ there exists a non-negative constant $M_K$ such that for any $k$ the following inequality be true

$$(3.1) \quad ||D^k u; K||_\infty \leq M_k^{k+1} (k!)^d$$

where $||u, K||_\infty = \sup_{t \in K} ||u(t)||$. 
In the following we denote by $G^d(\Delta, H)$ the space of all $d$-$H$-valued hypoanalytic functions on $\Delta$. If $H=R$ we omit $R$ and write $G^d(\Delta)$.

**Theorem 2.** Suppose that $R(\lambda, A^*)$ exists in a region

$$\sum: \{\lambda ; |Im \lambda| \leq C |Re \lambda|^{1/d}; \quad |Re \lambda| \geq N_0\}$$

$C, N_0 \geq 0, d \geq 1$ and that

$$||R(\lambda, A^*)|| \leq pol (|\lambda|) \exp (N |Im \lambda|);$$

for some $N \geq 0$. Let $u \in \mathcal{D}'(-a, a; H)$ be a solution of $(E)$ with $f \in G^d(-a, a; H)$. Then $u$ is $d$-hypoanalytic in the interval $|t| < a-N$.

**Proof.** We use the notations of the proof of theorem 1. First we assume that $d>1$. Then we may choose $\varphi \in \mathcal{D}(R) \cap G^d(R)$ so that $\varphi(t)=1$ for $|t| \leq a' + \delta$ and $\varphi(t)=0$ for $|t| \geq a' + \delta'$; $N < \delta < \delta' < N_1$. Hence $E^*f \varphi \in G^d(R, H)$ and

$$||D^k(E^*f \varphi)(\psi)|| \leq M^{k+1}(k!)^d ||\psi||_{L^2}$$

for every $\psi \in \mathcal{D}(-a, a; H)$.

Let $\rho(t)$ be a scalar $G^d(R)$-function such that $\rho(t)=1$ for $|t| < \varepsilon$ and $\rho(t)=0$ for $|t| > \varepsilon'$, where $0 < \varepsilon < \varepsilon'$. To estimate $|D^k((1-\rho)E^*g)(\psi)|$ we write it in the form

$$D^k((1-\rho)E^*g)(\psi) = (-1)^k(g(f^{(i)}) + g(f^{(2)}))$$

where

$$f^{(i)}(t) = (2\pi)^{-1} \int_{|\sigma| > N_0} R(-\sigma, A^*) \hat{\psi}_{i,1}^{(i)}(\sigma) d\sigma, \quad i=1,2.$$

Using the fact that $\rho \in G^d(R)$ we obtain the estimates

$$||\psi_{i,1}^{(i)}(\sigma - iC |\sigma|^{1/d})|| \leq M \exp \left(-C \varepsilon |\sigma|^{1/d}\right)$$

and similarly

$$||\psi_{i,2}^{(i)}(\sigma + iC |\sigma|^{1/d})|| \leq M \exp \left(-C \varepsilon |\sigma|^{1/d}\right).$$

By a contour deformation we may write

$$f^{(i)}(t) = (2\pi)^{-1} \int_{\Gamma^i} R(-\lambda, A^*) \hat{\psi}_{i,1}^{(i)}(\lambda) d\lambda$$

where $\Gamma^i = \{\lambda; \lambda = \sigma + iC |\sigma|^{1/d}\} \cup \{ |Re \lambda| = N_0; 0 \leq Im \lambda \leq CN_1^{1/d}\}$ and $\Gamma^d = \{\lambda; \lambda = \sigma - iC |\sigma|^{1/d}, |\sigma| \geq N_0\} \cup \{ |Re \lambda| = N_0; -CN_1^{1/d} \leq Im \lambda \leq 0\}$. 

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Using the estimates (3.6) and (3.7) we get

\[(3.9) \quad \|f_i^k(t)\| \leq M \|\psi\|_{L^2} \sum_{j=0}^i M^j(j!)^d \int |\sigma|^{k+j} \exp (N-\varepsilon) C |\sigma|^{1/d} d\sigma\]

for every \(\psi \in \mathcal{D}(-a', a'; D_{A^*})\). Choosing \(\varepsilon > N\), from Stirling's formula it follows

\[(3.10) \quad \|f_i^k(t)\| \leq M_1^{i+1}(k!)^d \|\psi\|_{L^2} \quad ; \quad \psi \in \mathcal{D}(-a', a'; H) \quad , \quad i=1,2 .\]

This implies that

\[(3.11) \quad |D^k((1-\rho)E^{g^k})(\psi)| \leq M_1^{k+1}(k!)^d \|\psi\|_{L^2} .\]

Hence

\[(3.12) \quad |D^k(u^{\varphi})(\psi)| \leq M_1^{k+1}(k!)^d \|\psi\|_{L^2} , \quad \text{for} \quad \psi \in \mathcal{D}(-a', a'; H)\]

where \(M_1\) is a non-negative constant independent of \(k\). Hence \(u \in G^d(-a', a'; H)\).

To prove theorem 2 in the analytic case \(d=1\), we consider instead of \(\varphi(t)\) and \(\rho(t)\) two sequences of \(C^\infty\) scalar functions \(\{\varphi_j(t)\}_{j=0}^m\) and \(\{\rho_j(t)\}_{j=0}^m\) such that (see Friberg [2])

\[(3.13) \quad |D^k\varphi_j(t)| \leq M_1^{k+1}j^k \quad ; \quad \text{for} \quad k \leq j ,\]

where \(\text{supp } \varphi_j \subset \{t; |t| \leq a' + \delta'\}\), \(\varphi_j(t)=1\) for \(|t| \leq a' + \delta\) and similarly

\[(3.14) \quad |D^k\rho_j(t)| \leq M_1^{k+1}j^k \quad \text{for} \quad k \leq j\]

\(\text{supp } \rho_j \subset \{t; |t| \leq \varepsilon'\}\) and \(\rho_j(t)=1\) for \(|t| \leq \varepsilon\).

Then denoting \(g_j = D^j\varphi_j\mu\), as above we obtain

\[(3.15) \quad |D^k((1-\rho_k)E^{g_k})(\psi)| \leq M_1^{k+1}k! \|\psi\|_{L^2}\]

for every \(\psi \in \mathcal{D}(-a', a'; H)\) and \(k=0,1,\ldots\)

Hence

\[\|D^k(u^{\varphi_k})\| \leq M_1^{k+1}k! , \quad k=0,1,\ldots\]

That is \(u \in G^1(-a', a'; H)\).

As consequence of theorem 2 we get the following result (see Agmon-Nirenberg [1])

**Corollary 1.** Suppose that \(R(\lambda, A^*)\) exists in the sector \(\Sigma: \{ |\arg (\pm \lambda) | \leq \alpha; \quad |\lambda| \geq N_0 \}, \quad 0<\alpha<\pi/2\), and

\[\|R(\lambda, A^*)\| \leq \text{pol} (|\lambda|) \exp (N |\text{Im } \lambda|) , \quad \text{for } \lambda \in \Sigma\]
where $N$ is a non-negative constant. Suppose that $f$ is analytic in $|t| < a$. Then every solution $u \in \mathcal{D}'(-a, a; H)$ of $(E)$ is analytic in the subinterval $|t| < a - N$.

By a slight modification of the preceding proof one easily verifies the following

REMARK. The conclusions of theorem 2 hold if we merely assume that $f \in \mathcal{D}'(-a, a; D_{A^*})$ and

$$(3.16) \quad |D^k f(\psi)| \leq M^{k+1}(k!)^{d}||\psi + A^*\psi||_{L^2}, \quad \psi \in \mathcal{D}(-a, a; D_{A^*}).$$

JASSY UNIVERSITY

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