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ON THE REGULARITY OF THE WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

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1. Preliminaries

Let H be a Hilbert space; ( ), $\|\ \|$ and are the notations for the scalar product and for the norm. Denote by R the real axis, $-\infty<t<\infty$, and $\mathcal{D}(R)$, $\mathcal{D}(R, H)$, $\mathcal{D}'(R, H)$ the spaces of infinitely differentiable scalar functions with compact support, infinitely differentiable H-valued functions with compact support and H-valued distribution, respectively, on R with their usual topologies (see L. Schwartz [7]). The space of H-valued distributions with compact support will be denoted by $\mathcal{E}'(R, H)$. If $u\in\mathcal{D}'(R, H)$ we may define $D^k u\in\mathcal{D}'(R, H)$ by the formula: $D^k u(\varphi)=(-1)^k u(D^k\varphi)$, $\forall\varphi\in\mathcal{D}(R, H)$. If $\varphi\in\mathcal{D}(R, H)$ then for each complex λ, $\hat{\varphi}(\lambda)$ denotes its Fourier-Laplace transform. (Here $D^1\varphi= =1/i\frac{d\varphi}{dt}$).

Let $A: D_A\subset H\to H$ be a closed linear operator with the domain D_A dense in H and let A^* be its adjoint. The domain D_{A^*} of the operator A^* is Banach space in the norm $|x|=\|x\|+\|A^*x\|$. We denote by $\mathcal{D}(R, D_{A^*})$ the space of infinitely differentiable D_{A^*}-valued functions with compact support on R and by $\mathcal{D}'(R, D_{A^*})$ its dual. Since $\mathcal{D}(R, D_{A^*})=\mathcal{D}(R)\hat{\otimes}D_{A^*}$ it is easy to see that the space $\mathcal{D}(R, D_{A^*})$ is dense in $\mathcal{D}(R, H)$. In an analogous manner, we may define the spaces $\mathcal{D}(a, b; H)$, $\mathcal{D}'(a, b; H)$ and $\mathcal{D}(a, b; D_{A^*})$. Let $L^*: \mathcal{D}(R, D_{A^*})\to \mathcal{D}(R, H)$ be the linear operator

$$L^*\varphi = -\left(\frac{1}{i}\frac{d\varphi}{dt}+A^*\varphi\right) \tag{1.1}$$

and let L: $\mathcal{D}'(R, H)\to\mathcal{D}'(R, D_{A^*})$ be its adjoint defined by

$$Lu(\varphi) = u(L^*\varphi)\,, \qquad \varphi\in\mathcal{D}(R, D_{A^*})\,. \tag{1.2}$$

Let $\mathcal{L}(\mathcal{D}, H)$ be the space of all vector H-valued distributions $\mathcal{L}(\mathcal{D}(R, H), H)$. For $E\in\mathcal{L}(\mathcal{D}, H)$ we define LE by the formula

$$LE(\varphi) = E(L^*\varphi)\,, \qquad \varphi\in\mathcal{D}(R, D_{A^*})\,, \tag{1.3}$$

and denote, for every $\varphi \in \mathcal{D}(R, H)$

$$(E*\varphi)(t) = E_s(\varphi(t-s)) .$$

It is easy to see that $E*\varphi \in C^\infty(R, H)$. If $u \in \mathcal{E}'(R, H)$ and $E \in \mathcal{L}(\mathcal{D}, H)$, $E*u$ denotes the distribution defined by

$$(E*u)(\varphi) = u_t(E_s(\varphi(t+s))) , \qquad \varphi \in \mathcal{D}(R, H) .$$

If $\varphi(t) \in \mathcal{D}(R)$, then as in the scalar case it follows immediately

$$\operatorname{supp} (\rho E*u) \subset \operatorname{supp} \rho + \operatorname{supp} u .$$

DEFINITION. We say that the distribution $u \in \mathcal{D}'(a, b; H)$ is a weak solution on (a, b) of the equation

$$\text{(E)} \qquad \frac{1}{i}\frac{du}{dt} - Au = f ,$$

where $f \in \mathcal{D}'(a, b; D_{A^*})$, if the following relation

$$\text{(1.4)} \qquad u(L^*\varphi) = f(\varphi)$$

holds for any $\varphi \in \mathcal{D}(a, b; D_{A^*})$.

The existence theorems for the weak solutions of the equation (E) have been obtained by T. Kato and H. Tanabe [3], S. Zaidman [8] and M. A. Malik [6]. We give in this paper some results concerning the regularity of the weak solutions of (E). For the strict solutions of (E) a similar result has been proved by S. Agmon and L. Nirenberg [1].

2. Differentiability of solutions

In the following we denote by $R(\lambda, A^*)$ the resolvent $(\lambda I - A^*)^{-1}$ of the operator A^*.

Theorem 1. *Suppose that for every $m>0$ there exists a number $C_m>0$ such that the resolvent $R(\lambda, A^*)$ exists in the domain*

$$\text{(2.1)} \qquad \Lambda_m = \{\lambda ; |Im\, \lambda| \leqslant m \log|Re\, \lambda| ;\ |Re\, \lambda| \geqslant C_m\}$$

and

$$\text{(2.2)} \qquad ||R(\lambda, A^*)|| \leqslant C_m^1 |\lambda|^M \exp(N|Im\, \lambda|), \quad \text{in } \Lambda m ,$$

where $M>0$, $N>0$ are constants independent of m and $C_m^1>0$. Then every weak solution $u \in \mathcal{D}'(-a, a; H)$ of (E) with $f \in C^\infty(-a, a; H)$ is infinitely differentiable on the interval $|t|<a-N$.

Proof. Let $E\in\mathcal{L}(\mathcal{D}, D_{A^*})$ be defined by the equality

$$E(\varphi)=-(2\pi)^{-1}\int_{|\sigma|\geqslant C_m} R(-\sigma, A^*)\hat{\varphi}(\sigma)\,d\sigma\,;\qquad \varphi\in\mathcal{D}(R, H) \tag{2.3}$$

Obviously

$$E(L^*\varphi)=\varphi(0)-\int_{|\sigma|\leqslant C_m}\hat{\varphi}(\sigma)\,d\sigma\,;\qquad \varphi\in\mathcal{D}(R, D_{A^*})\,. \tag{2.4}$$

We denote by Δ the interval $(-a', a')$ where $a'<a-N_1$, $N<N_1<a$, and consider $\varphi(t)\in\mathcal{D}(R)$ such that $\varphi(t)=1$ for $|t|\leqslant a'+\delta$ and $\varphi(t)=0$ in $|t|\geqslant a'+\delta'$. Assume that $N<\delta<\delta'<N_1$. If $u\in\mathcal{D}'(-a, a; H)$ is a weak solution of (E) then we have

$$L(u\varphi)(\psi)=(f\varphi)(\psi)+(D^1\varphi u)(\psi) \tag{2.5}$$

for every $\psi\in\mathcal{D}(-a, a; H)$. On the other hand since A^* is closed, from (2.2) it follows

$$(L(u\varphi)*E)(\psi)=(LE*u\varphi)(\psi)\,. \tag{2.6}$$

Let us denote by g the distribution $D^1\varphi\cdot u$. Then from (2.4) and (2.5) we get

$$u\varphi(\psi)=(E*f\varphi)(\psi)+(E*g)(\psi)+u\varphi(h_\psi) \tag{2.7}$$

for every $\psi\in\mathcal{D}(-a, a; H)$, where $h_\psi(t)=\int_{|\sigma|\leqslant C_m} e^{it\sigma}\hat{\psi}(\sigma)\,d\sigma$.

Obviously

$$\|D^k h_\psi(t)\|\leqslant M_k\|\psi\|_{L^2}\,,\qquad t\in R \tag{2.8}$$

for any $\psi\in\mathcal{D}(-a, a; H)$, where $\|\ \|_{L^2}$ denotes the norm in the space $L^2(R, H)$.

Since $f\varphi\in C^\infty(R, H)$ it follows that $E*f\varphi\in C^\infty(R, H)$ which implies that

$$|D^k(E*f\varphi)(\psi)|\leqslant M_k^1\|\psi\|_{L^2};\qquad \psi\in\mathcal{D}(-a, a; H)\,. \tag{2.9}$$

Let $\rho(t)$ be a scalar C^∞ function on the real line such that $\rho(t)=1$ for $|t|\leqslant\varepsilon$ and $\rho(t)=0$ for $|t|\geqslant\varepsilon'$; $0<\varepsilon<\varepsilon'$.

Since supp $g\subset\{t;\ a'+\delta<|t|\leqslant a'+\delta'\}$, taking ε so small such that $\varepsilon'<\delta$, from an above remark we deduce that $(\rho E*g)(\psi)=0$ for any $\psi\in\mathcal{D}(-a', a'; H)$.

Hence

$$(E*g)(\psi)=((1-\rho)E*g)(\psi)\,,\qquad \psi\in\mathcal{D}(-a', a'; H)\,. \tag{2.10}$$

Now we introduce the function $\psi_t^{(k)}(s)=(1-\rho(s))D^k\psi(t+s)$ and denote by $\hat{\psi}_t^{(k)}(\lambda)$ its Laplace transform. Let m be an arbitrary non-negative integer. We may write $\hat{\psi}_t^{(k)}(\lambda)$ in the form

$$\hat{\psi}_t^{(k)}(\lambda) = \hat{\psi}_{t,1}^{(k)}(\lambda) + \hat{\psi}_{t,2}^{(k)}(\lambda)$$

where

$$\hat{\psi}_{t,1}^{(k)}(\lambda) = \int_{s>\varepsilon} e^{-i\lambda s}(1-\rho(s))D^k\psi(t+s)\,ds$$

and

$$\hat{\psi}_{t,2}^{(k)}(\lambda) = \int_{s<-\varepsilon} e^{-i\lambda s}(1-\rho(s))D^k\psi(t+s)\,ds\,.$$

A simple computation shows that with another constant M_k, one must have the estimates,

$$\|\hat{\psi}_{t,1}^{(k)}(\sigma - i\,m\,\log|\sigma|)\| \leqslant M_k|\sigma|^{k-m\varepsilon}\|\psi\|_{L^2}\,, \qquad \sigma \in R \tag{2.11}$$

and

$$\|\hat{\psi}_{t,2}^{(k)}(\sigma + i\,m\,\log\,|\sigma|)\| \leqslant M_k|\sigma|^{k-m\varepsilon}\|\psi\|_{L^2}\,, \qquad \sigma \in R \tag{2.12}$$

Let $f_i^{(k)}(t)$ be the functions

$$f_i^{(k)}(t) = (2\pi)^{-1}\int_{|\sigma|\geqslant C_m} R(-\sigma, A^*)\hat{\psi}_{t,i}^{(k)}(\sigma)d\sigma\,; \qquad i=1,2;\ t\in R\,. \tag{2.13}$$

After a suitable deformation of contours in the complex plane, the functions $f_i^{(k)}(t)$ can be expressed in the following form

$$f_i^{(k)}(t) = (2\pi)^{-1}\int_{\Gamma_m^i} R(-\lambda, A^*)\hat{\psi}_{t,i}^{(k)}(\lambda)d\lambda\,; \qquad i=1,2\,; \tag{2.14}$$

where Γ_m^1 is the frontier of the domain $\{\lambda;\ Im\,\lambda \geqslant -m\log|Re\,\lambda|\,;|Re\,\lambda|\geqslant C_m\}$ and Γ_m^2 the frontier of $\{\lambda;\ Im\,\lambda \leqslant m\log|Re\,\lambda|\,;\ |Re\,\lambda|\geqslant C_m\}$. It is easy to see that the shift of the integration contour is legitimate. Now we have on Γ_m^i,

$$\|R(-\lambda,\ A^*)\hat{\psi}_{t,i}^{(k)}(\lambda)\| \leqslant M_k|\sigma|^{M+k-m(\varepsilon-N)}\|\psi\|_{L^2}; \qquad \sigma = Re\,\lambda\,.$$

Choosing ε so that $\varepsilon > N$ and m so large such that $M+k-m(\varepsilon-N)<-1$, one obtains

$$\|f_i^{(k)}(t)\| \leqslant M_k^1\|\psi\|_{L^2}; \qquad t\in R;\ \psi\in\mathcal{D}(-a',\ a';\ H)\,. \tag{2.15}$$

We remark that

$$D^k((1-\rho)E*g)(\psi) = (-1)^k(g(f_1^{(k)})+g(f_2^{(k)}))\,.$$

From (2.15) this implies that

$$\|D^k((1-\rho)E*g)(\psi)\| \leqslant M_k\|\psi\|_{L^2}\,; \qquad \psi\in\mathcal{D}(-a',\ a';\ D_{A^*})\,. \tag{2.16}$$

Using (2.7), (2.8), (2.9) and (2.16) we obtain

$$|D^k(u\varphi)(\psi)| \leqslant M_k\|\psi\|_{L^2}\,; \qquad \psi\in\mathcal{D}(-a',\ a';\ D_{A^*})\,. \tag{2.17}$$

Since the space $\mathscr{D}(-a', a'; D_{A^*})$ is dense in $\mathscr{D}(-a', a'; H)$ from the Hahn-Banach theorem it follows that $D^k(u\varphi)\in L^2(-a', a'; H)$ for any $k=0,1\cdots$. Hence $u\varphi\in C^\infty(-a', a'; H)$. Because the number $N_1>N$ is arbitrary, the proof is complete.

Corollary 1. *Suppose that there exist some non-negative numbers N, C, N_0 such that $R(\lambda, A^*)$ exists in the domain*

$$\Lambda = \{\lambda; |Im\,\lambda|\leqslant C\log|Re\,\lambda|\ ; \quad |Re\,\lambda|\geqslant N_0\} \tag{2.18}$$

and

$$||R(\lambda, A^*)||\leqslant \mathrm{pol}\,(|\lambda|)\exp(N|Im\lambda|)\,. \tag{2.19}$$

Then every solution $u\in\mathscr{D}(R, H)$ of (E), with $f\in C^\infty(R, H)$, is infinitely differentiable on R.

Corollary 2. *Suppose that $f\in\mathscr{D}'(-a, a; D_{A^*})$ such that*

$$|D^kf(\psi)|\leqslant M_k||\psi+A^*\psi||_{L^2}\ ; \qquad \forall\psi\in\mathscr{D}(-a, a; D_{A^*})\,. \tag{2.20}$$

If the hypotheses of theorem 1 are satisfied, then every solution $u\in\mathscr{D}'(-a, a; H)$ of (E) is infinitely differentiable on the subinterval $|t|<a-N$.

Proof. The proof in this case is very much the same, except the inequality (2.9). To estimate $|D^k(E*f\varphi)(\psi)|$ we remark that

$$||A^*(E*\psi)(t)||\leqslant M||D^l\psi||_{L^2}\ ; \qquad \psi\in\mathscr{D}(-a, a; H)$$

where l is a non-negative integer. From (2.6), (2.7) and (2.16) this implies that

$$|D^k(u\varphi)(\psi)|\leqslant M_k||D^l\psi||_{L^2}\ ; \qquad \psi\in\mathscr{D}(-a', a'; D_{A^*})\,. \tag{2.21}$$

As in the proof of theorem 1 this implies that $u\in C^\infty(-a', a'; H)$.

REMARK. If $u\in C^\infty(\Delta, H)$ is a weak solution of (E) with $f\in C^\infty(\Delta, H)$, then $u(t)$ is a strict solution of (E). To prove this it is enough to choose in the equality (1.2), $\varphi=\varphi_0\otimes x$ where $\varphi_0\in\mathscr{D}(\Delta)$ and $x\in H$. Hence the necessity results for differentiability, proved by Agmon-Nirenberg [1], are true in our case.

3. Hypoanaliticity of solutions

DEFINITIONS. A C^∞ H-valued function $u(t)$ is said to be d-hypoanalytic on $\Delta\subset R$ if for any compact subset $K\subset\Delta$ there exists a non-negative constant M_K such that for any k the following inequality be true

$$||D^ku; K||_\infty\leqslant M_K^{k+1}(k!)^d \tag{3.1}$$

where $||u, K||_\infty=\sup\limits_{t\in K}||u(t)||$.

In the following we denote by $G^d(\Delta, H)$ the space of all d-H-valued hypoanalytic functions on Δ. If $H=R$ we omit R and write $G^d(\Delta)$.

Theorem 2. *Suppose that $R(\lambda, A^*)$ exists in a region*

$$\sum: \{\lambda; |Im\,\lambda| \leqslant C|Re\,\lambda|^{1/d}; \qquad |Re\,\lambda| \geqslant N_0\}$$

$C, N_0 \geqslant 0$, $d \geqslant 1$ and that

$$\|R(\lambda, A^*)\| \leqslant \text{pol}\,(|\lambda|)\exp(N|Im\,\lambda|); \tag{3.2}$$

for some $N \geqslant 0$. Let $u \in \mathcal{D}'(-a, a; H)$ be a solution of (E) with $f \in G^d(-a, a; H)$. Then u is d-hypoanalytic in the interval $|t| < a-N$.

Proof. We use the notations of the proof of theorem 1. First we assume that $d>1$. Then we may choose $\varphi \in \mathcal{D}(R) \cap G^d(R)$ so that $\varphi(t)=1$ for $|t| \leqslant a' + \delta$ and $\varphi(t)=0$ for $|t| \geqslant a'+\delta'$; $N<\delta<\delta'<N_1$. Hence $E*f\varphi \in G^d(R, H)$ and

$$|D^k(E*f\varphi)(\psi)| \leqslant M^{k+1}(k!)^d\|\psi\|_{L^2} \tag{3.4}$$

for every $\psi \in \mathcal{D}(-a, a; H)$.

Let $\rho(t)$ be a scalar $G^d(R)$-function such that $\rho(t)=1$ for $|t| \leqslant \varepsilon$ and $\rho(t)=0$ for $|t| > \varepsilon'$, where $0<\varepsilon<\varepsilon'$. To estimate $|D^k((1-\rho)E*g)(\psi)|$ we write it in the form

$$D^k((1-\rho)E*g)(\psi) = (-1)^k(g(f_1^{(k)})+g(f_2^{(k)})) \tag{3.5}$$

where

$$f_i^{(k)}(t) = (2\pi)^{-1}\int_{|\sigma|\geqslant N_0} R(-\sigma, A^*)\hat{\psi}_{t,1}^{(k)}(\sigma)\,d\sigma\,, \qquad i=1,2\,.$$

Using the fact that $\rho \in G^d(R)$ we obtain the estimates

$$\|\psi_{t,1}^{(k)}(\sigma - iC|\sigma|^{1/d})\| \leqslant M\exp(-C\varepsilon|\sigma|^{1/d}) \|\psi\|_{L^2}\sum_{j=0}^{k} M^j(j!)^d|\sigma|^{k-j} \tag{3.6}$$

and similarly

$$\|\psi_{t,i}^{(k)}(\sigma + iC|\sigma|^{1/d})\| \leqslant \leqslant M\exp(-C\varepsilon|\sigma|^{1/d})\|\psi\|_{L^2}\sum_{j=0}^{k} M^j(j!)^d|\sigma|^{k-j}\,. \tag{3.7}$$

By a contour deformation we may write

$$f_i^{(k)}(t) = (2\pi)^{-1}\int_{\Gamma^i} R(-\lambda, A^*)\hat{\psi}_{t,i}^{(k)}(\lambda)\,d\lambda \tag{3.8}$$

where $\Gamma^1 = \{\lambda; \lambda=\sigma+iC|\sigma|^{1/d}\} \cup \{|Re\,\lambda|=N_0; 0 \leqslant \text{Im}\,\lambda \leqslant CN_0^{1/d}\}$ and $\Gamma^2 = \{\lambda; \lambda=\sigma-iC|\sigma|^{1/d}, |\sigma| \geqslant N_0\} \cup \{|Re\,\lambda|=N_0; -CN_0^{1/d} \leqslant \text{Im}\,\lambda \leqslant 0\}$.

Using the estimates (3.6) and (3.7) we get

$$(3.9) \qquad \|f_i^{(k)}(t)\| \leqslant M \|\psi\|_{L^2} \sum_{j=0}^{k} M^j (j!)^d \int |\sigma|^{p+k-j} \exp (N-\varepsilon) C |\sigma|^{1/d} d\sigma$$

for every $\psi \in \mathcal{D}(-a', a'; D_{A^*})$. Choosing $\varepsilon > N$, from Stirling's formula it follows

$$(3.10) \qquad \|f_i^{(k)}(t)\| \leqslant M_1^{k+1} (k!)^d \|\psi\|_{L^2} \; ; \; \psi \in \mathcal{D}(-a', a'; H) , \qquad i=1,2 .$$

This implies that

$$(3.11) \qquad |D^k((1-\rho)E*g)(\psi)| \leqslant M_1^{k+1}(k!)^d \|\psi\|_{L^2} .$$

Hence

$$(3.12) \qquad |D^k(u\varphi)(\psi)| \leqslant M_1^{k+1}(k!)^d \|\psi\|_{L^2} , \qquad \text{for} \quad \psi \in \mathcal{D}(-a', a'; H)$$

where M_1 is a non-negative constant independent of k. Hence $u \in G^d(-a', a'; H)$.

To prove theorem 2 in the analytic case $d=1$, we consider instead of $\varphi(t)$ and $\rho(t)$ two sequences of C^∞ scalar functions $\{\varphi_j(t)\}_{j=0}^\infty$ and $\{\rho_j(t)\}_{j=0}^\infty$ such that (see Friberg [2])

$$(3.13) \qquad |D^k \varphi_j(t)| \leqslant M^{k+1} j^k \; ; \qquad \text{for} \quad k \leqslant j ,$$

where supp $\varphi_j \subset \{t; |t| \leqslant a'+\delta'\}$, $\varphi_j(t)=1$ for $|t| \leqslant a'+\delta$ and similarly

$$(3.14) \qquad |D^k \rho_j(t)| \leqslant M^{k+1} j^k \qquad \text{for} \quad k \leqslant j$$

supp $\rho_j \subset \{t; \; |t| \leqslant \varepsilon'\}$ and $\rho_j(t)=1$ for $|t| \leqslant \varepsilon$.

Then denoting $g_j = D^1 \varphi_j u$, as above we obtain

$$(3.15) \qquad |D^k(1-\rho_k)E*g_k)(\psi)| \leqslant M_1^{k+1} k^k \|\psi\|_{L^2}$$

for every $\psi \in \mathcal{D}(-a', a'; H)$ and $k=0,1,\cdots$

Hence

$$\|D^k(u\varphi_k)\|_\infty \leqslant M_1^{k+1} k! , \qquad k=0,1,\cdots$$

That is $u \in G^1(-a', a'; H)$.

As consequence of theorem 2 we get the following result (see Agmon-Nirenberg [1])

Corollary 1. *Suppose that $R(\lambda, A^*)$ exists in the sector $\sum : \{ |\arg(\pm\lambda)| \leqslant \alpha ; |\lambda| \geqslant N_0 \}$, $0 < \alpha < \pi/2$, and*

$$\|R(\lambda, A^*)\| \leqslant \text{pol}(|\lambda|) \exp (N |Im\,\lambda|) , \qquad \textit{for} \quad \lambda \in \sum$$

where N is a non-negative constant. Suppose that f is analytic in $|t|<a$. Then every solution $u\in\mathcal{D}'(-a, a; H)$ of (E) is analytic in the subinterval $|t|<a-N$.

By a slight modification of the preceding proof one easily verifies the following

REMARK. The conclusions of theorem 2 hold if we merely assume that $f\in\mathcal{D}'(-a, a; D_{A^*})$ and

$$(3.16) \qquad |D^k f(\psi)| \leqslant M^{k+1}(k!)^d \|\psi+A^*\psi\|_{L^2}, \qquad \psi\in\mathcal{D}(-a, a; D_{A^*}).$$

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