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Author(s)	Barbu, Viorel
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ON THE REGULARITY OF THE WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

VIOREL BARBU

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1. Preliminaries

Let H be a Hilbert space; (\cdot, \cdot) , $\|\cdot\|$ and $\|\cdot\|$ are the notations for the scalar product and for the norm. Denote by R the real axis, $-\infty < t < \infty$, and $\mathcal{D}(R)$, $\mathcal{D}(R, H)$, $\mathcal{D}'(R, H)$ the spaces of infinitely differentiable scalar functions with compact support, infinitely differentiable H -valued functions with compact support and H -valued distribution, respectively, on R with their usual topologies (see L. Schwartz [7]). The space of H -valued distributions with compact support will be denoted by $\mathcal{E}'(R, H)$. If $u \in \mathcal{D}'(R, H)$ we may define $D^k u \in \mathcal{D}'(R, H)$ by the formula: $D^k u(\varphi) = (-1)^k u(D^k \varphi)$, $\forall \varphi \in \mathcal{D}(R, H)$. If $\varphi \in \mathcal{D}(R, H)$ then for each complex λ , $\hat{\varphi}(\lambda)$ denotes its Fourier-Laplace transform. (Here $D^1 \varphi = 1/i \frac{d\varphi}{dt}$).

Let $A: D_A \subset H \rightarrow H$ be a closed linear operator with the domain D_A dense in H and let A^* be its adjoint. The domain D_{A^*} of the operator A^* is Banach space in the norm $\|x\| = \|x\| + \|A^*x\|$. We denote by $\mathcal{D}(R, D_{A^*})$ the space of infinitely differentiable D_{A^*} -valued functions with compact support on R and by $\mathcal{D}'(R, D_{A^*})$ its dual. Since $\mathcal{D}(R, D_{A^*}) = \mathcal{D}(R) \hat{\otimes} D_{A^*}$ it is easy to see that the space $\mathcal{D}(R, D_{A^*})$ is dense in $\mathcal{D}(R, H)$. In an analogous manner, we may define the spaces $\mathcal{D}(a, b; H)$, $\mathcal{D}'(a, b; H)$ and $\mathcal{D}(a, b; D_{A^*})$. Let $L^*: \mathcal{D}(R, D_{A^*}) \rightarrow \mathcal{D}(R, H)$ be the linear operator

$$(1.1) \quad L^* \varphi = - \left(\frac{1}{i} \frac{d\varphi}{dt} + A^* \varphi \right)$$

and let $L: \mathcal{D}'(R, H) \rightarrow \mathcal{D}'(R, D_{A^*})$ be its adjoint defined by

$$(1.2) \quad Lu(\varphi) = u(L^* \varphi), \quad \varphi \in \mathcal{D}(R, D_{A^*}).$$

Let $\mathcal{L}(\mathcal{D}, H)$ be the space of all vector H -valued distributions $\mathcal{L}(\mathcal{D}(R, H), H)$. For $E \in \mathcal{L}(\mathcal{D}, H)$ we define LE by the formula

$$(1.3) \quad LE(\varphi) = E(L^* \varphi), \quad \varphi \in \mathcal{D}(R, D_{A^*}),$$

and denote, for every $\varphi \in \mathcal{D}(R, H)$

$$(E*\varphi)(t) = E_s(\varphi(t-s)).$$

It is easy to see that $E*\varphi \in C^\infty(R, H)$. If $u \in \mathcal{E}'(R, H)$ and $E \in \mathcal{L}(\mathcal{D}, H)$, $E*u$ denotes the distribution defined by

$$(E*u)(\varphi) = u_t(E_s(\varphi(t+s))), \quad \varphi \in \mathcal{D}(R, H).$$

If $\varphi(t) \in \mathcal{D}(R)$, then as in the scalar case it follows immediately

$$\text{supp } (\rho E*u) \subset \text{supp } \rho + \text{supp } u.$$

DEFINITION. We say that the distribution $u \in \mathcal{D}'(a, b; H)$ is a weak solution on (a, b) of the equation

$$(E) \quad \frac{1}{i} \frac{du}{dt} - Au = f,$$

where $f \in \mathcal{D}'(a, b; D_{A^*})$, if the following relation

$$(1.4) \quad u(L^*\varphi) = f(\varphi)$$

holds for any $\varphi \in \mathcal{D}(a, b; D_{A^*})$.

The existence theorems for the weak solutions of the equation (E) have been obtained by T. Kato and H. Tanabe [3], S. Zaidman [8] and M.A. Malik [6]. We give in this paper some results concerning the regularity of the weak solutions of (E). For the strict solutions of (E) a similar result has been proved by S. Agmon and L. Nirenberg [1].

2. Differentiability of solutions

In the following we denote by $R(\lambda, A^*)$ the resolvent $(\lambda I - A^*)^{-1}$ of the operator A^* .

Theorem 1. *Suppose that for every $m > 0$ there exists a number $C_m > 0$ such that the resolvent $R(\lambda, A^*)$ exists in the domain*

$$(2.1) \quad \Lambda_m = \{\lambda; |Im \lambda| \leq m \log |Re \lambda|; |Re \lambda| \geq C_m\}$$

and

$$(2.2) \quad \|R(\lambda, A^*)\| \leq C_m^1 |\lambda|^M \exp(N |Im \lambda|), \quad \text{in } \Lambda_m,$$

where $M > 0$, $N > 0$ are constants independent of m and $C_m^1 > 0$. Then every weak solution $u \in \mathcal{D}'(-a, a; H)$ of (E) with $f \in C^\infty(-a, a; H)$ is infinitely differentiable on the interval $|t| < a - N$.

Proof. Let $E \in \mathcal{L}(\mathcal{D}, D_A^*)$ be defined by the equality

$$(2.3) \quad E(\varphi) = -(2\pi)^{-1} \int_{|\sigma| \geq C_m} R(-\sigma, A^*) \hat{\phi}(\sigma) d\sigma; \quad \varphi \in \mathcal{D}(R, H)$$

Obviously

$$(2.4) \quad E(L^* \varphi) = \varphi(0) - \int_{|\sigma| \leq C_m} \hat{\phi}(\sigma) d\sigma; \quad \varphi \in \mathcal{D}(R, D_A^*).$$

We denote by Δ the interval $(-a', a')$ where $a' < a - N_1$, $N < N_1 < a$, and consider $\varphi(t) \in \mathcal{D}(R)$ such that $\varphi(t) = 1$ for $|t| \leq a' + \delta$ and $\varphi(t) = 0$ in $|t| \geq a' + \delta'$. Assume that $N < \delta < \delta' < N_1$. If $u \in \mathcal{D}'(-a, a; H)$ is a weak solution of (E) then we have

$$(2.5) \quad L(u\varphi)(\psi) = (f\varphi)(\psi) + (D^1 \varphi u)(\psi)$$

for every $\psi \in \mathcal{D}(-a, a; H)$. On the other hand since A^* is closed, from (2.2) it follows

$$(2.6) \quad (L(u\varphi) * E)(\psi) = (LE * u\varphi)(\psi).$$

Let us denote by g the distribution $D^1 \varphi \cdot u$. Then from (2.4) and (2.5) we get

$$(2.7) \quad u\varphi(\psi) = (E * f\varphi)(\psi) + (E * g)(\psi) + u\varphi(h_\psi)$$

for every $\psi \in \mathcal{D}(-a, a; H)$, where $h_\psi(t) = \int_{|\sigma| \leq C_m} e^{it\sigma} \hat{\psi}(\sigma) d\sigma$.

Obviously

$$(2.8) \quad \|D^k h_\psi(t)\| \leq M_k \|\psi\|_{L^2}, \quad t \in R$$

for any $\psi \in \mathcal{D}(-a, a; H)$, where $\|\cdot\|_{L^2}$ denotes the norm in the space $L^2(R, H)$.

Since $f\varphi \in C^\infty(R, H)$ it follows that $E * f\varphi \in C^\infty(R, H)$ which implies that

$$(2.9) \quad |D^k(E * f\varphi)(\psi)| \leq M_k^1 \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a, a; H).$$

Let $\rho(t)$ be a scalar C^∞ function on the real line such that $\rho(t) = 1$ for $|t| \leq \varepsilon$ and $\rho(t) = 0$ for $|t| \geq \varepsilon'$; $0 < \varepsilon < \varepsilon'$.

Since $\text{supp } g \subset \{t; a' + \delta < |t| \leq a' + \delta'\}$, taking ε so small such that $\varepsilon' < \delta$, from an above remark we deduce that $(\rho E * g)(\psi) = 0$ for any $\psi \in \mathcal{D}(-a', a'; H)$.

Hence

$$(2.10) \quad (E * g)(\psi) = ((1 - \rho)E * g)(\psi), \quad \psi \in \mathcal{D}(-a', a'; H).$$

Now we introduce the function $\psi_t^{(k)}(s) = (1 - \rho(s))D^k \psi(t + s)$ and denote by $\hat{\psi}_t^{(k)}(\lambda)$ its Laplace transform. Let m be an arbitrary non-negative integer. We may write $\hat{\psi}_t^{(k)}(\lambda)$ in the form

$$\hat{\psi}_t^{(k)}(\lambda) = \hat{\psi}_{t,1}^{(k)}(\lambda) + \hat{\psi}_{t,2}^{(k)}(\lambda)$$

where

$$\hat{\psi}_{t,1}^{(k)}(\lambda) = \int_{s>\varepsilon} e^{-i\lambda s} (1-\rho(s)) D^k \psi(t+s) ds$$

and

$$\hat{\psi}_{t,2}^{(k)}(\lambda) = \int_{s<-\varepsilon} e^{-i\lambda s} (1-\rho(s)) D^k \psi(t+s) ds.$$

A simple computation shows that with another constant M_k , one must have the estimates,

$$(2.11) \quad \|\hat{\psi}_{t,1}^{(k)}(\sigma - i m \log |\sigma|)\| \leq M_k |\sigma|^{k-m\varepsilon} \|\psi\|_{L^2}, \quad \sigma \in R$$

and

$$(2.12) \quad \|\hat{\psi}_{t,2}^{(k)}(\sigma + i m \log |\sigma|)\| \leq M_k |\sigma|^{k-m\varepsilon} \|\psi\|_{L^2}, \quad \sigma \in R$$

Let $f_t^{(k)}(t)$ be the functions

$$(2.13) \quad f_t^{(k)}(t) = (2\pi)^{-1} \int_{|\sigma| \geq C_m} R(-\sigma, A^*) \hat{\psi}_{t,i}^{(k)}(\sigma) d\sigma; \quad i=1, 2; t \in R.$$

After a suitable deformation of contours in the complex plane, the functions $f_t^{(k)}(t)$ can be expressed in the following form

$$(2.14) \quad f_t^{(k)}(t) = (2\pi)^{-1} \int_{\Gamma_m^i} R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda) d\lambda; \quad i=1, 2;$$

where Γ_m^1 is the frontier of the domain $\{\lambda; Im \lambda \geq -m \log |Re \lambda|; |Re \lambda| \geq C_m\}$ and Γ_m^2 the frontier of $\{\lambda; Im \lambda \leq m \log |Re \lambda|; |Re \lambda| \geq C_m\}$. It is easy to see that the shift of the integration contour is legitimate. Now we have on Γ_m^i ,

$$\|R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda)\| \leq M_k |\sigma|^{M+k-m(\varepsilon-N)} \|\psi\|_{L^2}; \quad \sigma = Re \lambda.$$

Choosing ε so that $\varepsilon > N$ and m so large such that $M+k-m(\varepsilon-N) < -1$, one obtains

$$(2.15) \quad \|f_t^{(k)}(t)\| \leq M_k^1 \|\psi\|_{L^2}; \quad t \in R; \psi \in \mathcal{D}(-a', a'; H).$$

We remark that

$$D^k((1-\rho)E*g)(\psi) = (-1)^k(g(f_1^{(k)}) + g(f_2^{(k)})).$$

From (2.15) this implies that

$$(2.16) \quad \|D^k((1-\rho)E*g)(\psi)\| \leq M_k \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_{A^*}).$$

Using (2.7), (2.8), (2.9) and (2.16) we obtain

$$(2.17) \quad |D^k(u\varphi)(\psi)| \leq M_k \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_{A^*}).$$

Since the space $\mathcal{D}(-a', a'; D_A^*)$ is dense in $\mathcal{D}(-a', a'; H)$ from the Hahn-Banach theorem it follows that $D^k(u\varphi) \in L^2(-a', a'; H)$ for any $k=0, 1, \dots$. Hence $u\varphi \in C^\infty(-a', a'; H)$. Because the number $N_1 > N$ is arbitrary, the proof is complete.

Corollary 1. *Suppose that there exist some non-negative numbers N, C, N_0 such that $R(\lambda, A^*)$ exists in the domain*

$$(2.18) \quad \Lambda = \{\lambda; |Im \lambda| \leq C \log |Re \lambda|; \quad |Re \lambda| \geq N_0\}$$

and

$$(2.19) \quad \|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N |Im \lambda|).$$

Then every solution $u \in \mathcal{D}(R, H)$ of (E), with $f \in C^\infty(R, H)$, is infinitely differentiable on R .

Corollary 2. *Suppose that $f \in \mathcal{D}'(-a, a; D_A^*)$ such that*

$$(2.20) \quad |D^k f(\psi)| \leq M_k \|\psi + A^* \psi\|_{L^2}; \quad \forall \psi \in \mathcal{D}(-a, a; D_A^*).$$

If the hypotheses of theorem 1 are satisfied, then every solution $u \in \mathcal{D}'(-a, a; H)$ of (E) is infinitely differentiable on the subinterval $|t| < a - N$.

Proof. The proof in this case is very much the same, except the inequality (2.9). To estimate $|D^k(E * f\varphi)(\psi)|$ we remark that

$$\|A^*(E * \psi)(t)\| \leq M \|D^l \psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a, a; H)$$

where l is a non-negative integer. From (2.6), (2.7) and (2.16) this implies that

$$(2.21) \quad |D^k(u\varphi)(\psi)| \leq M_k \|D^l \psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_A^*).$$

As in the proof of theorem 1 this implies that $u \in C^\infty(-a', a'; H)$.

REMARK. If $u \in C^\infty(\Delta, H)$ is a weak solution of (E) with $f \in C^\infty(\Delta, H)$, then $u(t)$ is a strict solution of (E). To prove this it is enough to choose in the equality (1.2), $\varphi = \varphi_0 \otimes x$ where $\varphi_0 \in \mathcal{D}(\Delta)$ and $x \in H$. Hence the necessity results for differentiability, proved by Agmon-Nirenberg [1], are true in our case.

3. Hypoanalyticity of solutions

DEFINITIONS. A $C^\infty H$ -valued function $u(t)$ is said to be d -hypoanalytic on $\Delta \subset R$ if for any compact subset $K \subset \Delta$ there exists a non-negative constant M_K such that for any k the following inequality be true

$$(3.1) \quad \|D^k u; K\|_\infty \leq M_K^{k+1} (k!)^d$$

where $\|u, K\|_\infty = \sup_{t \in K} \|u(t)\|$.

In the following we denote by $G^d(\Delta, H)$ the space of all d - H -valued hypo-analytic functions on Δ . If $H=R$ we omit R and write $G^d(\Delta)$.

Theorem 2. *Suppose that $R(\lambda, A^*)$ exists in a region*

$$\Sigma: \{\lambda; |Im \lambda| \leq C |Re \lambda|^{1/d}; \quad |Re \lambda| \geq N_0\}$$

$C, N_0 \geq 0, d \geq 1$ and that

$$(3.2) \quad \|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N |Im \lambda|);$$

for some $N \geq 0$. Let $u \in \mathcal{D}'(-a, a; H)$ be a solution of (E) with $f \in G^d(-a, a; H)$. Then u is d -hypoanalytic in the interval $|t| < a - N$.

Proof. We use the notations of the proof of theorem 1. First we assume that $d > 1$. Then we may choose $\varphi \in \mathcal{D}(R) \cap G^d(R)$ so that $\varphi(t) = 1$ for $|t| \leq a' + \delta$ and $\varphi(t) = 0$ for $|t| \geq a' + \delta'$; $N < \delta < \delta' < N_1$. Hence $E * f \varphi \in G^d(R, H)$ and

$$(3.4) \quad |D^k(E * f \varphi)(\psi)| \leq M^{k+1} (k!)^d \|\psi\|_{L^2}$$

for every $\psi \in \mathcal{D}(-a, a; H)$.

Let $\rho(t)$ be a scalar $G^d(R)$ -function such that $\rho(t) = 1$ for $|t| \leq \varepsilon$ and $\rho(t) = 0$ for $|t| > \varepsilon'$, where $0 < \varepsilon < \varepsilon'$. To estimate $|D^k((1 - \rho)E * g)(\psi)|$ we write it in the form

$$(3.5) \quad D^k((1 - \rho)E * g)(\psi) = (-1)^k (g(f_1^{(k)}) + g(f_2^{(k)}))$$

where

$$f_i^{(k)}(t) = (2\pi)^{-1} \int_{|\sigma| \geq N_0} R(-\sigma, A^*) \hat{\psi}_{i,1}^{(k)}(\sigma) d\sigma, \quad i=1,2.$$

Using the fact that $\rho \in G^d(R)$ we obtain the estimates

$$(3.6) \quad \|\psi_{i,1}^{(k)}(\sigma - iC|\sigma|^{1/d})\| \leq M \exp(-C\varepsilon|\sigma|^{1/d}) \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d |\sigma|^{k-j}$$

and similarly

$$(3.7) \quad \|\psi_{i,1}^{(k)}(\sigma + iC|\sigma|^{1/d})\| \leq M \exp(-C\varepsilon|\sigma|^{1/d}) \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d |\sigma|^{k-j}.$$

By a contour deformation we may write

$$(3.8) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{\Gamma^i} R(-\lambda, A^*) \hat{\psi}_{i,1}^{(k)}(\lambda) d\lambda$$

where $\Gamma^1 = \{\lambda; \lambda = \sigma + iC|\sigma|^{1/d}\} \cup \{|Re \lambda| = N_0; 0 \leq Im \lambda \leq CN_0^{1/d}\}$ and $\Gamma^2 = \{\lambda; \lambda = \sigma - iC|\sigma|^{1/d}, |\sigma| \geq N_0\} \cup \{|Re \lambda| = N_0; -CN_0^{1/d} \leq Im \lambda \leq 0\}$.

Using the estimates (3.6) and (3.7) we get

$$(3.9) \quad \|f_i^{(k)}(t)\| \leq M \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d \int |\sigma|^{p+k-j} \exp(N-\varepsilon) C |\sigma|^{1/d} d\sigma$$

for every $\psi \in \mathcal{D}(-a', a'; D_A^*)$. Choosing $\varepsilon > N$, from Stirling's formula it follows

$$(3.10) \quad \|f_i^{(k)}(t)\| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; H), \quad i=1,2.$$

This implies that

$$(3.11) \quad |D^k((1-\rho)E * g)(\psi)| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}.$$

Hence

$$(3.12) \quad |D^k(u\varphi)(\psi)| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}, \quad \text{for } \psi \in \mathcal{D}(-a', a'; H)$$

where M_1 is a non-negative constant independent of k . Hence $u \in G^d(-a', a'; H)$.

To prove theorem 2 in the analytic case $d=1$, we consider instead of $\varphi(t)$ and $\rho(t)$ two sequences of C^∞ scalar functions $\{\varphi_j(t)\}_{j=0}^\infty$ and $\{\rho_j(t)\}_{j=0}^\infty$ such that (see Friberg [2])

$$(3.13) \quad |D^k \varphi_j(t)| \leq M^{k+1} j^k; \quad \text{for } k \leq j,$$

where $\text{supp } \varphi_j \subset \{t; |t| \leq a' + \delta'\}$, $\varphi_j(t) = 1$ for $|t| \leq a' + \delta$ and similarly

$$(3.14) \quad |D^k \rho_j(t)| \leq M^{k+1} j^k \quad \text{for } k \leq j$$

$\text{supp } \rho_j \subset \{t; |t| \leq \varepsilon'\}$ and $\rho_j(t) = 1$ for $|t| \leq \varepsilon$.

Then denoting $g_j = D^1 \varphi_j u$, as above we obtain

$$(3.15) \quad |D^k(1-\rho_k)E * g_k(\psi)| \leq M_1^{k+1} k^k \|\psi\|_{L^2}$$

for every $\psi \in \mathcal{D}(-a', a'; H)$ and $k=0,1,\dots$

Hence

$$\|D^k(u\varphi_k)\|_\infty \leq M_1^{k+1} k!, \quad k=0,1,\dots$$

That is $u \in G^1(-a', a'; H)$.

As consequence of theorem 2 we get the following result (see Agmon-Nirenberg [1])

Corollary 1. *Suppose that $R(\lambda, A^*)$ exists in the sector $\Sigma: \{|\arg(\pm\lambda)| \leq \alpha; |\lambda| \geq N_0\}$, $0 < \alpha < \pi/2$, and*

$$\|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N|\text{Im } \lambda|), \quad \text{for } \lambda \in \Sigma$$

where N is a non-negative constant. Suppose that f is analytic in $|t| < a$. Then every solution $u \in \mathcal{D}'(-a, a; H)$ of (E) is analytic in the subinterval $|t| < a - N$.

By a slight modification of the preceding proof one easily verifies the following

REMARK. The conclusions of theorem 2 hold if we merely assume that $f \in \mathcal{D}'(-a, a; D_{A^*})$ and

$$(3.16) \quad |D^k f(\psi)| \leq M^{k+1} (k!)^d \|\psi + A^* \psi\|_{L^2}, \quad \psi \in \mathcal{D}(-a, a; D_{A^*}).$$

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