

Title	On the regularity of the weak solutions of abstract differential equations
Author(s)	Barbu, Viorel
Citation	Osaka Journal of Mathematics. 6(1) P.49-P.56
Issue Date	1969
Text Version	publisher
URL	<a href="https://doi.org/10.18910/12363">https://doi.org/10.18910/12363</a>
DOI	10.18910/12363
rights	
Note	

***Osaka University Knowledge Archive : OUKA***

<https://ir.library.osaka-u.ac.jp/repo/ouka/all/>

## ON THE REGULARITY OF THE WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

VIOREL BARBU

(Received November 13, 1968)

(Revised February 17, 1969)

### 1. Preliminaries

Let  $H$  be a Hilbert space;  $(\cdot, \cdot)$ ,  $\|\cdot\|$  and  $\|\cdot\|$  are the notations for the scalar product and for the norm. Denote by  $R$  the real axis,  $-\infty < t < \infty$ , and  $\mathcal{D}(R)$ ,  $\mathcal{D}(R, H)$ ,  $\mathcal{D}'(R, H)$  the spaces of infinitely differentiable scalar functions with compact support, infinitely differentiable  $H$ -valued functions with compact support and  $H$ -valued distribution, respectively, on  $R$  with their usual topologies (see L. Schwartz [7]). The space of  $H$ -valued distributions with compact support will be denoted by  $\mathcal{E}'(R, H)$ . If  $u \in \mathcal{D}'(R, H)$  we may define  $D^k u \in \mathcal{D}'(R, H)$  by the formula:  $D^k u(\varphi) = (-1)^k u(D^k \varphi)$ ,  $\forall \varphi \in \mathcal{D}(R, H)$ . If  $\varphi \in \mathcal{D}(R, H)$  then for each complex  $\lambda$ ,  $\hat{\varphi}(\lambda)$  denotes its Fourier-Laplace transform. (Here  $D^1 \varphi = 1/i \frac{d\varphi}{dt}$ ).

Let  $A : D_A \subset H \rightarrow H$  be a closed linear operator with the domain  $D_A$  dense in  $H$  and let  $A^*$  be its adjoint. The domain  $D_{A^*}$  of the operator  $A^*$  is Banach space in the norm  $\|x\| = \|x\| + \|A^*x\|$ . We denote by  $\mathcal{D}(R, D_{A^*})$  the space of infinitely differentiable  $D_{A^*}$ -valued functions with compact support on  $R$  and by  $\mathcal{D}'(R, D_{A^*})$  its dual. Since  $\mathcal{D}(R, D_{A^*}) = \mathcal{D}(R) \hat{\otimes} D_{A^*}$  it is easy to see that the space  $\mathcal{D}(R, D_{A^*})$  is dense in  $\mathcal{D}(R, H)$ . In an analogous manner, we may define the spaces  $\mathcal{D}(a, b; H)$ ,  $\mathcal{D}'(a, b; H)$  and  $\mathcal{D}(a, b; D_{A^*})$ . Let  $L^* : \mathcal{D}(R, D_{A^*}) \rightarrow \mathcal{D}(R, H)$  be the linear operator

$$(1.1) \quad L^* \varphi = - \left( \frac{1}{i} \frac{d\varphi}{dt} + A^* \varphi \right)$$

and let  $L : \mathcal{D}'(R, H) \rightarrow \mathcal{D}'(R, D_{A^*})$  be its adjoint defined by

$$(1.2) \quad Lu(\varphi) = u(L^* \varphi), \quad \varphi \in \mathcal{D}(R, D_{A^*}).$$

Let  $\mathcal{L}(\mathcal{D}, H)$  be the space of all vector  $H$ -valued distributions  $\mathcal{L}(\mathcal{D}(R, H), H)$ . For  $E \in \mathcal{L}(\mathcal{D}, H)$  we define  $LE$  by the formula

$$(1.3) \quad LE(\varphi) = E(L^* \varphi), \quad \varphi \in \mathcal{D}(R, D_{A^*}),$$

and denote, for every  $\varphi \in \mathcal{D}(R, H)$

$$(E*\varphi)(t) = E_s(\varphi(t-s)).$$

It is easy to see that  $E*\varphi \in C^\infty(R, H)$ . If  $u \in \mathcal{E}'(R, H)$  and  $E \in \mathcal{L}(\mathcal{D}, H)$ ,  $E*u$  denotes the distribution defined by

$$(E*u)(\varphi) = u_t(E_s(\varphi(t+s))), \quad \varphi \in \mathcal{D}(R, H).$$

If  $\varphi(t) \in \mathcal{D}(R)$ , then as in the scalar case it follows immediately

$$\text{supp } (\rho E*u) \subset \text{supp } \rho + \text{supp } u.$$

**DEFINITION.** We say that the distribution  $u \in \mathcal{D}'(a, b; H)$  is a weak solution on  $(a, b)$  of the equation

$$(E) \quad \frac{1}{i} \frac{du}{dt} - Au = f,$$

where  $f \in \mathcal{D}'(a, b; D_{A^*})$ , if the following relation

$$(1.4) \quad u(L*\varphi) = f(\varphi)$$

holds for any  $\varphi \in \mathcal{D}(a, b; D_{A^*})$ .

The existence theorems for the weak solutions of the equation (E) have been obtained by T. Kato and H. Tanabe [3], S. Zaidman [8] and M.A. Malik [6]. We give in this paper some results concerning the regularity of the weak solutions of (E). For the strict solutions of (E) a similar result has been proved by S. Agmon and L. Nirenberg [1].

## 2. Differentiability of solutions

In the following we denote by  $R(\lambda, A^*)$  the resolvent  $(\lambda I - A^*)^{-1}$  of the operator  $A^*$ .

**Theorem 1.** *Suppose that for every  $m > 0$  there exists a number  $C_m > 0$  such that the resolvent  $R(\lambda, A^*)$  exists in the domain*

$$(2.1) \quad \Lambda_m = \{\lambda; |Im \lambda| \leq m \log |Re \lambda|; |Re \lambda| \geq C_m\}$$

and

$$(2.2) \quad \|R(\lambda, A^*)\| \leq C_m^1 |\lambda|^M \exp(N |Im \lambda|), \quad \text{in } \Lambda_m,$$

where  $M > 0$ ,  $N > 0$  are constants independent of  $m$  and  $C_m^1 > 0$ . Then every weak solution  $u \in \mathcal{D}'(-a, a; H)$  of (E) with  $f \in C^\infty(-a, a; H)$  is infinitely differentiable on the interval  $|t| < a - N$ .

Proof. Let  $E \in \mathcal{L}(\mathcal{D}, D_{A^*})$  be defined by the equality

$$(2.3) \quad E(\varphi) = -(2\pi)^{-1} \int_{|\sigma| \geq c_m} R(-\sigma, A^*) \hat{\varphi}(\sigma) d\sigma; \quad \varphi \in \mathcal{D}(R, H)$$

Obviously

$$(2.4) \quad E(L^*\varphi) = \varphi(0) - \int_{|\sigma| \leq c_m} \hat{\varphi}(\sigma) d\sigma; \quad \varphi \in \mathcal{D}(R, D_{A^*}).$$

We denote by  $\Delta$  the interval  $(-a', a')$  where  $a' < a - N_1$ ,  $N < N_1 < a$ , and consider  $\varphi(t) \in \mathcal{D}(R)$  such that  $\varphi(t) = 1$  for  $|t| \leq a' + \delta$  and  $\varphi(t) = 0$  in  $|t| \geq a' + \delta'$ . Assume that  $N < \delta < \delta' < N_1$ . If  $u \in \mathcal{D}'(-a, a; H)$  is a weak solution of (E) then we have

$$(2.5) \quad L(u\varphi)(\psi) = (f\varphi)(\psi) + (D^1\varphi u)(\psi)$$

for every  $\psi \in \mathcal{D}(-a, a; H)$ . On the other hand since  $A^*$  is closed, from (2.2) it follows

$$(2.6) \quad (L(u\varphi)*E)(\psi) = (LE*u\varphi)(\psi).$$

Let us denote by  $g$  the distribution  $D^1\varphi \cdot u$ . Then from (2.4) and (2.5) we get

$$(2.7) \quad u\varphi(\psi) = (E*f\varphi)(\psi) + (E*g)(\psi) + u\varphi(h_\psi)$$

for every  $\psi \in \mathcal{D}(-a, a; H)$ , where  $h_\psi(t) = \int_{|\sigma| \leq c_m} e^{it\sigma} \hat{\psi}(\sigma) d\sigma$ .

Obviously

$$(2.8) \quad \|D^k h_\psi(t)\| \leq M_k \|\psi\|_{L^2}, \quad t \in R$$

for any  $\psi \in \mathcal{D}(-a, a; H)$ , where  $\|\cdot\|_{L^2}$  denotes the norm in the space  $L^2(R, H)$ .

Since  $f\varphi \in C^\infty(R, H)$  it follows that  $E*f\varphi \in C^\infty(R, H)$  which implies that

$$(2.9) \quad |D^k(E*f\varphi)(\psi)| \leq M_k^1 \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a, a; H).$$

Let  $\rho(t)$  be a scalar  $C^\infty$  function on the real line such that  $\rho(t) = 1$  for  $|t| \leq \varepsilon$  and  $\rho(t) = 0$  for  $|t| \geq \varepsilon'$ ;  $0 < \varepsilon < \varepsilon'$ .

Since  $\text{supp } g \subset \{t; a' + \delta < |t| \leq a' + \delta'\}$ , taking  $\varepsilon$  so small such that  $\varepsilon' < \delta$ , from an above remark we deduce that  $(\rho E*g)(\psi) = 0$  for any  $\psi \in \mathcal{D}(-a', a'; H)$ .

Hence

$$(2.10) \quad (E*g)(\psi) = ((1-\rho)E*g)(\psi), \quad \psi \in \mathcal{D}(-a', a'; H).$$

Now we introduce the function  $\psi_t^{(k)}(s) = (1-\rho(s))D^k\psi(t+s)$  and denote by  $\hat{\psi}_t^{(k)}(\lambda)$  its Laplace transform. Let  $m$  be an arbitrary non-negative integer. We may write  $\hat{\psi}_t^{(k)}(\lambda)$  in the form

$$\hat{\psi}_t^{(k)}(\lambda) = \hat{\psi}_{t,1}^{(k)}(\lambda) + \hat{\psi}_{t,2}^{(k)}(\lambda)$$

where

$$\hat{\psi}_{t,1}^{(k)}(\lambda) = \int_{s>\varepsilon} e^{-i\lambda s} (1-\rho(s)) D^k \psi(t+s) ds$$

and

$$\hat{\psi}_{t,2}^{(k)}(\lambda) = \int_{s<-\varepsilon} e^{-i\lambda s} (1-\rho(s)) D^k \psi(t+s) ds.$$

A simple computation shows that with another constant  $M_k$ , one must have the estimates,

$$(2.11) \quad \|\hat{\psi}_{t,1}^{(k)}(\sigma - i m \log |\sigma|)\| \leq M_k |\sigma|^{k-m\varepsilon} \|\psi\|_{L^2}, \quad \sigma \in R$$

and

$$(2.12) \quad \|\hat{\psi}_{t,2}^{(k)}(\sigma + i m \log |\sigma|)\| \leq M_k |\sigma|^{k-m\varepsilon} \|\psi\|_{L^2}, \quad \sigma \in R$$

Let  $f_i^{(k)}(t)$  be the functions

$$(2.13) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{|\sigma| \geq C_m} R(-\sigma, A^*) \hat{\psi}_{t,i}^{(k)}(\sigma) d\sigma; \quad i=1, 2; t \in R.$$

After a suitable deformation of contours in the complex plane, the functions  $f_i^{(k)}(t)$  can be expressed in the following form

$$(2.14) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{\Gamma_m^i} R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda) d\lambda; \quad i=1, 2;$$

where  $\Gamma_m^1$  is the frontier of the domain  $\{\lambda; \text{Im } \lambda \geq -m \log |\text{Re } \lambda|; |\text{Re } \lambda| \geq C_m\}$  and  $\Gamma_m^2$  the frontier of  $\{\lambda; \text{Im } \lambda \leq m \log |\text{Re } \lambda|; |\text{Re } \lambda| \geq C_m\}$ . It is easy to see that the shift of the integration contour is legitimate. Now we have on  $\Gamma_m^i$ ,

$$\|R(-\lambda, A^*) \hat{\psi}_{t,i}^{(k)}(\lambda)\| \leq M_k |\sigma|^{M+k-m(\varepsilon-N)} \|\psi\|_{L^2}; \quad \sigma = \text{Re } \lambda.$$

Choosing  $\varepsilon$  so that  $\varepsilon > N$  and  $m$  so large such that  $M+k-m(\varepsilon-N) < -1$ , one obtains

$$(2.15) \quad \|f_i^{(k)}(t)\| \leq M_k^1 \|\psi\|_{L^2}; \quad t \in R; \psi \in \mathcal{D}(-a', a'; H).$$

We remark that

$$D^k((1-\rho)E * g)(\psi) = (-1)^k (g(f_1^{(k)}) + g(f_2^{(k)})).$$

From (2.15) this implies that

$$(2.16) \quad \|D^k((1-\rho)E * g)(\psi)\| \leq M_k \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_{A^*}).$$

Using (2.7), (2.8), (2.9) and (2.16) we obtain

$$(2.17) \quad |D^k(u\varphi)(\psi)| \leq M_k \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_{A^*}).$$

Since the space  $\mathcal{D}(-a', a'; D_{A^*})$  is dense in  $\mathcal{D}(-a', a'; H)$  from the Hahn-Banach theorem it follows that  $D^k(u\varphi) \in L^2(-a', a'; H)$  for any  $k=0, 1, \dots$ . Hence  $u\varphi \in C^\infty(-a', a'; H)$ . Because the number  $N_1 > N$  is arbitrary, the proof is complete.

**Corollary 1.** *Suppose that there exist some non-negative numbers  $N, C, N_0$  such that  $R(\lambda, A^*)$  exists in the domain*

$$(2.18) \quad \Lambda = \{\lambda; |Im \lambda| \leq C \log |Re \lambda|; \quad |Re \lambda| \geq N_0\}$$

and

$$(2.19) \quad \|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N|Im \lambda|).$$

Then every solution  $u \in \mathcal{D}(R, H)$  of (E), with  $f \in C^\infty(R, H)$ , is infinitely differentiable on  $R$ .

**Corollary 2.** *Suppose that  $f \in \mathcal{D}'(-a, a; D_{A^*})$  such that*

$$(2.20) \quad |D^k f(\psi)| \leq M_k \|\psi + A^* \psi\|_{L^2}; \quad \forall \psi \in \mathcal{D}(-a, a; D_{A^*}).$$

If the hypotheses of theorem 1 are satisfied, then every solution  $u \in \mathcal{D}'(-a, a; H)$  of (E) is infinitely differentiable on the subinterval  $|t| < a - N$ .

Proof. The proof in this case is very much the same, except the inequality (2.9). To estimate  $|D^k(E^* f \varphi)(\psi)|$  we remark that

$$\|A^*(E^* \psi)(t)\| \leq M \|D^l \psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a, a; H)$$

where  $l$  is a non-negative integer. From (2.6), (2.7) and (2.16) this implies that

$$(2.21) \quad |D^k(u\varphi)(\psi)| \leq M_k \|D^l \psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; D_{A^*}).$$

As in the proof of theorem 1 this implies that  $u \in C^\infty(-a', a'; H)$ .

REMARK. If  $u \in C^\infty(\Delta, H)$  is a weak solution of (E) with  $f \in C^\infty(\Delta, H)$ , then  $u(t)$  is a strict solution of (E). To prove this it is enough to choose in the equality (1.2),  $\varphi = \varphi_0 \otimes x$  where  $\varphi_0 \in \mathcal{D}(\Delta)$  and  $x \in H$ . Hence the necessity results for differentiability, proved by Agmon-Nirenberg [1], are true in our case.

### 3. Hypoanalyticity of solutions

DEFINITIONS. A  $C^\infty H$ -valued function  $u(t)$  is said to be  $d$ -hypoanalytic on  $\Delta \subset R$  if for any compact subset  $K \subset \Delta$  there exists a non-negative constant  $M_K$  such that for any  $k$  the following inequality be true

$$(3.1) \quad \|D^k u; K\|_\infty \leq M_K^{k+1} (k!)^d$$

where  $\|u, K\|_\infty = \sup_{t \in K} \|u(t)\|$ .

In the following we denote by  $G^d(\Delta, H)$  the space of all  $d$ - $H$ -valued hypo-analytic functions on  $\Delta$ . If  $H=R$  we omit  $R$  and write  $G^d(\Delta)$ .

**Theorem 2.** *Suppose that  $R(\lambda, A^*)$  exists in a region*

$$\Sigma: \{\lambda; |Im \lambda| \leq C |Re \lambda|^{1/d}; \quad |Re \lambda| \geq N_0\}$$

$C, N_0 \geq 0, d \geq 1$  and that

$$(3.2) \quad \|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N |Im \lambda|);$$

for some  $N \geq 0$ . Let  $u \in \mathcal{D}'(-a, a; H)$  be a solution of (E) with  $f \in G^d(-a, a; H)$ . Then  $u$  is  $d$ -hypoanalytic in the interval  $|t| < a - N$ .

*Proof.* We use the notations of the proof of theorem 1. First we assume that  $d > 1$ . Then we may choose  $\varphi \in \mathcal{D}(R) \cap G^d(R)$  so that  $\varphi(t) = 1$  for  $|t| \leq a' + \delta$  and  $\varphi(t) = 0$  for  $|t| \geq a' + \delta'$ ;  $N < \delta < \delta' < N_1$ . Hence  $E^*f\varphi \in G^d(R, H)$  and

$$(3.4) \quad |D^k(E^*f\varphi)(\psi)| \leq M^{k+1}(k!)^d \|\psi\|_{L^2}$$

for every  $\psi \in \mathcal{D}(-a, a; H)$ .

Let  $\rho(t)$  be a scalar  $G^d(R)$ -function such that  $\rho(t) = 1$  for  $|t| \leq \varepsilon$  and  $\rho(t) = 0$  for  $|t| > \varepsilon'$ , where  $0 < \varepsilon < \varepsilon'$ . To estimate  $|D^k((1-\rho)E^*g)(\psi)|$  we write it in the form

$$(3.5) \quad D^k((1-\rho)E^*g)(\psi) = (-1)^k(g(f_1^{(k)}) + g(f_2^{(k)}))$$

where

$$f_i^{(k)}(t) = (2\pi)^{-1} \int_{|\sigma| \geq N_0} R(-\sigma, A^*) \hat{\psi}_{i,1}^{(k)}(\sigma) d\sigma, \quad i=1,2.$$

Using the fact that  $\rho \in G^d(R)$  we obtain the estimates

$$(3.6) \quad \|\psi_{i,1}^{(k)}(\sigma - iC|\sigma|^{1/d})\| \leq M \exp(-C\varepsilon|\sigma|^{1/d}) \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d |\sigma|^{k-j}$$

and similarly

$$(3.7) \quad \|\psi_{i,2}^{(k)}(\sigma + iC|\sigma|^{1/d})\| \leq M \exp(-C\varepsilon|\sigma|^{1/d}) \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d |\sigma|^{k-j}.$$

By a contour deformation we may write

$$(3.8) \quad f_i^{(k)}(t) = (2\pi)^{-1} \int_{\Gamma_i} R(-\lambda, A^*) \hat{\psi}_{i,i}^{(k)}(\lambda) d\lambda$$

where  $\Gamma^1 = \{\lambda; \lambda = \sigma + iC|\sigma|^{1/d}\} \cup \{|Re \lambda| = N_0; 0 \leq Im \lambda \leq CN_0^{1/d}\}$  and  $\Gamma^2 = \{\lambda; \lambda = \sigma - iC|\sigma|^{1/d}, |\sigma| \geq N_0\} \cup \{|Re \lambda| = N_0; -CN_0^{1/d} \leq Im \lambda \leq 0\}$ .

Using the estimates (3.6) and (3.7) we get

$$(3.9) \quad \|f_i^{(k)}(t)\| \leq M \|\psi\|_{L^2} \sum_{j=0}^k M^j (j!)^d \int |\sigma|^{p+k-j} \exp(N-\varepsilon) C |\sigma|^{1/d} d\sigma$$

for every  $\psi \in \mathcal{D}(-a', a'; D_A^*)$ . Choosing  $\varepsilon > N$ , from Stirling's formula it follows

$$(3.10) \quad \|f_i^{(k)}(t)\| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}; \quad \psi \in \mathcal{D}(-a', a'; H), \quad i=1,2.$$

This implies that

$$(3.11) \quad |D^k((1-\rho)E*g)(\psi)| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}.$$

Hence

$$(3.12) \quad |D^k(u\varphi)(\psi)| \leq M_1^{k+1} (k!)^d \|\psi\|_{L^2}, \quad \text{for } \psi \in \mathcal{D}(-a', a'; H)$$

where  $M_1$  is a non-negative constant independent of  $k$ . Hence  $u \in G^d(-a', a'; H)$ .

To prove theorem 2 in the analytic case  $d=1$ , we consider instead of  $\varphi(t)$  and  $\rho(t)$  two sequences of  $C^\infty$  scalar functions  $\{\varphi_j(t)\}_{j=0}^\infty$  and  $\{\rho_j(t)\}_{j=0}^\infty$  such that (see Friberg [2])

$$(3.13) \quad |D^k \varphi_j(t)| \leq M^{k+1} j^k; \quad \text{for } k \leq j,$$

where  $\text{supp } \varphi_j \subset \{t; |t| \leq a' + \delta'\}$ ,  $\varphi_j(t) = 1$  for  $|t| \leq a' + \delta$  and similarly

$$(3.14) \quad |D^k \rho_j(t)| \leq M^{k+1} j^k \quad \text{for } k \leq j$$

$\text{supp } \rho_j \subset \{t; |t| \leq \varepsilon'\}$  and  $\rho_j(t) = 1$  for  $|t| \leq \varepsilon$ .

Then denoting  $g_j = D^1 \varphi_j u$ , as above we obtain

$$(3.15) \quad |D^k(1-\rho_k)E*g_k(\psi)| \leq M_1^{k+1} k^k \|\psi\|_{L^2}$$

for every  $\psi \in \mathcal{D}(-a', a'; H)$  and  $k=0,1,\dots$

Hence

$$\|D^k(u\varphi_k)\|_\infty \leq M_1^{k+1} k!, \quad k=0,1,\dots$$

That is  $u \in G^1(-a', a'; H)$ .

As consequence of theorem 2 we get the following result (see Agmon-Nirenberg [1])

**Corollary 1.** *Suppose that  $R(\lambda, A^*)$  exists in the sector  $\Sigma: \{|\arg(\pm\lambda)| \leq \alpha; |\lambda| \geq N_0\}$ ,  $0 < \alpha < \pi/2$ , and*

$$\|R(\lambda, A^*)\| \leq \text{pol}(|\lambda|) \exp(N|\text{Im } \lambda|), \quad \text{for } \lambda \in \Sigma$$



where  $N$  is a non-negative constant. Suppose that  $f$  is analytic in  $|t| < a$ . Then every solution  $u \in \mathcal{D}'(-a, a; H)$  of  $(E)$  is analytic in the subinterval  $|t| < a - N$ .

By a slight modification of the preceding proof one easily verifies the following

REMARK. The conclusions of theorem 2 hold if we merely assume that  $f \in \mathcal{D}'(-a, a; D_{A^*})$  and

$$(3.16) \quad |D^k f(\psi)| \leq M^{k+1} (k!)^d \|\psi + A^* \psi\|_{L^2}, \quad \psi \in \mathcal{D}(-a, a; D_{A^*}).$$

JASSY UNIVERSITY

---

### Bibliography

- [1] S. Agmon and L. Nirenberg: *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16** (1963), 121–239.
- [2] J. Friberg: *Estimates for partially hypoelliptic differential operators*, Thesis, Lund, 1963.
- [3] T. Kato and H. Tanabe: *On the abstract evolution equations*, Osaka Math. J. **13** (1962), 107–133.
- [4] L. Hörmander: *Linear Partial Differential Operators*, Springer, Berlin, 1963.
- [5] J.L. Lions: *Equations Différentielles Opérationnelles*, Springer, Berlin, 1961.
- [6] M.A. Malik: *Weak solutions of abstract differential equations*, Proc. Nat. Acad. Sci. U.S.A. **57** (1967), 883–884.
- [7] L. Schwartz: *Théorie des Distributions*, Hermann, Paris, 1967.
- [8] S. Zaidman: *A global existence theorem for some differential equations in Hilbert spaces*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1019–1022.