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A SIMPLE PROOF OF THE RATIO ERGODIC THEOREM

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1. Introduction

Throughout the development of ergodic theory, attention has been devoted by many authors, beginning with the classical struggles of Birkhoff and von Neumann, to proofs of different forms of ergodic theorems. Recenttly, a standard principle has begun to emerge (see [6], but also [4], [5]). The aim of this short note is to apply the principle to obtain a proof of Hopf's ratio ergodic theorem ([1]).

Fundamental Lemma. *Let $(a_n)_{n=0,1,\cdots}$ and $(b_n)_{n=0,1,\cdots}$ be sequences of non-negative real numbers for which there exists a positive integer M such that for any $n = 0, 1, \cdots$ there exists an integer m with $1 \le m \le M$ satisfying that*

$$\sum_{0 \le i < m} a_{n+i} \ge \sum_{0 \le i < m} b_{n+i}.$$

Then for any integer N with $N > M$,

$$\sum_{0 \le n < N} a_n \ge \sum_{0 \le n < N-M} b_n.$$

Proof. The proof of the Fundamental Lemma is easy. By the assumption, we can take integers $0 = m_0 < m_1 < \cdots < m_k < N$ with $m_{i+1} - m_i \le M \quad (i = 0, 1, \cdots, k-1)$ and $N - m_k < M$ such that

$$\sum_{m_i \le n < m_{i+1}} a_n \ge \sum_{m_i \le n < m_{i+1}} b_n.$$

for any $i = 0, 1, \cdots, k-1$. Then, by adding these inequalities, we have

$$\sum_{0 \le n < N} a_n \ge \sum_{0 \le n < m_k} a_n \ge \sum_{0 \le n < m_k} b_n \ge \sum_{0 \le n < N-M} b_n.$$

We apply this Lemma to prove the Ratio Ergodic Theorem [8]. Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space and $T : \Omega \to \Omega$ be a measure preserving transformation.

Let f and g be integrable functions on Ω such that

(1) $$g(\omega) \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} g(T^n\omega) = \infty$$

for all $\omega \in \Omega$. Then we have the following theorem. □

Ratio Ergodic Theorem. *The following limit exists for almost all* $\omega \in \Omega$:

$$r(\omega) := \lim_{n\to\infty} \frac{f(\omega) + f(T\omega) + \cdots + f(T^{n-1}\omega)}{g(\omega) + g(T\omega) + \cdots + g(T^{n-1}\omega)}.$$

Moreover, r is T-invariant and

(2) $$\int f d\mu = \int r g d\mu$$

In the special case that μ is a finite measure, the above theorem is nothing but a consequence of the individual Ergodic theorem applied to f and g separately.

2. The proof of the Theorem

We may and do assume without loss of generality that $f(\omega) \geq 0$ for any $\omega \in \Omega$. For any function h on Ω and a positive integer n, we denote

$$h_n(\omega) := h(\omega) + h(T\omega) + \cdots + h(T^{n-1}\omega).$$

Let

$$\overline{r}(\omega) := \overline{\lim}_{n\to\infty} \frac{f_n(\omega)}{g_n(\omega)}$$

and

$$\underline{r}(\omega) := \underline{\lim}_{n\to\infty} \frac{f_n(\omega)}{g_n(\omega)},$$

where we admit $+\infty$ as a value. Then, $\overline{r}$ and $\underline{r}$ are T-invariant, that is, $\overline{r}(\omega) = \overline{r}(T\omega)$ and $\underline{r}(\omega) = \underline{r}(T\omega)$ for any $\omega \in \Omega$, in virtue of (1).

We fix $\varepsilon > 0$ and $L > 0$. For any $\omega \in \Omega$, there exists a positive integer m such that

$$\frac{f_m(\omega)}{g_m(\omega)} \geq (\overline{r}(\omega)\hat{\ }L)(1 - \varepsilon),$$

where we denote $a\hat{}b := \min\{a, b\}$. Let ν be the finite measure on Ω defined by $d\nu(\omega) = g(\omega)d\mu(\omega)$. Then for any $\delta > 0$, there exists a positive integer M such that $v(\Omega_0) > v(\Omega) - \delta$, where

$$\Omega_0 := \{\omega \in \Omega; \text{there exists } m \text{ with } 1 \le m \le M \text{ such that}$$
$$f_m(\omega) \le (\overline{r}(\omega)\hat{}L)g_m(\omega) \cdot (1 - \varepsilon)\}.$$

Let

$$F(\omega) := \begin{cases} f(\omega) & \omega \in \Omega_0 \\ Lg(\omega) & \omega \notin \Omega_0. \end{cases}$$

Then, for $a_n = F(T^n\omega)$ and $b_n = (\overline{r}(\omega)\hat{}L)g(T^n\omega) \cdot (1 - \varepsilon)$ $(n = 0, 1, \cdots)$, the assumption of the Fundamental Lemma holds since if $T^n\omega \in \Omega_0$, then we can take the same m as in the definition of Ω_0, nothing that $f \le F$ as well as $\overline{r}\hat{}L$ is T-invariant, and if $T^n\omega \notin \Omega_0$, then we can take $m = 1$. Therefore, by the Fundamental Lemma, for any $\omega \in \Omega$ and any positive integer $N > M$,

$$F_N(\omega) \ge (\overline{r}(\omega)\hat{}L)g_{N-M}(\omega) \cdot (1 - \varepsilon),$$

where we use the fact that $\overline{r}$ is T-invariant. We integrate the both sides in the above inequality by $d\mu(\omega)$ and use the fact that μ as well as $\overline{r}$ is T-invariant. We have

$$N\int Fd\mu = \int F_N d\mu \ge \int (\overline{r}\hat{}L)g_{N-M} \cdot (1 - \varepsilon)d\mu = (N - M)\int (\overline{r}\hat{}L)gd\mu \cdot (1 - \varepsilon).$$

On the other hand, since

$$\int fd\mu \ge \int Fd\mu - \int_{\Omega\setminus\Omega_0} Lgd\mu \ge \int Fd\mu - L\delta,$$

we have

$$\int fd\mu \ge \frac{N - M}{N}\int (\overline{r}\hat{}L)gd\mu \cdot (1 - \varepsilon) - L\delta.$$

Here, letting $N \to \infty$, $\delta \downarrow 0$, $\varepsilon \downarrow 0$ and $L \to \infty$ in this order, we have

$$\int fd\mu \ge \int \overline{r}gd\mu. \tag{3}$$

This also implies that $\overline{r}g$ is integrable and by (1), $\overline{r}(\omega) < \infty$ for almost all $\omega \in \Omega$.

Now, we prove equality in (3) by establishing the reverse inequality. Fix $\varepsilon > 0$. Then for any $\omega \in \Omega$, there exists a positive integer m such that

$$\frac{f_m(\omega)}{g_m(\omega)} \le \underline{r}(\omega) + \varepsilon.$$

Let ν be the finite measure on Ω defined by $d\nu(\omega) = f(\omega)d\mu(\omega)$. Then for any $\delta > 0$, there exists a positive integer M such that $\nu(\Omega_0) > \nu(\Omega) - \delta$, where

$$\Omega_0 := \{\omega \in \Omega; (1 \leq {}^\exists m \leq M) \ [f_m(\omega) \leq (\underline{r}(\omega) + \varepsilon)g_m(\omega)]\}.$$

Let

$$F(\omega) := \begin{cases} f(\omega) & \omega \in \Omega \\ 0 & \omega \notin \Omega_0. \end{cases}$$

Then, for $b_n = F(T^n\omega)$ and $a_n = (\underline{r}(\omega) + \varepsilon)g(T^n\omega)$ $(n = 0, 1, \cdots)$, the assumption of the Fundamental Lemma holds since if $T^n\omega \in \Omega_0$, then we can take the same m as in the definition of Ω_0, nothing that $f \geq F$, and if $T^n\omega \notin \Omega_0$, then we can take $m = 1$. Therefore, by the Fundamental Lemma, it holds that for any $\omega \in \Omega$ and any positive integer $N > M$,

$$F_{N-M}(\omega) \leq (\underline{r}(\omega) + \varepsilon)g_N(\omega),$$

where we use the fact that $\underline{r}$ is T-invariant. We integrate both sides in the above inequality by $d\mu(\omega)$ and use the fact that μ as well as $\underline{r}$ is T-invariant. We have

$$(N - M)\int F d\mu = \int F_{N-M} d\mu \leq \int (\underline{r} + \varepsilon)g_N d\mu = N\int (\underline{r} + \varepsilon)g d\mu.$$

On the other hand, since

$$\int f d\mu \leq \int F d\mu + \int_{\Omega\setminus\Omega_0} f d\mu \leq \int F d\mu + \delta,$$

we have

$$\int f d\mu \leq \frac{N}{N - M}\int (\underline{r} + \varepsilon)g d\mu + \delta.$$

Here, letting $N \to \infty$, $\delta \downarrow 0$ and $\varepsilon \downarrow 0$ in this order, we have

$$\int f d\mu \leq \int \underline{r}\, g d\mu. \tag{4}$$

By (1), (3), (4) and the trivial inequality that $\overline{r}(\omega) \geq \underline{r}(\omega)$ for any $\omega \in \Omega$, it follows that $\overline{r}(\omega) = \underline{r}(\omega) < \infty$ for almost all $\omega \in \Omega$. Hence, $r(\omega)$ exists and $r(\omega) = \overline{r}(\omega) = \underline{r}(\omega)$ for almost all $\omega \in \Omega$. The equality (2) also follows, the proof is complete.

REMARK. The basic idea used here originates from [4], which is developed in [5] so that a new type of proofs of Birkhoff type ergodic theorems without using the maximal ergodic lemma is created. The article [9] has escaped attention until now for obvious reasons, being published in an engineering journal. It seems to be an early instance of an application of the basic idea, but is still cluttered with unnecessary details, as is the development in [5]. The clearest and simplest exposition is, in our opinion, contained in [7] and based on [6]. Other relevant references include [2], [3] and [10], and there may well be earlier ones which we have not yet found.

References

[1] E. Hopf: *Ergodentheorie*, Ergebnisse der Mathematik, Springer-Verlag, Berlin, **5** (1937).
[2] R.L. Jones: *New proofs for the maximal ergodic theorem and the Hardy-Littlewood maximal theorem*, Proc. **87** (1983), 681–684.
[3] R.L. Jones: *Ornstein's* $L\log^{+} L$ *theorem*, Proc. **91** (1984), 262–264.
[4] T. Kamae: *A simple proof of the ergodic theorem using nonstandard analysis*, Israel J. Math. **42-4** (1982), 284–290.
[5] Y. Katznelson and B. Weiss: *A simple proof of some ergodic theorems*, Israel J Math. **42-4** (1982), 291–296.
[6] M. Keane: Ergodic Theory, Symbolic Dynamics and Hyperbolic Space, Chapter 2, T. Bedford et al.(ed), Oxford Science Publications, (1991).
[7] M. Keane: The Essence of the Law of Large Numbers, in Algorithms, Fractals and Dynamics, Plenum Press, N.Y. (1995), 125–129.
[8] K. Petersen: *Ergodic Theory*, Cambridge Studies in Advanced Mathematics 2, Cambridge University Press, (1983).
[9] P.C. Shields: *The ergodic and entropy theorems revisited*, IEEE Transactions on Information Theory, **33** (1987), 263–266.
[10] J.M. Steele: *Kingman's subadditive ergodic theorem*, Ann. Inst. H. Poincare Probab. Statist. **25** (1989), 93–98.

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