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# NON UNIFORM DECAY OF THE TOTAL ENERGY OF THE DISSIPATIVE WAVE EQUATION

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**Abstract**

Kawashita, Nakazawa, and Soga [3] give a necessary condition for the uniform energy decay of the dissipative wave equation whose principal term has constant coefficients. In their proof, they construct asymptotic solutions for a suitable family of the Cauchy data. In this paper, instead of the asymptotic solutions, we consider the semiclassical measure associated with the family and extend this result to the variable coefficient case. Moreover we give some lower bound estimate for the energy decay.

## 1. Introduction

We consider the Cauchy problem of the dissipative wave equation of the form

$$\begin{cases} \partial_t^2 u(t,x) - \sum_{i,j} \partial_{x_i}\{a_{ij}(x)\partial_{x_j} u(t,x)\} + a(t,x)\partial_t u(t,x) = 0 & \text{in} \quad [0,\infty)\times \mathbf{R}^n, \\ u|_{t=0} = g_1(x),\ \partial_t u|_{t=0} = g_2(x) & \text{on} \quad \mathbf{R}^n, \end{cases} \tag{1.1}$$

where the functions $a_{ij}(x) \in \mathcal{B}^\infty(\mathbf{R}^n)$ and $a(t,x) \in \mathcal{B}^\infty([0,\infty)\times\mathbf{R}^n)$ are real-valued; $\mathcal{B}^\infty(X)$ is the space of smooth functions on $X$ whose all derivatives are bounded, and $(a_{ij}(x))$ is positive-definite for each $x$. Moreover we assume the uniform ellipticity condition: $(a_{ij}(x)) \geq cId$ in $\mathbf{R}^n$ for some $c > 0$, and a dissipative condition: $a(t,x) \geq 0$ in $[0,\infty)\times\mathbf{R}^n$. As is well known, under these conditions, for any Cauchy data $(g_1, g_2) \in H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$ there exists a solution $u \in C^0([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$ of (1.1) which is unique in the space $C^1([0,\infty); L^2(\mathbf{R}^n))$ and satisfies the energy equation

$$E(u,t) + \int_0^t \int_{\mathbf{R}^n} a|u_t|^2(t,x)\,dx\,dt = E(u,0) \tag{1.2}$$

for any $t \geq 0$. Here $E(u,t)$ is the total energy

$$E(u,t) := \frac{1}{2}\int_{\mathbf{R}^n} |\partial_t u|^2 + \sum_{i,j} a_{ij}\ \partial_{x_j} u\ \overline{\partial_{x_i} u}\,dx.$$

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From this energy equation, $a \geq 0$ means dissipation; if $t_1 \geq t_2$, then $E(u, t_1) \leq E(u, t_2)$. Therefore we expect that the total energy converges to zero as $t$ tends to infinity if the dissipative term is sufficiently large. For example, if $a$ is a positive constant, then $E(u, t)$ decays uniformly with respect to the norm $(\|g_1\|^2_{H^1(\mathbf{R}^n)} + \|g_2\|^2_{L^2(\mathbf{R}^n)})$.

In the paper [3], Kawashita, Nakazawa, and Soga define the uniform energy decay property

DEFINITION (Uniform decay property). We say that the equation (1.1) has the uniform decay property if and only if for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ independent of the Cauchy data such that the inequality

$$(1.3)\qquad \begin{aligned} &E(u, t) \leq \varepsilon(\|g_1\|^2_{H^1(\mathbf{R}^n)} + \|g_2\|^2_{L^2(\mathbf{R}^n)} + \|(g_1, g_2)\|_{E_\varphi(\mathbf{R}^n)}),\\ &\|(g_1, g_2)\|_{E_\varphi(\mathbf{R}^n)} := \int (1 + |x|)\{|g_2|^2 + |\nabla g_1|^2\}\, dx \end{aligned}$$

holds for any $t \geq T(\varepsilon)$ and for any solution $u$ of (1.1) with the Cauchy data $(g_1, g_2)$ satisfying $\|(g_1, g_2)\|_{E_\varphi(\mathbf{R}^n)} < \infty$.

They proved following theorem for a necessary condition of the uniform decay property.

**Theorem 1.1.** *In the equation* (1.1), *we assume* $a_{ij} = \delta_{ij}$, $a(t, x) = a(x)$. *Assume also that there exist some points* $y_0 \in \mathbf{R}^n$, $\eta_0 \in S^{n-1}$ *and a neighborhood* $U$ *of* $y_0$ *satisfying*

$$\sup_{y \in U} \int_0^\infty a(y + s\eta_0)\, ds < \infty \quad \text{or} \quad \sup_{y \in U} \int_{-\infty}^0 a(y + s\eta_0)\, ds < \infty.$$

*Then the equation* (1.1) *does not have the uniform decay property.*

REMARK. For a sufficient condition of uniform energy decay, in [6], under the assumption that $a_{ij} = \delta_{ij}$,

$$a(t, x) \geq a_0(1 + |x| + t)^{-1}, \quad \partial_t a(t, x) \leq 0, \quad a_0 > 1,$$

it is proved that the equation (1.1) has uniform decay property.

In their proof, they construct a family of asymptotic solutions whose energy is concentrated on the ray. Instead of asymptotic solutions, we consider the semiclassical measure associated with a family and extend this result to the equation with variable coefficients. In this case, we should consider a curve $(y_+(t, (y_0, \eta_0)), \eta_+(t, (y_0, \eta_0)))$

which is the solution to the following the Hamilton equation

$$(1.4)\qquad \begin{cases} \dfrac{d}{dt} y_+(t,(y_0,\eta_0)) = \dfrac{\partial}{\partial \xi} p(y_+(t,(y_0,\eta_0)), \eta_+(t,(y_0,\eta_0))), & t>0 \\ \dfrac{d}{dt} \eta_+(t,(y_0,\eta_0)) = -\dfrac{\partial}{\partial x} p(y_+(t,(y_0,\eta_0)), \eta_+(t,(y_0,\eta_0))), & t>0 \\ (y_+(0,(y_0,\eta_0)), \eta_+(0,(y_0,\eta_0))) = (y_0,\eta_0) \end{cases}$$

where $p = \sqrt{\sum_{i,j} a_{ij}(x)\xi^i\xi^j}$. Similarly, $(y_-(t,(y_0,\eta_0)), \eta_-(t,(y_0,\eta_0)))$ denotes the solution of the Hamilton equation for $-p$.

Our main result is the following lower bound estimate for the energy decay.

**Theorem 1.2.** *Let $(y_0,\eta_0) \in \mathbf{R}^n \times \mathbf{R}^n \setminus \{0\}$ and $U$ be an neighbourhood of $y_0$, then we have*

$$\sup_{u\in\mathcal{A}(U)} \left\{ \frac{E(u,t)}{E(u,0)} \right\} \geq \exp\left\{ -\int_0^t a(s, y_\pm(t-s,(y_0,\eta_0)))\,ds \right\}.$$

*Here $\mathcal{A}(U)$ is the set of solutions to* (1.1) *satisfying $u|_{t=0}, u_t|_{t=0} \in C_0^\infty(U)$ and $\pm$ means that the above inequality holds for each $+$, $-$.*

From this theorem and the Poincaré's inequality, we have

**Corollary 1.3.** *Assume that*

$$(1.5)\qquad \liminf_{t\to\infty} \int_0^t a(s, y_+(t-s,(y_0,\eta_0)))\,ds < \infty \quad \text{or} \quad \liminf_{t\to\infty} \int_0^t a(s, y_-(t-s,(y_0,\eta_0)))\,ds < \infty$$

*for some $(y_0,\eta_0) \in \mathbf{R}^n \times \mathbf{R}^n \setminus \{0\}$. Then the equation* (1.1) *does not have the uniform decay property.*

REMARK. If $a_{ij} = \delta_{ij}$, $a(t,x) = a(x)$, then the assumption becomes

$$\int_0^\infty a(y_0 + s\eta_0)\,ds < \infty \quad \text{or} \quad \int_{-\infty}^0 a(y_0 + s\eta_0)\,ds < \infty.$$

So in the assumption of Theorem 1.1, it is sufficient that an integration of $a(x)$ on some ray is bounded.

Thus a magnitude of the dissipative term can be measured by an integration of $a(x)$ on the Hamilton flow. This is well-known for the wave equation on a compact Riemannian manifold. In this case, there are many works for the energy decay and more detailed results are known, e.g. [4], [7]. Especially in [4], the semiclassical measure is used but the framework is a little different from this paper.

## 2. Preliminaries

NOTATION. $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{R}^n_{|\xi|>C} = \{\xi \in \mathbf{R}^n;\ |\xi| > C\}$ $(C > 0)$. For points $x, y \in \mathbf{R}^n$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$, we write $\partial_{x_j} = \partial/\partial x_j$, $\nabla_x = \partial/\partial x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $D_{x_j} = (1/i)\partial_{x_j}$, $D_x = (1/i)\nabla_x$, $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle D\rangle^s u = \mathcal{F}^{-1}[\langle\xi\rangle^s \mathcal{F}u(\xi)]$. Here $\mathcal{F}u(\xi) = (1/(2\pi)^{n/2}) \int e^{-ix\cdot\xi} u(x)\,dx$.

$C^k(U; V)$ is the set of all $C^k$ maps from $U$ to $V$ $(k \in \mathbf{N}_0 \cup \{\infty\})$. $L(E, F)$ is the set of all continuous linear operators from $E$ to $F$. The symbol $(\,\cdot\,,\,\cdot\,)$ denotes the inner product of $L^2(\mathbf{R}^n)$.

**2.1. h-pseudodifferential operators.** We recall basic facts about h-pseudodifferential operators. For $a \in \mathcal{S}'(\mathbf{R}^{2n})$, its Weyl quantization $a_h^w = a^w(x, hD)$ is defined by

$$a_h^w u(x) = \frac{1}{(2\pi h)^n} \iint e^{i((x-y)\cdot\xi)/h} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\,dy\,d\xi \tag{2.1}$$

where $h \in (0, 1]$ and $u \in \mathcal{S}(\mathbf{R}^n)$. We interpret this integral in the sense of the temperate distribution. The correspondence $\mathrm{Op}_h$: $a \in \mathcal{S}'(\mathbf{R}^{2n}) \mapsto a_h^w = a^w(x, hD) \in L(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ is an isomorphism for each $h \in (0, 1]$. For $A \in L(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$, $\sigma_h(A) := \mathrm{Op}_h^{-1}(A)$ denotes the Weyl symbol of A. We define the adjoint $A^* \in L(\mathcal{S}(\mathbf{R}^{2n}), \mathcal{S}'(\mathbf{R}^{2n}))$ by $(Au, v) = (u, A^* v)$, $u, v \in \mathcal{S}(\mathbf{R}^n)$, where $(u, v) = \int u\bar{v}\,dx$. Then $A^* = \mathrm{Op}_h(\bar{a})$, $a = \sigma_h(A)$. So if $a \in \mathcal{S}'(\mathbf{R}^{2n})$ is real, then $A = \mathrm{Op}_h(a)$ is formally self-adjoint.

DEFINITION. The symbol space $S^k$, $k \in \mathbf{R}$ is the set of $h$-dependent $C^\infty$ functions $a(x, \xi; h)$ on $\mathbf{R}^{2n}$ satisfying

$$|\partial^\alpha a(x, \xi; h)| \le C_\alpha \langle\xi\rangle^k \quad \text{on} \quad \mathbf{R}^{2n} \times (0, 1] \quad \text{for any multi-index} \quad \alpha.$$

We denote by $OPS^k$ the space of operators whose symbol is in the space $S^k$.

Following theorems are fundamental tools for pseudodifferential operators.

**Theorem 2.1.** (i). *If $a(x, \xi; h) \in S^k$ for $k \in \mathbf{R}$, then $\mathrm{Op}_h(a)$ is an element of $L(\mathcal{S}, \mathcal{S})$, and $L(\mathcal{S}', \mathcal{S}')$, for each $h \in (0, 1]$.*
(ii) (*composition*). *If $a_j \in S^{k_j}$, $j = 1, 2$, then $\mathrm{Op}_h(a_1)\,\mathrm{Op}_h(a_2) = \mathrm{Op}_h(a_1 \sharp_h a_2)$ with*

$$\begin{aligned}\mathrm{Op}_h(a_1 \sharp_h a_2) &= e^{(ih/2)[D_\xi D_y - D_x D_\eta]} a_1(x, \xi) a_2(y, \eta)|_{y=x, \eta=\xi}\\ &= \sum_{j=0}^{N-1} \frac{1}{j!}\Big(\frac{ih}{2}[D_\xi D_y - D_x D_\eta]\Big)^j a_1(x, \xi) a_2(y, \eta)|_{y=x, \eta=\xi}\\ &\quad + \mathrm{Op}_h(r_N(a_1, a_2)(x, \xi));\end{aligned}$$

$$r_N(a_1, a_2)(x, \xi) \in h^N S^{k_1+k_2}.$$

*Here $N \in \mathbf{N}$. Set $r_0(a_1, a_2) = a_1 \sharp_h a_2$.*

(iii) ($L^2$ *boundedness*). *If* $a \in S^0$, *then* $\mathrm{Op}_h(a)$ *is* $L^2$ *bounded and there is a constant* $C > 0$ *independent of* $h$ *such that* $\|\mathrm{Op}_h(a)\|_{L(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))} \le C$ *for* $0 < h \le 1$.
(iv) (*the sharp Gårding inequality*). *Let* $a \in S^0 \otimes M_d(\mathbf{C})$ $(d \in \mathbf{N})$ *satisfy* $(a + a^*)/2 \ge 0$ *on* $\mathbf{R}^{2n}$. *Then there exists* $C > 0$ *such that*

$$(\mathrm{Op}_h\,(a)u, u)_{(L^2)^d} \ge -Ch\|u\|^2_{(L^2)^d},$$

*for every* $u \in (L^2(\mathbf{R}^n))^d$ *and all* $h \in (0, 1]$.

See [1] or [2] for the proof. Next lemmas follow from the above theorem.

**Lemma 2.2.** *Let* $a_j \in S^{k_j}$, $j = 1, 2$ *and* $\mathrm{supp}(a_1) \cap \mathrm{supp}(a_2) = \emptyset$. *Then* $a_1 \sharp_h a_2 \in h^N S^{k_1+k_2}$ *for any* $N \in \mathbf{R}$.

From now on, we write $h^\infty S^k := \bigcap_{r \in \mathbf{R}} h^r S^k$ and $h^\infty OPS^k$ in a similar way.

DEFINITION. Let $V$ be a subset of $\mathbf{R}^{2n}$. We say that $a \in S^k$ is elliptic in $S^k$ on $V$, if there exist $h_0 > 0$ and $C > 0$ such that

$$|a(x, \xi; h)| \ge C\langle \xi \rangle^k \quad \text{on} \quad (0, h_0] \times V.$$

**Lemma 2.3.** (i). *Let* $a \ge 0 \in S^k$ *be elliptic in* $S^k$ *on* $\mathbf{R}^{2n}$. *Then there exists* $\alpha \in S^{k/2}$ *which is real, elliptic in* $S^{k/2}$ *on* $\mathbf{R}^{2n}$ *and satisfies*

$$\mathrm{Op}_h(a) \equiv \mathrm{Op}_h(\alpha)\,\mathrm{Op}_h(\alpha) \mod h^\infty OPS^k.$$

*Moreover* $\alpha$ *is the form* $\alpha = \sqrt{a} + ha_1$, $a_1 \in S^{k/2}$.
(ii). *Let* $a \in S^k$, $b \in S^0$ *and assume that* $a$ *is elliptic in* $S^k$ *on* $\mathrm{supp}(b)$. *Then there exists* $c \in S^{-k}$ *such that*

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(c) \equiv \mathrm{Op}_h(b) \mod h^\infty OPS^0.$$

**Proposition 2.4.** *Let* $a \in S^k$ *and assume that* $a$ *is elliptic in* $S^k$ *on* $\mathbf{R}^{2n}$. *Then there exists* $h_0 > 0$ *and* $b \in S^{-k}$ *such that*

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(b)\,\mathrm{Op}_h(a) = Id,$$

*for any* $h \in (0, h_0]$.

See [1] for the proof.

**2.2. Semiclassical defect measures.** In this section, we explain some properties of the semiclassical measure. We consider the family of functions on $\mathbf{R}^n$ $\{u_h\}_{0<h\leq h_0}$ that is bounded in $L^2$;

$$\sup_{0<h\leq h_0} \|u_h\|_{L^2} < \infty.$$

**Lemma 2.5.** *Let $a \in S^0$. Then*

$$\|a^w(x, hD)\|_{\mathcal{L}(L^2,L^2)} \leq \sup_{\mathbf{R}^{2n}}|a| + \mathcal{O}(h^{1/2}). \tag{2.2}$$

Proof. We write $A_h = a^w(x, hD)$. From Theorem 2.1, we have $A_h^* A_h = \mathrm{Op}_h(\bar{a}a) + h\,\mathrm{Op}_h^w(r)$, $r \in S^0$. Since $\sup_{\mathbf{R}^{2n}}|a|^2 - \bar{a}a \geq 0$ and the sharp Gårding inequality,

$$\|Au\|_{L^2}^2 \leq \sup_{\mathbf{R}^{2n}}|a|^2\|u\|_{L^2}^2 + Ch\|u\|_{L^2}^2. \qquad \square$$

**Theorem 2.6** (Existance of the semiclassical defect measure). *There exists a* Radon *measure $\mu$ on $\mathbf{R}^{2n}$ and a sequence $h_j \to 0$ such that*

$$(a^w(x, h_j D)u_{h_j}, u_{h_j}) \to \int_{\mathbf{R}^{2n}} a(x, \xi)\, d\mu$$

*for all symbols $a \in C_0^\infty(\mathbf{R}^{2n})$.*

See [2] for the proof.

DEFINITION. We call $\mu$ a semiclassical defect measure associated with the sequence $\{u_{h_j}\}$.

REMARK. In general, the semiclassical defect measure depend on how to take a sequence $\{h_j\}$.

We give examples of semiclassical measures which are used in the proof of the main theorem.

EXAMPLE 1. Let $\mu$ be a semiclassical defect measure associated with a sequence $\{u_{h_j}\}$ and $\chi \in S^0$. Then $\{\chi^w(x, h_j D)u_{h_j}\}$ has a semiclassical defect measure $|\chi|^2\mu$.

EXAMPLE 2. Let $x_0, x_1, \xi_0 \in \mathbf{R}^n$, $\varphi \in L^2(\mathbf{R}^n)$. Take

$$u_h = h^{-n/4}\varphi\left(\frac{x - x_0}{h^{1/2}}\right)e^{i((x-x_1)\cdot\xi_0)/h}.$$

Then there exists precisely one associated semiclassical measure

$$\mu = \|\varphi\|_{L^2}^2 \delta_{(x_0,\xi_0)}.$$

Proof. We prove it here for $a \in C_0^\infty(\mathbf{R}^{2n})$, $\varphi \in C_0^\infty(\mathbf{R}^n)$ since we only use this case in this paper.

$$\begin{aligned}
&(a(x, hD)u_h, u_h)\\
&= \frac{1}{(2\pi h)^n} \int a\Big(\frac{x+y}{2}, \xi\Big) e^{(i/h)(x-y)\cdot\xi} u_h(y)\overline{u_h(x)}\, dy\, d\xi\, dx\\
&= \frac{1}{(2\pi)^n h^{3n/2}} \int a\Big(\frac{x+y}{2}, \xi\Big) e^{(i/h)(x-y)\cdot(\xi-\xi_0)} \varphi\Big(\frac{y-x_0}{h^{1/2}}\Big)\overline{\varphi\Big(\frac{x-x_0}{h^{1/2}}\Big)}\, dy\, d\xi\, dx
\end{aligned}$$

After the change of variables $(x-x_0)/h^{1/2} = X$, $(y-x_0)/h^{1/2} = Y$, $(\xi-\xi_0)/h^{1/2} = \Xi$, we obtain.

$$= \frac{1}{(2\pi)^n} \int a\Big(x_0 + h^{1/2}\frac{X+Y}{2}, \xi_0 + h^{1/2}\Xi\Big) e^{i(X-Y)\cdot\Xi} \varphi(Y)\overline{\varphi(X)}\, dY\, d\Xi\, dX.$$

Now we apply the Lebesgue's convergence theorem.

$$\begin{aligned}
&\lim_{h\to 0}(a(x, hD)u_h, u_h)\\
&= \lim_{h\to 0} \frac{1}{(2\pi)^n} \int a\Big(x_0 + h^{1/2}\frac{X+Y}{2}, \xi_0 + h^{1/2}\Xi\Big) e^{i(X-Y)\cdot\Xi} \varphi(Y)\overline{\varphi(X)}\, dY\, d\Xi\, dX\\
&= \frac{1}{(2\pi)^{n/2}} \int a(x_0, \xi_0) e^{iX\cdot\Xi} (\mathcal{F}\varphi)(\Xi)\overline{\varphi(X)}\, d\Xi\, dX\\
&= a(x_0, \xi_0)\|\varphi\|_{L^2}^2.
\end{aligned}$$

□

For the proof of the main theorem, we consider a semiclassical measure associated with a function $u(x,t;h)$ on $\mathbf{R}^n \times [0,\infty) \times (0,h_0]$ for some $h_0 > 0$. Here $t$ is a parameter and assume that

$$\sup_{0<h\le h_0, 0\le t\le T} \|u_h\|_{L^2(\mathbf{R}_x^n)} < \infty \quad \text{for any} \quad T > 0.$$

We define the symbol class $S_t^k$ which is the space of functions $a(t, x, \xi; h)$ satisfying

$$\sup_{(t,x,\xi,h)\in[0,\infty)\times\mathbf{R}^{2n}\times(0,1]} \frac{|\partial_t^l \partial_{x,\xi}^\alpha a(t, x, \xi; h)|}{\langle\xi\rangle^k} < \infty, \quad \text{for all} \quad l \in \mathbf{N}_0 \quad \text{and multi-index} \quad \alpha.$$

Similarly $OPS_t^k$ is the space of operators whose symbol is in the space $S_t^k$. Let $r(x,t;h)$ be the function on $\mathbf{R}^n \times [0,\infty) \times (0,h_0]$ such that $r(x,t;h) \in H^d(\mathbf{R}_x^n)$ for some $d \in \mathbf{R}$

and any $(t, h) \in [0, \infty) \times (0, h_0]$. We write $r(x; t, h) = o_t^U(h)$ when

$$\left|\frac{(\mathrm{Op}_h(a)r_{t,h}, r_{t,h})}{h}\right| \to 0, \quad h \to 0 \quad \text{uniformly on} \quad [0, T]$$

for any $T > 0$ and $a \in C_0^\infty(U)$. Here $U \subset \mathbf{R}^{2n}$ is an open set.

We now assume that a family $\{u_h\}_{0<h\leq h_0}$ is an approximate solution of a certain equation. In this case, we have a propagation theorem for a corresponding semiclassical defect measure.

**Theorem 2.7.** *Let $p = p_0 + hp_1$, $p_0, p_1 \in S_t^k$ where $p_0, p_1$ are independent of the $h$-variable and $p_0$ is real-valued. Assume that $u_h(t) \in C^1([0, \infty); L^2(\mathbf{R}^n))$ satisfy*

$$\begin{cases} (hD_t + P_h)u_h = r_{t,h}, \\ u_h|_{t=0} = v_h, \end{cases} \tag{2.3}$$

*where $r_{t,h} = o_t^U(h)$ for some open set $U$. Moreover we assume that*

$$\sup_{0<h\leq 1, 0\leq t\leq T} \|u_h(t)\|_{L^2(\mathbf{R}^n)} < \infty$$

*for any $T > 0$, and $\{v_{h_j}\}$ has a semiclassical measure $\nu$. Then there exists a subsequence of $\{h_j\}$ such that $\{u_{h_{j_k}}(t)\}$ has a semiclassical measure $\mu(t) \in C^1([0,\infty); \mathcal{D}'(U))$ satisfying*

$$\begin{cases} \dfrac{d}{dt}\mu(t) + \{p_0, \mu\}(t) - 2(\mathrm{Im}\, p_1)\mu = 0, \quad \text{in} \quad [0, \infty) \times U, \\ \mu(0) = \nu. \end{cases} \tag{2.4}$$

*Here $\{a, b\}$ denotes the Poisson bracket of $a$ and $b$, defined by*

$$\{a, b\} = \frac{\partial a}{\partial \xi}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial \xi}.$$

Proof. By usual existence theorem and diagonal argument, there is a subsequence $\{h_{j_k}\} \subset \{h_j\}$ such that $\{u_{h_{j_k}}\}$ has an associated semiclassical defect measure $\mu(t) \in \mathcal{D}'$ for any $t$ in a dense set $\mathcal{T} \subset [0, \infty)$. we extend this semiclassical defect measure to any $t \in [0, \infty)$.

For any sequence $\{t_i\} \subset \mathcal{T}$, $\lim_{i\to\infty} t_i = t$ and $a \in C_0^\infty(U)$, we show the existence of $\lim_{i\to\infty} \int a\, d\mu(t_i)$. We estimate $(A_h u_h, u_h)(t_i)$ where we write $A_h = a^w(x, hD)$

$$\begin{aligned} &|(A_h u_h, u_h)(t_i + l) - (A_h u_h, u_h)(t_i)| \\ &= \left|\int_{t_i}^{t_i+l} \frac{d}{dt}(A_h u_h, u_h)(t)\, dt\right| \\ &= \left|\int_{t_i}^{t_i+l} (A_h \partial_t u_h, u_h)\, dt + \int_{t_i}^{t_i+l} (A_h u_h, \partial_t u_h)\, dt\right| \end{aligned}$$

$$= \left| \int_{t_i}^{t_i+l} \left\{ -\frac{1}{h}(A_h i P_{0h} u_h, u_h)(t) - (A_h i P_{1h} u_h, u_h)(t) + \frac{i}{h}(A_h r_{h,t}, u_h)(t) \right\} dt \right.$$
$$\left. + \int_{t_i}^{t_i+l} \left\{ -\frac{1}{h}(A_h u_h, i P_{0h} u_h)(t) - (A_h u_h, i P_{1h} u_h)(t) - \frac{i}{h}(A_h u_h, r_{h,t})(t) \right\} dt \right|$$
$$\leq \left| \int_{t_i}^{t_i+l} \left\{ \frac{1}{h}(i[P_{0h}, A_h]u_h, u_h)(t) - (A_h i P_{1h} + (i P_{1h})^* A_h u_h, u_h)(t) \right\} dt \right|$$
$$+ \left| \int_{t_i}^{t_i+l} \frac{i}{h}\{(A_h r_{h,t}, u_h)(t) - (u_h, A_h^* r_{h,t})(t)\} \, dt \right|$$

since $u_h$ is a solution of (2.3). By $r_{t,h} = o_t^U(h)$ and Lemma 2.5, we have
(2.5)
$$|(A_h u_h, u_h)(t_i + l) - (A_h u_h, u_h)(t_i)| \leq C\{(\sup|\{p_0, a\}| + 2\sup|ap_1|) + o(1)\}|l|, \quad h \to 0.$$

We now take $t_i + l \in \mathcal{T}$, $h = h_{j_k}$. Since $\{u_{j_k}(t)\}$ has a semiclassical measure for $t \in \mathcal{T}$, by taking $k \to \infty$, we obtain

$$\left| \int a \, d\mu(t_i + l) - \int a \, d\mu(t_i) \right| \leq C(\sup|\{p_0, a\}| + 2\sup|ap_1|)|l|. \tag{2.6}$$

From this inequality, $\left\{\int a \, d\mu(t_i)\right\}$ is the Cauchy sequence so $\lim_{i\to\infty} \int a \, d\mu(t_i)$ exists. We now define $\int a \, d\mu(t)$ by this limit. We have $\mu \in C([0, \infty), \mathcal{D}'(U))$ by (2.6).

Next we show $\lim_{k\to\infty}(A_h u_{h_{j_k}}, u_{h_{j_k}})(t) \to \int a \, d\mu(t)$. This can be seen from the following inequality

$$\left| \left(A_h u_{h_{j_k}}, u_{h_{j_k}}\right)(t) - \int a \, d\mu(t) \right|$$
$$\leq \left| \left(A_h u_{h_{j_k}}, u_{h_{j_k}}\right)(t) - \left(A_h u_{h_{j_k}}, u_{h_{j_k}}\right)(t_i) \right|$$
$$+ \left| \left(A_h u_{h_{j_k}}, u_{h_{j_k}}\right)(t_i) - \int a \, d\mu(t_i) \right| + \left| \int a \, d\mu(t_i) - \int a \, d\mu(t) \right|.$$

Finally we show $\mu(t)$ satisfies (2.4). From previous calculus, we have

$$(A_h u_{h_{j_k}}, u_{h_{j_k}})(t + l) - (A_h u_{h_{j_k}}, u_{h_{j_k}})(t)$$
$$= \int_t^{t+l} \left(\{\mathrm{Op}_h^w(\{p_0, a\}) + \mathrm{Op}_h^w(2a(\mathrm{Im}\, p_1))\} u_{h_{j_k}}, u_{h_{j_k}}\right) dt + o(1), \quad k \to \infty.$$

Take $k \to \infty$. Then we have

$$\int a \, d\mu(t + l) - \int a \, d\mu(t) = \int_t^{t+l} \int (\{p_0, a\} + 2a(\mathrm{Im}\, p_1)) \, d\mu(t) \, dt.$$

Dividing this equation by $l$ and letting $l \to 0$, we obtain

$$\frac{d}{dt} \int a \, d\mu(t) = \int (\{p_0, a\} + 2a(\mathrm{Im}\, p_1)) \, d\mu(t).$$

So $\mu(t)$ satisfies (2.4) and $\mu \in C^1([0, \infty], \mathcal{D}'(U))$. □

## 3. Proof of Theorem 1.2

**3.1. Systemization.** We transform the equation (1.1) to the first order system by using h-pseudodifferntial operators. We multiply (1.1) by $h$

$$h\,\partial_t^2 u + \frac{1}{h}\sum_{i,j} hD_i a_{ij}(x)hD_j u + a(t,x)h\,\partial_t u = 0. \tag{3.1}$$

Put $q = q_0 + q_1 \in S^2$. Here

$$q_0 = \sum_{i,j} a_{ij}\xi^i\xi^j, \quad q_1 = \frac{h^2}{4}\sum_{i,j}\partial_i\partial_j a_{ij}.$$

We can rewrite (3.1) as

$$h\,\partial_t^2 u + \frac{1}{h}q_h^w u + a(t,x)h\,\partial_t u = 0. \tag{3.2}$$

Take $\chi \in C_0^\infty(\mathbf{R}^n)$ which has a sufficiently small support near $\xi = 0$, $\chi(0) > 0$ and $0 \le \chi(\xi) \le 1$. After adding $(1/h)\chi_h^w u$ to both side of the equation, we have

$$h\,\partial_t^2 u + \frac{1}{h}(q+\chi)_h^w u + a(t,x)h\,\partial_t u = \frac{1}{h}\chi_h^w u. \tag{3.3}$$

Then $q + \chi > 0$ and it is elliptic in $S^2$. So from Lemma 2.3 (i), we can take $\lambda \in S^1$ that is elliptic in $S^1$ and satisfies

$$(q+\chi)_h^w \equiv \lambda_h^w \circ \lambda_h^w \mod h^\infty OPS^2.$$

Moreover $\lambda$ is of the form $\lambda = \lambda_0 + h\lambda_1$, $\lambda_0 = \sqrt{q_0 + \chi}$, $\lambda_1 \in S^1$. Set

$$\begin{pmatrix} \partial_t + \frac{i}{h}\lambda_h^w \\ \partial_t - \frac{i}{h}\lambda_h^w \end{pmatrix} u = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.4}$$

Then (3.3) can be written in the following form

(3.5)

$$h\partial_t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{h}{2}a(t,x)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \left(\frac{1}{h}\chi_h^w + r_h^w\right)\begin{pmatrix} u \\ u \end{pmatrix}$$

where $r_h^w \in h^\infty S^2$. We take $Q(t) \in S_t^0 \otimes M_2(\mathbf{C})$ of the form $Q(t) = I + hQ_1(t)$ to diagonalise (3.5). Here

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1 = \frac{1}{4i\lambda}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.$$

We have

$$
\begin{aligned}
h\partial_t Q_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} Q_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{h}{2}\begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix} Q_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&\quad - \left( Q_h^w \frac{1}{h}\chi_h^w + \tilde{r}_h^w \right)\begin{pmatrix} u \\ u \end{pmatrix} + R_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\end{aligned} \tag{3.6}
$$

where $\tilde{r} \in h^\infty S_t^2$, $R \in h^2 S_t^0 \otimes M_2(\mathbf{C})$. Here we use the $t$-differentiability of $a$. Let

$$
Q_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}.
$$

We obtain

$$
\begin{aligned}
h\partial_t \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} &= \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} - \frac{h}{2}\begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\quad - \left( Q \frac{1}{h}\chi_h^w + \tilde{r}_h^w \right)\begin{pmatrix} u \\ u \end{pmatrix} + R_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\end{aligned} \tag{3.7}
$$

Choose $\tilde{\chi} \in C_0^\infty$ such that $0 \le \tilde{\chi} \le 1$,

$$
\tilde{\chi} \equiv \begin{cases} 0 & (|\xi| : \text{sufficiently large}), \\ 1 & (|\xi| : \text{sufficiently small}), \end{cases} \tag{3.8}
$$

and $\operatorname{supp}(1-\tilde{\chi}) \cap \operatorname{supp}(\chi) = \emptyset$. Multiply (3.7) by $(1-\tilde{\chi})_h^w$ and the equation becomes

$$
\begin{aligned}
&h\partial_t (1-\tilde{\chi})_h^w \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\equiv (1-\tilde{\chi})_h^w \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} - \frac{h}{2}(1-\tilde{\chi})_h^w \begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\equiv \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} (1-\tilde{\chi})_h^w \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} - \frac{h}{2}\begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix} (1-\tilde{\chi})_h^w \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} \left[ (1-\tilde{\chi})_h^w, i\lambda_h^w - \frac{h}{2}a \right] & 0 \\ 0 & \left[ (1-\tilde{\chi})_h^w, -i\lambda_h^w - \frac{h}{2}a \right] \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\mod h^\infty OPS_t^2 u, h^2 OPS_t^2 \otimes M_2(\mathbf{C}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\end{aligned}
$$

We write $[(1-\tilde{\chi})_h^w, i\lambda_h^w - (h/2)a] = {\varphi_1}_h^w$, $[(1-\tilde{\chi})_h^w, -i\lambda_h^w - (h/2)a] = {\varphi_2}_h^w$. Then $\varphi_1, \varphi_2 \in hS_t^1$, $\operatorname{supp}(\varphi_1), \operatorname{supp}(\varphi_2) \subset \operatorname{supp}(\tilde{\chi}) \mod h^\infty S_t^1$.

We set

$$(1-\tilde{\chi})_h^w \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Finally we get

$$\begin{aligned} h\partial_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\equiv \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \frac{h}{2}\begin{pmatrix} a(t,x) & 0 \\ 0 & a(t,x) \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} {\varphi_1}_h^w & 0 \\ 0 & {\varphi_2}_h^w \end{pmatrix}\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\ &\quad \bmod h^\infty OPS_t^2 u,\ h^2 OPS_t^1 \otimes M_2(\mathbf{C})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{aligned} \tag{3.9}$$

Here $\varphi_1, \varphi_2 \in S_t^1$ and $\mathrm{supp}(\varphi_1), \mathrm{supp}(\varphi_2) \subset \mathrm{supp}(\tilde{\chi}) \bmod h^\infty S_t^1$.

**3.2. Proof of Theorem 1.2.** First we prepare some lemmas for the proof.

**Lemma 3.1.** *Let $u$ be the solution of the equation* (1.1) *and $v_1, v_2$ be the functions defined in* (3.4). *Then for any $N \in \mathbf{N}$, there exists a constant $C > 0$ such that*

$$\|((1-\tilde{\chi})_h^w v_1\|_{L^2(\mathbf{R}^n)}^2 + \|(1-\tilde{\chi})_h^w v_2\|_{L^2(\mathbf{R}^n)}^2 \le 4E(u,t) + C\big(hE(u,t) + h^N \|u\|_{L^2(\mathbf{R}^n)}^2\big). \tag{3.10}$$

Here $\tilde{\chi}$ is the function in (3.8).

Proof. We write

$$\begin{aligned} \|(1-\tilde{\chi})_h^w v_1\|_{L^2}^2 + \|(1-\tilde{\chi})_h^w v_2\|_{L^2}^2 &= 2\left\{\|(1-\tilde{\chi})_h^w \partial_t u\|_{L^2}^2 + \frac{1}{h^2}\|(1-\tilde{\chi})_h^w \lambda_h^w u\|_{L^2}^2\right\} \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

We easily have an estimate $\mathrm{I} = \|(1-\tilde{\chi})_h^w \partial_t u\|_{L^2}^2 \le \|\partial_t u\|_{L^2}^2$.

Next we shall estimate II

$$\mathrm{II} = \frac{1}{h^2}((1-\tilde{\chi})_h^w \lambda_h^w u, (1-\tilde{\chi})_h^w \lambda_h^w u) = \frac{1}{h^2}\langle \lambda_h^w (1-\tilde{\chi})_h^w (1-\tilde{\chi})_h^w \lambda_h^w u, u\rangle_{H^{-1},H^1}.$$

Thanks to the composition formula, we have

$$\begin{aligned} &\lambda_h^w (1-\tilde{\chi})_h^w (1-\tilde{\chi})_h^w \lambda_h^w = \{(1-\tilde{\chi})_h^w\}^2 \lambda_h^w \lambda_h^w + h\psi_h^w, \\ &\psi \in S^2,\ \ \mathrm{supp}(\psi) \subset \mathrm{supp}(1-\tilde{\chi}) \quad \bmod h^\infty S^2. \end{aligned}$$

So we estimate II by dividing three parts; $h^\infty S^2$, $h\psi_h^w$ and $\{(1-\tilde{\chi})_h^w\}^2 \lambda_h^w \lambda_h^w$.

Let us begin with $h^\infty S^2$ part. Suppose $r \in h^\infty S^2$. We obtain

$$\begin{aligned}\langle r_h^w u, u\rangle_{H^{-1},H^1} &= (\langle D\rangle^{-1} r_h^w u, \langle D\rangle u)_{L^2} \\ &\le Ch^N(\|u\|_{H^1}^2) \le Ch^N(\|u\|_{L^2}^2 + E(u,t))\end{aligned}$$

for all $N \in \mathbf{N}$. By this estimate, we can ignore mod$h^\infty S^2$ term appeared in II.

$h\psi_h^w$ part; Since $\psi$ vanishes near $\xi = 0$, we can take $\tilde\psi \in S^0$ satisfying $-h^2\Delta\tilde\psi_h^w \equiv \psi_h^w$. For $-h^2\Delta\tilde\psi_h^w$, we have

$$\begin{aligned}\langle -h^2\Delta\tilde\psi_h^w u, u\rangle_{H^{-1},H^1} &= (hD\tilde\psi_h^w u, hDu) \\ &= (\tilde\psi_h^w hDu, hDu) + ([hD, \tilde\psi_h^w]u, hDu) \\ &= \mathcal{O}(h^2\|Du\|_{L^2(\mathbf{R}^n)}^2) + ([hD\tilde\psi_h^w]u, hDu) \\ &= \mathcal{O}(h^2 E(u,t)) + \langle hD\cdot[hD, \tilde\psi_h^w]u, u\rangle_{H^{-1},H^1}.\end{aligned}$$

Let $\sigma(hD\cdot[hD, \tilde\psi_h^w]) = h\psi_1$. Then $\psi_1 \in S^2$, $\mathrm{supp}(\psi_1) \subset \mathrm{supp}(1-\tilde\chi)$. So we can apply similar argument to $\psi_1, \psi_2, \ldots$ and get sufficient estimate.

$\{(1-\tilde\chi)_h^w\}^2\lambda_h^w\lambda_h^w$ part; Recall $\lambda_h^w\lambda_h^w \equiv q_h^w + \chi_h^w$ and $\mathrm{supp}\,\chi \cap \mathrm{supp}(1-\tilde\chi) = \emptyset$. Then Lemma 2.3 implies $\{(1-\tilde\chi)_h^w\}^2\lambda_h^w\lambda_h^w \equiv \{(1-\tilde\chi)_h^w\}^2 q_h^w$. So we estimate $\{(1-\tilde\chi)_h^w\}^2 q_h^w$

$$\begin{aligned}\langle\{(1-\tilde\chi)_h^w\}^2 q_h^w u, u\rangle_{H^{-1},H^1} &= \sum_{i,j}(\{(1-\tilde\chi)_h^w\}^2 hD_i a_{ij} hD_j u, u) \\ &= \sum_{i,j}(\{(1-\tilde\chi)_h^w\}^2 a_{ij} hD_j u, hD_i u) \\ &= \sum_{i,j}(a_{ij} hD_j u, hD_i u) \\ &\quad + \sum_{i,j}(((1-\tilde\chi)^2 - 1)_h^w a_{ij} hD_j u, hD_i u).\end{aligned}$$

Since $(\{(1-\tilde\chi)^2 - 1\}a_{ij})$ is negative, we apply the sharp Gårding inequality. We have

$$\sum_{i,j}(((1-\tilde\chi)^2-1)_h^w a_{ij} hD_j u, hD_i u) \le Ch\|hDu\|_{L^2}^2 \le Ch^3 E(u,t).$$

This completes the proof. □

By definition, we have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - (1-\tilde\chi)_h^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1-\tilde\chi)_h^w hQ_{1h}^w \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We take $\check\chi \in C_0^\infty$ satisfying $\mathrm{supp}\,\check\chi \subset \mathrm{supp}\,\tilde\chi$ and $\check\chi \equiv 1$ near the origin. Then $(1-\tilde\chi)_h^w hQ_{1h}^w = (1-\tilde\chi)_h^w hQ_{1h}^w (1-\check\chi)_h^w$ mod $h^\infty OPS_t$. Using Lemma 3.1, we have

$$\|w_1 - (1-\tilde\chi)_h^w v_1\| = \mathcal{O}_t(h\|u\|_E(t)^{1/2} + h^\infty\|u\|_{L^2(\mathbf{R}^n)}). \tag{3.11}$$

From $\|u\|_E(t) \leq \|u\|_E(0)$, we can replace $\|u\|_E(t)$ with $\|u\|_E(0)$. The same estimate follows for $w_2$.

Next we consider a family of solutions $\{u_h\}_{0<h<h_0}$ of the equation (1.1) with a family of the Cauchy data $\{(g_{1,h}, g_{2,h})\}_{0<h<h_0}$. We systematise $\{u_h\}_{0<h<h_0}$ as in the previous section and use notations, for example, $w_{1,h}, w_{2,h}, \dots$ in an analogous way.

**Lemma 3.2.** *Let $\{u_h\}_{0<h<h_0}$ be the family of solutions of the equation* (1.1) *and assume that $E(u_h, 0)$ is uniformly bounded on $h$. Then*

$$\|u_h\|_{L^2(\mathbf{R}^n)}(t) \leq Ct + \|u_h\|_{L^2(\mathbf{R}^n)}(0). \tag{3.12}$$

Proof. This lemma follows from the inequality

$$\begin{aligned} &2\|u_h\|_{L^2(\mathbf{R}^n)} \frac{d}{dt}\|u_h\|_{L^2(\mathbf{R}^n)} \\ &= \frac{d}{dt}\{\|u_h\|^2_{L^2(\mathbf{R}^n)}\} = 2\operatorname{Re}\left(\frac{d}{dt}u_h, u_h\right) \\ &\leq 2\left\|\frac{d}{dt}u_h\right\|_{L^2(\mathbf{R}^n)} \|u_h\| L^2(\mathbf{R}^n) \leq 2\sqrt{2\|u_h\|_E(t)}\,\|u_h\|_{L^2(\mathbf{R}^n)} \leq C\|u_h\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

□

**Theorem 3.3.** *Let $\{u_h\}$ be a family of solutions to* (1.1). *Assume that $\|g_{1,h}\|_{L^2} = \mathcal{O}(h^{-\infty})$ and $\sup_{h\in(0,h_0]} E(u_h, 0) < \infty$. For some $C > 0$, we can take a subsequence $\{h_j\} \subset (0, h_0]$ such that $w_{1,h_j}, w_{2,h_j}$ have semiclassical measures $\nu_1, \nu_2$ on $\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}$ satisfying the equation*

$$\begin{cases} \dfrac{d}{dt}\nu_1 = \{\lambda_0, \nu_1\} - a\nu_1 & \textit{in} \quad [0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}, \\ \dfrac{d}{dt}\nu_2 = -\{\lambda_0, \nu_2\} - a\nu_2 & \textit{in} \quad [0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}. \end{cases} \tag{3.13}$$

Here $\mathcal{O}(h^{-\infty})$ means $\mathcal{O}(h^{-m})$ for some $m \in \mathbf{N}$. We can take $C$ arbitrary small by shrinking $\operatorname{supp}(\tilde{\chi})$.

Proof. We prove this theorem for $w_{1,h}$. The proof for $w_{2,h}$ is similar. By (3.9), we have

$$hD_t w_{1,h} \equiv \left(\lambda_h^w + \frac{ih}{2}a(t,x)\right) w_{1,h} - i\varphi_{1h}^{\,w}\tilde{v}_{1,h}$$

$$\bmod\ h^\infty OPS_t^2 u_h,\ h^2 OPS_t^1 \otimes M_2(\mathbf{C}) \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix}.$$

Here $\varphi_1$ and $\operatorname{supp}(\varphi_1) \subset \operatorname{supp}(\tilde{\chi}) \bmod h^\infty S_t^1$. We take $C$ such that $\operatorname{supp}(\tilde{\chi}) \subset \{\xi \in \mathbf{R}^n; |\xi| < C\}$. This theorem follows from Lemma 3.1, Lemma 3.2 and Theorem 2.7.

□

Now we give the proof of Theorem 1.2.

Proof. We show Theorem 1.2 for $(y_+(t, (y_0, \eta_0)), \eta_+(t, (y_0, \eta_0)))$. In this case, we consider $w_{1,h}$. For $(y_-(t, (y_0, \eta_0)), \eta_-(t, (y_0, \eta_0)))$, we can apply the same argument to $w_{2,h}$.

We take a family of the Cauchy data $(g_{1,h}, g_{2,h}) = ((h/i)(\lambda_h^w)^{-1}(1/2)g_h, (1/2)g_h)$ for $h \in (0, h_0]$. Then we have $(v_{1,h}, v_{2,h})|_{t=0} = (g_h, 0)$. We set $g_h = h^{-n/4}\varphi((x-y_0)/h^{1/2})e^{ix\cdot\eta_0/h}$ for $\varphi \in C_0^\infty$ and $\|\varphi\|_{L^2(\mathbf{R}^n)} = 1$. We saw $g_h$ has a semiclassical measure $\delta_{(y_0,\eta_0)}$ in Example 2 of the semiclassical measure. If $\operatorname{supp}(\tilde{\chi})$ is sufficiently small, then $(1-\tilde{\chi})_h^w v_1$ and $w_1$ have a semiclassical measure $\delta_{(y_0,\eta_0)}$ by Example 1 and (3.11).

Let $u_h$ be the solution (1.1) for this family of the Cauchy data. Then $u_h \in \mathcal{A}(U)$ if $h$ is small enough. We estimate $E(u_h, 0)$ as

$$
\begin{aligned}
E(u_h, 0) &= \frac{1}{2}\int_{\mathbf{R}^n} \sum_{i,j} a_{i,j}\, \partial_{x_j} g_{1,h}\, \overline{\partial_{x_i} g_{1,h}} + |g_{2,h}|^2\, dx \\
&= \frac{1}{8}\sum_{i,j} (a_{i,j} hD_{x_j}(\lambda_h^w)^{-1} g_h,\, hD_{x_i}(\lambda_h^w)^{-1} g_h)_{L^2} + \frac{1}{8} \\
&= \frac{1}{8}\sum_{i,j} ((\lambda_h^w)^{-1} hD_{x_i} a_{i,j} hD_{x_j}(\lambda_h^w)^{-1} g_h,\, g_h)_{L^2} + \frac{1}{8} \\
&= \frac{1}{4} - (\chi_h^w(\lambda_h^w)^{-1} g_h, (\lambda_h^w)^{-1} g_h)_{L^2} + \mathcal{O}(h^\infty) \le \frac{1}{4} + \mathcal{O}(h^\infty).
\end{aligned}
$$

By Theorem 3.3, we can take a subsequence $\{h_j\}$ and $C > 0$ such that a semiclassical measure $\nu_1$ of $w_{1,h}$ exists and satisfies

$$
\begin{cases}
\dfrac{d}{dt}\nu_1 = \{\lambda_0, \nu_1\} - a\nu_1 \quad \text{in} \quad [0, \infty) \times \mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}, \\
\nu_1|_{t=0} = \delta_{(y_0,\eta_0)}.
\end{cases}
$$

We solve this equation. By the ellipticity of $\lambda_0$, the Hamilton vector field for the Hamiltonian $\lambda_0$ is complete. So the solution is unique and decided on $(y_+(t, (y_0, \eta_0)), \eta_+(t, (y_0, \eta_0)))$. Since the Hamilton flow conserve its Hamiltonian and $|\eta_0| > c$ for some $c > 0$, we can assume that $\lambda_0 = p$ on $(y_+(t, (y_0, \eta_0)), \eta_+(t, (y_0, \eta_0)))$ by changing $\chi$ to smaller one. Then this equation has the following solution.

$$
\begin{aligned}
&d\nu_1 = \exp\left\{-\int_0^t a(s, y_+(t-s, (y_0, \eta_0)))\, ds\right\}\delta_{(y_+(t,(y_0,\eta_0)),\eta_+(t,(y_0,\eta_0)))} \\
&\text{in} \quad [0, \infty) \times \mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\int_{\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}} d\nu_1 &= \int_{\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>C}} \exp\left\{-\int_0^t a(s, y_+(t-s, (y_0, \eta_0)))\, ds\right\}\delta_{(y_+,\eta_+)}\, dx\, d\xi \\
&\ge \exp\left\{-\int_0^t a(s, y_+(t-s, (y_0, \eta_0)))\, ds\right\}.
\end{aligned}
$$

By (3.11), we have $\|w_{1,h}\|_{L^2}^2 \leq 4E(u_h, t) + \mathcal{O}(h)$ which implies

$$\int_{\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>c}} d\nu_1 \leq 4 \limsup_{h\to 0} E(u_h, t).$$

We have

$$4 \limsup_{h\to 0} E(u_h, t) \geq \exp\left\{-\int_0^t a(s, y_+(t-s, (y_0, \eta_0)))\, ds\right\} \quad \text{for any} \quad t > 0. \tag{3.14}$$

By $E(u_h, 0) \leq 1/4 + \mathcal{O}(h^\infty)$, for any $\varepsilon > 0$ there exists $\tilde{h} > 0$ such that

$$\frac{E(u_h, t)}{E(u_h, 0)} \geq \frac{4}{1+\epsilon} E(u_h, t) \quad \text{for any} \quad h \in (0, \tilde{h}].$$

This estimate and (3.14) imply

$$\sup_{u\in\mathcal{A}(U)} \left\{\frac{E(u, t)}{E(u, 0)}\right\} \geq \exp\left\{-\int_0^t a(s, y_+(t-s, (y_0, \eta_0)))\, ds\right\}.$$

We have proved the theorem. □

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