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ON FINITE POINT TRANSITIVE AFFINE PLANES WITH TWO ORBITS ON l_∞

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1. Introduction

Kallaher [3] proposed the following conjecture.

Conjecture. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π . If G has two orbits on the line at infinity, then one of the following statements holds :*

- (i) *The plane π is a translation plane, and the group G contains the group of translations of π .*
- (ii) *The plane π is a dual translation plane, and the group G contains the group of dual translations of π .*

The purpose of this paper is to study this conjecture. When G_A has two orbits of length 1 and n on the line at infinity, where A is an affine point of π , some work has been done on this conjecture. (See Johnson and Kallaher [2].)

Our notation is largely standard and taken from [3]. Let $\mathcal{P} = \pi \cup l_\infty$ be the projective extension of an affine plane π , and G a collineation group of \mathcal{P} . If P is a point of \mathcal{P} and l is a line of \mathcal{P} , then $G(P, l)$ is the subgroup of G consisting of all perspectivities in G with center P and axis l . If m is a line of \mathcal{P} , then $G(m, m)$ is the subgroup consisting of all elations in G with axis m .

In § 2 we prove the following theorem.

Theorem 1. *Let π be a finite affine plane of order n with a collineation group G and let Δ be a subset of l_∞ such that $|\Delta| = t \geq 2$, $(n, t) = 1$ and $(n, t-1) = 1$. If there is an integer $k_1 > 1$ such that $|G(P, l_\infty)| = k_1$ for all $P \in \Delta$ and there is an integer $k_2 > 1$ such that $|G(Q, l_\infty)| = k_2$ for all $Q \in l_\infty - \Delta$, then π is a translation plane, and G contains the group T of translations of π .*

In § 3 and § 4, we prove the following theorem by using Theorem 1.

Theorem 2. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π . If G has two orbits of length 2 and $n-1$ on l_∞ , then one of the following statements holds :*

- (i) *The plane π is a translation plane, and the group G contains the group T of translations of π .*
- (ii) *$|G(\ell_\infty, \ell_\infty)| = n = 2^m$ for some $m \geq 1$, $G(P_1, \ell_\infty) = G(P_2, \ell_\infty) = 1$ and $|G(P, \ell_\infty)| = 2$ for all $P \in \ell_\infty - \{P_1, P_2\}$.*

The planes which are not André planes, satisfying the hypothesis of Theorem 2, include a class of translation planes of order q^3 , where q is an odd prime power. (See Suetake [4] and Hiramine [1].)

2. The proof of Theorem 1

In this section, we prove Theorem 1.

Let π be a finite affine plane of order n with a collineation group G , satisfying the hypothesis of Theorem 1. By Theorem 4.5 of [3], $G(\ell_\infty, \ell_\infty)$ is an elementary abelian r -group for some prime r dividing n . Hence there exist positive integers m and s such that $k_1 = r^m$ and $k_2 = r^s$. Let P be a point of π such that $P \in \Delta$. Let ℓ be an affine line of π such that $\ell \ni P$. Since $G(P, \ell_\infty)$ is semi-regular on $\ell - \{P\}$, $r^m | n$. Similarly, $r^s | n$. By definition, $G(\ell_\infty, \ell_\infty) = \bigcup_{P \in \ell_\infty} G(P, \ell_\infty)$ and $G(P, \ell_\infty) \cap G(Q, \ell_\infty) = 1$ for distinct points $P, Q \in \ell_\infty$. Thus

$$\begin{aligned}
 |G(\ell_\infty, \ell_\infty)| &= 1 + \sum_{P \in \Delta} (|G(P, \ell_\infty)| - 1) + \sum_{Q \in \ell_\infty - \Delta} (|G(Q, \ell_\infty)| - 1) \\
 &= 1 + t(r^m - 1) + (n + 1 - t)(r^s - 1).
 \end{aligned}$$

Since $r^m || G(\ell_\infty, \ell_\infty)|$, it follows $0 \equiv 1 - t + (1 - t)r^s - 1 + t \pmod{r^m}$. Therefore $(t - 1)r^s \equiv 0 \pmod{r^m}$. Since $(t - 1, r) = 1$, this implies $r^m | r^s$. Thus $m \leq s$. On the other hand, since $r^s || G(\ell_\infty, \ell_\infty)|$, it follows $0 \equiv 1 + t(r^m - 1) - 1 + t \pmod{r^s}$. Therefore $tr^m \equiv 0 \pmod{r^s}$. Since $(t, r) = 1$, this implies $r^s | r^m$. Thus $m \geq s$. Therefore $m = s$ and $k_1 = k_2$. By a result of Gleason (See Theorem 5.2 of [3].), the theorem holds.

3. The proof of Theorem 2 when n is odd

In this section, we prove Theorem 2 when n is odd.

Let π be a finite affine plane of odd order n with a collineation group G which is transitive on the affine points of π , satisfying the hypothesis of Theorem 2. Then G has an orbit $\Delta = \{P_1, P_2\}$ of length 2 on ℓ_∞ . Let A be an affine point of π . Let Φ be the set of the affine points of π , and let $\Omega = \Phi \cup \ell_\infty$. Then G induces a permutation group on Ω . Φ, Δ and $\ell_\infty - \Delta$ are orbits of G . Since $(|\Phi|, |\Delta|) = (n^2, 2) = 1$ and $(|\Phi|, |\ell_\infty - \Delta|) = (n^2, n - 1) = 1$, by Theorem 3.3 of [3] Δ and $\ell_\infty - \Delta$ are orbits of G_A .

Lemma 3.1. *G_A includes an involutory homology of π .*

Proof. G_A induces a permutation group on $\ell_\infty - \{P_1, P_2\}$. Since n is odd, $|\ell_\infty - \{P_1, P_2\}| = n - 1$ is even. Let S be a Sylow 2-subgroup of G_A . As G_A is transitive on $\ell_\infty - \{P_1, P_2\}$, $n - 1 \mid |G_A|$. Hence $S \neq 1$. There exists an involution σ in the center of S . Suppose that σ is a Baer involution. If $P_1\sigma = P_1$, then $P_2\sigma = P_2$ and so $|\{P \in \ell_\infty - \Delta \mid P\sigma = P\}| = \sqrt{n} - 1$. This contradicts a result of Lüneburg. (See Corollary 3.6.1 of [3].) If $P_1\sigma \neq P_1$, then $P_2\sigma \neq P_2$ and so $|\{P \in \ell_\infty - \Delta \mid P\sigma = P\}| = \sqrt{n} + 1$. This is again a contradiction by Corollary 3.6.1 of [3]. Therefore σ is an involutory homology.

Lemma 3.2. *Let σ be an involutory homology of π such that $\sigma \in G_A$. If $P_1\sigma = P_1$, then π is a translation plane, and G contains the group T of translations of π .*

Proof. Since $P_1\sigma = P_1$, $P_2\sigma = P_2$. Assume that ℓ_∞ is the axis of σ . Then $\sigma \in G(A, \ell_\infty)$. By a result of André (See Corollary 10.1.3 of [3].), the lemma holds. Assume that ℓ_∞ is not the axis of σ . We may assume that AP_1 is the axis of σ . Then $\sigma \in G(P_2, AP_1)$. There exists $\tau \in G_A$ such that $P_1\tau = P_2$. Clearly $P_2\tau = P_1$. Since $P_2\tau = P_1$ and $(AP_1)\tau = AP_2$, $\tau^{-1}\sigma\tau \in G(P_1, AP_2)$. Therefore $\sigma(\tau^{-1}\sigma\tau) \in G(A, \ell_\infty) - \{1\}$, by a result of Ostrom. (See Lemma 4.13 of [3].) Thus the lemma holds by Corollary 10.1.3 of [3].

Lemma 3.3. *If G_A includes an involutory homology of π which does not fix P_1 , then the following statements hold:*

- (i) *If $P \in \ell_\infty - \{P_1, P_2\}$, then there exist $Q \in \ell_\infty - \{P_1, P_2, P\}$ and $\sigma \in G(Q, AP)$ such that $|\sigma| = 2$.*
- (ii) *If $Q \in \ell_\infty - \{P_1, P_2\}$, then there exist $P \in \ell_\infty - \{P_1, P_2, Q\}$ and $\tau \in G(Q, AP)$ such that $|\tau| = 2$.*

Proof. By assumption, there exists an involutory homology σ of π such that $\sigma \in G_A$ and $P_1\sigma \neq P_1$. Clearly $P_2\sigma \neq P_2$. There exists $P_0 \in \ell_\infty - \{P_1, P_2\}$ such that AP_0 is the axis of σ . Let Q_0 be the center of σ . Then $Q_0 \in \ell_\infty - \{P_1, P_2, P_0\}$. Let $P \in \ell_\infty - \{P_1, P_2\}$. Then there exists $\varphi \in G_A$ such that $P = P_0\varphi$. Set $Q = Q_0\varphi$. Clearly $Q \notin \{P_1, P_2\}$. Since $\sigma \in G(Q_0, AP_0)$ and $(AP_0)\varphi = AP$, $\varphi^{-1}\sigma\varphi \in G(Q, AP)$. This yields the statement (i). Similarly, we have the statement (ii).

Lemma 3.4. *If G_A includes an involutory homology of π which does not fix P_1 , then one of the following statements holds:*

- (i) *The plane π is a translation plane and G contains the group T of translations of π .*
- (ii) *If $P \in \ell_\infty - \{P_1, P_2\}$, then $G(P, AP) \neq 1$.*

Proof. Let $P \in \mathcal{L}_\infty - \{P_1, P_2\}$. By Lemma 3.3 (i), there exist $Q \in \mathcal{L}_\infty - \{P_1, P_2, P\}$ and $\sigma \in G(Q, AP)$ such that $|\sigma| = 2$. On the other hand, by Lemma 3.3 (ii) there exist $R \in \mathcal{L}_\infty - \{P_1, P_2, Q\}$ and $\tau \in G(R, AQ)$ such that $|\tau| = 2$. Assume that $R = P$. Then $\sigma \in G(Q, AP)$ and $\tau \in G(P, AQ)$. By Lemma 4.13 of [3], $\sigma\tau \in G(A, \mathcal{L}_\infty) - \{1\}$. Thus the statement (i) holds by Corollary 10.1.3 of [3]. Assume that $R \neq P$. Then since $\tau \in G(R, AQ)$ and $(AQ)\sigma = AQ$, $\sigma^{-1}\tau\sigma \in G(R\sigma, AQ)$. As $R \neq R\sigma$, $\tau(\sigma^{-1}\tau\sigma) \in G(Q, AQ) - \{1\}$ by a result of Baer. (See Lemma 4.12 of [3].) Thus $G(Q, AQ) \neq 1$. On the other hand, since G_A acts transitively on $\mathcal{L}_\infty - \{P_1, P_2\}$, the statement (ii) holds.

Lemma 3.5. *If $G(P, AP) \neq 1$ for all $P \in \mathcal{L}_\infty - \{P_1, P_2\}$, then there is an integer $k > 1$ such that $|G(P, \mathcal{L}_\infty)| = k$ for all $P \in \mathcal{L}_\infty - \{P_1, P_2\}$.*

Proof. Let $P \in \mathcal{L}_\infty - \{P_1, P_2\}$. Let ℓ be an affine line of π such that $\ell \ni P$. By a result of Ostrom and Wagner (See Theorem 4.3 of [3].), there exists $\tau \in G_P$ such that $(AP)\tau = \ell$. Since $G(P, AP) \neq 1$, $\tau^{-1}G(P, AP)\tau = G(P\tau, (AP)\tau) = G(P, \ell) \neq 1$. Therefore by the dual of Corollary 4.6.1 of [3], $G(P, \mathcal{L}_\infty) \neq 1$. On the other hand, since G_A acts transitively on $\mathcal{L}_\infty - \{P_1, P_2\}$, the lemma holds.

Lemma 3.6. *If $G(P, AP) \neq 1$ for all $P \in \mathcal{L}_\infty - \{P_1, P_2\}$, then $|G(P_1, \mathcal{L}_\infty)| = |G(P_2, \mathcal{L}_\infty)| > 1$.*

Proof. Since the order n of π is odd, by Lemma 3.5 $|G(P, \mathcal{L}_\infty)| \geq 3$ for all $P \in \mathcal{L}_\infty - \{P_1, P_2\}$. Therefore

$$\begin{aligned} & \left| \bigcup_{P \in \mathcal{L}_\infty - \{P_1, P_2\}} G(P, \mathcal{L}_\infty) \right| \\ &= 1 + \sum_{P \in \mathcal{L}_\infty - \{P_1, P_2\}} (|G(P, \mathcal{L}_\infty)| - 1) \\ &\geq 1 + 2(n-1) \\ &= 2n-1 \\ &> n. \end{aligned}$$

Thus $|G(\mathcal{L}_\infty, \mathcal{L}_\infty)| > n$. Hence by a result of Ostrom (See Theorem 4.6 of [3].), $G(P, \mathcal{L}_\infty) \neq 1$ for all $P \in \mathcal{L}_\infty$. In particular $G(P_1, \mathcal{L}_\infty) \neq 1$. There exists $\tau \in G_A$ such that $P_2\tau = P_1$. Thus $|G(P_2, \mathcal{L}_\infty)| = |\tau^{-1}G(P_2, \mathcal{L}_\infty)\tau| = |G(P_1, \mathcal{L}_\infty)| > 1$. Hence the lemma holds.

Proof of Theorem 2 when n is odd: By Lemmas 3.2, 3.4, 3.5, 3.6 and Theorem 1, the theorem holds.

4. The proof of Theorem 2 when n is even

In this section, we prove Theorem 2 when n is even.

Let π be a finite affine plane of even order n with a collineation group G

which is transitive on the affine points of π satisfying the hypothesis of Theorem 2. Then G has an orbit $\Delta = \{P_1, P_2\}$ of length 2 on ℓ_∞ .

Lemma 4.1. *G includes a translation of order 2 of π .*

Proof. Since $n^2 \mid |G|, 2 \mid |G|$. Let S be a Sylow 2-subgroup of G . Then there exists an involution σ in the center of S . By Corollary 3.6.1 of [3] the involution σ is neither a Baer involution, nor an affine elation. It follows that σ is a translation of π .

Lemma 4.2. *$G(\ell_\infty, \ell_\infty)$ is an elementary abelian 2-group and $|G(\ell_\infty, \ell_\infty)| \geq 2$.*

Proof. If $n=2$, then the lemma holds. Let $n \neq 2$. Considering the action of G on ℓ_∞ , by Lemma 4.1 there exist distinct points $Q_1, Q_2 \in \ell_\infty$ such that $G(Q_1, \ell_\infty) \neq 1$ and $G(Q_2, \ell_\infty) \neq 1$. By Theorem 4.5 of [3], the lemma holds.

Lemma 4.3. *If $G(P_1, \ell_\infty) \neq 1$, then the plane π is a translation plane, and the group G contains the group T of translations of π .*

Proof. There exists an involution σ_i such that $\sigma_i \in G(P_i, \ell_\infty)$ for $i \in \{1, 2\}$. Then $\sigma_1 \sigma_2 \in G(\ell_\infty, \ell_\infty)$ and $|\sigma_1 \sigma_2| = 2$. Let Q be the center of $\sigma_1 \sigma_2$. Then $Q \in \ell_\infty - \{P_1, P_2\}$. Since G acts transitively on $\ell_\infty - \{P_1, P_2\}$, there exists $r \geq 1$ such that $|G(P, \ell_\infty)| = 2^r$ for all $P \in \ell_\infty - \{P_1, P_2\}$. There exists $s \geq 1$ such that $|G(P_1, \ell_\infty)| = |G(P_2, \ell_\infty)| = 2^s$. Let $|G(\ell_\infty, \ell_\infty)| = 2^t$. Then $t \geq r + s$. Since

$$|G(\ell_\infty, \ell_\infty)| = 1 + \sum_{P \in \ell_\infty - \{P_1, P_2\}} (|G(P, \ell_\infty)| - 1) + \sum_{Q \in \{P_1, P_2\}} (|G(Q, \ell_\infty)| - 1),$$

$$2^t = 1 + (n-1)(2^r - 1) + 2(2^s - 1). \tag{*}$$

By the same argument as in the proof of Theorem 1, $2^r \equiv 0 \pmod{2^s}$ and $2^{s+1} \equiv 0 \pmod{2^r}$. Thus $s \leq r \leq s + 1$.

Suppose that $r = s + 1$. From (*), $2^t = 1 + (n-1)(2^{s+1} - 1) + 2(2^s - 1)$ follows. Therefore $n = 2^t (2^{s+1} - 1)^{-1}$. As n is an integer, this is a contradiction. Hence $r = s$. By Theorem 5.2 of [3], the lemma holds.

Lemma 4.4. *If $G(P_1, \ell_\infty) = 1$, then $|G(\ell_\infty, \ell_\infty)| = n = 2^m$ for some $m \geq 1$, $G(P_1, \ell_\infty) = 1$ and $|G(P, \ell_\infty)| = 2$ for all $P \in \ell_\infty - \{P_1, P_2\}$.*

Proof. By assumption, $G(P_2, \ell_\infty) = 1$ follows. If $P \in \ell_\infty - \{P_1, P_2\}$, then $G(P, \ell_\infty) \neq 1$. Therefore there exists an integer $r \geq 1$ such that $|G(Q, \ell_\infty)| = 2^r$ for all $Q \in \ell_\infty - \{P_1, P_2\}$. Suppose that $r \geq 2$. Then

$$|G(\ell_\infty, \ell_\infty)|$$

$$= \sum_{Q \in \ell_\infty - \{P_1, P_2\}} (|G(Q, \ell_\infty)| - 1) + 1$$

$$= (2^r - 1)(n - 1) + 1$$

$$\begin{aligned}
 &\geq 3(n-1)+1 \\
 &= 3n-2 \\
 &> n.
 \end{aligned}$$

By Theorem 4.6 of [3], it follows that $G(Q, \mathcal{L}_\infty) \neq 1$ for all $Q \in \mathcal{L}_\infty$. In particular $G(P_1, \mathcal{L}_\infty) \neq 1$, a contradiction. Hence $r=1$. Therefore $|G(\mathcal{L}_\infty, \mathcal{L}_\infty)| = (2-1) \cdot (n-1) + 1 = n$. Therefore there exists an integer $m \geq 1$ such that $n=2^m$. Thus the lemma holds.

Proof of Theorem 2 when n is even: By Lemmas 4.3 and 4.4, the theorem holds.

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