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MATRICES OVER GROUP RINGS WHICH ARE ALEXANDER MATRICES[1]

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Introduction

Let $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ be a presentation of a group G. Then an Alexander matrix of G can be obtained by mapping the $n\times m$ matrix $(\partial r_i/\partial x_j)$ into a matrix with coefficients in the group ring JH of some homomorphic image H of G. (We are using i for the row index and j for the column index. Moreover, what we call Alexander matrices are called in Fox [4] 'Homomorphisms of the Jacobian'.) In this note, we consider the reverse of the above procedure. We start with a matrix A over a group ring, and look for groups with an Alexander matrix equal to A.

Let F be the free group on the set of m letters $\{x_1, \cdots, x_m\}$, and JF be the integral group ring on F. Let χ: $F\rightarrow H$ be an epimorphism from F onto a group H, and let $\tilde{\chi}$: $JF\rightarrow JH$ be the extension of χ to group rings. Then for an $n\times m$ matrix A with entries $\tilde{f}^i_j$ over JH, if G is such that

$$\begin{tikzcd} F \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & H \\ & G \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes and $(\partial r_i/\partial x_j)^{\tilde{\chi}}=A$, we say G *realizes* A w.r.t. χ. Here ϕ is the canonical projection, and ψ is the epimorphism induced by χ. Let R denote Ker χ. We show

Theorem I. *Given an $n\times m$ matrix A with entries $\tilde{f}^i_j$ over JH, there is a group G realizing A w.r.t. χ iff $\sum_{j=1}^m \tilde{f}^i_j\tilde{\chi}(x_j-1)=0$, $i=1, \cdots, n$. Further if the entries of A satisfy this condition and G is a group with presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\chi}=A$, the collection of all groups realizing A w.r.t χ is*

$$\{(x_1, \cdots, x_m: a_1r_1, \cdots, a_nr_n)\,|\,a_1, \cdots, a_n\in[R, R]\}\,.$$

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Thus 'up to $[R, R]$', groups realizing A w.r.t. χ are unique, a result in effect established in Crowell [1] (by a different method) and attributed there to Blanchfield.

For the proof, we consider the set $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ defined with respect to $\tilde{f}_1, \cdots, \tilde{f}_m$ in JH by

$$\Delta(\tilde{f}_1, \cdots, \tilde{f}_m) = \{w \in F \colon \tilde{\chi}(\partial w / \partial x_j) = \tilde{f}_j,\ j = 1, \cdots, m\} .$$

We also use the following condition (*):

$$\textstyle\sum_{j=1}^{m} \tilde{f}_j \tilde{\chi}(x_j - 1) = h - 1_{JH}, \qquad \text{for some } h \in H \qquad (*)$$

and show

Theorem II. *$\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ is non empty iff $\tilde{f}_1, \cdots, \tilde{f}_m$ satisfy (*), in which case, for $w \in \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$, $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m) = w[R, R]$.*

As an immediate corollary, we give a description of

$$\theta(A) = \{(w^1, \cdots, w^n) \in F^n \colon (\partial w^i / \partial x_j)^{\chi} = A\} .$$

When a group G realizing A w.r.t. χ satisfies a certain condition, we say that A is the pseudo Fox Alexander matrix of G w.r.t. χ. (See section 1) We give necessary and sufficient conditions on A for A to be the pseudo Fox Alexander matrix of some group w.r.t χ.

In order to compare matrices of different size, we introduce the concept of a satisfactory (matrix, homomorphism) pair (A, χ), where (A, χ) is satisfactory iff A can be realized by some group w.r.t. χ. To every satisfactory pair there is uniquely associated a (group, homomorphism) pair we call the associate. We define an equivalence relation in the spirit of [4] between satisfactory pairs, and an equivalence relation between associates, such that equivalent satisfactory pairs have equivalent associates, and satisfactory pairs with equivalent associates are equivalent. Further, we consider satisfactory pairs (A, χ) such that A is the pseudo Fox Alexander matrix of some group G w.r.t. χ, and show that in this case the associate of (A, χ) has group G/G'', where G'' is the second commutator of G. In the special case H is the trivial group, the uniqueness of χ renders the concept of a pair redundant. The satisfactory pairs are effectively all matrices A over $JH = J$. The associate to A turns out to be the abelian group with relation matrix A. Moreover, the equivalence relation on satisfactory pairs reduces to the usual equivalence relation on relation matrices of abelian groups, and the equivalence relation on associates is that of group isomorphism. We have thus generalized the well known abelian group—integral matrix correspondence.

In some cases, it is a simple matter to determine whether polynomials $\tilde{f}_1, \cdots, \tilde{f}_m$ satisfy (*). Good examples are the abelianizer $F \to F/[F, F]$, and the epimorphism $F \to \langle t \rangle$ onto the free group on one element defined by $x_j \to t$

for all j. Here $\operatorname{Ker} \chi$ equals $\{w \in F$: exponent sum over each generator is zero$\}$ (Lyndon [5], corollary 4.2), and $\{w \in F$: exponent sum equals zero$\}$, respectively. Moreover, in the proof of sufficiency of $(*)$ for $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ to be non empty, we explicitly construct an element w in $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$.

Section 1 contains the proof of the main result and corollaries. Section 2 deals with satisfactory pairs and their associates. In section 3 we give some examples of the construction of a w in $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ for $\tilde{f}_1, \cdots, \tilde{f}_m$ satisfying $(*)$.

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1. The main results

Suppose there is a $w \in F$ such that $\tilde{\chi}(\partial w/\partial x_j)=\tilde{f}_j$, $j=1, \cdots, m$. By the fundamental formula (Fox [3], 2.3), we have

$$\textstyle\sum_{j=1}^m (\partial w/\partial x_j)(x_j-1) = w-1\,.$$

Then applying $\tilde{\chi}$,

$$\textstyle\sum_{j=1}^m \tilde{f}_j\tilde{\chi}(x_j-1) = \tilde{\chi}(w-1) = \chi(w)-1_{JH}\,.$$

Noting that $\chi(w) \in H$, condition $(*)$ in the introduction is seen to be necessary for $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ to be non empty. We proceed to show sufficiency.

Lemma 1.1. *Let $\tilde{f}_1, \cdots, \tilde{f}_m$ be elements of JH, and h be an element of H, such that $\sum_{j=1}^m \tilde{f}_j\tilde{\chi}(x_j-1)=h-1_{JH}$. Then there is a word $w \in F$ such that $\chi(w)=h$, and $\tilde{\chi}(\partial w/\partial x_j)=\tilde{f}_j, j=1, \cdots, m$.*

Proof. Let $\mathfrak{R}$ denote $\operatorname{Ker} \chi$. Let $w^* \in F$ be an element of $\chi^{-1}(h)$. (recall χ is assumed to be onto.) Let f_j^* be representatives of $\chi^{-1}(\tilde{f}_j)$, $j=1, \cdots, m$, and set $f=\sum_{j=1}^m f_j(x_j-1)+1$. Then $\tilde{\chi}(\partial f/\partial x_j)=\tilde{f}_j$, $j=1, \cdots, m$. For $\partial(f^*(x_j-1))/\partial x_j=f^*\delta_{ij}$, for any $f^* \in JF$. Further, $\tilde{\chi}(f-w^*)=(\sum_{j=1}^m \tilde{f}_j\tilde{\chi}(x_j-1)+1)-h=0$. So $f-w^* \in \mathfrak{R}$. Hence $fw^{*^{-1}}-1 \in \mathfrak{R}$, since $\mathfrak{X}$ is an ideal. Let denote the fundamental ideal ([3]) of JF. By [3, 4.10], there is an element $r \in R$ such that $fw^{*^{-1}}-r \in \mathfrak{R}\mathfrak{X}$;moreover, since $\mathfrak{R}\mathfrak{X}$ is an ideal, we have $f-rw^* \in \mathfrak{R}\mathfrak{X}$. Set $w=rw^* \in F$. Then $\chi(w)=h$, and by [3, 4.5], $\tilde{\chi}(\partial(f-w)/\partial x_j)=0$, or $\tilde{\chi}(\partial w/\partial x_j)=\tilde{\chi}(\partial f/\partial x_j)=\tilde{f}_j, j=1, \cdots, m$, as required.

Corollary 1.2. *$\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ is non empty iff $\tilde{f}_1, \cdots, \tilde{f}_m$ satisfy* $(*)$.

The proof is immediate.

We now turn to the question of structure in $\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$.

Lemma 1.3. *For $a \in [R, R]$, $aw \in \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ whenever $w \in \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$.*

Proof. By [3, 4.9], the ideal $\mathfrak{R}\mathfrak{X}$ determines the commutator subgroup

$[R, R]$ of R. Hence $\{a \in F: a-1 \in \mathcal{R}\mathcal{X}\}=[R, R]$. But by [3, 4.5], for $a \in F$, an element $a-1$ is in $\mathcal{R}\mathcal{X}$ iff all derivatives of a belong to $\mathcal{R}$. Hence for $a \in [R, R]$, $\tilde{\chi}(\partial a/\partial x_j)=0, j=1, \cdots, m$. So as $\partial(aw)/\partial x_j = \partial a/\partial x_j + a(\partial w/\partial x_j)$, we see $\tilde{\chi}(\partial(aw)/\partial x_j)=0+\tilde{\chi}(\partial w/\partial x_j)=f_j, j=1, \cdots, m$, completing the proof.

Lemma 1.3 implies that $w[R, R]=[R, R]w \subset \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ for $w \in \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$.

Lemma 1.4. *Any two elements $w, w' \in \Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$ differ by an element of $[R, R]$, so that $w[R, R]=[R, R]w=\Delta(\tilde{f}_1, \cdots, \tilde{f}_m)$.*

Proof. By assumption, $\tilde{\chi}(\partial w/\partial x_j)=\tilde{f}_j=\tilde{\chi}(\partial w'/\partial x_j)$, $j=1, \cdots, m$. Hence $\tilde{\chi}(\partial(w-w')/\partial x_j)=0$, $j=1, \cdots, m$. Then, by [3, 4.5], $w-w' \in \mathcal{R}\mathcal{X}$. But $w-w'$ in $\mathcal{R}\mathcal{X}$ implies $ww'^{-1}-1 \in \mathcal{R}\mathcal{X}$, since $\mathcal{R}\mathcal{X}$ is an ideal. So by [3, 4.9], $ww'^{-1} \in [R, R]$, and the proof is complete.

Proof of Theorem II. Corollary 1.2 and Lemmas 1.3 and 1.4.

Corollary 1.5. *Given an $n \times m$ matrix A with entries $\tilde{f}^i_j$ over JH, $\theta(A)=\prod_{i=1}^n \Delta(\tilde{f}^i_1, \cdots, \tilde{f}^i_m)$.*

Proof of Theorem I. Assume G is a group realizing A w.r.t. χ, with presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\tilde{\chi}}=A$. By the fundamental formula $r_i-1=\sum_{j=1}^m (\partial r_1/x\partial_j)(x_j-1)$, $i=1, \cdots, n$. Hence $\tilde{\chi}(r_i-1)=\sum_{j=1}^m \tilde{\chi}(\partial r_i/\partial x_j)(x_j-1)$, $i=1, \cdots, n$. But by our assumption on G, $\chi(r_i)=1$, and we have $0=\sum_{j=1}^m \tilde{\chi}(\partial r_i/\partial x_j)\tilde{\chi}(x_j-1)=\sum_{j=1}^m \tilde{f}^i_j\tilde{\chi}(x_j-1)$, $i=1, \cdots, n$, as required.

Conversely, suppose the entries of A satisfy $\sum_{j=1}^m \tilde{f}^i_j\tilde{\chi}(x_j-1)=0$, $i=1, \cdots, n$. Since this is just (*) with $h=1_{JH}$, Theorem II gives elements $r_i \in \Delta(\tilde{f}^i_1, \cdots, \tilde{f}^i_m)$, $i=1, \cdots, n$. Using the definition of Δ and the fundamental formula, we see

$$0=\sum_{j=1}^m \tilde{f}^i_j\tilde{\chi}(x_j-1)=\sum_{j=1}^m \tilde{\chi}(\partial r_i/\partial x_j)\tilde{\chi}(x_j-1)=\tilde{\chi}(r_i-1),$$

whence $\chi(r_i)=1$, $i=1, \cdots, n$. It is now easy to see the group G presented by $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ realizes A w.r.t. χ. That all groups G^* realizing A w.r.t. χ, with presentation $(x_1, \cdots, x_m: r^*_1, \cdots, r^*_n)$ such that $(\partial r^*_i/\partial x_j)^{\tilde{\chi}}=A$ have the stated form follows from Corollary 1.5 and Theorem II, after noting that $(r^*_1, \cdots, r^*_n) \subset \theta(A)$. This completes the proof.

Let λ: $F \to G/[G, G]$ be the composite $F \xrightarrow{\phi} G \to G/[G, G]$, and $\tilde{\lambda}$ be its extension to group rings. Fox ([4, § 4]) calls the $n \times m$ matrix $(\partial r_i/\partial x_j)^{\tilde{\lambda}}$ an Alexander matrix of G. This leads to the question of when, for a group G realizing A w.r.t. χ, there is an isomorphism σ: $G/[G, G] \to H$ such that the diagram

$$\begin{tikzcd} F \arrow[rr, "\chi"] \arrow[dr, "\lambda"'] & & H \\ & G/[G, G] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes. When such a diagram exists, we say that G *has* A *as its Alexander matrix in the pseudo Fox sense* w.r.t. χ, or that A *is the pseudo Fox Alexander matrix of* G w.r.t. χ.

Lemma 1.6. *Let A be an $n\times m$ matrix with entries $\tilde{f}^i_j$ over JH, and G be group realizing A w.r.t. χ, with presentation $(x_1, \cdots, x_m\colon r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\tilde{\chi}}=A$. Then G has A as its Alexander matrix in the pseudo Fox sense w.r.t. χ iff H is abelian and A° is a relation matrix for H, where $\circ$ denotes the trivializer $JH\to J$.*

Proof. Observe that $A^\circ=(\tilde{f}^i_j)^\circ=(\partial r_i/\partial x_j)^\circ$. Suppose that G has A as its Alexander matrix in the pseudo Fox sense w.r.t. χ. Then $G/[G, G]\cong H$. H is therefore abelian, and by [4, 3.5], A° is a relation matrix for H. This proves sufficiency.

Conversely, suppose H is abelian and A° is a relation matrix for H. Since H is abelian, there is an epimorphism $\sigma\colon G/[G, G]\to H$ such that

$$\begin{tikzcd} F \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & H \\ & G \arrow[ur, "\psi"] \arrow[d] & \\ & G/[G, G] \arrow[uur, "\sigma"'] & \end{tikzcd}$$

commutes. Moreover, by [4, 3.5], A° is also a relation matrix for $G/[G, G]$, whence $G/[G, G]\cong H$. But any homomorphism from a finitely generated group onto itself is an isomorphism, from which we deduce σ to be an isomorphism. This completes the proof.

2. Satisfactory pairs and their associates

We say that the pair (A, χ) is *satisfactory* when there exists a group G realizing A w.r.t. χ. Recall that if G is a group realizing A w.r.t. χ, there is a diagram

$$\begin{tikzcd} F \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & H \\ & G \arrow[ur, "\psi"'] & \end{tikzcd}$$

Define the subgroup G^{++} of G to be $[\phi R, \phi R]$. Since $G^{++}\subset \operatorname{Ker}\psi$, ψ induces an epimorphism $\bar{\psi}\colon G/G^{++}\to H$. Let $\bar{G}=G/G^{++}$. Then

Theorem 3.1. *The pair $(\bar{G}$, $\bar{\psi}$ is determined uniquely by the satisfactory pair (A, χ).*

Proof. The quotient $\bar{G}$ has a presentation which may be obtained from a presentation of G by adding $\{a\colon a\in[R, R]\}$ as relators. The proof now follows from the description of all groups realizing A w.r.t. χ given in Theorem I.

$(\bar{G}, \bar{\psi})$ is called the *associate* of (A, χ). The associates $(\bar{G}, \bar{\psi})$ and $(\bar{G}_*, \bar{\psi}_*)$ of satisfactory pairs (A, χ) and (A_*, χ_*) are said to be *equal* in case there is an isomorphism $\bar{\rho}\colon \bar{G}\rightarrow\bar{G}_*$ such that

$$\begin{tikzcd} \bar{G} \arrow[dd, "\rho"'] & \\ & H \arrow[ul, "\bar{\psi}"'] \\ \bar{G}_* \arrow[ur, "\bar{\psi}_*"'] & \end{tikzcd}$$

commutes. Equality is the equivalence relation between associates mentioned in the introduction. We define an equivalence relation between satisfactory pairs as follows:

DEFINITION (compare [4], p. 199.). Two satisfactory pairs are equivalent if one can be obtained from the other by a finite number of elementary transformations (I), (II), $(\mathrm{I})^{-1}$, $(\mathrm{II})^{-1}$, defined as follows:

(I) Replace (A, χ) by (A', χ'), where A' is obtained from A by adjoining a new row equal to a left linear combination of the rows of A, and $\chi'=\chi$.

(II) Replace (A, χ) by (A', χ'), where A' is the result of adjoining to A a new row and a new column (say the pth and qth respectively) such that:

(a) The entry in the intersection of the row and column is 1.

(b) The remaining entries in the new column are all 0.

(c) The remaining entries $\tilde{f}_1^p, \cdots, \hat{\tilde{f}_q^p}, \cdots, \tilde{f}_{m+1}^p$ in the new row satisfy '$-(*)$'; that is

$$\sum_{\substack{j=1\\ j\neq q}}^{m+1} \tilde{f}_j^p \tilde{\chi}(x_j-1) = -h+1_{JH}, \qquad \text{for some } h\in H,$$

and the epimorphism χ' from the free group F' on $m+1$ letter $\{x_1, \cdots, x_q, \cdots, x_{m+1}\}$ onto H is defined by $\chi'(x_j)=\chi(x_j)$, $j=1, \cdots, \hat{q}, \cdots, m+1$; $\chi'(x_q)=h$.

$(\mathrm{I})^{-1}$ The inverse operation to (I).

$(\mathrm{II})^{-1}$ Replace (A, χ) by (A', χ'), where A' is obtained from A by removal of the pth row and the qth column, and χ' is the restriction of χ to the free group F' generated by $\{x_1, \cdots, \hat{x}_q, \cdots, x_m\}$. Here A satisfies

(a) The entry in the intersection of the pth row and the qth column is 1.

(b) The remaining entries in the qth column are all 0.

If (A, χ) is satisfactory, by Theorem I $\sum_{j=1}^{m} \tilde{f}_j^p \tilde{\chi}(x_j-1)=0$. If in addition A satisfies condition (a) of $(\mathrm{II})^{-1}$,

$$\sum_{\substack{i=1\\ j\neq q}}^{m} \tilde{f}_j^p \tilde{\chi}(x_j-1) = 1-\chi(x_q).$$

So $\tilde{f}_1^p, \cdots, \hat{\tilde{f}_q^p}, \cdots, \tilde{f}_m^p$ satisfy '$-(*)$' for $h=\chi(x_q)$. This together with the next lemma justifies the label $(\mathrm{II})^{-1}$.

Since the interchange of any two rows or any two columns is easily seen to be an elementary transformation, we shall assume throughout the rest of this paper that the pth row and qth column are the bottom row and extreme right hand column respectively.

Lemma 2.2. *The result of applying* (I), (II), $(\mathrm{I})^{-1}$, *or* $(\mathrm{II})^{-1}$ *to a satisfactory pair is again a satisfactory pair.*

Proof. For the elementary transformations (I), $(\mathrm{I})^{-1}$, and (II), this is clear. In the case of $(\mathrm{II})^{-1}$ we must show the map χ' from F' into H is onto, or $\chi(x_m)$ can be written in terms of $\chi(x_1), \cdots, \chi(x_{m-1})$. By Lemma 1.1, there is a word $v \in F$ such that $\chi(v)=1$, and $\tilde{\chi}(\partial v/\partial x_j)=\tilde{f}_j^n$, $j=1, \cdots, m$. In the special case the number of times x_m occurs in v is one, the relation $\chi(v)=1$ shows $\chi(x_m)$ is expressible in terms of $\chi(x_1), \cdots, \chi(x_{m-1})$, and χ' is onto. So suppose the number of times x_m occurs in v is $p>1$. As $\tilde{\chi}(\partial v/\partial x_m)=1$, p is odd, and among the p terms in this derivative, there are $(p-1)/2$ cancelling pairs corresponding to particular occurrences of x_m and x_m^{-1} in v. Focussing on one pair, write $v=ax_m^\varepsilon b x_m^{-\varepsilon} c$, $\varepsilon=\pm 1$, and note the assumption

$$\tilde{\chi}[(a-ax_m^\varepsilon b x_m^{-\varepsilon})(\partial x_m^\varepsilon/\partial x_m)] = 0$$

implies $b \in \operatorname{Ker} \chi$. Consequently $v_1=abc$ is such that $\chi(v_1)=1$, and $\tilde{\chi}(\partial v_1/\partial x_m)=1$. Moreover, v_1 has fewer cancelling pairs by one. Repeating this argument enough times brings us back to the special case, and χ' is seen to be onto. This completes the proof.

Theorem 2.3. *Equivalent satisfactory pairs have equal associates.*

Proof. Let (A, χ) be a satisfactory pair, and G be a group realizing A w.r.t. χ with presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\tilde{\chi}}=A$. By Lemma 2.2, it suffices to show the satisfactory pair obtained from (A, χ) by applying any one of (I), (II), $(\mathrm{I})^{-1}$, or $(\mathrm{II})^{-1}$ has associate equal to that of (A, χ). For this, let G' be any group realizing A' w.r.t. χ'. Let ϕ': $F' \to G'$ be canonical projection, and ψ': $G' \to H$ be the epimorphism induced by χ'. We find a G' and an isomorphism ρ: $G \to G'$ such that

$$\begin{tikzcd} G \arrow[dr, "\psi"] \arrow[d, "\rho"'] & \\ G' \arrow[r, "\psi'"'] & H \end{tikzcd} \qquad \cdots\cdots\cdots(\mathrm{a})$$

commutes. Then $\rho\phi \operatorname{Ker} \chi = \phi' \operatorname{Ker} \chi'$, and ρ induces an isomorphism $\bar{\rho}$: $\bar{G} \to \bar{G}'$ such that

$$\begin{tikzcd} \bar{G} \arrow[d, "\bar{\rho}"'] \arrow[dr, "\bar{\psi}"] & \\ \bar{G}' \arrow[r, "\bar{\psi}'"'] & H \end{tikzcd}$$

commutes. Hence this suffices to show (A, χ) and (A', χ') have equal associates. The isomorphism $\rho: G \to G'$ is defined as follows. A group G' realizing A' w.r.t. χ' can be obtained by applying a Tietze transformation ([4], p. 197) to $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ of the same type as the type of the elementary transformation used to obtain (A', χ') from (A, χ). For an elementary transformation of type (I) or $(\mathrm{I})^{-1}$, this follows from [4], p. 199. For a type (II) elementary transformation, by Lemma 1.1, there is a $w \in F$ such that $\chi(w) = h = \chi(x_{m+1})$, and $\tilde{\chi}(\partial w/\partial x_j) = -\tilde{f}_j^{n+1}, j = 1, \cdots, m$. But then

$$\tilde{\chi}'(\partial(x_{m+1}w^{-1})/\partial x_j) = \begin{cases} -\tilde{\chi}'((x_{m+1}w^{-1})(\partial w/\partial x_j)) = \tilde{f}_j^{n+1}, & j = 1, \cdots, m\,, \\ 1_{JH} & j = m+1\,. \end{cases}$$

So we may take G' to be the group presented by $(x_1, \cdots, x_m, x_{m+1}: r_1, \cdots, r_n, x_{m+1}w^{-1})$. For a type $(\mathrm{II})^{-1}$ elementary transformation, recall that in Lemma 2.2 we showed $\chi(x_1), \cdots, \chi(x_{m-1})$ generate H. Hence there is a word $w \in F'$ such that $\chi(w) = \chi(x_m)$, and $\tilde{\chi}(\partial w/\partial x_j) = \tilde{f}_j^n, j = 1, \cdots, m-1$. We may thus assume the presentation of G is $(x_1, \cdots, x_m: r_1, \cdots, r_{m-1}, x_m w^{-1})$. The isomoıphism $\rho: G \to G'$ induced by the Tietze transformation is easily seen to satisfy diagram (a), and the proof is complete.

Next we establish the converse to this theorem.

Theorem 2.4. *Satisfactory pairs with equal associates are equivalent.*

Proof. Let (A, χ) and (A_*, χ_*) be satisfactory pairs, where A_* is a $u \times t$ matrix over JH and χ_* is an epimorphism from the free group F_* on t letters $\{y_1, \cdots, y_t\}$ onto H. Further, let G be a group realizing A w.r.t. χ, with presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\tilde{\chi}} = A$; let G_* be a group realizing A_* w.r.t. χ_*, with presentation $(y_1, \cdots, y_t: s_1, \cdots, s_u)$ such that $(\partial s_k/\partial y_l)^{\tilde{\chi}_*} = A_*$; and let $\bar{\rho}: \bar{G} \to \bar{G}_*$ be an isomorphism such that

$$\begin{tikzcd} \bar{G} \arrow[d, "\bar{\rho}"'] \arrow[dr, "\bar{\psi}"] & \\ \bar{G}_* \arrow[r, "\bar{\psi}_*"'] & H \end{tikzcd}$$

commutes. We must show (A, χ) and (A_*, χ_*) are equivalent. We begin with two observations. First, a Tietze transformation T applied to $(x_1, \cdots, x_m:$

$r_1, \cdots, r_n)$ induces an elementary transformation on (A, χ) of the same type as follows. Let the result of applying T to $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ be $(x'_1, \cdots, x'_o: r'_1, \cdots, r'_p)$. Denote the group presented by $(x'_1, \cdots, x'_o: r'_1, \cdots, r'_p)$ by G', and let $\rho: G \to G'$ be the isomorphism induced by T. Denote by ϕ' the canonical projection from the free group F' on $\{x'_1, \cdots, x'_o\}$ onto G', and define the epimorphism $\chi': F' \to H$ to be the composite

$$F' \xrightarrow{\phi'} G' \xrightarrow{\bar{\rho}} G \xrightarrow{\psi} H .$$

Set $A'=(\partial r'_i/\partial x'_j)^{\chi'}$, $i=1, \cdots, p$; $j=1, \cdots, o$. Then it is easy to see that the satisfactory pair (A', χ') differs from (A, χ) by an elementary transformation $\mathcal{S}$ of the same type as that of T. We say $\mathcal{S}$ was induced by T. Second, let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be subsets of F and F_* respectively which generate $[\operatorname{Ker}\chi, \operatorname{Ker}\chi]$ and $[\operatorname{Ker}\chi_*, \operatorname{Ker}\chi_*]$ respectively. Then $\bar{G}$ is presented by $(x_1, \cdots, x_m: r_1, \cdots, r_n, a_1, a_2, \cdots)$ and $\bar{G}_*$ is presented by $(y_1, \cdots, y_t: s_1, \cdots, s_u, b_1, b_2, \cdots)$. Moreover, $\tilde{\chi}(\partial a/\partial x_j)=0$, for all $a \in \{a_1, a_2, \cdots\}$ and $j=1, \cdots, m$, by Lemma 1.3 with $w=1$, and $\tilde{f}_1=\cdots=\tilde{f}_m=0$. So denoting by a_F any finite subset of $\{a_1, a_2, \cdots\}$, the Alexander matrix A' of the presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n, a_F)$ at χ is just A with finitely many zero rows added. Hence (A, χ) and (A', χ) differ by an elementary transformation of type (I), so are equivalent. Similarly, $\tilde{\chi}_*(\partial b/\partial y_k)=0$, for all $b \in \{b_1, b_2, \cdots\}$. And denoting by b_F any finite subset of $\{b_1, b_2, \cdots\}$, the Alexander matrix A'_* of the presentation $(y_1, \cdots, y_t: s_1, \cdots, s_u, b_F)$ at χ_* is just A_* with finitely many zero rows added. Hence (A'_*, χ_*) and (A_*, χ_*) are equivalent. Our method of proof is to give a finite sequence of Tietze transformations starting from $(x_1, \cdots, x_m: r_1, \cdots, r_n, a_F)$ such that the induced sequence of elementary transformations applied to (A', χ) gives (A'_*, χ_*). Here a_F is the smallest subset of $\{a_1, a_2, \cdots\}$ necessary for the manipulations which follow. It will be clear that a_F is finite.

Let $\nu: F \to \bar{G}$ denote the composite $F \to G \to \bar{G}$ of canonical projections. Define $\nu_*: F_* \to \bar{G}_*$ similarly. Pick a representative p_l in $\nu^{-1}\bar{\rho}^{-1}\nu_*(y_l)$ and add a new generator y_l and a new relator $y_l p_l^{-1}$ for each l, $l=1, \cdots, t$, obtaining

$$(x_1, \cdot, x_m, y_1, \cdot, y_t: r_1, \cdot, r_n, y_1 p_1^{-1}, \cdot, y_t p_t^{-1}, a_F) \qquad \cdots\cdots\cdots(1).$$

Let (A_1, χ_1) be the result of applying the induced elementary transformations to (A', χ). The epimorphism χ_1 maps from the free group on $\{x_1, \cdot, x_m, y_1, \cdot, y_t\}$ onto H, and is defined by $\chi_1(x_j)=\chi(x_j)$, $j=1, \cdots, m$; $\chi_1(y_l)=\chi(p_l)$, $l=1, \cdots, t$. For all Tietze transformations, hence all induced elementary transformations, are of type (II). The matrix A_1 is the Alexander matrix of the presentation (1) at χ_1.

Now $\{\bar{\rho}^{-1}\nu_*(y_l): l=1, \cdots, t\}$ generates $\bar{G}$. Hence $\nu(x_j)=\bar{\rho}^{-1}\nu_*(q_j)$ for some word q_j in F_*. Using the yp^{-1} type relators, q can be rewritten as a word w_j in F. Then $x_j w_j^{-1}$ is in $\operatorname{Ker}\nu$, the consequence of $\{r_1, \cdot, r_n, a_1, a_2, \cdots\}$, $j=1, \cdots, m$. But the number of times elements of $\{a_1, a_2, \cdots\}$ appear in $x_j w_j^{-1}$ is finite, whence there is a finite subset of $\{a_1, a_2, \cdots\}$ (assumed to be in a_F) which together with $r_1, \cdots, r_n$ have $x_j w_j^{-1}$ in its consequence, $j=1, \cdots, m$. Use this, together with the yp^{-1} type relators, to add in relators $x_j q_j^{-1}$, $j=1, \cdots, m$. Moreover, s_k can be rewritten as a word v_k in F, and v_k is easily seen to be in $\operatorname{Ker} \nu$. Use this, together with the arguments above, to add in the relators s_k, $k=1, \cdots, u$. The result is the presentation

$$(x, y: r, s, yp^{-1}, xq^{-1}, a_F) \qquad \cdots\cdots\cdots(2)$$

where we have suppressed the subscripts of the generators and relators. Let (A_2, χ_2) be the result of applying the induced elementary transformations to (A_1, χ_1). The epimorphism χ_2 is equal to χ_1, since all Tietze transformations are of type (I). The matrix A_2 is the Alexander matrix of the presentation (2) at χ_2.

Reversing the roles of G and G_*, it follows that $(y_1, \cdots, y_t: s_1, \cdots, s_u, b_F)$ is equivalent to

$$(x, y: r, s, yp^{-1}, xq^{-1}, b_F) \qquad \cdots\cdots\cdots(3).$$

Here b_F is the smallest subset of $\{b_1, b_2, \cdots\}$ necessary for the proof. It will be clear b_F is finite. Let (A_3, χ_3) be the result of applying the induced elementary transformations to (A'_*, χ_*). The epimorphism χ_3 from the free group on $\{x_1, \cdot, x_m, y_1, \cdot, y_t\}$ onto H is defined by $\chi_3(x_j)=\chi_*(q_j)$, $j=1, \cdots, m$, $\chi_3(y_l)=\chi_*(y_l)$, $l=1, \cdots, t$. The matrix A_3 is the Alexander matrix of the presentation (3) at χ_3. The arguments used to add the relators s_k, $k=1, \cdots, u$, to (1) are easily adapted to first add b_F to (2) and then add the resulting a_F to (3). The result in both cases is a presentation of the form

$$(x, y: r, s, yp^{-1}, xq^{-1}, a_F, b_F).$$

All transformations are of type (1), so if $\chi_2=\chi_3$, (A_2, χ_2) is seen to be equivalent to (A_3, χ_3), and the proof will be complete. But by associate equality

$$\chi_2(x_j) = \chi_1(x_j) = \chi(x_j) = \bar{\Psi}\nu(x_j) = \bar{\Psi}\bar{\rho}^{-1}\nu_*(q_j) = \chi_*(q_j) = \chi_3(x_j), \quad j=1, \cdots, m,$$

and

$$\chi_2(y_l) = \chi_1(y_l) = \chi(p_l) = \bar{\Psi}\nu(p_l) = \bar{\Psi}\bar{\rho}^{-1}\nu_*(y_l) = \chi_*(y_l) = \chi_3(y_l), \quad l=1, \cdots, u.$$

This completes the proof.

In Lemma 1.6, we showed that if H is abelian, A is a matrix over JH such that A° is a relation matrix for H, and G is a group realizing A w.r.t. χ, then G has A as its Alexander matrix in the pseudo Fox sense w.r.t. χ. It is easy to see that the equivalence relation on satisfactory pairs is such that the matrices A° and A_1° obtained from equivalent pairs (A, χ_1) and (A, χ_1) are relation matrices of isomorphic groups. So if the matrix of one pair in an equivalence class is a pseudo Fox Alexander matrix with respect to its epimorphism, all are. We claim that in such an equivalence class, the associate has group equal to G/G''. This will follow if $G^{++}=G''$. But if A has G as its Alexander matrix in the pseudo Fox sense w.r.t. χ, $\operatorname{Ker}\lambda=\operatorname{Ker}\chi$. (Section 1.) Then $[\operatorname{Ker}\lambda, \operatorname{Ker}\lambda]=[\operatorname{Ker}\chi, \operatorname{Ker}\chi]$, and $G''=[\operatorname{Ker}\lambda, \operatorname{Ker}\lambda]=[\operatorname{Ker}\chi, \operatorname{Ker}\chi]=G^{++}$, as required.

3. Examples

EXAMPLE 1. Let F be the free group on $\{x_1, x_2, x_3\}$. Let $\chi\colon F\to\langle t\rangle$ be defined by $x_j\to t$, $j=1, 2, 3$, and $\tilde{\chi}$ be the extension of to group rings. Set

$$\begin{aligned}\tilde{f}_1&=-4t^{-1}+1+3t+5t^3,\\ \tilde{f}_2&=2t^{-1}-2t-5t^3,\\ \tilde{f}_3&=2t^{-1}+t^2.\end{aligned}$$

Then $\tilde{f}_1+\tilde{f}_2+\tilde{f}_3=1+t+t^2$, and $\sum_{j=1}^3 \tilde{f}_j(t-1)=t^3-1$. By Theorem I there is no group realizing $A=(\tilde{f}_1\tilde{f}_2\tilde{f}_3)$ w.r.t. χ. But by Lemma 1.1, $\Delta(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ is non empty; we use the method of proof to construct an element. Set

$$\begin{aligned}w&=x_1^3,\\ f_1^*&=-4x_1^{-1}+1+3x_1+5x_1^3,\\ f_2^*&=2x_1^{-1}-2x_1-5x_1^3,\\ f_3^*&=2x_1^{-1}+x_1^2.\end{aligned}$$

Then $f=\sum_{j=1}^3 f_j^*(x_j-1)+1=-4+2x_1^2+5x_1^4+2x_1^{-1}x_2-2x_1x_2-5x_1^3x_2+2x_1^{-1}x_3+x_1^2x_3$. $\tilde{\chi}(\partial f/\partial x_j)=\tilde{f}_j$, $j=1$ 2, 3. Further, $f-w^*$ is in $\operatorname{Ker}\chi$, whence by [3], p. 549, we can write

$$f-w^*=\sum_{k=1}^{l}\varepsilon_k d_k(s_k-1)e_k=\sum_{k=1}^{l}(r_k-1)c_k,$$

where $\varepsilon_k=\pm1$; d_k, e_k, and c_k are in F; s_k, r_k are in $\operatorname{Ker}\chi$, and $r_k=d_k s_k^{\varepsilon_k} d_k^{-1}$, and $c_k=d_k s^{(1-\varepsilon_k)/2}e_k$. Let $w=(\prod_{k=1}^{l} r_k)w^*$. Then w is in $\Delta(f_1, f_2, f_3)$; this is the essence of the proofs of Lemma 1.1 and [3, 4.10]. A method for determining the r_k is to be found in [3], p. 549; we write:

$$\begin{aligned}-4+2x_1^{-1}x_2+2x_1^{-1}x_3&=2(x_1^{-1}x_2-1)+2(x_1^{-1}x_3-1),\\ 2x_1^2-2x_1x_2&=2(x_1^2x_2^{-1}x_1^{-1}-1)x_1x_2,\\ x_1^2x_3-x_1^3&=(x_1^2x_3x_1^{-3}-1)x_1^3,\\ 5x_1^4-5x_1^3x_2&=5(x_1^4x_2^{-1}x_1^{-3}-1)x_1^3x_2,\end{aligned}$$

and put

$$w=(x_1^{-1}x_2)^2(x_1^{-1}x_3)^2(x_1^2x_2^{-1}x_1^{-1})^2(x_1^2x_3x_1^{-3})(x_1^4x_2^{-1}x_1^{-3})^5(x_1^3)\,.$$

(order is immaterial among the r_k's.) $\tilde{\chi}(\partial w/\partial x_j)=\tilde{f}_j$, $j=1, 2, 3$.

EXAMPLE 2. Let F be the free group on $\{x_1, x_2\}$. Let $\chi\colon F\to F/[F, F]$ be the abelianizer, let $\chi(x_1)=\tilde{x}_1$, $\chi(x_2)=\tilde{x}_2$, and let $\tilde{\chi}$ be the extension of χ to group rings. Set

$$\tilde{f}_1=(2\tilde{x}_1^2\tilde{x}_2-4\tilde{x}_2^{-2})(1-\tilde{x}_2)\,,$$
$$\tilde{f}_2=(2\tilde{x}_1^2\tilde{x}_2-4\tilde{x}_2^{-2})(\tilde{x}_1-1)\,.$$

Then $\tilde{f}_1(\tilde{x}_1-1)+\tilde{f}_2(\tilde{x}_2-1)=0$, so by Theorem I there is a group G realizing $A=(\tilde{f}_1\ \tilde{f}_2)$ w.r.t. χ. Further, since $A^\circ=(0\ 0)$ is a relation matrix for $F/[F, F]$, it follows by Lemma 1.6 that G has A as its Alexander matrix in the pseudo Fox sense w.r.t. χ. To find a G realizing A w.r.t. χ, set

$$\begin{aligned} w^* &= 1\\ f_1^* &= -4x_2^{-2}+4x_2^{-1}+2x_1^2x_2-2x_1^2x_2^2\\ f_2^* &= 2x_1^3x_2-4x_1x_2^{-2}-2x_1^2x_2-4x_2^{-2}\,. \end{aligned}$$

Then $f=\sum_{j=1}^2 f_j^*(x_j-1)+1=-4x_2^{-2}x_1+4x_2^{-1}x_1+2x_1^2x_2x_1-2x_1^2x_2^2x_1+2x_1^3x_2^2-4x_1x_2^{-1}+4x_1x_2^{-2}-2x_1^3x_2+1$. Proceeding as in the previous example, we write

$f-1=4(x_1x_2^{-2}x_1^{-1}x_2^2-1)x_2^{-2}x_1+4(x_2^{-1}x_1x_2x_1^{-1}-1)x_1x_2^{-1}+2(x_1^2x_2x_1x_2^{-1}x_1^{-3}-1)x_1^3x_2+2(x_1^3x_2^2x_1^{-1}x_2^{-2}x_1^{-2}-1)x_1^2x_2^2x_1$, and set

$$w=(x_1x_2^{-2}x_1^{-2}x_2^2)^4(x_2^{-1}x_1x_2x_1^{-1})^4(x_1^2x_2x_1x_2^{-1}x_1^{-3})^2(x_1^3x_2^2x_2^{-1}x_1^{-1}x_1^{-2})^2\,.$$

The group G presented by $(x_1, x_2\colon w)$ has A as its pseudo Fox Alexander matrix w.r.t. χ.

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