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Introduction

Let \((x_1, \ldots, x_m; r_1, \ldots, r_n)\) be a presentation of a group \(G\). Then an Alexander matrix of \(G\) can be obtained by mapping the \(n \times m\) matrix \((\partial r_i/\partial x_j)\) into a matrix with coefficients in the group ring \(JH\) of some homomorphic image \(H\) of \(G\). (We are using \(i\) for the row index and \(j\) for the column index. Moreover, what we call Alexander matrices are called in Fox [4] 'Homomorphisms of the Jacobian'.) In this note, we consider the reverse of the above procedure. We start with a matrix \(A\) over a group ring, and look for groups with an Alexander matrix equal to \(A\).

Let \(F\) be the free group on the set of \(m\) letters \(\{x_1, \ldots, x_m\}\), and \(JF\) be the integral group ring on \(F\). Let \(\chi: F \twoheadrightarrow H\) be an epimorphism from \(F\) onto a group \(H\), and let \(\bar{\chi}: JF \twoheadrightarrow JH\) be the extension of \(\chi\) to group rings. Then for an \(n \times m\) matrix \(A\) with entries \(f_{ij}\) over \(JH\), if \(G\) is such that \(\phi: G \rightarrow H\) commutes and \((\partial r_i/\partial x_j)\bar{\chi} = A\), we say \(G\) realizes \(A\) w.r.t. \(\chi\). Here \(\phi\) is the canonical projection, and \(\psi\) is the epimorphism induced by \(\chi\). Let \(R\) denote \(\text{Ker } \chi\). We show

**Theorem I.** Given an \(n \times m\) matrix \(A\) with entries \(f_{ij}\) over \(JH\), there is a group \(G\) realizing \(A\) w.r.t. \(\chi\) iff \(\sum_{j=1}^{m} f_{ij}\bar{\chi}(x_j-1) = 0\), \(i = 1, \ldots, n\). Further if the entries of \(A\) satisfy this condition and \(G\) is a group with presentation \((x_1, \ldots, x_m; r_1, \ldots, r_n)\) such that \((\partial r_i/\partial x_j)\bar{\chi} = A\), the collection of all groups realizing \(A\) w.r.t \(\chi\) is

\[
\{(x_1, \ldots, x_m; a_1r_1, \ldots, a_nr_n)|a_1, \ldots, a_n \in [R, R]\}.
\]

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Thus ‘up to \([R, R]\)’, groups realizing \(A\) w.r.t. \(\chi\) are unique, a result in effect established in Crowell [1] (by a different method) and attributed there to Blanchfield.

For the proof, we consider the set \(\Delta(f_1, \ldots, f_m)\) defined with respect to \(f_1, \ldots, f_m\) in \(JH\) by
\[
\Delta(f_1, \ldots, f_m) = \{w \in F : \bar{\chi}(\partial w/\partial x_j) = f_j, j = 1, \ldots, m\}.
\]
We also use the following condition (\(\ast\)):
\[
\sum_{j=1}^m f_j \bar{\chi}(x_j - 1) = h - 1_{JH}, \quad \text{for some } h \in H
\]
and show

**Theorem II.** \(\Delta(f_1, \ldots, f_m)\) is non empty iff \(f_1, \ldots, f_m\) satisfy (\(\ast\)), in which case, for \(w \in \Delta(f_1, \ldots, f_m)\), \(\Delta(f_1, \ldots, f_m) = w[R, R]\).

As an immediate corollary, we give a description of
\[
\Theta(A) = \{(w', \ldots, w^n) \in F^n : (\partial w'/\partial x_j)^x = A\}.
\]

When a group \(G\) realizing \(A\) w.r.t. \(\chi\) satisfies a certain condition, we say that \(A\) is the pseudo Fox Alexander matrix of \(G\) w.r.t. \(\chi\). (See section 1) We give necessary and sufficient conditions on \(A\) for \(A\) to be the pseudo Fox Alexander matrix of some group w.r.t \(\chi\).

In order to compare matrices of different size, we introduce the concept of a satisfactory (matrix, homomorphism) pair \((A, \chi)\), where \((A, \chi)\) is satisfactory iff \(A\) can be realized by some group w.r.t. \(\chi\). To every satisfactory pair there is uniquely associated a (group, homomorphism) pair we call the associate. We define an equivalence relation in the spirit of [4] between satisfactory pairs, and an equivalence relation between associates, such that equivalent satisfactory pairs have equivalent associates, and satisfactory pairs with equivalent associates are equivalent. Further, we consider satisfactory pairs \((A, \chi)\) such that \(A\) is the pseudo Fox Alexander matrix of some group \(G\) w.r.t. \(\chi\), and show that in this case the associate of \((A, \chi)\) has group \(G/G''\), where \(G''\) is the second commutator of \(G\). In the special case \(H\) is the trivial group, the uniqueness of \(\chi\) renders the concept of a pair redundant. The satisfactory pairs are effectively all matrices \(A\) over \(JH = J\). The associate to \(A\) turns out to be the abelian group with relation matrix \(A\). Moreover, the equivalence relation on satisfactory pairs reduces to the usual equivalence relation on relation matrices of abelian groups, and the equivalence relation on associates is that of group isomorphism. We have thus generalized the well known abelian group—integral matrix correspondence.

In some cases, it is a simple matter to determine whether polynomials \(f_1, \ldots, f_m\) satisfy (\(\ast\)). Good examples are the abelianizer \(F \to F/[F, F]\), and the epimorphism \(F \to \langle t \rangle\) onto the free group on one element defined by \(x_j \to t\).
for all $j$. Here $\text{Ker} \chi$ equals \{$w \in F$: exponent sum over each generator is zero\} (Lyndon [5], corollary 4.2), and \{$w \in F$: exponent sum equals zero\}, respectively. Moreover, in the proof of sufficiency of (*)& for $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$ to be non empty, we explicitly construct an element $w$ in $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$.

Section 1 contains the proof of the main result and corollaries. Section 2 deals with satisfactory pairs and their associates. In section 3 we give some examples of the construction of a $w$ in $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$ for $\bar{f}_1, \ldots, \bar{f}_m$ satisfying (*). It gives me much pleasure to thank my supervisors Professor J. Tao and Professor A. Kawauchi, for all their help and encouragement.

1. The main results

Suppose there is a $w \in F$ such that $\tilde{\chi}(\partial w/\partial x_j) = \bar{f}_j$, $j = 1, \ldots, m$. By the fundamental formula (Fox [3], 2.3), we have

$$\sum_{j=1}^m (\partial w/\partial x_j)(x_j - 1) = w - 1.$$  

Then applying $\tilde{\chi}$,

$$\sum_{j=1}^m \bar{f}_j \tilde{\chi}(x_j - 1) = \tilde{\chi}(w - 1) = \chi(w) - 1_{JH}.$$  

Noting that $\chi(w) \in H$, condition (*) in the introduction is seen to be necessary for $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$ to be non empty. We proceed to show sufficiency.

**Lemma 1.1.** Let $\bar{f}_1, \ldots, \bar{f}_m$ be elements of $JH$, and $h$ be an element of $H$, such that $\sum_{j=1}^m \bar{f}_j \tilde{\chi}(x_j - 1) = h - 1_{JH}$. Then there is a word $w \in F$ such that $\chi(w) = h$, and $\tilde{\chi}(\partial w/\partial x_j) = \bar{f}_j$, $j = 1, \ldots, m$.

**Proof.** Let $R$ denote $\text{Ker} \chi$. Let $w^* \in F$ be an element of $\chi^{-1}(h)$. (recall $\chi$ is assumed to be onto.) Let $f^*$ be representatives of $\chi^{-1}(\bar{f}_j)$, $j = 1, \ldots, m$, and set $f = \sum_{j=1}^m f_j(x_j - 1) + 1$. Then $\tilde{\chi}(\partial f^*/\partial x_j) = \bar{f}_j$, $j = 1, \ldots, m$. For $\partial(f^*(x_j - 1))/\partial x_j = f^* \delta_{jj}$, for any $f^* \in JF$. Further, $\tilde{\chi}(f - w^*) = (\sum_{j=1}^m \bar{f}_j \tilde{\chi}(x_j - 1) + 1) - h = 0$. So $f - w^* \in R$. Hence $f w^* - 1 \in R$, since $\mathcal{X}$ is an ideal. Let denote the fundamental ideal ([3]) of $JF$. By [3, 4.10], there is an element $r \in R$ such that $f w^* - r \in \mathcal{R} \mathcal{X}$; moreover, since $\mathcal{R} \mathcal{X}$ is an ideal, we have $f - r w^* \in \mathcal{R} \mathcal{X}$. Set $w = r w^* \in F$. Then $\chi(w) = h$, and by [3, 4.5], $\tilde{\chi}(\partial(f - w)/\partial x_j) = 0$, or $\tilde{\chi}(\partial w/\partial x_j) = \tilde{\chi}(\partial f/\partial x_j) = \bar{f}_j$, $j = 1, \ldots, m$, as required.

**Corollary 1.2.** $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$ is non empty iff $\bar{f}_1, \ldots, \bar{f}_m$ satisfy (*).

The proof is immediate.

We now turn to the question of structure in $\Delta(\bar{f}_1, \ldots, \bar{f}_m)$.

**Lemma 1.3.** For $a \in [R, R]$, $a w \in \Delta(\bar{f}_1, \ldots, \bar{f}_m)$ whenever $w \in \Delta(\bar{f}_1, \ldots, \bar{f}_m)$.

**Proof.** By [3, 4.9], the ideal $\mathcal{R} \mathcal{X}$ determines the commutator subgroup
Lemma 1.3 implies that $w[R, R] = [R, R]w \subseteq \Delta(f_1, \ldots, f_m)$ for $w \in \Delta(f_1, \ldots, f_m)$.

**Lemma 1.4.** Any two elements $w, w' \in \Delta(f_1, \ldots, f_m)$ differ by an element of $[R, R]$, so that $w[R, R] = [R, R]w = \Delta(f_1, \ldots, f_m)$.

**Proof.** By assumption, $\tilde{\chi}(\partial w/\partial x_j) = \tilde{\chi}(\partial w'/\partial x_j)$, $j = 1, \ldots, m$. Hence $\tilde{\chi}(\partial (w-w')/\partial x_j) = 0$, $j = 1, \ldots, m$. Then, by [3, 4.5], $w-w' \in \mathcal{R}$. But $w-w'$ in $\mathcal{R}$ implies $ww^{-1} \in \mathcal{R}$, since $\mathcal{R}$ is an ideal. So by [3, 4.9], $ww^{-1} \in [R, R]$, and the proof is complete.

**Proof of Theorem II.** Corollary 1.2 and Lemmas 1.3 and 1.4.

**Corollary 1.5.** Given an $n \times m$ matrix $A$ with entries $f_{ij}$ over $\mathcal{F}$, $\Theta(A) = \prod_{i=1}^{n} \Delta(f_1, \ldots, f_m)$.

**Proof of Theorem I.** Assume $G$ is a group realizing $A$ w.r.t. $\chi$, with presentation $(x_1, \ldots, x_n; r_1, \ldots, r_n)$ such that $(\partial r_i/\partial x_j)\tilde{\chi} = A$. By the fundamental formula $r_i = \sum_{j=1}^{n} (\partial r_i/\partial x_j)(x_j-1)$, $i = 1, \ldots, n$. Hence $\tilde{\chi}(r_i) = \sum_{j=1}^{n} \tilde{\chi}(\partial r_i/\partial x_j)(x_j-1)$, $i = 1, \ldots, n$. But by our assumption on $G$, $\chi(r_i) = 1$, and we have $0 = \sum_{j=1}^{n} \tilde{\chi}(\partial r_i/\partial x_j)(x_j-1) = \sum_{j=1}^{n} f_{ij}\tilde{\chi}(x_j-1)$, $i = 1, \ldots, n$, as required.

Conversely, suppose the entries of $A$ satisfy $\sum_{j=1}^{n} f_{ij}\tilde{\chi}(x_j-1) = 0$, $i = 1, \ldots, n$. Since this is just $(\ast)$ with $h=1$, Theorem II gives elements $r_i \in \Delta(f_1, \ldots, f_m)$, $i = 1, \ldots, n$. Using the definition of $\Delta$ and the fundamental formula, we see

$$0 = \sum_{j=1}^{n} f_{ij}\tilde{\chi}(x_j-1) = \sum_{j=1}^{n} \tilde{\chi}(\partial r_i/\partial x_j)(x_j-1) = \tilde{\chi}(r_i) - 1,$$

whence $\chi(r_i) = 1$, $i = 1, \ldots, n$. It is now easy to see the group $G$ presented by $(x_1, \ldots, x_n; r_1, \ldots, r_n)$ realizes $A$ w.r.t. $\chi$. That all groups $G^*$ realizing $A$ w.r.t. $\chi$, with presentation $(x_1, \ldots, x_n; r_1^*, \ldots, r_n^*)$ such that $(\partial r_i^*/\partial x_j)\tilde{\chi} = A$ have the stated form follows from Corollary 1.5 and Theorem II, after noting that $(r_1^*, \ldots, r_n^*) \subseteq \Theta(A)$. This completes the proof.

Let $\lambda: F \rightarrow [G, G]$ be the composite $F \rightarrow G \rightarrow G/[G, G]$, and $\tilde{\lambda}$ be its extension to group rings. Fox ([4, §4]) calls the $n \times m$ matrix $(\partial r_i/\partial x_j)\tilde{\chi}$ an Alexander matrix of $G$. This leads to the question of when, for a group $G$ realizing $A$ w.r.t. $\chi$, there is an isomorphism $\sigma: G/[G, G] \rightarrow H$ such that the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\chi} & H \\
\lambda \downarrow & & \sigma \\
G/[G, G] & & 
\end{array}
$$
commutes. When such a diagram exists, we say that \( G \) has \( A \) as its \textit{Alexander matrix in the pseudo Fox sense} w.r.t. \( \chi \), or that \( A \) is the \textit{pseudo Fox Alexander matrix} of \( G \) w.r.t. \( \chi \).

\textbf{Lemma 1.6.} Let \( A \) be an \( n \times m \) matrix with entries \( \frac{\partial f_i}{\partial x_j} \) over \( JH \), and \( G \) be group realizing \( A \) w.r.t. \( \chi \), with presentation \( \langle x_1, \ldots, x_n; r_1, \ldots, r_s \rangle \) such that \( \frac{\partial r_i}{\partial x_j} \chi = A \). Then \( G \) has \( A \) as its Alexander matrix in the pseudo Fox sense w.r.t. \( \chi \) iff \( H \) is abelian and \( A^o \) is a relation matrix for \( H \), where \( \circ \) denotes the \textit{trivializer} \( JH \to J \).

Proof. Observe that \( A^o = (\frac{\partial f_i}{\partial x_j})^o = (\frac{\partial r_i}{\partial x_j})^o \). Suppose that \( G \) has \( A \) as its Alexander matrix in the pseudo Fox sense w.r.t. \( \chi \). Then \( G/\langle G, G \rangle \simeq H \). \( H \) is therefore abelian, and by [4, 3.5], \( A^o \) is a relation matrix for \( H \). This proves sufficiency.

Conversely, suppose \( H \) is abelian and \( A^o \) is a relation matrix for \( H \). Since \( H \) is abelian, there is an epimorphism \( \sigma : G/\langle G, G \rangle \to H \) such that

\[
\begin{array}{ccc}
F & \xrightarrow{\chi} & H \\
\downarrow{\phi} & & \downarrow{\sigma} \\
G & \xrightarrow{\psi} & G/\langle G, G \rangle
\end{array}
\]

commutes. Moreover, by [4, 3.5], \( A^o \) is also a relation matrix for \( G/\langle G, G \rangle \), whence \( G/\langle G, G \rangle \simeq H \). But any homomorphism from a finitely generated group onto itself is an isomorphism, from which we deduce \( \sigma \) to be an isomorphism. This completes the proof.

2. Satisfactory pairs and their associates

We say that the pair \( (A, \chi) \) is \textit{satisfactory} when there exists a group \( G \) realizing \( A \) w.r.t. \( \chi \). Recall that if \( G \) is a group realizing \( A \) w.r.t. \( \chi \), there is a diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\chi} & H \\
\downarrow{\phi} & & \\
G
\end{array}
\]

Define the subgroup \( G^{++} \) of \( G \) to be \( \langle \phi R, \phi R \rangle \). Since \( G^{++} \subset \text{Ker} \psi \), \( \psi \) induces an epimorphism \( \bar{\psi} : G/G^{++} \to H \). Let \( \bar{G} = G/G^{++} \). Then

\textbf{Theorem 3.1.} The pair \( (\bar{G}, \bar{\psi}) \) is determined uniquely by the satisfactory pair \( (A, \chi) \).
Proof. The quotient $\overline{G}$ has a presentation which may be obtained from a presentation of $G$ by adding $\{a: a\in [R, R]\}$ as relators. The proof now follows from the description of all groups realizing $A$ w.r.t. $\chi$ given in Theorem I. $(\overline{G}, \overline{\psi})$ is called the associate of $(A, \chi)$. The associates $(\overline{G}, \overline{\psi})$ and $(\overline{G}_*, \overline{\psi}_*)$ of satisfactory pairs $(A, \chi)$ and $(A_*, \chi_*)$ are said to be equal in case there is an isomorphism $\rho: \overline{G}\rightarrow \overline{G}_*$ such that

\[
\begin{array}{ccc}
\overline{G} & \xrightarrow{\overline{\psi}} & H \\
\downarrow\rho & & \downarrow\overline{\psi}_* \\
\overline{G}_* & \xrightarrow{\overline{\psi}_*} & H
\end{array}
\]

commutes. Equality is the equivalence relation between associates mentioned in the introduction. We define an equivalence relation between satisfactory pairs as follows:

**DEFINITION** (compare [4], p. 199.). Two satisfactory pairs are equivalent if one can be obtained from the other by a finite number of elementary transformations (I), (II), (I)$^{-1}$, (II)$^{-1}$, defined as follows:

(I) Replace $(A, \chi)$ by $(A', \chi')$, where $A'$ is obtained from $A$ by adjoining a new row equal to a left linear combination of the rows of $A$, and $\chi'=\chi$.

(II) Replace $(A, \chi)$ by $(A', \chi')$, where $A'$ is the result of adjoining to $A$ a new row and a new column (say the $p$th and $q$th respectively) such that:

(a) The entry in the intersection of the row and column is 1.

(b) The remaining entries in the new column are all 0.

(c) The remaining entries $f_p^q, \ldots, f_{p+1}^q$ in the new row satisfy $-(*')$; that is

\[
\sum_{j=1}^{m+1} f_p^q x_j - 1 = -h + 1_{f,j}^{H},
\]

for some $h\in H$.

and the epimorphism $\chi'$ from the free group $F'$ on $m+1$ letter $\{x_1, \ldots, x_q, \ldots, x_{m+1}\}$ onto $H$ defined by $\chi'(x_j) = \chi(x_j), f=1, \ldots, q, \ldots, m+1; \chi'(x_q) = h$.

(I)$^{-1}$ The inverse operation to (I).

(II)$^{-1}$ Replace $(A, \chi)$ by $(A', \chi')$, where $A'$ is obtained from $A$ by removal of the $p$th row and the $q$th column, and $\chi'$ is the restriction of $\chi$ to the free group $F'$ generated by $\{x_1, \ldots, x_q, \ldots, x_{m}\}$. Here $A$ satisfies

(a) The entry in the intersection of the $p$th row and the $q$th column is 1.

(b) The remaining entries in the $q$th column are all 0.

If $(A, \chi)$ is satisfactory, by Theorem I $\sum_{j=1}^{m} f_p^q \chi(x_j - 1) = 0$. If in addition $A$ satisfies condition (a) of (II)$^{-1}$,

\[
\sum_{j=1}^{m} f_p^q \chi(x_j - 1) = 1 - \chi(x_q).
\]

So $f_p^q, \ldots, f_{q-1}^q, \ldots, f_m^q$ satisfy $-(*')$ for $h=\chi(x_q)$. This together with the next
lemma justifies the label \((\Pi)^{-1}\).

Since the interchange of any two rows or any two columns is easily seen to be an elementary transformation, we shall assume throughout the rest of this paper that the \(p\)th row and \(q\)th column are the bottom row and extreme right hand column respectively.

**Lemma 2.2.** The result of applying (I), (II), \((\Pi)^{-1}\), or \((\Pi)^{-1}\) to a satisfactory pair is again a satisfactory pair.

Proof. For the elementary transformations (I), \((\Pi)^{-1}\), and (II), this is clear. In the case of \((\Pi)^{-1}\) we must show the map \(\chi'\) from \(F'\) into \(H\) is onto, or \(\chi(x_m)\) can be written in terms of \(\chi(x_1), \ldots, \chi(x_{m-1})\). By Lemma 1.1, there is a word \(w \in F\) such that \(\chi(w) = 1\), and \(\tilde{\chi}(\partial w/\partial x_j) = \delta_j^\rho\), \(j = 1, \ldots, m\). In the special case the number of times \(x_m\) occurs in \(w\) is one, the relation \(\chi(x_m) = 1\) shows \(\chi(x_m)\) is expressible in terms of \(\chi(x_1), \ldots, \chi(x_{m-1})\), and \(\chi'\) is onto. So suppose the number of times \(x_m\) occurs in \(w\) is \(p > 1\). As \(\tilde{\chi}(\partial w/\partial x_m) = 1\), \(p\) is odd, and among the \(p\) terms in this derivative, there are \((p-1)/2\) cancelling pairs corresponding to particular occurrences of \(x_m\) and \(x_m^{-1}\) in \(w\). Focussing on one pair, write \(w = ax_m^p bx_m^{-p}c\), \(\varepsilon = \pm 1\), and note the assumption

\[
\tilde{\chi}[(a-ax_m^p bx_m^{-p})(\partial x_m^p/\partial x_m)] = 0
\]

implies \(b \in \text{Ker } \chi\). Consequently \(v_1 = abc\) is such that \(\chi(v_1) = 1\), and \(\tilde{\chi}(\partial v_1/\partial x_m) = 1\). Moreover, \(v_1\) has fewer cancelling pairs by one. Repeating this argument enough times brings us back to the special case, and \(\chi'\) is seen to be onto. This completes the proof.

**Theorem 2.3.** Equivalent satisfactory pairs have equal associates.

Proof. Let \((A, \chi)\) be a satisfactory pair, and \(G\) be a group realizing \(A\) w.r.t. \(\chi\) with presentation \((x_1, \ldots, x_m: r_1, \ldots, r_n)\) such that \((\partial r_i/\partial x_j)^\chi = A\). By Lemma 2.2, it suffices to show the satisfactory pair obtained from \((A, \chi)\) by applying any one of (I), (II), \((\Pi)^{-1}\), or \((\Pi)^{-1}\) has associate equal to that of \((A, \chi)\). For this, let \(G'\) be any group realizing \(A'\) w.r.t. \(\chi'\). Let \(\phi': F' \to G'\) be canonical projection, and \(\psi': G' \to H\) be the epimorphism induced by \(\chi'\). We find a \(G'\) and an isomorphism \(\rho: G \to G'\) such that

\[
\text{commutes. Then } \rho \phi \text{ Ker } \chi = \phi' \text{ Ker } \chi', \text{ and } \rho \text{ induces an isomorphism } \rho: G \to G' \text{ such that}
\]
commutes. Hence this suffices to show \((A, \chi)\) and \((A', \chi')\) have equal associates. The isomorphism \(\rho: G \rightarrow G'\) is defined as follows. A group \(G'\) realizing \(A'\) w.r.t. \(\chi'\) can be obtained by applying a Tietze transformation ([4], p. 197) to \((x_1, \ldots, x_m; r_1, \ldots, r_n)\) of the same type as the type of the elementary transformation used to obtain \((A', \chi')\) from \((A, \chi)\). For an elementary transformation of type (I) or (I)', this follows from [4], p. 199. For a type (II) elementary transformation, by Lemma 1.1, there is a \(w^F\) such that \(\chi(w) = h = \chi(x_{m+1})\), and \(\tilde{\chi}(\partial w/\partial x_j) = -\tilde{f}'_{j+1}, j = 1, \ldots, m\). But then

\[
\tilde{\chi}'((x_{m+1}w^{-1})(\partial w/\partial x_j)) = \tilde{f}'_j, j = 1, \ldots, m.
\]

So we may take \(G'\) to be the group presented by \((x_1, \ldots, x_m, x_{m+1}; r_1, \ldots, r_n, x_{m+1}w^{-1})\). For a type (II)' elementary transformation, recall that in Lemma 2.2 we showed \(\chi(x_1), \ldots, \chi(x_{m-1})\) generate \(H\). Hence there is a word \(w \in F'\) such that \(\chi(w) = \chi(x_m)\), and \(\tilde{\chi}(\partial w/\partial x_j) = \tilde{f}'_j, j = 1, \ldots, m-1\). We may thus assume the presentation of \(G\) is \((x_1, \ldots, x_m; r_1, \ldots, r_{m-1}, x_mw^{-1})\). The isomorphism \(\rho: G \rightarrow G'\) induced by the Tietze transformation is easily seen to satisfy diagram (a), and the proof is complete.

Next we establish the converse to this theorem.

**Theorem 2.4.** Satisfactory pairs with equal associates are equivalent.

Proof. Let \((A, \chi)\) and \((A_*, \chi_*)\) be satisfactory pairs, where \(A_*\) is a \(u \times t\) matrix over \(JH\) and \(\chi_*\) is an epimorphism from the free group \(F_*\) on \(t\) letters \(\{y_1, \ldots, y_t\}\) onto \(H\). Further, let \(G\) be a group realizing \(A\) w.r.t. \(\chi\), with presentation \((x_1, \ldots, x_m; r_1, \ldots, r_n)\) such that \((\partial r_i/\partial x_j)^{\chi} = A\); let \(G_*\) be a group realizing \(A_*\) w.r.t. \(\chi_*\), with presentation \((y_1, \ldots, y_t; s_1, \ldots, s_u)\) such that \((\partial s_k/\partial y_j)^{\chi_*} = A_*\); and let \(\tilde{\rho}: \tilde{G} \rightarrow \tilde{G}_*\) be an isomorphism such that

\[
\tilde{G} \quad \tilde{\Psi} \\
\downarrow \tilde{\rho} \quad \downarrow \tilde{\Psi}_* \\
\tilde{G}_* \quad \tilde{H}_*
\]

commutes. We must show \((A, \chi)\) and \((A_*, \chi_*)\) are equivalent. We begin with two observations. First, a Tietze transformation \(T\) applied to \((x_1, \ldots, x_m)\):
Let the result of applying \( T \) to \((x_1, \cdots, x_n: r_1, \cdots, r_n)\) be \((x_1', \cdots, x_n': r_1', \cdots, r_n')\). Denote the group presented by \((x_1', \cdots, x_n': r_1', \cdots, r_n')\) by \(G'\), and let \(p: G \rightarrow G'\) be the isomorphism induced by \(T\). Denote by \(\phi'\) the canonical projection from the free group \(F'\) on \(\{x_1', \cdots, x_n'\}\) onto \(G'\), and define the epimorphism \(\chi': F' \rightarrow H\) to be the composite

\[
F' \xrightarrow{\phi'} G' \xrightarrow{\psi} G \xrightarrow{\psi} H.
\]

Set \(A'=(\partial r_i/\partial x_i)^{x_i}, i=1, \cdots, p; j=1, \cdots, o\). Then it is easy to see that the satisfactory pair \((A', \chi')\) differs from \((A, \chi)\) by an elementary transformation \(T\). We say \(T\) was induced by \(\phi'\). Second, let \(\{a_1, a_2, \cdots\}\) and \(\{b_1, b_2, \cdots\}\) be subsets of \(F\) and \(F^*\) respectively which generate \([\ker \chi, \ker \chi]\) and \([\ker \chi_\#, \ker \chi_\#]\) respectively. Then \(\bar{G}\) is presented by \((x_1, \cdots, x_n, r_1, \cdots, r_n, a_1, a_2, \cdots)\) and \(\bar{G}_*\) is presented by \((y_1, \cdots, y_t: s_1, \cdots, s_n, b_1, b_2, \cdots)\). Moreover, \(\bar{\chi}(\partial a_i/\partial x_i)=0\), for all \(a_i \in \{a_1, a_2, \cdots\}\) and \(j=1, \cdots, m\), by Lemma 1.3 with \(\omega=1\), and \(f_1=\cdots=f_m=0\). So denoting by \(a_\Sigma\) any finite subset of \(\{a_1, a_2, \cdots\}\), the Alexander matrix \(A'\) of the presentation \((x_1, \cdots, x_n: r_1, \cdots, r_n, a_\Sigma)\) at \(\chi\) is just \(A\) with finitely many zero rows added. Hence \((A, \chi)\) and \((A', \chi')\) differ by an elementary transformation of type (I), so are equivalent. Similarly, \(X'(dbldy_i)=0\), for all \(b^{b_1, b_2, \cdots}\). And denoting by \(b_\Sigma\) any finite subset of \(\{b_1, b_2, \cdots\}\), the Alexander matrix \(A_*\) of the presentation \((y_1, \cdots, y_t: s_1, \cdots, s_n, b_\Sigma)\) at \(\chi_*\) is just \(A_*\) with finitely many zero rows added. Hence \((A_\#, \chi_\#)\) and \((A'_\#, \chi'_\#)\) are equivalent. Our method of proof is to give a finite sequence of Tietze transformations starting from \((x_1, \cdots, x_n: r_1, \cdots, r_n, a_\Sigma)\) such that the induced sequence of elementary transformations applied to \((A', \chi')\) gives \((A'_\#, \chi'_\#)\). Here \(a_\Sigma\) is the smallest subset of \(\{a_1, a_2, \cdots\}\) necessary for the manipulations which follow. It will be clear that \(a_\Sigma\) is finite.

Let \(v: F \rightarrow \bar{G}\) denote the composite \(F \rightarrow G \rightarrow \bar{G}\) of canonical projections. Define \(v_*: F_* \rightarrow \bar{G}_*\) similarly. Pick a representative \(p_i\) in \(v^{-1}p^{-1}p_*^{-1}(y_i)\) and add a new generator \(y_i\) and a new relator \(y_ip_i^{-1}\) for each \(l, l=1, \cdots, t\), obtaining

\[
(x_1, \cdots, x_n, y_1, \cdots, y_t: r_1, \cdots, r_n, y_ip_i^{-1}, a_\Sigma) \quad \cdots \quad (1).
\]

Let \((A, \chi)\) be the result of applying the induced elementary transformations to \((A', \chi')\). The epimorphism \(\chi_1\) maps from the free group on \(\{x_1, \cdots, x_n, y_1, \cdots, y_t\}\) onto \(H\), and is defined by \(\chi_1(x_j)=\chi(x_j), j=1, \cdots, m; \chi_1(y_l)=\chi(p_l), l=1, \cdots, t\). For all Tietze transformations, hence all induced elementary transformations, are of type (II). The matrix \(A_1\) is the Alexander matrix of the presentation (1) at \(\chi_1\).

Now \(\{p^{-1}p_*^{-1}(q)\} : l=1, \cdots, t\) generates \(\bar{G}\). Hence \(v(x_j)=p^{-1}p_*^{-1}(q)\) for some word \(q\) in \(F_*\). Using the \(yp^{-1}\) type relators, \(q\) can be rewritten as a word \(w_j\) in \(F\). Then \(x_jw_j^{-1}\) is in \(\ker v\), the consequence of \(\{r_1, \cdots, r_n, a_1, a_2, \cdots\}, j=1, \cdots, m\). But the number of times elements of \(\{a_1, a_2, \cdots\}\) appear in \(x_jw_j^{-1}\) is
finite, whence there is a finite subset of \{a_1, a_2, \ldots\} (assumed to be in \(a_F\)) which together with \(r_1, \ldots, r_s\) have \(x_i, w_i^{r_i^{-1}}\) in its consequence, \(j = 1, \ldots, m\). Use this, together with the \(yp^{-1}\) type relators, to add in relators \(x_i q_j^{-1}, j = 1, \ldots, m\). Moreover, \(s_j\) can be rewritten as a word \(v_k\) in \(F\), and \(v_k\) is easily seen to be in \(\text{Ker} \ \nu\). Use this, together with the arguments above, to add in the relators \(s_k, k = 1, \ldots, u\). The result is the presentation

\[(x, y: r, s, yp^{-1}, xq^{-1}, a_f)\] ............(2)

where we have suppressed the subscripts of the generators and relators. Let \((A_2, \chi_2)\) be the result of applying the induced elementary transformations to \((A, \chi)\). The epimorphism \(\chi_2\) is equal to \(\chi_1\), since all Tietze transformations are of type (I). The matrix \(A_2\) is the Alexander matrix of the presentation (2) at \(\chi_2\).

Reversing the roles of \(G\) and \(G_\ast\), it follows that \((y_1, \ldots, y_t: s_1, \ldots, s_u, b_F)\) is equivalent to

\[(x, y: r, s, yp^{-1}, xq^{-1}, b_F)\] ............(3).

Here \(b_F\) is the smallest subset of \{\(b_1, b_2, \ldots\}\} necessary for the proof. It will be clear \(b_F\) is finite. Let \((A_3, \chi_3)\) be the result of applying the induced elementary transformations to \((A_\ast, \chi_\ast)\). The epimorphism \(\chi_3\) from the free group on \{\(x_1, \ldots, x_m, y_1, \ldots, y_t\)\} onto \(H\) is defined by \(\chi_3(x_i) = \chi_\ast(q_j), j = 1, \ldots, m, \chi_3(y_i) = \chi_\ast(y_i), l = 1, \ldots, t\). The matrix \(A_3\) is the Alexander matrix of the presentation (3) at \(\chi_3\). The arguments used to add the relators \(s_k, k = 1, \ldots, u\), to (1) are easily adapted to first add \(b_F\) to (2) and then add the resulting \(a_F\) to (3). The result in both cases is a presentation of the form

\[(x, y: r, s, yp^{-1}, xq^{-1}, a_F, b_F).\]

All transformations are of type (1), so if \(\chi_2 = \chi_3\), \((A_2, \chi_2)\) is seen to be equivalent to \((A_3, \chi_3)\), and the proof will be complete. But by associate equality

\[\chi_3(x_i) = \chi_\ast(x_i) = \chi(x_i) = \frac{\nu}{\nu} \chi(x_i) = \frac{\nu}{\nu} \chi_\ast(q_j) = \chi_\ast(q_j) = \chi_\ast(x_i), j = 1, \ldots, m,\]

and

\[\chi_3(y_i) = \chi_\ast(y_i) = \chi(y_i) = \frac{\nu}{\nu} \chi(y_i) = \frac{\nu}{\nu} \chi_\ast(y_i) = \chi_\ast(y_i), l = 1, \ldots, u.\]

This completes the proof.

In Lemma 1.6, we showed that if \(H\) is abelian, \(A\) is a matrix over \(JH\) such that \(A^\circ\) is a relation matrix for \(H\), and \(G\) is a group realizing \(A\) w.r.t. \(\chi\), then \(G\) has \(A\) as its Alexander matrix in the pseudo Fox sense w.r.t. \(\chi\). It is easy to see that the equivalence relation on satisfactory pairs is such that the matrices \(A^\circ\) and \(A_\ast^\circ\) obtained from equivalent pairs \((A, \chi)\) and \((A, \chi)\) are rela-
tion matrices of isomorphic groups. So if the matrix of one pair in an equivalence class is a pseudo Fox Alexander matrix with respect to its epimorphism, all are. We claim that in such an equivalence class, the associate has group equal to \( G/G'' \). This will follow if \( G^{++} = G'' \). But if \( A \) has \( G \) as its Alexander matrix in the pseudo Fox sense w.r.t. \( \chi \), Ker \( \lambda = \text{Ker} \chi \). (Section 1.) Then \([\text{Ker} \lambda, \text{Ker} \lambda] = [\text{Ker} \chi, \text{Ker} \chi] \), and \( G'' = [\text{Ker} \lambda, \text{Ker} \lambda] = [\text{Ker} \chi, \text{Ker} \chi] = G^{++} \), as required.

3. Examples

Example 1. Let \( F \) be the free group on \( \{x_1, x_2, x_3\} \). Let \( \chi: F \to \langle t \rangle \) be defined by \( x_j \mapsto t \), \( j = 1, 2, 3 \), and \( \bar{\chi} \) be the extension of to group rings. Set

\[
\begin{align*}
\bar{f}_1 &= -4t^{-1} + 1 + 3t + 5t^2, \\
\bar{f}_2 &= 2t^{-1} - 2t - 5t^2, \\
\bar{f}_3 &= 2t^{-1} + t^2.
\end{align*}
\]

Then \( \bar{f}_1 + \bar{f}_2 + \bar{f}_3 = 1 + t + t^2 \), and \( \sum_{j=1}^3\bar{f}_j(t-1) = t^2 - 1 \). By Theorem I there is no group realizing \( A = (f_1, f_2, f_3) \) w.r.t. \( \chi \). But by Lemma 1.1, \( \Delta(f_1, f_2, f_3) \) is nonempty; we use the method of proof to construct an element. Set

\[
\begin{align*}
w &= x_1^3, \\
f_1 &= -4x_1^{-1} + 1 + 3x_1 + 5x_1^2, \\
f_2 &= 2x_1^{-1} - 2x_1 - 5x_1^2, \\
f_3 &= 2x_1^{-1} + x_1^2.
\end{align*}
\]

Then \( f = \sum_{j=1}^3 f_j(x_j-1) + 1 = -4 + 2x_1 + 5x_1 + 2x_1^{-1}x_2 - 2x_1x_2 - 5x_1^2x_2 + 2x_1^{-1}x_3 + x_1^2x_3 \). \( \bar{\chi}(\partial f/\partial x_j) = \bar{f}_j, j = 1, 2, 3 \). Further, \( f - w^* \) is in Ker \( \chi \), whence by [3], p. 549, we can write

\[
f - w^* = \sum_{k=1}^3 \varepsilon_k d_k(s_k - 1)e_k = \sum_{k=1}^3 (r_k - 1)c_k,
\]

where \( \varepsilon_k = \pm 1 \); \( d_k, e_k \), and \( c_k \) are in \( F \); \( s_k, r_k \) are in Ker \( \chi \), and \( r_k = d_k s_k d_k^{-1} \), and \( c_k = d_k^{(1-\varepsilon_k)/2} e_k \). Let \( w = (\prod_{k=1}^3 r_k)w^* \). Then \( w \) is in \( \Delta(f_1, f_2, f_3) \); this is the essence of the proofs of Lemma 1.1 and [3, 4.10]. A method for determining the \( r_k \) is to be found in [3], p. 549; we write:

\[
\begin{align*}
-4 + 2x_1^{-1}x_2 + 2x_1^{-1}x_3 &= 2(x_1^{-1}x_2 - 1) + 2(x_1^{-1}x_3 - 1), \\
2x_1^2 - 2x_1x_2 &= 2(x_1^2x_2^{-1}x_1^{-1} - 1)x_1x_2, \\
x_1^2x_3 - x_1 &= (x_1^2x_3x_1^{-3} - 1)x_1, \\
5x_1^4 - 5x_1^3x_2 &= 5(x_1^4x_2^{-1}x_1^{-3} - 1)x_1x_2,
\end{align*}
\]

and put
\[ w = (x_1^{-1}x_2)^2(x_1^{-1}x_3)^2(x_1^{-1}x_4)^2(x_1^{-1}x_5)^2(x_1^{-1}x_6)^2(x_1^{-1}x_7)^2(x_1^{-1}x_8)^2(x_1^{-1}x_9)^2(x_1^{-1}x_{10})^2(x_1^{-1}x_{11})^2 \cdot \]

(order is immaterial among the \( r_s \)'s.) \( \bar{x}(\partial w/\partial x_j) = \bar{f}_j, j = 1, 2, 3. \)

**Example 2.** Let \( F \) be the free group on \( \{x_1, x_2\} \). Let \( \chi: F \rightarrow \mathbb{F}/[F, F] \) be the abelianizer, let \( \chi(x_1) = \bar{x}_1, \chi(x_2) = \bar{x}_2 \), and let \( \bar{x} \) be the extension of \( \chi \) to group rings. Set

\[
\begin{align*}
\bar{f}_1 &= (2\bar{x}_1^2\bar{x}_2 - 4\bar{x}_2^2)(1 - \bar{x}_1), \\
\bar{f}_2 &= (2\bar{x}_1^2\bar{x}_2 - 4\bar{x}_2^2)(\bar{x}_1 - 1).
\end{align*}
\]

Then \( \bar{f}_1(\bar{x}_1 - 1) + \bar{f}_2(\bar{x}_2 - 1) = 0 \), so by Theorem I there is a group \( G \) realizing \( A = (\bar{f}_1, \bar{f}_2) \) w.r.t. \( \bar{x} \). Further, since \( A^\circ = (0, 0) \) is a relation matrix for \( \mathbb{F}/[F, F] \), it follows by Lemma 1.6 that \( G \) has \( A \) as its Alexander matrix in the pseudo Fox sense w.r.t. \( \bar{x} \). To find a \( G \) realizing \( A \) w.r.t. \( \bar{x} \), set

\[
\begin{align*}
w^* &= 1, \\
f_1^* &= -4x_2^2 + 4x_1^{-1}x_1 + 2x_1^2x_1 - 2x_2^2x_2, \\
f_2^* &= 2x_1^2x_2 - 4x_1x_2^2 - 2x_1^3x_1 - 2x_1x_2^2 - 2x_2^3 + 1.
\end{align*}
\]

Then \( f = \sum_{j=1}^3 f_j^*(x_j - 1) + 1 = -4x_2^2x_1 + 4x_1^{-1}x_1 + 2x_1^2x_1 - 2x_2^2x_2 + 2x_1^3x_1 - 4x_1x_2^2 + 2x_1^3x_2 - 2x_1^2x_1 + 1 \). Proceeding as in the previous example, we write

\[
f - 1 = 4(x_1x_2^2x_1^{-1}x_2^2 - 1)x_2^{-2}x_1 + 4(x_2^{-1}x_2x_1x_2^{-1} - 1)x_1x_2^{-1} + 2(x_1^2x_1x_2x_2^{-1}x_1^{-3} - 1)x_1^2x_2 + 2(x_1^2x_2x_2^{-1}x_2^{-1}x_1^{-3} - 1)x_1^2x_2 + \]

and set

\[
w = (x_1x_2^2x_1^{-1}x_2^2)^4(x_2^{-1}x_1x_2x_1^{-1})^4(x_2^2x_1x_2x_2^{-1}x_1^{-3})^2(x_2^2x_2x_2^{-1}x_1^{-1}x_2^2)^2.
\]

The group \( G \) presented by \( (x_1, x_2; w) \) has \( A \) as its pseudo Fox Alexander matrix w.r.t. \( \bar{x} \).

**References**