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MODULES OVER DEDEKIND PRIME RINGS. V

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Let R be a Dedekind prime ring, let F be a non-trivial right additive topology on R and let F_l be the left additive topology corresponding to F (cf. [8]). For any positive integer n , let F^n be the set of all right ideals containing a finite intersection of elements in F , each of which has at most n as the length of composition series of its factor module. An exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of right R -modules is EF^n -pure if the induced sequence $0 \rightarrow \text{Ext}(Q_{F^n}/R, L) \rightarrow \text{Ext}(Q_{F^n}/R, M) \rightarrow \text{Ext}(Q_{F^n}/R, N) \rightarrow 0$ is splitting exact, where $Q_{F^n} = \varinjlim I^{-1}$ (I ranges over all elements in F^n). If R is the ring of integers, p is a prime number and F is the topology of all powers of p , then EF^n -purity is equivalent to p^n -purity in the sense of [12].

The aim of this paper is to investigate the structure of EF^n -pure injective modules. In Section 1, a notion of maximal F^n -torsion modules will be introduced. It is shown, in Theorem 1.10, that there is a duality between all maximal F^n -torsion modules and all direct summands of direct products of copies of $\hat{R}_{F_l^n}$ by using the results in [9], where $\hat{R}_{F_l^n} = \varprojlim R/J$ ($J \in F_l^n$). In Section 2, we shall study the category $C(F^n)$ of F^n -reduced, EF^n -pure injective modules. After discussing some properties of EF^n -purities and F^n -purities we shall give, in Theorem 2.9, characterizations of projective objects in the category $C(F^n)$. In particular, it is established that a module is a direct summand of a direct product of copies of $\hat{R}_{F_l^n}$ if and only if it is a projective object in $C(F^n)$. F is bounded if each element of F contains a non-zero ideal of R . If F is bounded, then $\hat{R}_{F_l^n} = \prod R/P^n$, where P ranges over all prime ideals contained in F . So our results may essentially be interesting in case F contains completely faithful right ideals of R in the sense of [3].

1. The Harrison duality

Throughout this paper, R will be a Dedekind prime ring with the two-sided quotient ring Q and $K = Q/R \neq 0$. By a module we shall understand a unitary right R -module. In place of \otimes_R , Hom_R , Ext_R and Tor^R , we shall just write \otimes , Hom , Ext and Tor , respectively. Since R is hereditary, $\text{Tor}_n = 0 = \text{Ext}^n$ for all $n > 1$ and so we shall use Ext for Ext^1 and Tor for Tor_1 . Let I be

an essential right ideal of R . Then R/I is an artinian module by Theorem 1.3 of [3]. So the length of the composition series of the module R/I is finite. We call it the *length* of I . Let F be any non-trivial right additive topology, then F consists of essential right ideals of R (cf. p. 548 in [8]). For any positive integer n , let F^n be the set of all right ideals containing a finite intersection of elements in F , each of which has at most n as the length. Let M be a module. An element m of M is said to be F^n -torsion if $O(m) = \{r \in R \mid mr = 0\} \in F^n$, and we denote the set of all F^n -torsion elements in M by M_{F^n} . M_{F^n} is a submodule of M , because F^n is a pretopology on R . Following [8], we shall denote the left additive topology corresponding to F by F_l . In a similar way we can define the concepts of F_l^n -torsion elements and F_l^n -torsion submodules for left modules. We put $Q_{F^n} = \varinjlim (I \in F^n) I^{-1}$ and $Q_{F_l^n} = \varinjlim (J \in F_l^n) J^{-1}$.

Concerning the terminology we refer to [8] and [9].

Lemma 1.1. (1) $Q_{F^n} = Q_{F_l^n}$ and so Q_{F^n} is an (R, R) -bimodule.
 (2) $K_{F^n} = Q_{F^n}/R = K_{F_l}$

Proof. (1) We shall prove that $Q_{F^n} \cong Q_{F_l^n}$. To prove this let J be any element of F_l^n with length $J \leq n$. Then the length of the composition series of the module J^{-1}/R is at most n . By Proposition 1.4 of [8], J^{-1}/R is F -torsion. Hence, for every element $q \in J^{-1}$, we have $qI_q \subseteq R$ for some $I_q \in F$ with length $I_q \leq n$. Hence $q \in qI_q I_q^{-1} \subseteq R I_q^{-1} \subseteq Q_{F^n}$ and thus $J^{-1} \subseteq Q_{F^n}$. So $Q_{F_l^n} \subseteq Q_{F^n}$ by Lemma 4.8 of [5]. Similarly $Q_{F^n} \subseteq Q_{F_l^n}$ and thus $Q_{F^n} = Q_{F_l^n}$.

(2) is evident from (1).

The exact sequence $0 \rightarrow R \xrightarrow{\iota} Q_{F^n} \rightarrow K_{F^n} \rightarrow 0$ yields the exact sequences:

$$\begin{aligned} 0 \rightarrow \text{Tor}(M, K_{F^n}) \rightarrow M \xrightarrow{\iota^*} M \otimes Q_{F^n} \rightarrow M \otimes K_{F^n} \rightarrow 0, \\ \text{Hom}(K_{F^n}, M) \rightarrow \text{Hom}(Q_{F^n}, M) \xrightarrow{\iota^*} M \rightarrow \text{Ext}(K_{F^n}, M), \end{aligned}$$

where $\iota_*(m) = m \otimes 1$ and $\iota^*(f) = f(1)$ ($m \in M$ and $f \in \text{Hom}(Q_{F^n}, M)$).

Lemma 1.2. (1) $\text{Tor}(M, K_{F^n}) \cong M_{F^n}$.
 (2) If M is F^n -torsion, then $M \otimes Q_{F^n} \cong M \otimes K_{F^n}$.
 (3) $\text{Im } \iota^* \subseteq \cap MJ$ ($J \in F_l^n$).

Proof. (1) is obtained by the similar way as in Theorem 3.2 of [11], and (2) is evident from (1) and the above exact sequence.

(3) Let J be any element of F_l^n . Then from the exact sequence $0 \rightarrow R \rightarrow J^{-1} \rightarrow J^{-1}/R \rightarrow 0$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \text{Hom}(Q_{F^n}, M) & \rightarrow & M & \rightarrow & \text{Ext}(K_{F^n}, M) \\ \downarrow & & \parallel & & \downarrow \\ \text{Hom}(J^{-1}, M) & \rightarrow & M & \rightarrow & \text{Ext}(J^{-1}/R, M). \end{array}$$

From this diagram and Proposition 3.2 of [13], we get $\text{Im } \iota^* \subseteq MJ$ and so $\text{Im } \iota^* \subseteq \cap MJ$.

We denote the submodule $\text{Im } \iota^*$ of the module M by MF^n , and if $MF^n = 0$, then M is said to be F^n -reduced. If M is F^n -reduced, then it is F -reduced in the sense of [9].

Lemma 1.3. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence such that $\text{Ext}(Q_{F^n}, L) = 0$ and M is F^n -reduced. Then N is F^n -reduced.*

Proof. This is evident from the following commutative diagram with exact columns:

$$\begin{array}{ccc} \text{Hom}(Q_{F^n}, M) & \rightarrow & M \\ \downarrow & & \downarrow \\ \text{Hom}(Q_{F^n}, N) & \rightarrow & N \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

For any module M we denote by M_F the submodule of F -torsion elements in M . If $M_F = 0$, then we say that M is F -torsion-free.

Following [9], an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is F^∞ -pure if the induced sequence $0 \rightarrow L_F \rightarrow M_F \rightarrow N_F \rightarrow 0$ is splitting exact. A module is F^∞ -pure injective if it has the injective property to the class of F^∞ -pure exact sequences. We denote the injective hull of a module M by $E(M)$, and the F -injective hull of M by $E_F(M)$. By the results in §1 of [9], we have the following:

- (1) A module G is F^∞ -pure injective if and only if $G \cong E(GF^\infty) \oplus \text{Ext}(K_F, G)$, where GF^∞ is the maximal F -divisible submodule of G .
- (2) For a module G , the following are equivalent:
 - (i) G is F -reduced and F^∞ -pure injective.
 - (ii) $\delta: G \cong \text{Ext}(K_F, G)$, where δ is the connecting homomorphism.
 - (iii) G is F -reduced and $\text{Ext}(Q_F, G) = 0$.
 - (iv) G is F -reduced and $\text{Ext}(X, G) = 0$ for every F -torsion-free module X .

These results will be used in this paper without references.

Lemma 1.4. *Let M be a module. Then $H_n = \text{Hom}(K_{F^n}, M)$ and $G_n = \text{Ext}(K_{F^n}, M)$ are both F^n -reduced and F^∞ -pure injective.*

Proof. (1) H_n is F^∞ -pure injective by Proposition 5.1 of [13] and Proposition 1.4 of [9]. Further, from the exact sequence $Q_{F^n} \rightarrow K_{F^n} \rightarrow 0$ and Proposition 5.2' of [2, Chap. II], we get the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 \rightarrow \text{Hom}(K_{F^n}, H_n) & \longrightarrow & \text{Hom}(Q_{F^n}, H_n) & & \\ & \Downarrow & f & \Downarrow & \\ 0 \rightarrow \text{Hom}(K_{F^n} \otimes K_{F^n}, M) & \rightarrow & \text{Hom}(Q_{F^n} \otimes K_{F^n}, M) & & \end{array}$$

By Lemma 1.2, f is an isomorphism and so H_n is F^n -reduced.

(2) From the exact sequence $0 \rightarrow M \rightarrow E(M) \xrightarrow{g} E(M)/M \rightarrow 0$ we derive an exact sequence $0 \rightarrow H_n \rightarrow \text{Hom}(K_{F^n}, E(M)) \xrightarrow{g_*} \text{Hom}(K_{F^n}, E(M)/M) \rightarrow G_n \rightarrow 0$. Since $\text{Hom}(K_{F^n}, E(M))$ is F -reduced and F^∞ -pure injective, $\text{Ext}(Q_{F^n}, \text{Hom}(K_{F^n}, E(M))) = 0$. So $\text{Ext}(Q_{F^n}, \text{Im } g_*) = 0$ and thus G_n is F^n -reduced by (1) and Lemma 1.3. By Proposition 3.5a of [2, Chap. VI], we have $\text{Ext}(Q_F, G_n) \cong \text{Ext}(\text{Tor}(Q_F, K_{F^n}), M) = 0$. Therefore G_n is F^∞ -pure injective.

Let M be any module. From the exact sequence $0 \rightarrow R \rightarrow Q_{F^n} \rightarrow K_{F^n} \rightarrow 0$ we have the exact sequences:

$$\begin{aligned} \text{Ext}(K_{F^n}, M) &\xrightarrow{\delta'} \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M)) \rightarrow \text{Ext}(Q_{F^n}, \text{Ext}(K_{F^n}, M)), \\ M &\xrightarrow{\delta} \text{Ext}(K_{F^n}, M) \rightarrow \text{Ext}(Q_{F^n}, M). \end{aligned}$$

The second exact sequence yields a homomorphism $\delta_*: \text{Ext}(K_{F^n}, M) \rightarrow \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M))$.

Lemma 1.5. δ' and δ_* are both isomorphisms.

Proof. From the exact sequence $0 \rightarrow R \rightarrow Q_{F^n} \rightarrow K_{F^n} \rightarrow 0$ we have the isomorphism: $\text{Tor}(K_{F^n}, K_{F^n}) \cong R \otimes K_{F^n}$. Applying Theorem 2.1 of [11] we get the commutative diagram:

$$\begin{array}{ccc} \text{Ext}(R \otimes K_{F^n}, M) & \cong & \text{Ext}(\text{Tor}(K_{F^n}, K_{F^n}), M) \\ \parallel & & \parallel \\ \text{Hom}(R, \text{Ext}(K_{F^n}, M)) & \xrightarrow{\delta'} & \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M)). \end{array}$$

Hence δ' is an isomorphism.

From Theorem 1.5 of [9] we obtain the following commutative diagram with exact row:

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{h} & E(MF^\infty) \oplus \text{Ext}(K_F, M) & \rightarrow & \text{Coker } h \rightarrow 0 \\ & & \parallel & & \downarrow p & & \\ & & M & \xrightarrow{\delta_1} & \text{Ext}(K_F, M), & & \end{array}$$

where p is the projection and δ_1 is the connecting homomorphism. Since $\text{Coker } h$ is F -torsion-free and injective, applying $\text{Ext}(K_{F^n}, _)$ to the diagram we have the isomorphism $\delta_{1*}: \text{Ext}(K_{F^n}, M) \cong \text{Ext}(K_{F^n}, \text{Ext}(K_F, M))$. From the exact sequence $0 \rightarrow K_{F^n} \xrightarrow{\theta} K_F$ we have the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\delta_1} & \text{Ext}(K_F, M) \\ \parallel & & \downarrow \theta^* \\ M & \xrightarrow{\delta} & \text{Ext}(K_{F^n}, M). \end{array}$$

From this diagram we get the commutative diagram:

$$\begin{array}{ccc} \text{Ext}(K_{F^n}, M) & \xrightarrow{\delta_{1*}} & \text{Ext}(K_{F^n}, \text{Ext}(K_F, M)) \\ \parallel & & \downarrow (\theta^*)_* \\ \text{Ext}(K_{F^n}, M) & \xrightarrow{\delta_*} & \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M)) \end{array}$$

By Proposition 3.5a of [2, Chap. VI] and Lemma 1.2, $(\theta^*)_*$ is an isomorphism and so δ_* is also an isomorphism.

From the inclusion map $\theta: K_{F^n} \rightarrow K_F$, we get the epimorphism $\theta^*: \text{Ext}(K_F, M) \rightarrow \text{Ext}(K_{F^n}, M)$ for any module M .

Lemma 1.6. $\text{Ker } \theta^* = \text{Ext}(K_F, M)F^n$.

Proof. We consider the following commutative diagram with exact row:

$$\begin{array}{ccccccc} 0 \rightarrow \text{Ker } \theta^* & \longrightarrow & \text{Ext}(K_F, M) & \longrightarrow & \text{Ext}(K_{F^n}, M) & \rightarrow & 0 \\ & & \downarrow & & \downarrow \delta' & & \\ & & \text{Ext}(K_{F^n}, \text{Ker } \theta^*) & \rightarrow & \text{Ext}(K_{F^n}, \text{Ext}(K_F, M)) & \xrightarrow{(\theta^*)_*} & \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M)) \end{array}$$

By Lemma 1.5 and its proof, δ' and $(\theta^*)_*$ are both isomorphisms. Hence $\text{Ker } \theta^* = \text{Ext}(K_F, M)F^n$.

Lemma 1.7. Let N be an F_I -torsion left module. Then $E_{F_I}(N)$ is a direct summand of a direct sum of copies of K_F .

Proof. Since $E(N)$ is a torsion left module, there is a torsion left module L such $E(N) \oplus L = \Sigma \oplus K$. So $\Sigma \oplus K_F = \Sigma \oplus K_{F_I} = (\Sigma \oplus K)_{F_I} = E(N)_{F_I} \oplus L_{F_I}$ by Proposition 1.4 of [8]. It is evident that $E(N)_{F_I} \subseteq E_{F_I}(N)$ by Proposition 6.3 of [14]. Since N is F_I -torsion, the converse inclusion also holds. Thus $E_{F_I}(N)$ is a direct summand of $\Sigma \oplus K_F$.

An F -torsion module D is said to be *maximal F^n -torsion* provided $(E(D))_{F^n} = D$. This is clearly equivalent to $(E_F(D))_{F^n} = D$. For any module M we define $\hat{M}_{F_I^n} = \varprojlim M/MJ$ ($J \in F_I^n$). Then $\hat{R}_{F_I^n}$ becomes a ring and $\hat{M}_{F_I^n}$ becomes an $\hat{R}_{F_I^n}$ -module by the similar way as in §4 of [13]. Let $\alpha: M \rightarrow \hat{M}_{F_I^n}$ be the canonical map. Then $\text{Ker } \alpha = \cap MJ$ ($J \in F_I^n$).

Lemma 1.8. Let M be an F -torsion-free module. Then

- (1) $M \otimes K_{F^n}$ is maximal F^n -torsion.
- (2) There are isomorphism $\hat{M}_{F_I^n} \cong \text{Hom}(K_{F^n}, M \otimes K_{F^n}) \cong \text{Ext}(K_{F^n}, M)$ such that the diagram

$$\begin{array}{ccccc} M & \xlongequal{\quad} & M & \xlongequal{\quad} & M \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \delta \\ \hat{M}_{F_I^n} & \cong & \text{Hom}(K_{F^n}, M \otimes K_{F^n}) & \cong & \text{Ext}(K_{F^n}, M) \end{array}$$

commutes, where $\beta(m)(\bar{q}) = m \otimes \bar{q}$ ($m \in M$, $\bar{q} \in K_{F^n}$) and δ is the connecting homomorphism.

Proof. (1) The commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & R & \rightarrow & Q_{F^n} & \rightarrow & K_{F^n} \rightarrow 0 \\ & & \parallel & & \downarrow \kappa & & \downarrow \theta \\ 0 & \rightarrow & R & \rightarrow & Q_F & \rightarrow & K_F \rightarrow 0 \end{array}$$

yields the commutative diagram with exact rows:

$$(A): \begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & M \otimes Q_{F^n} & \rightarrow & M \otimes K_{F^n} \rightarrow 0 \\ & & \parallel & & \downarrow \kappa_* & & \downarrow \theta_* \\ 0 & \rightarrow & M & \rightarrow & M \otimes Q_F & \rightarrow & M \otimes K_F \rightarrow 0. \end{array}$$

By Proposition 1.1, Lemma 2.5 of [8] and Lemma 1.7, $\text{Tor}(M, Q_F/Q_{F^n})=0$. Hence κ_* is a monomorphism and so θ_* is also a monomorphism. Let x be any element in $M \otimes K_F$ such that $I=O(x) \in F^n$ and let y be an element in $M \otimes Q_F$ mapping on x . Then $yI \subseteq M$ and so $y \in M \otimes I^{-1}$ in $M \otimes Q_F$. This implies that $y \in M \otimes Q_{F^n}$ and thus $x \in M \otimes K_{F^n}$. Hence $(M \otimes K_F)_{F^n} = M \otimes K_{F^n}$. Therefore $M \otimes K_{F^n}$ is maximal F^n -torsion, because $M \otimes K_F$ is F -injective.

(2) By the similar way as in Lemma 2.7 of [8], we have $\text{Hom}(K_{F^n}, M \otimes K_F) \cong \text{Ext}(K_{F^n}, M)$ such that the diagram

$$\begin{array}{ccc} M & \xlongequal{\quad} & M \\ \downarrow & & \downarrow \\ \text{Hom}(K_F, M \otimes K_F) & \cong & \text{Ext}(K_F, M) \\ \downarrow & & \downarrow \\ \text{Hom}(K_{F^n}, M \otimes K_F) & \cong & \text{Ext}(K_{F^n}, M) \end{array}$$

commutes. Since $(M \otimes K_F)_{F^n} = M \otimes K_{F^n}$, we have $\text{Hom}(K_{F^n}, M \otimes K_F) \cong \text{Hom}(K_{F^n}, M \otimes K_{F^n})$. This is the proof of the assertion to the right diagram. Next we consider the following commutative diagram with exact right column:

$$\begin{array}{ccc} \hat{M}_{F_I} & \xrightarrow{\eta} & \text{Hom}(K_F, M \otimes K_F) \\ \downarrow & & \downarrow \\ \hat{M}_{F_I^n} & \xrightarrow{\eta'} & \text{Hom}(K_{F^n}, M \otimes K_F) (= \text{Hom}(K_{F^n}, M \otimes K_{F^n})) \\ & & \downarrow \\ & & 0, \end{array}$$

where $\eta(\hat{m})(\bar{q}) = m_J \otimes \bar{q}$ ($\hat{m} = ([m_J + MJ])$, $\bar{q} = [q + R]$ and $q \in J^{-1}$), η' is the homomorphism induced by η and the map: $\hat{M}_{F_I} \rightarrow \hat{M}_{F_I^n}$ is the natural homomorphism. By Lemma 2.7 of [8], η is an isomorphism. If $\eta'(\hat{m})=0$, where

$\hat{m} = ([m_J + MJ]) \in \hat{M}_{F^n}$, then $m_J \otimes J^{-1}/R = 0$ in $M \otimes K_{F^n}$. This implies that $m_J \otimes J^{-1} \subseteq M$ by the diagram (A) and so $m_J \in MJ$. Hence $\hat{m} = 0$ and thus η' is an isomorphism. The commutativity of the left diagram is clear.

Lemma 1.9. *Let M be an F -torsion-free module. Then $\text{Ext}(K_{F^n}, M)$ is isomorphic to a direct summand of a direct product of copies of \hat{R}_{F^n} .*

Proof. Since $M_F = 0$, the exact sequence $0 \rightarrow \text{Ker } f \rightarrow \Sigma \oplus R \xrightarrow{f} M \rightarrow 0$ is F^∞ -pure and so the sequence $0 \rightarrow \text{Ext}(K_F, \text{Ker } f) \rightarrow \text{Ext}(K_F, \Sigma \oplus R) \rightarrow \text{Ext}(K_F, M) \rightarrow 0$ is splitting exact by Lemma 1.3 of [9]. By Proposition 3.5a of [2, Chap. VI] and Lemma 1.2, this sequence yields the splitting exact sequence $0 \rightarrow \text{Ext}(K_{F^n}, \text{Ker } f) \rightarrow \text{Ext}(K_{F^n}, \Sigma \oplus R) \rightarrow \text{Ext}(K_{F^n}, M) \rightarrow 0$. So it suffices to prove that $\text{Hom}(K_{F^n}, \Sigma \oplus K_{F^n})$ is a direct summand of $\text{Hom}(K_{F^n}, \Pi K_{F^n})$ by Lemma 1.8. To prove this we consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ 0 \rightarrow \Sigma \oplus K_{F^n} & \rightarrow & (\Pi K_{F^n})_{F^n} & & \\ & \downarrow & & \downarrow & \\ 0 \rightarrow \Sigma \oplus K_F & \rightarrow & (\Pi K_F)_F & & \end{array}$$

The second row splits, because $\Sigma \oplus K_F$ is F -injective. Since $(\Sigma \oplus K_F)_{F^n} = \Sigma \oplus K_{F^n}$, the splitting map induces an splitting map of the first row. Hence $\text{Hom}(K_{F^n}, \Sigma \oplus K_{F^n})$ is a direct summand of $\text{Hom}(K_{F^n}, (\Pi K_{F^n})_{F^n}) = \text{Hom}(K_{F^n}, \Pi K_{F^n})$.

Theorem 1.10 (The Harrison duality). *The correspondence*

$$(B) \quad D \rightarrow G = \text{Hom}(K_{F^n}, D)$$

is one-to-one between all maximal F^n -torsion modules D and all direct summands G of direct products of copies of \hat{R}_{F^n} . The inverse of (B) is given by the correspondence $G \rightarrow G \otimes K_{F^n}$.

Proof. (i) Let D be maximal F^n -torsion and let $H = \text{Hom}(K_F, E_F(D))$. Then H is F -torsion-free, F^∞ -pure injective and $\eta: H \otimes K_F \cong E_F(D)$ by Theorem 2.2 of [9], where $\eta(x \otimes \bar{q}) = x(\bar{q})$ ($x \in H$ and $\bar{q} \in K_F$). From the exact sequence $0 \rightarrow K_{F^n} \xrightarrow{\theta} K_F$, we get the exact sequence: $H \xrightarrow{\theta^*} \text{Hom}(K_{F^n}, E_F(D)) (= \text{Hom}(K_{F^n}, D)) \rightarrow 0$. This yields the commutative diagram:

$$\begin{array}{ccc} H \otimes K_F & \cong & E_F(D) \\ \uparrow \theta_* & & \uparrow \eta' \\ H \otimes K_{F^n} & \xrightarrow{\eta'} & D \\ \searrow (\theta^*)_* \nearrow \varphi & & \\ \text{Hom}(K_{F^n}, D) \otimes K_{F^n}, & & \end{array}$$

where $\varphi(x \otimes \bar{q}) = x(\bar{q})$ and η' is the map induced by η . Since H is F -torsion-free, θ_* is a monomorphism and $H \otimes K_{F^n}$ is maximal F^n -torsion. Hence η' is an isomorphism, and $\varphi(\theta^*)_* = \eta'$ implies that φ is also an isomorphism, because $(\theta^*)_*$ is an epimorphism. From $D \cong H \otimes K_{F^n}$, we have $\text{Hom}(K_{F^n}, D) \cong \text{Hom}(K_{F^n}, H \otimes K_{F^n}) \cong \text{Ext}(K_{F^n}, H)$ by Lemma 1.8. Hence $\text{Hom}(K_{F^n}, D)$ is a direct summand of a direct product of copies of \hat{R}_{F^n} by Lemma 1.9.

(ii) Let G be a direct summand of a direct product of copies of \hat{R}_{F^n} . Then we may assume from Lemma 1.8 that $G \oplus X = \text{Hom}(K_{F^n}, \Pi K_{F^n}) = \text{Hom}(K_{F^n}, (\Pi K_{F^n})_{F^n})$, where X is a module. Since $(\Pi K_{F^n})_{F^n}$ is maximal F^n -torsion, we get, by (i), the isomorphism $\varphi: (G \otimes K_{F^n}) \oplus (X \otimes K_{F^n}) \cong (\Pi K_{F^n})_{F^n}$. Hence $G \otimes K_{F^n}$ is maximal F^n -torsion. Applying $\text{Hom}(K_{F^n}, _)$ to the isomorphism we obtain the isomorphism $\varphi_*: \text{Hom}(K_{F^n}, G \otimes K_{F^n}) \oplus \text{Hom}(K_{F^n}, X \otimes K_{F^n}) \cong G \oplus X$. We may define $\lambda: G \oplus X \rightarrow \text{Hom}(K_{F^n}, G \otimes K_{F^n}) \oplus \text{Hom}(K_{F^n}, X \otimes K_{F^n})$ by $\lambda(g+x)(\bar{q}) = (g \otimes \bar{q}) + (x \otimes \bar{q})$, where $g \in G$, $x \in X$ and $\bar{q} \in K_{F^n}$. Then it follows that $\varphi_* \lambda = 1$ and that $\lambda(G) \subseteq \text{Hom}(K_{F^n}, G \otimes K_{F^n})$, $\lambda(X) \subseteq \text{Hom}(K_{F^n}, X \otimes K_{F^n})$. Hence $G \cong \text{Hom}(K_{F^n}, G \otimes K_{F^n})$.

This duality was first exhibited by Harrison in [4] between all divisible, torsion abelian groups and all reduced, torsion-free cotorsion abelian groups. This duality was generalized by Matlis [10] to modules over commutative integral domains. To modules over non-commutative complete discrete valuation rings the result was established by Liebert [6]. The author generalized it in [9] to the case of modules over Dedekind prime rings.

2. Projective objects of the category of F^n -reduced, EF^n -pure injective modules

In this section we shall define a notion of EF^n -pure injective modules and give characterizations of direct summands of direct products of copies of \hat{R}_{F^n} which were discussed in §1.

A short exact sequence

$$(E): 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

of modules is said to be F^n -pure if $MJ \cap L = LJ$ for all $J \in F^n$. (E) is said to be Ext - F^n -pure (abbr. EF^n -pure) if the induced sequence $0 \rightarrow \text{Ext}(K_{F^n}, L) \rightarrow \text{Ext}(K_{F^n}, M) \rightarrow \text{Ext}(K_{F^n}, N) \rightarrow 0$ is splitting exact. A module G is EF^n -pure injective if it has the injective property relative to the class of EF^n -pure exact sequences. A right additive topology F is bounded if any element of F contains a non-zero ideal of R .

Lemma 2.1. (1) If (E) is EF^n -pure, then it is F^n -pure.

(2) If (E) is F^∞ -pure, then it is EF^n -pure.

(3) If F is bounded and (E) is F^n -pure, then it is EF^n -pure.

Proof. (1) If (E) is EF^n -pure, then the induced sequence $0 \rightarrow \text{Ext}(K_{F^n}, L) \rightarrow \text{Ext}(K_{F^n}, M)$ is splitting exact. Let J be any element of F^n . Then we get the following commutative diagram with splitting exact rows:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Ext}(J^{-1}/R, \text{Ext}(K_{F^n}, L)) & \rightarrow & \text{Ext}(J^{-1}/R, \text{Ext}(K_{F^n}, M)) & & \\
 & & \Downarrow & & \Downarrow & & \\
 0 & \rightarrow & \text{Ext}(\text{Tor}(J^{-1}/R, K_{F^n}), L) & \rightarrow & \text{Ext}(\text{Tor}(J^{-1}/R, K_{F^n}), M) & & \\
 & & \Downarrow & & \Downarrow & & \\
 0 & \longrightarrow & \text{Ext}(J^{-1}/R, L) & \longrightarrow & \text{Ext}(J^{-1}/R, M) & & \\
 & & \Downarrow & & \Downarrow & & \\
 0 & \longrightarrow & L/LJ & \longrightarrow & M/MJ & &
 \end{array}$$

Hence (E) is F^n -pure.

(2) If (E) is F^∞ -pure, then the induced sequence $0 \rightarrow \text{Ext}(K_F, L) \rightarrow \text{Ext}(K_F, M)$ is splitting exact by Lemma 1.3 of [9]. Applying to $\text{Ext}(K_{F^n}, \quad)$ the above sequence we get the splitting exact sequence $0 \rightarrow \text{Ext}(K_{F^n}, L) \rightarrow \text{Ext}(K_{F^n}, M)$ by the same way as in (1). Therefore (E) is EF^n -pure.

(3) Let P be a prime ideal of R . Then the set $F_P = \{I \mid I \supseteq P^k \text{ for some non-negative integer } k, I \text{ is a right ideal of } R\}$ is a right additive topology. We shall prove that $F_P^n = \{I \mid I \supseteq P^n \text{ and } I \in F_P\}$. It is well known that $R/P^n = (D)_n$, where D is a completely primary ring with the Jacobson radical $J(D)$ such that one-sided ideals of D are only $\{J(D)^l \mid l = 0, 1, 2, \dots, n\}$ (cf. Theorem 4.32 of [1]). we can easily see that $P^n \in F_P^n$. Let I be any element of F_P^n . To prove that $I \supseteq P^n$, it suffices to prove it in case the length of I is k ($k \leq n$). If $k=1$ and $I \not\supseteq P$. Then $I+P=R$. Since $I \in F_P$, we may assume that $I \not\supseteq P^{i-1}$ and $I \supseteq P^i$ for some natural number i . It follows that $P^{i-1} = (I+P)P^{i-1} \subseteq I$, which is a contradiction. Hence $I \supseteq P$. Assume that $k > 1$. Let I_0 be any element of F_P such that $I_0 \not\supseteq I$ and the length of I_0 is $k-1$. By induction assumption, we have $I_0 \supseteq P^{k-1}$. Write $I_0 = aR + I$. Then we have $a^{-1}I \supseteq P$, since the length of $a^{-1}I$ is 1. Thus we get $P^k = P^{k-1}P \subseteq I_0P \subseteq I$. Hence $I \supseteq P^n$, as desired.

Now let F be a bounded right additive topology. Then by Proposition 1.2 of [8], F is determined by a class $\{S_\gamma \mid \gamma \in \Gamma\}$ of simple modules and each S_γ is annihilated by a prime ideal P_γ of R , since F is bounded. Further, we obtain that a prime ideal P of R is an element in F if and only if simple modules in R/P are isomorphic to S_γ for some $\gamma \in \Gamma$, because R/P is a simple and artinian ring. So $F = \{I \mid I \supseteq P_1^{n_1} \cap \dots \cap P_k^{n_k}, \text{ where } P_i \in F \text{ and } n_1, \dots, n_k \text{ are non-negative integers}\}$ by Proposition 1.2 of [8]. Thus we have $F^n = \{I \mid I \supseteq P_1^{n_1} \cap \dots \cap P_k^{n_k}, \text{ where } P_i \in F\}$ and so $K_{F^n} = \Sigma \oplus P^{-n}/R$, where P ranges over all prime ideals contained in F . If (E) is F^n -pure, then it is P^n -pure in the sense

of [7] for any $P \in F$ and so the sequence $0 \rightarrow L/LP^n \rightarrow M/MP^n$ splits by Lemma 1.1 of [7]. Hence we get the commutative diagram with splitting exact rows:

$$\begin{array}{ccc} 0 \rightarrow \prod L/LP^n & \longrightarrow & \prod M/MP^n \\ \parallel & & \parallel \\ 0 \rightarrow \text{Ext}(K_{F^n}, L) & \rightarrow & \text{Ext}(K_{F^n}, M). \end{array}$$

Therefore (E) is EF^n -pure.

Lemma 2.2. *The following conditions of a short exact sequence (E): $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ are equivalent:*

- (1) (E) is F^n -pure.
- (2) For any finitely generated F^n -torsion module T , the natural homomorphism $\text{Hom}(T, M) \rightarrow \text{Hom}(T, N) \rightarrow 0$ is exact.
- (3) For any F_i^n -torsion left module T , the natural homomorphism $0 \rightarrow L \otimes T \rightarrow M \otimes T$ is exact.

Proof. Let I be any element of F^n . Then $I^{-1}/R \subseteq K_{F^n}$ by Lemma 1.1 and it is finitely generated. So $I^{-1}/R \cong \Sigma \oplus R/J_i$ for some $J_i \in F_i^n$ by Theorem 3.11 of [3]. Applying $\text{Hom}(_, K_F)$ to this isomorphism we get $R/I \cong \Sigma \oplus J_i^{-1}/R$ by Proposition 3.3 of [13], because $\text{Hom}(R/J_i, K_F) \cong J_i^{-1}/R$. Further any finitely generated F^n -torsion module is a finite direct sum of modules R/I ($I \in F^n$). Combining these facts with Lemma 5.2 of [11], we get the equivalence of (1) and (2). For any module X and any left ideal J , $(X/XJ) \cong X \otimes R/J$ and \otimes commutes with direct limits. So the equivalence of (1) and (3) is also evident.

Lemma 2.3. *If a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is F^n -pure, then the induced sequence $0 \rightarrow L_{F^n} \rightarrow M_{F^n} \rightarrow N_{F^n} \rightarrow 0$ is exact.*

Proof. It follows from Lemmas 1.2 and 2.2.

Lemma 2.4. *For a module G , the following are equivalent:*

- (1) G is F^n -reduced and EF^n -pure injective.
- (2) G is F^n -reduced and F^∞ -pure injective.
- (3) The connecting homomorphism $\delta: G \rightarrow \text{Ext}(K_{F^n}, G)$ is an isomorphism.

Proof. (1) \Rightarrow (2): It is evident from Lemma 2.1.

(2) \Rightarrow (3): We consider the commutative diagram:

$$\begin{array}{ccc} G \cong \text{Ext}(K_F, G) & & \\ \parallel \delta & \downarrow \theta^* & \\ G \rightarrow \text{Ext}(K_{F^n}, G) & & \\ \downarrow & & \\ 0. & & \end{array}$$

By Lemma 1.6 and the assumption, θ^* is an isomorphism and so δ is an iso-

morphism.

(3) \Rightarrow (1): It is evident from Lemma 1.4 and definition.

Let $f: M \rightarrow E(MF^n)$ be an extension of the inclusion map $MF^n \rightarrow M$ and $\delta: M \rightarrow \text{Ext}(K_{F^n}, M)$ be the connecting homomorphism. We define a map $g: M \rightarrow E(MF^n) \oplus \text{Ext}(K_{F^n}, M)$ by $g(m) = (f(m), \delta(m))$ for every $m \in M$.

Lemma 2.5. *The sequence*

$$0 \rightarrow M \xrightarrow{g} E(MF^n) \oplus \text{Ext}(K_{F^n}, M) \rightarrow \text{Coker } g \rightarrow 0$$

is exact and EF^n -pure. Further $E(MF^n) \oplus \text{Ext}(K_{F^n}, M)$ is EF^n -pure injective and $\text{Coker } g$ is injective.

Proof. By the similar way as in Lemma 2.7 of [7], $\text{Coker } g$ is injective. The other assertions follow from Lemmas 1.5 and 2.4.

Lemma 2.6. *Let M be any module. Then the natural homomorphism $\eta: M \rightarrow M/MF^n$ induces the following commutative diagram:*

$$(C) \quad \begin{array}{ccccc} M & \xrightarrow{\delta} & \text{Ext}(K_{F^n}, M) & \xrightarrow{f} & \text{Ext}(Q_{F^n}, M) \\ \downarrow \eta & & \Downarrow \eta_1 & & \Downarrow \eta_2 \\ M/MF^n & \rightarrow & \text{Ext}(K_{F^n}, M/MF^n) & \rightarrow & \text{Ext}(Q_{F^n}, M/MF^n) \end{array}$$

Proof. It is evident that f, η_1, η_2 are all epimorphisms. δ induces the homomorphism $\bar{\delta}: M/MF^n \rightarrow \text{Ext}(K_{F^n}, M)$ such that $\bar{\delta}\eta = \delta$. Hence we get the commutative diagram with exact column:

$$\begin{array}{ccc} \text{Ext}(K_{F^n}, M) & \xrightarrow{\delta_*} & \text{Ext}(K_{F^n}, \text{Ext}(K_{F^n}, M)) \\ \downarrow \eta_1 & (\bar{\delta})_* \nearrow & \\ \text{Ext}(K_{F^n}, M/MF^n) & & \\ \downarrow & & \\ 0 & & \end{array}$$

By Lemma 1.5, δ_* is an isomorphism. Therefore η_1 is an isomorphism. So it follows from the diagram (C) that η_2 is also an isomorphism.

Corollary 2.7. *For any module M , M/MF^n is F^n -reduced.*

Proof. It is clear from the diagram (C).

Let $C(F^n)$ be the category of F^n -reduced and EF^n -pure injective modules together with their homomorphisms. We note that a module G is an element in $C(F^n)$ if and only if $\text{Ext}(Q_F, G) = 0 = GF^n$ by Proposition 1.4 of [9] and Lemma 2.4.

Proposition 2.8. *$C(F^n)$ is an Abelian category.*

Proof. Let M, N be modules in $C(F^n)$ and $f: M \rightarrow N$ be a homomorphism. Then the exact sequence $0 \rightarrow \text{Ker } f \rightarrow M \rightarrow \text{Im } f \rightarrow 0$ yields an exact sequence: $0 = \text{Hom}(Q_F, \text{Im } f) \rightarrow \text{Ext}(Q_F, \text{Ker } f) \rightarrow \text{Ext}(Q_F, M) \rightarrow \text{Ext}(Q_F, \text{Im } f) \rightarrow 0$. The first term is zero, because Q_F is F -injective and $\text{Im } f$ is F -reduced. Therefore $\text{Ext}(Q_F, \text{Ker } f) = 0 = \text{Ext}(Q_F, \text{Im } f)$, because $\text{Ext}(Q_F, M) = 0$ and so $\text{Ker } f, \text{Im } f \in C(F^n)$. Next we consider the exact sequence $0 \rightarrow \text{Im } f \rightarrow N \rightarrow \text{Coker } f \rightarrow 0$. By Lemma 1.3, $\text{Coker } f$ is F^n -reduced. Since $\text{Ext}(Q_F, N) \rightarrow \text{Ext}(Q_F, \text{Coker } f) \rightarrow 0$ is exact, it follows that $\text{Coker } f \in C(F^n)$. It is easy to prove the other axioms for Abelian categories.

A module in $C(F^n)$ is said to be $C(F^n)$ -projective if it is a projective object in the category $C(F^n)$.

Theorem 2.9. *Let G be a module. Then the following conditions are equivalent:*

- (1) G is $C(F^n)$ -projective.
- (2) G is a direct summand of $\text{Ext}(K_{F^n}, \Sigma \oplus R)$.
- (3) G is a direct summand of $\Pi \hat{R}_{F^n}$.
- (4) G is isomorphic to $\text{Ext}(K_{F^n}, M)$, where M is an F -torsion-free module.
- (5) G is a direct summand of $\text{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F^n})$.

Proof. We shall give the following implications: (2) \Leftrightarrow (5) and (1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2).

(2) \Leftrightarrow (5): By Lemmas 1.8 and 2.5, the exact sequence $0 \rightarrow R \xrightarrow{g} E(RF^n) \oplus \hat{R}_{F^n} \rightarrow \text{Coker } g \rightarrow 0$ is EF^n -pure and $\text{Coker } g$ is divisible. So it is F^n -pure by Lemma 2.1. Hence the exact sequence $0 \rightarrow \Sigma \oplus R \xrightarrow{\Sigma \oplus g} \Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F^n} \xrightarrow{k} \text{Coker } (\Sigma \oplus g) \rightarrow 0$ is F^n -pure and $\text{Coker } (\Sigma \oplus g)$ is divisible. Applying $\text{Hom}(K_{F^n}, _)$ to the exact sequence we obtain the exact sequence $0 \rightarrow \text{Hom}(K_{F^n}, (\Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F^n})) \xrightarrow{k_*} \text{Hom}(K_{F^n}, \text{Coker } (\Sigma \oplus g)) \rightarrow \text{Ext}(K_{F^n}, \Sigma \oplus R) \rightarrow \text{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F^n}) \rightarrow 0$. On the other hand F^n -purity of the exact sequence yields the isomorphism $\bar{k}: (\Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F^n})_{F^n} \cong (\text{Coker } (\Sigma \oplus g))_{F^n}$ by Lemma 2.3. So k_* is an isomorphism and thus we get $\text{Ext}(K_{F^n}, \Sigma \oplus R) \cong \text{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F^n})$.

(1) \Rightarrow (2): An exact sequence $0 \rightarrow \text{Ker } f \rightarrow \Sigma \oplus R \xrightarrow{f} G \rightarrow 0$ yields the exact sequence $0 \rightarrow \text{Ker } f_* \rightarrow \text{Ext}(K_{F^n}, \Sigma \oplus R) \xrightarrow{f_*} \text{Ext}(K_{F^n}, G) (\cong G) \rightarrow 0$. Since $\text{Ext}(K_{F^n}, \Sigma \oplus R), G \in C(F^n)$, we have $\text{Ker } f_* \in C(F^n)$ by Proposition 2.8. Hence, by assumption, the sequence splits.

(2) \Rightarrow (3): This is clear from Lemma 1.9.

(3) \Rightarrow (4): By Theorem 1.10 $G \cong \text{Hom}(K_{F^n}, D)$, where D is a maximal F^n -torsion module. We let $H = \text{Hom}(K_F, E_F(D))$. Then it is F -torsion-free and $G \cong \text{Hom}(K_{F^n}, H \otimes K_{F^n}) \cong \text{Ext}(K_{F^n}, H)$ by Lemma 1.8 and the proof of

Theorem 1.10.

(4) \Rightarrow (2): Let M be an F -torsion-free module. Then an exact sequence $0 \rightarrow \text{Ker } k \rightarrow \Sigma \oplus R \xrightarrow{k} M \rightarrow 0$ is F^∞ -pure and so it is EF^n -pure. Hence the induced sequence $0 \rightarrow \text{Ext}(K_{F^n}, \text{Ker } k) \rightarrow \text{Ext}(K_{F^n}, \Sigma \oplus R) \rightarrow \text{Ext}(K_{F^n}, M) \rightarrow 0$ is splitting exact.

(2) \Rightarrow (1): It suffices to prove that $\text{Ext}(K_{F^n}, \Sigma \oplus R)$ is $C(F^n)$ -projective. We consider a diagram of the form

$$(D) \quad \begin{array}{ccccccc} & & \Sigma \oplus R & & & & \\ & \swarrow \eta & & \searrow \delta & & & \\ 0 \rightarrow \Sigma \oplus R / (\Sigma \oplus R)F^n & \xrightarrow{\delta} & \text{Ext}(K_{F^n}, \Sigma \oplus R) & \rightarrow & \text{Ext}(Q_{F^n}, \Sigma \oplus R) & \rightarrow & 0 \\ & & \downarrow g & & & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0, & & \end{array}$$

where M and $N \in C(F^n)$, f is an epimorphism, δ is the connecting homomorphism and η is the natural homomorphism. Then there is a homomorphism $h: \Sigma \oplus R \rightarrow M$ such that $g\delta = fh$. The homomorphism h and η yield the commutative diagram by Lemmas 2.4 and 2.6:

$$\begin{array}{ccc} M & \xrightarrow{\delta_1} & \text{Ext}(K_{F^n}, M) \\ \uparrow h & & \uparrow h_* \\ \Sigma \oplus R & \xrightarrow{\delta} & \text{Ext}(K_{F^n}, \Sigma \oplus R) \\ \downarrow \eta & & \downarrow \eta_* \\ (\Sigma \oplus R) / (\Sigma \oplus R)F^n & \xrightarrow{\delta_2} & \text{Ext}(K_{F^n}, (\Sigma \oplus R) / (\Sigma \oplus R)F^n). \end{array}$$

We put $\bar{h} = \delta_1 h_* \eta_*^{-1} \delta_2$. Then $h = \bar{h} \eta$. The upper row in the diagram (D) is exact and is EF^n -pure by Lemmas 2.5, 2.6 and Corollary 2.7. So we have a homomorphism $k: \text{Ext}(K_{F^n}, \Sigma \oplus R) \rightarrow M$ such that $k\delta = \bar{h}$. Hence $g\delta = fh = f\bar{h}\eta = fk\delta\eta = fk\delta$. So $g - fk$ induces a homomorphism $\overline{g - fk}: \text{Ext}(K_{F^n}, \Sigma \oplus R) / \delta(\Sigma \oplus R) \rightarrow N$. On the other hand $\text{Ext}(K_{F^n}, \Sigma \oplus R) / \delta(\Sigma \oplus R) \cong \text{Ext}(Q_{F^n}, \Sigma \oplus R)$ and $\text{Ext}(Q_{F^n}, \Sigma \oplus R)$ is a homomorphic image of $\text{Ext}(Q, \Sigma \oplus R)$. Hence $\text{Ext}(Q_{F^n}, \Sigma \oplus R)$ is injective. Since N is reduced, i.e., the injective submodule of N is zero, $\overline{g - fk} = \bar{0}$ and so $g = fk$. Hence $\text{Ext}(K_{F^n}, \Sigma \oplus R)$ is $C(F^n)$ -projective.

Corollary 2.10. *There is one-to-one between all maximal F^n -torsion modules and all $C(F^n)$ -projective objects in the category $C(F^n)$.*

REMARK. (1) Let $C(F^\infty)$ be the category of F -reduced and F^∞ -pure injective modules together with their homomorphisms. Then the corresponding results to Theorem 2.9 also hold for the category $C(F^\infty)$, where $\hat{R}_{F_i^\infty} = \hat{R}_{F_i}$ and $K_{F^\infty} = K_F$. Further a module G is $C(F^\infty)$ -projective if and only if $G \in C(F^\infty)$ and G is F -torsion-free by Proposition 2.3 of [9].

(2) $C(F^n)$ -projective objects are not necessarily F -torsion-free. For example, let P be a prime ideal of R and let $F_P = \{I \mid I \supseteq P^k \text{ for some } k\}$. Then the $C(F_P^n)$ -projective object $\text{Ext}(K_{F_P^n}, R)$ is F_P -torsion, because $\text{Ext}(K_{F_P^n}, R) \cong R/P^n$ by Proposition 3.2 of [13].

(3) $C(F^n)$ -projective objects are not necessarily F -torsion. For example, let R be a simple hereditary noetherian prime ring and let F be any non trivial right additive topology. Then $0 \rightarrow R \rightarrow \text{Ext}(K_{F^n}, R)$ is exact, because $RF^n = 0$. Thus the $C(F^n)$ -projective object $\text{Ext}(K_{F^n}, R)$ is not F -torsion.

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