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MODULES OVER DEDEKIND PRIME RINGS. V

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Let R be a Dedekind prime ring, let F be a non-trivial right additive topology on R and let F_l be the left additive topology corresponding to F (cf. [8]). For any positive integer n, let F^n be the set of all right ideals containing a finite intersection of elements in F, each of which has at most n as the length of composition series of its factor module. An exact sequence $0\rightarrow L\rightarrow M\rightarrow N\rightarrow 0$ of right R-modules is EF^n-pure if the induced sequence $0\rightarrow \mathrm{Ext}(\underset{\sim}{Q}_{F^n}/R, L)\rightarrow \mathrm{Ext}(Q_{F^n}/R, M)\rightarrow \mathrm{Ext}(Q_{F^n}/R, N)\rightarrow 0$ is splitting exact, where $Q_{F^n}=\varinjlim I^{-1}$ (I ranges over all elements in F^n). If R is the ring of integers, p is a prime number and F is the topology of all powers of p, then EF^n-purity is equivalent to p^n-purity in the sense of [12].

The aim of this paper is to investigate the structure of EF^n-pure injective modules. In Section 1, a notion of maximal F^n-torsion modules will be introduced. It is shown, in Theorem 1.10, that there is a duality between all maximal F^n-torsion modules and all direct summands of direct products of copies of $\hat{R}_{F_l^n}$ by using the results in [9], where $\hat{R}_{F_l^n}=\varprojlim R/J$ ($J\in F_l^n$). In Section 2, we shall study the category $C(F^n)$ of F^n-reduced, EF^n-pure injective modules. After discussing some properties of EF^n-puries and F^n-purities we shall give, in Theorem 2.9, characterizations of projective objects in the category $C(F^n)$. In particular, it is established that a module is a direct summand of a direct product of copies of $\hat{R}_{F_l^n}$ if and only if it is a projective object in $C(F^n)$. F is bounded if each element of F contains a non-zero ideal of R. If F is bounded, then $\hat{R}_{F_l^n}=\Pi R/P^n$, where P ranges over all prime ideals contained in F. So our results may essentially be interesting in case F contains completely faithful right ideals of R in the sense of [3].

1. The Harrison duality

Throughout this paper, R will be a Dedekind prime ring with the two-sided quotient ring Q and $K=Q/R\neq 0$. By a module we shall understand a unitary right R-module. In place of $\otimes_R$, Hom_R, Ext_R and Tor^R, we shall just write $\otimes$, Hom, Ext and Tor, respectively. Since R is hereditary, $\mathrm{Tor}_n=0$ $=\mathrm{Ext}^n$ for all $n>1$ and so we shall use Ext for Ext^1 and Tor for Tor_1. Let I be an essential right ideal of R. Then R/I is an artinian module by Theorem 1.3 of [3]. So the length of the composition series of the module R/I is finite. We call it the *length* of I. Let F be any non-trivial right additive topology, then F consists of essential right ideals of R (cf. p. 548 in [8]). For any positive integer n, let F^n be the set of all right ideals containing a finite intersection of elements in F, each of which has at most n as the length. Let M be a module. An element m of M is said to be *F^n-torsion* if $O(m)=\{r\in R\,|\,mr=0\}\in F^n$, and we denote the set of all F^n-torsion elements in M by M_{F^n}. M_{F^n} is a submodule of M, because F^n is a pretopology on R. Following [8], we shall denote the left additive topology corresponding to F by F_l. In a similar way we can define the concepts of *F_l^n-torsion elements* and *F_l^n-torsion sumbodules* for left modules. We put $Q_{F^n}=\varinjlim I^{-1}$ $(I\in F^n)$ and $Q_{F_l^n}=\varinjlim J^{-1}$ $(J\in F_l^n)$.

Concerning the terminology we refer to [8] and [9].

Lemma 1.1. (1) *$Q_{F^n}=Q_{F_l^n}$ and so Q_{F^n} is an (R, R)-bimodule.*
(2) $K_{F^n}=Q_{F^n}/R=K_{F_l}$

Proof. (1) We shall prove that $Q_{F_n}\supseteqq Q_{F_l^n}$. To prove this let J be any element of F_l^n with length $J\leqq n$. Then the length of the composition series of the module J^{-1}/R is at most n. By Proposition 1.4 of [8], J^{-1}/R is F-torsion. Hence, for every element $q\in J^{-1}$, we have $qI_q\subseteqq R$ for some $I_q\in F$ with length $I_q\leqq n$. Hence $q\in qI_qI_q^{-1}\subseteqq RI_q^{-1}\subseteqq Q_{F^n}$ and thus $J^{-1}\subseteqq Q_{F^n}$. So $Q_{F_l^n}\subseteqq Q_{F^n}$ by Lemma 4.8 of [5]. Similarly $Q_{F_l^n}\supseteqq Q_{F^n}$ and thus $Q_{F_l^n}=Q_{F^n}$.
(2) is evident from (1).

The exact sequence $0\to R\xrightarrow{\iota} Q_{F^n}\to K_{F^n}\to 0$ yields the exact sequences:

$$0\to \mathrm{Tor}(M, K_{F^n})\to M\xrightarrow{\iota_*} M\otimes Q_{F^n}\to M\otimes K_{F^n}\to 0\,,$$
$$\mathrm{Hom}(K_{F^n}, M)\to \mathrm{Hom}(Q_{F^n}, M)\xrightarrow{\iota^*} M\to \mathrm{Ext}(K_{F^n}, M)\,,$$

where $\iota_*(m)=m\otimes 1$ and $\iota^*(f)=f(1)$ ($m\in M$ and $f\in \mathrm{Hom}(Q_{F^n}, M)$).

Lemma 1.2. (1) *$Tor(M, K_{F_n})\simeq M_{F^n}$.*
(2) *If M is F^n-torsion, then $M\otimes Q_{F^n}\simeq M\otimes K_{F^n}$.*
(3) *$Im\ \iota^*\subseteqq \cap MJ$ $(J\in F_l^n)$.*

Proof. (1) is obtained by the similar way as in Theorem 3.2 of [11], and (2) is evident from (1) and the above exact sequence.

(3) Let J be any element of F_l^n. Then from the exact sequence $0\to R\to J^{-1}\to J^{-1}/R\to 0$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}(Q_{F^n}, M) & \to & M & \to & \mathrm{Ext}(K_{F^n}, M) \\ \downarrow & & \| & & \downarrow \\ \mathrm{Hom}(J^{-1}, M) & \to & M & \to & \mathrm{Ext}(J^{-1}/R, M)\,. \end{array}$$

From this diagram and Proposition 3.2 of [13], we get $\operatorname{Im} \iota^* \subseteq MJ$ and so $\operatorname{Im} \iota^* \subseteq \cap MJ$.

We denote the submodule $\operatorname{Im} \iota^*$ of the module M by MF^n, and if $MF^n = 0$, then M is said to be *F^n-reduced*. If M is F^n-reduced, then it is F-reduced in the sense of [9].

Lemma 1.3. *Let $0 \to L \to M \to N \to 0$ be an exact sequence such that Ext $(Q_{F^n}, L) = 0$ and M is F^n-reduced. Then N is F^n-reduced.*

Proof. This is evident from the following commutative diagram with exact columns:

$$\begin{array}{ccc}
\operatorname{Hom}(Q_{F^n}, M) & \to & M \\
\downarrow & & \downarrow \\
\operatorname{Hom}(Q_{F^n}, N) & \to & N \\
\downarrow & & \downarrow \\
0 & & 0 .
\end{array}$$

For any module M we denote by M_F the submodule of F-torsion elements in M. If $M_F = 0$, then we say that M is *F-torsion-free*.

Following [9], an exact sequence $0 \to L \to M \to N \to 0$ is *F^∞-pure* if the induced sequence $0 \to L_F \to M_F \to N_F \to 0$ is splitting exact. A module is *F^∞-pure injective* if it has the injective property to the class of F^∞-pure exact sequences. We denote the injective hull of a module M by $E(M)$, and the F-injective hull of M by $E_F(M)$. By the results in §1 of [9], we have the following:

(1) A module G is F^∞-pure injective if and only if $G \cong E(GF^\infty) \oplus \operatorname{Ext}(K_F, G)$, where GF^∞ is the maximal F_l-divisible submodule of G.

(2) For a module G, the following are equivalent:

(i) G is F-reduced and F^∞-pure injective.

(ii) δ: $G \cong \operatorname{Ext}(K_F, G)$, where δ is the connecting homomorphism.

(iii) G is F-reduced and $\operatorname{Ext}(Q_F, G) = 0$.

(iv) G is F-reduced and $\operatorname{Ext}(X, G) = 0$ for every F-torsion-free module X.

These results will be used in this paper without references.

Lemma 1.4. *Let M be a module. Then $H_n = Hom(K_{F^n}, M)$ and $G_n = Ext(K_{F^n}, M)$ are both F^n-reduced and F^∞-pure injective.*

Proof. (1) H_n is F^∞-pure injective by Proposition 5.1 of [13] and Proposition 1.4 of [9]. Further, from the exact sequence $Q_{F^n} \to K_{F^n} \to 0$ and Proposition 5.2′ of [2, Chap. II], we get the following commutative diagram with exact rows:

$$\begin{array}{ccc}
0 \to \operatorname{Hom}(K_{F^n}, H_n) & \longrightarrow & \operatorname{Hom}(Q_{F^n}, H_n) \\
\wr\Vert & f & \wr\Vert \\
0 \to \operatorname{Hom}(K_{F^n} \otimes K_{F^n}, M) & \to & \operatorname{Hom}(Q_{F^n} \otimes K_{F^n}, M) .
\end{array}$$

By Lemma 1.2, f is an isomorphism and so H_n is F^n-reduced.

(2) From the exact sequence $0\to M\to E(M)\xrightarrow{g} E(M)/M\to 0$ we derive an exact sequence $0\to H_n\to \mathrm{Hom}(K_{F^n}, E(M))\xrightarrow{g_*}\mathrm{Hom}(K_{F^n}, E(M)/M)\to G_n\to 0$. Since $\mathrm{Hom}(K_{F^n}, E(M))$ is F-reduced and F^∞-pure injective, $\mathrm{Ext}(Q_{F^n}, \mathrm{Hom}(K_{F^n}, E(M)))=0$. So $\mathrm{Ext}(Q_{F^n}, \mathrm{Im}\, g_*)=0$ and thus G_n is F^n-reduced by (1) and Lemma 1.3. By Proposition 3.5a of [2, Chap. VI], we have $\mathrm{Ext}(Q_F, G_n)\cong \mathrm{Ext}(\mathrm{Tor}(Q_F, K_{F^n}), M)=0$. Therefore G_n is F^∞-pure injective.

Let M be any module. From the exact sequence $0\to R\to Q_{F^n}\to K_{F^n}\to 0$ we have the exact sequences:

$$\mathrm{Ext}(K_{F^n}, M)\xrightarrow{\delta'}\mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M))\to \mathrm{Ext}(Q_{F^n}, \mathrm{Ext}(K_{F^n}, M)),$$
$$M\xrightarrow{\delta}\mathrm{Ext}(K_{F^n}, M)\to \mathrm{Ext}(Q_{F^n}, M).$$

The second exact sequence yields a homorphism δ_*: $\mathrm{Ext}(K_{F^n}, M)\to \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M))$.

Lemma 1.5. *δ' and δ_* are both isomorphisms.*

Proof. From the exact sequence $0\to R\to Q_{F^n}\to K_{F^n}\to 0$ we have the isomorphism: $\mathrm{Tor}(K_{F^n}, K_{F^n})\cong R\otimes K_{F^n}$. Applying Theorem 2.1 of [11] we get the commutative diagram:

$$\begin{array}{ccc}\mathrm{Ext}(R\otimes K_{F^n}, M) & \cong & \mathrm{Ext}(\mathrm{Tor}(K_{F^n}, K_{F^n}), M)\\ \wr\| & & \wr\| \\ \mathrm{Hom}(R, \mathrm{Ext}(K_{F^n}, M)) & \xrightarrow{\delta'} & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M)).\end{array}$$

Hence δ' is an isomorphism.

From Theorem 1.5 of [9] we obtain the following commutative diagram with exact row:

$$\begin{array}{ccccccccc}0 & \to & M & \xrightarrow{h} & E(MF^\infty)\oplus \mathrm{Ext}(K_F, M) & \to & \mathrm{Coker}\, h & \to & 0\\ & & \| & & \downarrow p & & & & \\ & & M & \xrightarrow{\delta_1} & \mathrm{Ext}(K_F, M), & & & & \end{array}$$

where p is the projection and δ_1 is the connecting homomorphism. Since $\mathrm{Coker}\, h$ is F-torsion-free and injective, applying $\mathrm{Ext}(K_{F^n},\ )$ to the diagram we have the isomorphism δ_{1*}: $\mathrm{Ext}(K_{F^n}, M)\cong \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_F, M))$. From the exact sequence $0\to K_{F^n}\xrightarrow{\theta} K_F$ we have the commutative diagram:

$$\begin{array}{ccc}M & \xrightarrow{\delta_1} & \mathrm{Ext}(K_F, M)\\ \| & & \downarrow \theta^* \\ M & \xrightarrow{\delta} & \mathrm{Ext}(K_{F^n}, M).\end{array}$$

From this diagram we get the commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}(K_{F^n}, M) & \overset{\delta_{1*}}{\cong} & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_F, M)) \\ \| & & \downarrow (\theta^*)_* \\ \mathrm{Ext}(K_{F^n}, M) & \overset{\delta_*}{\to} & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M)) . \end{array}$$

By Proposition 3.5a of [2, Chap. VI] and Lemma 1.2, $(\theta^*)_*$ is an isomorphism and so δ_* is also an isomorphism.

From the inclusion map θ: $K_{F^n} \to K_F$, we get the epimorphism θ^*: Ext $(K_F, M) \to \mathrm{Ext}(K_{F^n}, M)$ for any module M.

Lemma 1.6. *Ker* $\theta^* = \mathit{Ext}(K_F, M)F^n$.

Proof. We consider the following commutative diagram with exact row:

$$\begin{array}{ccccc} 0 \to \mathrm{Ker}\,\theta^* & \longrightarrow & \mathrm{Ext}(K_F, M) & \longrightarrow & \mathrm{Ext}(K_{F^n}, M) \to 0 \\ \downarrow & & \downarrow & & \downarrow \delta' \\ \mathrm{Ext}(K_{F^n}, \mathrm{Ker}\,\theta^*) & \to & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_F, M)) & \xrightarrow{(\theta^*)_*} & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M)) . \end{array}$$

By Lemma 1.5 and its proof, δ' and $(\theta^*)_*$ are both isomorphisms. Hence Ker $\theta^* = \mathrm{Ext}(K_F, M)F^n$.

Lemma 1.7. *Let N be an F_l-torsion let module. Then $E_{F_l}(N)$ is a direct summand of a direct sum of copies of K_F.*

Proof. Since $E(N)$ is a torsion left module, there is a torsion left module L such $E(N) \oplus L = \Sigma \oplus K$. So $\Sigma \oplus K_F = \Sigma \oplus K_{F_l} = (\Sigma \oplus K)_{F_l} = E(N)_{F_l} \oplus L_{F_l}$ by Proposition 1.4 of [8]. It is evident that $E(N)_{F_l} \subseteq E_{F_l}(N)$ by Proposition 6.3 of [14]. Since N is F_l-torsion, the converse inclusion also holds. Thus $E_{F_l}(N)$ is a direct summand of $\Sigma \oplus K_F$.

An F-torsion module D is said to be *maximal F^n-torsion* provided $(E(D))_{F^n} = D$. This is clearly equivalent to $(E_F(D))_{F^n} = D$. For any module M we define $\hat{M}_{F_l^n} = \varprojlim M/MJ$ $(J \in F_l^n)$. Then $\hat{R}_{F_l^n}$ becomes a ring and $\hat{M}_{F_l^n}$ becomes an $\hat{R}_{F_l^n}$-module by the similar way as in §4 of [13]. Let α: $M \to \hat{M}_{F_l^n}$ be the canonical map. Then Ker $\alpha = \cap MJ$ $(J \in F^n)$.

Lemma 1.8. *Let M be an F-torsion-free module. Then*

(1) *$M \otimes K_{F^n}$ is maximal F^n-torsion.*

(2) *There are isomorphism $\hat{M}_{F_l^n} \cong Hom(K_{F^n}, M \otimes K_{F^n}) \cong Ext(K_{F^n}, M)$ such that the diagram*

$$\begin{array}{ccccc} M & = & M & = & M \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \delta \\ \hat{M}_{F_l^n} & \cong & Hom(K_{F^n}, M \otimes K_{F^n}) & \cong & Ext(K_{F^n}, M) \end{array}$$

commutes, where $\beta(m)\,(\bar{q})=m\otimes\bar{q}$ $(m\in M,\ \bar{q}\in K_{F^n})$ *and* δ *is the connecting homomorphism.*

Proof. (1) The commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & R & \to & Q_{F^n} & \to & K_{F^n} & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\theta} & & \\ 0 & \to & R & \to & Q_F & \to & K_F & \to & 0 \end{array}$$

yields the commutative diagram with exact rows:

$$(A):\quad \begin{array}{ccccccccc} 0 & \to & M & \to & M\otimes Q_{F^n} & \to & M\otimes K_{F^n} & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\kappa_*} & & \downarrow{\scriptstyle\theta_*} & & \\ 0 & \to & M & \to & M\otimes Q_F & \to & M\otimes K_F & \to & 0\,. \end{array}$$

By Proposition 1.1, Lemma 2.5 of [8] and Lemma 1.7, $\mathrm{Tor}(M,\ Q_F/Q_{F^n})=0$. Hence κ_* is a monomorphism and so θ_* is also a monomorphism. Let x be any element in $M\otimes K_F$ such that $I=O(x)\in F^n$ and let y be an element in $M\otimes Q_F$ mapping on x. Then $yI\subseteq M$ and so $y\in M\otimes I^{-1}$ in $M\otimes Q_F$. This implies that $y\in M\otimes Q_{F^n}$ and thus $x\in M\otimes K_{F^n}$. Hence $(M\otimes K_F)_{F^n}=M\otimes K_{F^n}$. Therefore $M\otimes K_{F^n}$ is maximal F^n-torsion, because $M\otimes K_F$ is F-injective.

(2) By the similar way as in Lemma 2.7 of [8], we have $\mathrm{Hom}(K_{F^n},\ M\otimes K_F)\cong\mathrm{Ext}(K_{F^n},\ M)$ such that the diagram

$$\begin{array}{ccc} M & = & M \\ \downarrow & & \downarrow \\ \mathrm{Hom}(K_F,\ M\otimes K_F) & \cong & \mathrm{Ext}(K_F,\ M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(K_{F^n},\ M\otimes K_F) & \cong & \mathrm{Ext}(K_{F^n},\ M) \end{array}$$

commutes. Since $(M\otimes K_F)_{F^n}=M\otimes K_{F^n}$, we have $\mathrm{Hom}(K_{F^n},\ M\otimes K_F)\cong \mathrm{Hom}(K_{F^n},\ M\otimes K_{F^n})$. This is the proof of the assertion to the right diagram. Next we consider the following commutative diagram with exact right column:

$$\begin{array}{ccl} \hat{M}_{F_l} & \xrightarrow{\eta} & \mathrm{Hom}(K_F,\ M\otimes K_F) \\ \downarrow & & \downarrow \\ \hat{M}_{F_l^n} & \xrightarrow{\eta'} & \mathrm{Hom}(K_{F^n},\ M\otimes K_F)\ (=\mathrm{Hom}(K_{F^n},\ M\otimes K_{F^n})) \\ & & \downarrow \\ & & 0\,, \end{array}$$

where $\eta(\hat{m})\,(\bar{q})=m_J\otimes\bar{q}$ $(\hat{m}=([m_J+MJ]),\ \bar{q}=[q+R]$ and $q\in J^{-1})$, η' is the homomorphism induced by η and the map: $\hat{M}_{F_l}\to\hat{M}_{F_l^n}$ is the natural homomorphism. By Lemma 2.7 of [8], η is an isomorphism. If $\eta'(\hat{m})=0$, where

$\hat{m}=([m_J+MJ])\in\hat{M}_{F_I^n}$, then $m_J\otimes J^{-1}/R=0$ in $M\otimes K_{F^n}$. This implies that $m_J\otimes J^{-1}\subseteq M$ by the diagram (A) and so $m_J\in MJ$. Hence $\hat{m}=0$ and thus η' is an isomorphism. The commutativity of the left diagram is clear.

Lemma 1.9. *Let M be an F-torsion-free module. Then $Ext(K_{F^n}, M)$ is isomorphic to a direct summand of a direct product of copies of $\hat{R}_{F_I^n}$.*

Proof. Since $M_F=0$, the exact sequence $0\to\mathrm{Ker}\,f\to\Sigma\oplus R\xrightarrow{f}M\to 0$ is F^∞-pure and so the sequence $0\to\mathrm{Ext}(K_F, \mathrm{Ker}\,f)\to\mathrm{Ext}(K_F, \Sigma\oplus R)\to\mathrm{Ext}(K_F, M)\to 0$ is splitting exact by Lemma 1.3 of [9]. By Proposition 3.5a of [2, Chap. VI] and Lemma 1.2, this sequence yields the splitting exact sequence $0\to\mathrm{Ext}(K_{F^n}, \mathrm{Ker}\,f)\to\mathrm{Ext}(K_{F^n}, \Sigma\oplus R)\to\mathrm{Ext}(K_{F^n}, M)\to 0$. So it suffices to prove that $\mathrm{Hom}(K_{F^n}, \Sigma\oplus K_{F^n})$ is a direct summand of $\mathrm{Hom}(K_{F^n}, \Pi\, K_{F^n})$ by Lemma 1.8. To prove this we consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \to & \Sigma\oplus K_{F^n} & \to & (\Pi\, K_{F^n})_{F^n} \\
 & & \downarrow & & \downarrow \\
0 & \to & \Sigma\oplus K_F & \to & (\Pi\, K_F)_F\,.
\end{array}$$

The second row splits, because $\Sigma\oplus K_F$ is F-injective. Since $(\Sigma\oplus K_F)_{F^n}=\Sigma\oplus K_{F^n}$, the splitting map induces an splitting map of the first row. Hence $\mathrm{Hom}(K_{F^n}, \Sigma\oplus K_{F^n})$ is a direct summand of $\mathrm{Hom}(K_{F^n}, (\Pi\, K_{F^n})_{F^n})=\mathrm{Hom}(K_{F^n}, \Pi\, K_{F^n})$.

Theorem 1.10 (The Harrison duality). *The correspondence*

$$(B)\qquad D\to G=Hom(K_{F^n}, D)$$

is one-to-one between all maximal F^n-torsion modules D and all direct summands G of direct products of copies of $\hat{R}_{F_I^n}$. The inverse of (B) is given by the correspondence $G\to G\otimes K_{F^n}$.

Proof. (i) Let D be maximal F^n-torsion and let $H=\mathrm{Hom}(K_F, E_F(D))$. Then H is F-torsion-free, F^∞-pure injective and $\eta\colon H\otimes K_F\cong E_F(D)$ by Theorem 2.2 of [9], where $\eta(x\otimes\bar{q})=x(\bar{q})$ ($x\in H$ and $\bar{q}\in K_F$). From the exact sequence $0\to K_{F^n}\xrightarrow{\theta}K_F$, we get the exact sequence: $H\xrightarrow{\theta^*}\mathrm{Hom}(K_{F^n}, E_F(D))$ $(=\mathrm{Hom}(K_{F^n}, D))\to 0$. This yields the commutative diagram:

$$\begin{array}{ccc}
H\otimes K_F & \cong & E_F(D) \\
\uparrow\theta_* & & \uparrow \\
H\otimes K_{F^n} & \xrightarrow{\eta'} & D \\
 & \searrow(\theta^*)_*\quad\nearrow\varphi & \\
 & \mathrm{Hom}(K_{F^n}, D)\otimes K_{F^n}\,, &
\end{array}$$

where $\varphi(x\otimes\bar{q})=x(\bar{q})$ and η' is the map induced by η. Since H is F-torsion-free, θ_* is a monomorphism and $H\otimes K_{F^n}$ is maximal F^n-torsion. Hence η' is an isomorphism, and $\varphi(\theta^*)_*=\eta'$ implies that φ is also an isomorphism, because $(\theta^*)_*$ is an epimorphism. From $D\cong H\otimes K_{F^n}$, we have $\mathrm{Hom}(K_{F^n}, D)$ $\cong\mathrm{Hom}(K_{F^n}, H\otimes K_{F^n})\cong\mathrm{Ext}(K_{F^n}, H)$ by Lemma 1.8. Hence $\mathrm{Hom}(K_{F^n}, D)$ is a direct summand of a direct product of copies of $\hat{R}_{F^n_l}$ by Lemma 1.9.

(ii) Let G be a direct summand of a direct product of copies of $\hat{R}_{F^n_l}$. Then we may assume from Lemma 1.8 that $G\oplus X=\mathrm{Hom}(K_{F^n}, \Pi K_{F^n})=\mathrm{Hom}(K_{F^n}, (\Pi K_{F^n})_{F^n})$, where X is a module. Since $(\Pi K_{F^n})_{F^n}$ is maximal F^n-torsion, we get, by (i), the isomorphism φ: $(G\otimes K_{F^n})\oplus(X\otimes K_{F^n})\cong(\Pi K_{F^n})_{F^n}$. Hence $G\otimes K_{F^n}$ is maximal F^n-torsion. Applying $\mathrm{Hom}(K_{F^n},\ )$ to the isomorphism we obtain the isomorphism φ_*: $\mathrm{Hom}(K_{F^n}, G\otimes K_{F^n})\oplus\mathrm{Hom}(K_{F^n}, X\otimes K_{F^n})\cong G\oplus X$. We may define λ: $G\oplus X\rightarrow\mathrm{Hom}(K_{F^n}, G\otimes K_{F^n})\oplus\mathrm{Hom}(K_{F^n}, X\otimes K_{F^n})$ by $\lambda(g+x)(\bar{q})=(g\otimes\bar{q})+(x\otimes\bar{q})$, where $g\in G$, $x\in X$ and $\bar{q}\in K_{F^n}$. Then it follows that $\varphi_*\lambda=1$ and that $\lambda(G)\subseteq\mathrm{Hom}(K_{F^n}, G\otimes K_{F^n})$, $\lambda(X)\subseteq\mathrm{Hom}(K_{F^n}, X\otimes K_{F^n})$. Hence $G\cong\mathrm{Hom}(K_{F^n}, G\otimes K_{F^n})$.

This duality was first exhibited by Harrison in [4] between all divisible, torsion abelian groups and all reduced, torsion-free cotorsion abelian groups. This duality was generalized by Matlis [10] to modules over commutative integral domains. To modules over non-commutative complete discrete valuation rings the result was established by Liebert [6]. The author generalized it in [9] to the case of modules over Dedekind prime rings.

2. Projective objects of the category of F^n-reduced, EF^n-pure injective modules

In this section we shall define a notion of EF^n-pure injective modules and give characterizations of direct summands of direct products of copies of $\hat{R}_{F^n_l}$ which were discussed in §1.

A short exact sequence

$$(\mathrm{E}):\quad 0\rightarrow L\rightarrow M\rightarrow N\rightarrow 0$$

of modules is said to be *F^n-pure* if $MJ\cap L=LJ$ for all $J\in F^n_l$. (E) is said to be *Ext-F^n-pure* (abbr. *EF^n-pure*) if the induced sequence $0\rightarrow\mathrm{Ext}(K_{F^n}, L)\rightarrow\mathrm{Ext}(K_{F^n}, M)\rightarrow\mathrm{Ext}(K_{F^n}, N)\rightarrow 0$ is splitting exact. A module G is *EF^n-pure injective* if it has the injective property relative to the class of EF^n-pure exact sequences. A right additive topology F is *bounded* if any element of F contains a non-zero ideal of R.

Lemma 2.1. (1) *If (E) is EF^n-pure, then it is F^n-pure.*
(2) *If (E) is F^∞-pure, then it is EF^n-pure.*

(3) *If F is bounded and (E) is F^n-pure, then it is EF^n-pure.*

Proof. (1) If (E) is EF^n-pure, then the induced sequence $0 \to \mathrm{Ext}(K_{F^n}, L) \to \mathrm{Ext}(K_{F^n}, M)$ is splitting exact. Let J be any element of F_l^n. Then we get the following commutative diagram with splitting exact rows:

$$\begin{array}{ccccc}
0 \to & \mathrm{Ext}(J^{-1}/R, \mathrm{Ext}(K_{F^n}, L)) & \to & \mathrm{Ext}(J^{-1}/R, \mathrm{Ext}(K_{F^n}, M)) \\
& \wr\| & & \wr\| \\
0 \to & \mathrm{Ext}(\mathrm{Tor}(J^{-1}/R, K_{F^n}), L) & \to & \mathrm{Ext}(\mathrm{Tor}(J^{-1}/R, K_{F^n}), M) \\
& \wr\| & & \wr\| \\
0 \longrightarrow & \mathrm{Ext}(J^{-1}/R, L) & \longrightarrow & \mathrm{Ext}(J^{-1}/R, M) \\
& \wr\| & & \wr\| \\
0 \longrightarrow & L/LJ & \longrightarrow & M/MJ\,.
\end{array}$$

Hence (E) is F^n-pure.

(2) If (E) is F^∞-pure, then the induced sequence $0 \to \mathrm{Ext}(K_F, L) \to \mathrm{Ext}(K_F, M)$ is splitting exact by Lemma 1.3 of [9]. Applying to $\mathrm{Ext}(K_{F^n},\ )$ the above sequence we get the splitting exact sequence $0 \to \mathrm{Ext}(K_{F^n}, L) \to \mathrm{Ext}(K_{F^n}, M)$ by the same way as in (1). Therefore (E) is EF^n-pure.

(3) Let P be a prime ideal of R. Then the set $F_P = \{I \mid I \supseteq P^k$ for some non-negative integer k, I is a right ideal of $R\}$ is a right additive topology. We shall prove that $F_P^n = \{I \mid I \supseteq P^n$ and $I \in F_P\}$. It is well known that $R/P^n = (D)_m$, where D is a completely primary ring with the Jacobson radical $J(D)$ such that one-sided ideals of D are only $\{J(D)^l \mid l = 0, 1, 2, \cdots, n\}$ (cf. Theorem 4.32 of [1]). we can easily see that $P^n \in F_P^n$. Let I be any element of F_P^n. To prove that $I \supseteq P^n$, it suffices to prove it in case the length of I is k ($k \leqq n$). If $k=1$ and $I \not\supseteq P$. Then $I+P=R$. Since $I \in F_P$, we may assume that $I \not\supseteq P^{i-1}$ and $I \supseteq P^i$ for some natural number i. It follows that $P^{i-1} = (I+P)P^{i-1} \subseteq I$, which is a contradiction. Hence $I \supseteq P$. Assume that $k > 1$. Let I_0 be any element of F_P such that $I_0 \supsetneqq I$ and the length of I_0 is $k-1$. By induction assumption, we have $I_0 \supseteq P^{k-1}$. Write $I_0 = aR + I$. Then we have $a^{-1}I \supseteq P$, since the length of $a^{-1}I$ is 1. Thus we get $P^k = P^{k-1}P \subseteq I_0 P \subseteq I$. Hence $I \supseteq P^n$, as desired.

Now let F be a bounded right additive topology. Then by Proposition 1.2 of [8], F is determined by a class $\{S_\gamma \mid \gamma \in \Gamma\}$ of simple modules and each S_γ is annihilated by a prime ideal P_γ of R, since F is bounded. Further, we obtain that a prime ideal P of R is an element in F if and only if simple modules in R/P are isomorphic to S_γ for some $\gamma \in \Gamma$, because R/P is a simple and artinian ring. So $F = \{I \mid I \supseteq P_1^{n_1} \cap \cdots \cap P_k^{n_k}$, where $P_i \in F$ and $n_1, \cdots, n_k$ are non-negative integers$\}$ by Proposition 1.2 of [8]. Thus we have $F^n = \{I \mid I \supseteq P_1^n \cap \cdots \cap P_k^n$, where $P_i \in F\}$ and so $K_{F^n} = \Sigma \oplus P^{-n}/R$, where P ranges over all prime ideals contained in F. If (E) is F^n-pure, then it is P^n-pure in the sense of [7] for any $P\in F$ and so the sequence $0\to L/LP^n\to M/MP^n$ splits by Lemma 1.1 of [7]. Hence we get the commutative diagram with splitting exact rows:

$$\begin{array}{ccc} 0\to \Pi\, L/LP^n & \longrightarrow & \Pi\, M/MP^n \\ \wr\| & & \wr\| \\ 0\to \mathrm{Ext}(K_{F^n}, L) & \to & \mathrm{Ext}(K_{F^n}, M)\,. \end{array}$$

Therefore (E) is EF^n-pure.

Lemma 2.2. *The following conditions of a short exact sequence* (E): $0\to L\to M\to N\to 0$ *are equivalent*:

(1) (E) *is* F^n-*pure.*

(2) *For any finitely generated* F^n-*torsion module* T, *the natural homomorphism* $Hom(T, M)\to Hom(T, N)\to 0$ *is exact.*

(3) *For any* F_l^n-*torsion left module* T, *the natural homomorphism* $0\to L\otimes T\to M\otimes T$ *is exact.*

Proof. Let I be any element of F^n. Then $I^{-1}/R\subseteq K_{F^n}$ by Lemma 1.1 and it is finitely generated. So $I^{-1}/R\cong\Sigma\oplus R/J_i$ for some $J_i\in F_l^n$ by Theorem 3.11 of [3]. Applying Hom(, K_F) to this isomorphism we get $R/I\cong\Sigma\oplus J_i^{-1}/R$ by Proposition 3.3 of [13], because $\mathrm{Hom}(R/J_i, K_F)\cong J_i^{-1}/R$. Further any finitely generated F^n-torsion module is a finite direct sum of modules $R/I\ (I\in F^n)$. Combining these facts with Lemma 5.2 of [11], we get the equivalence of (1) and (2). For any module X and any left ideal J, $(X/XJ)\cong X\otimes R/J$ and $\otimes$ commutes with direct limits. So the equivalence of (1) and (3) is also evident.

Lemma 2.3. *If a short exact sequence* $0\to L\to M\to N\to 0$ *is* F^n-*pure, then the induced sequence* $0\to L_{F^n}\to M_{F^n}\to N_{F^n}\to 0$ *is exact.*

Proof. It follows from Lemmas 1.2 and 2.2.

Lemma 2.4. *For a module* G, *the following are equivalent*:

(1) G *is* F^n-*reduced and* EF^n-*pure injective.*

(2) G *is* F^n-*reduced and* F^∞-*pure injective.*

(3) *The connecting homomorphism* $\delta\colon G\to Ext(K_{F^n}, G)$ *is an isomorphism.*

Proof. (1)$\Rightarrow$(2): It is evident from Lemma 2.1.

(2)$\Rightarrow$(3): We consider the commutative diagram:

$$\begin{array}{ccc} G & \cong & \mathrm{Ext}(K_F, G) \\ \| & & \downarrow \theta^* \\ G & \xrightarrow{\delta} & \mathrm{Ext}(K_{F^n}, G) \\ & & \downarrow \\ & & 0\,. \end{array}$$

By Lemma 1.6 and the assumption, θ^* is an isomorphism and so δ is an isomorphism.

(3)$\Rightarrow$(1): It is evident from Lemma 1.4 and definition.

Let $f: M \to E(MF^n)$ be an extension of the inclusion map $MF^n \to M$ and δ: $M \to \mathrm{Ext}(K_{F^n}, M)$ be the connecting homomorphism. We define a map $g: M \to E(MF^n) \oplus \mathrm{Ext}(K_{F^n}, M)$ by $g(m)=(f(m), \delta(m))$ for every $m \in M$.

Lemma 2.5. *The sequence*

$$0 \to M \xrightarrow{g} E(MF^n) \oplus Ext(K_{F^n}, M) \to Coker\, g \to 0$$

is exact and EF^n-pure. Further $E(MF^n) \oplus Ext(K_{F^n}, M)$ is EF^n-pure injective and Coker g is injective.

Proof. By the similar way as in Lemma 2.7 of [7], Coker g is injective. The other assertions follow from Lemmas 1.5 and 2.4.

Lemma 2.6. *Let M be any module. Then the natural homomorphism η: $M \to M/MF^n$ induces the following commutative diagram:*

$$\text{(C)} \quad \begin{array}{ccccc} M & \xrightarrow{\delta} & \mathrm{Ext}(K_{F^n}, M) & \xrightarrow{f} & \mathrm{Ext}(Q_{F^n}, M) \\ \downarrow \eta & & \wr\| \eta_1 & & \wr\| \eta_2 \\ M/MF^n & \to & \mathrm{Ext}(K_{F^n}, M/MF^n) & \to & \mathrm{Ext}(Q_{F^n}, M/MF^n). \end{array}$$

Proof. It is evident that f, η_1, η_2 are all epimorphisms. δ induces the homomorphism $\bar{\delta}$: $M/MF^n \to \mathrm{Ext}(K_{F^n}, M)$ such that $\bar{\delta}\eta=\delta$. Hence we get the commutative diagram with exact column:

$$\begin{array}{ccc} \mathrm{Ext}(K_{F^n}, M) & \xrightarrow{\delta_*} & \mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_{F^n}, M)) \\ \downarrow \eta_1 & (\bar{\delta})_* \nearrow & \\ \mathrm{Ext}(K_{F^n}, M/MF^n) & & \\ \downarrow & & \\ 0. & & \end{array}$$

By Lemma 1.5, δ_* is an isomorphism. Therefore η_1 is an isomorphism. So it follows from the diagram (C) that η_2 is also an isomorphism.

Corollary 2.7. *For any module M, M/MF^n is F^n-reduced.*

Proof. It is clear from the diagram (C).

Let $C(F^n)$ be the category of F^n-reduced and EF^n-pure injective modules together with their homomorphisms. We note that a module G is an element in $C(F^n)$ if and only if $\mathrm{Ext}(Q_F, G)=0=GF^n$ by Proposition 1.4 of [9] and Lemma 2.4.

Proposition 2.8. *$C(F^n)$ is an Abelian category.*

Proof. Let M, N be modules in $C(F^n)$ and $f: M \to N$ be a homomorphism. Then the exact sequence $0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$ yields an exact sequence: $0 = \operatorname{Hom}(Q_F, \operatorname{Im} f) \to \operatorname{Ext}(Q_F, \operatorname{Ker} f) \to \operatorname{Ext}(Q_F, M) \to \operatorname{Ext}(Q_F, \operatorname{Im} f) \to 0$. The first term is zero, because Q_F is F-injective and $\operatorname{Im} f$ is F-reduced. Therefore $\operatorname{Ext}(Q_F, \operatorname{Ker} f) = 0 = \operatorname{Ext}(Q_F, \operatorname{Im} f)$, because $\operatorname{Ext}(Q_F, M) = 0$ and so $\operatorname{Ker} f$, $\operatorname{Im} f \in C(F^n)$. Next we consider the exact sequence $0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0$. By Lemma 1.3, Coker f is F^n-reduced. Since $\operatorname{Ext}(Q_F, N) \to \operatorname{Ext}(Q_F, \operatorname{Coker} f) \to 0$ is exact, it follows that $\operatorname{Coker} f \in C(F^n)$. It is easy to prove the other axioms for Abelian categories.

A module in $C(F^n)$ is said to be *$C(F^n)$-projective* if it is a projective object in the category $C(F^n)$.

Theorem 2.9. *Let G be a module. Then the following conditions are equivalent*:

(1) *G is $C(F^n)$-projective.*

(2) *G is a direct summand of $\operatorname{Ext}(K_{F^n}, \Sigma \oplus R)$.*

(3) *G is a direct summand of $\Pi \hat{R}_{F_l^n}$.*

(4) *G is isomorphic to $\operatorname{Ext}(K_{F^n}, M)$, where M is an F-torsion-free module.*

(5) *G is a direct summand of $\operatorname{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F_l^n})$.*

Proof. We shall give the following implications: (2)$\Leftrightarrow$(5) and (1)$\Leftrightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(2).

(2)$\Leftrightarrow$(5): By Lemmas 1.8 and 2.5, the exact sequence $0 \to R \xrightarrow{g} E(RF^n) \oplus \hat{R}_{F_l^n} \to \operatorname{Coker} g \to 0$ is EF^n-pure and Coker g is divisible. So it is F^n-pure by Lemma 2.1. Hence the exact sequence $0 \to \Sigma \oplus R \xrightarrow{\Sigma \oplus g} \Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F_l^n} \xrightarrow{k} \operatorname{Coker}(\Sigma \oplus g) \to 0$ is F^n-pure and $\operatorname{Coker}(\Sigma \oplus g)$ is divisible. Applying $\operatorname{Hom}(K_{F^n},\ )$ to the exact sequence we obtain the exact sequence $0 \to \operatorname{Hom}(K_{F^n}, (\Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F^n})) \xrightarrow{k_*} \operatorname{Hom}(K_{F^n}, \operatorname{Coker}(\Sigma \oplus g)) \to \operatorname{Ext}(K_{F^n}, \Sigma \oplus R) \to \operatorname{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F_l^n}) \to 0$. On the other hand F^n-purity of the exact sequence yields the isomorphism $\bar{k}: (\Sigma \oplus E(RF^n) \oplus \Sigma \oplus \hat{R}_{F_l^n})_{F^n} \cong (\operatorname{Coker}(\Sigma \oplus g))_{F^n}$ by Lemma 2.3. So k_* is an isomorphism and thus we get $\operatorname{Ext}(K_{F^n}, \Sigma \oplus R) \cong \operatorname{Ext}(K_{F^n}, \Sigma \oplus \hat{R}_{F_l})$.

(1)$\Rightarrow$(2): An exact sequence $0 \to \operatorname{Ker} f \to \Sigma \oplus R \xrightarrow{f} G \to 0$ yields the exact sequence $0 \to \operatorname{Ker} f_* \to \operatorname{Ext}(K_{F^n}, \Sigma \oplus R) \xrightarrow{f_*} \operatorname{Ext}(K_{F^n}, G)\ (\cong G) \to 0$. Since $\operatorname{Ext}(K_{F^n}, \Sigma \oplus R)$, $G \in C(F^n)$, we have $\operatorname{Ker} f_* \in C(F^n)$ by Proposition 2.8. Hence, by assumption, the sequence splits.

(2)$\Rightarrow$(3): This is clear from Lemma 1.9.

(3)$\Rightarrow$(4): By Theorem 1.10 $G \cong \operatorname{Hom}(K_{F^n}, D)$, where D is a maximal F^n-torsion module. We let $H = \operatorname{Hom}(K_F, E_F(D))$. Then it is F-torsion-free and $G \cong \operatorname{Hom}(K_{F^n}, H \otimes K_{F^n}) \cong \operatorname{Ext}(K_{F^n}, H)$ by Lemma 1.8 and the proof of

Theorem 1.10.

(4)$\Rightarrow$(2): Let M be an F-torsion-free module. Then an exact sequence $0\to\mathrm{Ker}\,k\to\Sigma\oplus R\xrightarrow{k}M\to 0$ is F^{∞}-pure and so it is EF^{n}-pure. Hence the induced sequence $0\to\mathrm{Ext}(K_{F^n},\,\mathrm{Ker}\,k)\to\mathrm{Ext}(K_{F^n},\,\Sigma\oplus R)\to\mathrm{Ext}(K_{F^n},\,M)\to 0$ is splitting exact.

(2)$\Rightarrow$(1): It suffices to prove that $\mathrm{Ext}(K_{F^n},\,\Sigma\oplus R)$ is $C(F^n)$-projective. We consider a diagram of the form

$$\begin{tikzcd}
 & \Sigma\oplus R \arrow[dl, "\eta"'] \arrow[d, phantom, "\delta"] \arrow[dr, "\delta"] & & \\
\text{(D)}\quad 0 \to \Sigma\oplus R/(\Sigma\oplus R)F^n \arrow[rr, "\bar{\delta}"] & & \mathrm{Ext}(K_{F^n},\Sigma\oplus R) \arrow[r] \arrow[d, "g"] & \mathrm{Ext}(Q_{F^n},\Sigma\oplus R) \to 0 \\
M \arrow[rr, "f"] & & N \arrow[r] & 0,
\end{tikzcd}$$

where M and $N\in C(F^n)$, f is an epimorphism, δ is the connecting homomorphism and η is the natural homomorphism. Then there is a homomorphism $h\colon \Sigma\oplus R\to M$ such that $g\delta=fh$. The homomorphism h and η yield the commutative diagram by Lemmas 2.4 and 2.6:

$$\begin{tikzcd}
M \arrow[r, "\delta_1", "\cong"'] & \mathrm{Ext}(K_{F^n}, M) \\
\Sigma\oplus R \arrow[u, "h"] \arrow[d, "\eta"'] \arrow[r, "\delta"] & \mathrm{Ext}(K_{F^n}, \Sigma\oplus R) \arrow[u, "h_*"'] \arrow[d, "\eta_*", "\wr\|"'] \\
(\Sigma\oplus R)/(\Sigma\oplus R)F^n \arrow[r, "\delta_2"] & \mathrm{Ext}(K_{F^n}, (\Sigma\oplus R)/(\Sigma\oplus R)F^n)\,.
\end{tikzcd}$$

We put $\bar{h}=\delta_1 h_* \eta_*^{-1}\delta_2$. Then $h=\bar{h}\eta$. The upper row in the diagram (D) is exact and is EF^n-pure by Lemmas 2.5, 2.6 and Corollary 2.7. So we have a a homomorphism $k\colon \mathrm{Ext}(K_{F^n},\,\Sigma\oplus R)\to M$ such that $k\bar{\delta}=\bar{h}$. Hence $g\delta=fh$ $=f\bar{h}\eta=fk\bar{\delta}\eta=fk\delta$. So $g-fk$ induces a homomorphism $\overline{g-fk}\colon \mathrm{Ext}(K_{F^n},\Sigma\oplus R)/\delta(\Sigma\oplus R)\to N$. On the other hand $\mathrm{Ext}(K_{F^n},\,\Sigma\oplus R)/\delta(\Sigma\oplus R)\cong\mathrm{Ext}(Q_{F^n},\,\Sigma\oplus R)$ and $\mathrm{Ext}(Q_{F^n},\Sigma\oplus R)$ is a homomorphic image of $\mathrm{Ext}(Q,\,\Sigma\oplus R)$. Hence $\mathrm{Ext}(Q_{F^n},\,\Sigma\oplus R)$ is injective. Since N is reduced, i.e., the injective submodule of N is zero, $\overline{g-fk}=\bar{0}$ and so $g=fk$. Hence $\mathrm{Ext}(K_{F^n},\,\Sigma\oplus R)$ is $C(F^n)$-projective.

Corollary 2.10. *There is one-to-one between all maximal F^n-torsion modules and all $C(F^n)$-projective objects in the category $C(F^n)$.*

REMARK. (1) Let $C(F^{\infty})$ be the category of F-reduced and F^{∞}-pure injective modules together with their homomorphisms. Then the corresponding results to Theorem 2.9 also hold for the category $C(F^{\infty})$, where $\hat{R}_{F_l^{\infty}}=\hat{R}_{F_l}$ and $K_{F^{\infty}}=K_F$. Further a module G is $C(F^{\infty})$-projective if and only if $G\in C(F^{\infty})$ and G is F-torsion-free by Proposition 2.3 of [9].

(2) $C(F^n)$-projective objects are not necessaily F-torsion-free. For example, let P be a prime ideal of R and let $F_P=\{I \mid I \supseteqq P^k \text{ for some } k\}$. Then the $C(F_p^n)$-projective object $\mathrm{Ext}(K_{F_p^n}, R)$ is F_P-torsion, because $\mathrm{Ext}(K_{F_p^n}, R) \cong R/P^n$ by Proposition 3.2 of [13].

(3) $C(F^n)$-projective objects are not necessarily F-torsion. For example, let R be a simple hereditary noethenian prime ring and let F be any non trivial right additive topology. Then $0 \to R \to \mathrm{Ext}(K_{F^n}, R)$ is exact, because $RF^n=0$. Thus the $C(F^n)$-projective object $\mathrm{Ext}(K_{F^n}, R)$ is not F-torsion.

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