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THE BERRY-ESSEEN BOUND FOR MAXIMUM LIKELIHOOD ESTIMATES OF TRANSLATION PARAMETER OF TRUNCATED DISTRIBUTION

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1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables with common density $f(x-\theta)$, $-\infty < x$, $\theta < \infty$, where θ is an unknown translation parameter. We shall consider here the case that $f(x)$ is a uniformly continuous density which vanishes on the interval $(-\infty, 0)$ and is positive on the interval $(0, \infty)$ and particularly

$$f(x) \sim \alpha x \quad \text{as} \quad x \to +0$$

with $0 < \alpha < \infty$. Let $\hat{\theta}_n = \hat{\theta}_n(X_1, \cdots, X_n)$ be a m.l.e. (maximum likelihood estimate) of θ for the sample size n. Woodroofe [1] showed that $(\frac{1}{2}\alpha n \log n)^{1/2} \times (\hat{\theta}_n - \theta)$ has an asymptotic standard normal distribution. The purpose of the present paper is to estimate the speed of convergence of $a_n(\hat{\theta}_n - \theta)$ to the standard normal distribution. Here $2a_n^2 = \alpha n(\log n + \log \log n)$. Similar results for minimum contrast estimates in the regular case were given by Michel and Pfanzagl [2] and Pfanzagl [3]. More precisely, Pfanzagl [3] showed that for every compact K there exists a constant c_K such that for all $\theta \in K$, $n \geq 1$ and $t \in R$

$$\left| P_\theta \left\{ \frac{n^{1/2}(\theta_n^* - \theta)}{\beta(\theta)} < t \right\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp\left(-\frac{r^2}{2}\right) dr \right| \leq c_K n^{-1/2},$$

where θ_n^* denotes a minimum contrast estimate for the sample size n.

2. Conditions and the main result. We shall impose the following regularity conditions on $f(x)$. These conditions are stronger than those made by Woodroofe [1].

CONDITIONS
(i) $f(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0)$ and is positive on $(0, \infty)$.
(ii) $f(x)$ is continuously differentiable on $(0, \infty)$ with derivative $f'(x)$ and $f'(x)$ is absolutely continuous on every compact subinterval of $(0, \infty)$ with de-

rivative $f''(x)$.

   (iii) For some α and r, $0<\alpha$, $r<\infty$

$$f'(x) = \alpha + O(x^r) \quad \text{and} \quad f''(x) = O(x^{r-1}) \qquad \text{as} \quad x \to +0 .$$

Let $g(x) = \log f(x)$ for $x>0$. Then $g(x)$ will be continuously differentiable on $(0, \infty)$ with derivative $g'=f'/f$ and $g'(x)$ will be absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $g''=(ff''-f'^2)/f^2$.

   (iv) For every $t \geq 0$

$$\int_0^\infty \{g(x+t)\}^2 f(x)\, dx < \infty .$$

   (v) For every $a>0$, there is a $\delta>0$, for which

$$\int_a^\infty \sup_{|u|\leq\delta} |g'(x+u)|^3 f(x)\, dx < \infty .$$

   (vi) For every $a>0$, there is a $\delta>0$, for which

$$\int_a^\infty \sup_{|u|\leq\delta} \{g''(x+u)\}^2 f(x)\, dx < \infty .$$

REMARK. Under conditions (i) and (ii), condition (iii) is equivalent to the following condition (iii)'.

   (iii)' For some α and r, $0<\alpha$, $r<\infty$

$$f(x) = \alpha x + O(x^{1+r}),\ g'(x) = x^{-1} + O(x^{r-1})\ \text{and}\ g''(x) = -x^{-2} + O(x^{r-2})$$

$$\text{as} \quad x \to +0 .$$

EXAMPLES ([1]). Let

$$f(x) = r\left[\Gamma\left(\frac{2}{r}\right)\right]^{-1} x \exp(-x^r), \quad x>0, \quad \text{for some } r>0 ,$$

$$\text{or } f(x) = \frac{1}{d(1+d)} \frac{x}{(1+x)^{2+d}}, \qquad x>0, \quad \text{for some } d>0 ,$$

then conditions (i)—(vi) are all satisfied.

Let $M_n = \min(X_1, \cdots, X_n)$ and $G_n(t) = \sum_{i=1}^n g(X_i - t)$ for $t<M_n$. Condition (i) insures that m.l.e.'s exist in the interval $(-\infty, M_n)$. Let $\hat{\theta}_n$, $n\geq 1$, be a sequence of m.l.e.'s. If conditions (i) and (ii) are satisfied, then

$$-\infty < \hat{\theta}_n < M_n \quad \text{and} \quad G_n'(\hat{\theta}_n) = 0$$

with probability 1.

   Theorem. *Suppose that conditions* (i)—(vi) *are all satisfied. Let $\hat{\theta}_n$, $n\geq 1$,*

denote a sequence of m.l.e.'s for $\prod_{i=1}^n f(X_i-\theta)$ and let $2a_n^2 = \alpha n(\log n + \log\log n)$.

Then there exists a constant c_1 such that for all $\theta \in R$, $n\geq 1$ and $t\leq 0$

(2.1) $$|P_\theta\{a_n(\hat{\theta}_n-\theta)\leq t\}-\Phi(t)|\leq c_1(\log n)^{-1}.$$

Also, for every s, $0<s<1$, *there exists a constant* c_2 *such that for all* $\theta\in R$, $n\geq 1$ *and* $t>0$

(2.2) $$|P_\theta\{a_n(\hat{\theta}_n-\theta)\leq t\}-\Phi(t)|\leq c_2(\log n)^{s-1}.$$

Here

$$\Phi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{x^2}{2}\right)dx.$$

REMARK. (1) The assertion of (2.2) holds with $(\log n)^{-1}$ instead of $(\log n)^{s-1}$ provided t is restricted to a finite interval $(0, M]$.

(2) We used $\{\frac{1}{2}\alpha n(\log n+\log\log n)\}^{1/2}$ as the convergence order of m.l.e. to the true parameter θ. However our result is true for any a_n, $n\geq 1$, satisfying that $\alpha na_n^{-2}\log a_n=1+O((\log n)^{-1})$. Obviously, this condition includes the case that $a_n=\{\frac{1}{2}\alpha n(\log n+\log\log n)\}^{1/2}$ but excludes the case that $a_n=(\frac{1}{2}\alpha n\log n)^{1/2}$.

3. Some lemmas.

Since θ is a translation parameter, it will suffice to prove our result in the special case that $\theta=0$. Hereafter, suppose that $\theta=0$. The following Lemma 1 refines the result of Woodroofe [1].

Lemma 1. *Let conditions* (i)–(iii) *and* (vi) *be satisfied. Then, for sufficiently small* $\varepsilon>0$, *there exists* $c\geq 0$ *such that*

(3.1) $$P\left\{\sup_{-\varepsilon\leq t<M_n}\frac{1}{n}G_n''(t)\geq -1\right\}\leq cn^{-1}$$

for all $n\geq 1$.

Proof. Let $a>0$ be so small that $g''(x)\leq-\frac{1}{2}x^{-2}$ for $0<x\leq 2a$. There is a sufficiently small number $0<\varepsilon<a$ such that

$$\int_0^a(x+\varepsilon)^{-2}f(x)dx>2\int_a^\infty\sup_{|t|\leq\varepsilon}|g''(x+t)|f(x)dx+5$$

because the left-hand integral diverges to ∞ as $\varepsilon\to 0$. Then the event $M_n\leq\varepsilon$ implies that

$$\sup_{-\varepsilon\leq t<M_n}\frac{1}{n}G_n''(t)\leq\frac{-1}{2n}\sum_0^a(X_i+\varepsilon)^{-2}+\frac{1}{n}\sum_a^\infty\sup_{|t|\leq\varepsilon}|g''(X_i+t)|$$

where $\sum_u^v$ denotes summation over $i\leq n$ for which $u\leq X_i<v$. Hence the relations $M_n\leq\varepsilon$ and $\sup_{-\varepsilon\leq t<M_n}\frac{1}{n}G_n''(t)\geq -1$ imply

$$\left| \frac{1}{n} \sum_{0}^{a} (X_i + \varepsilon)^{-2} - \int_0^a (x+\varepsilon)^{-2} f(x) dx \right| \geqq 1$$

or

$$\left| \frac{1}{n} \sum_{a}^{\infty} \sup_{|t| \leqq \varepsilon} |g''(X_i + t)| - \int_a^\infty \sup_{|t| \leqq \varepsilon} |g''(x+t)| f(x) dx \right| \geqq 1 \ .$$

Hence we have

$$P\left\{ \sup_{-\varepsilon \leqq t < \mathbf{M}_n} \frac{1}{n} G_n''(t) \geqq -1 \right\}$$

$$\leqq P\{M_n > \varepsilon\} + P\left\{ \left| \frac{1}{n} \sum_{0}^{a} (X_i + \varepsilon)^{-2} - \int_0^a (x+\varepsilon)^{-2} f(x) dx \right| \geqq 1 \right\}$$

$$+ P\left\{ \left| \frac{1}{n} \sum_{a}^{\infty} \sup_{|t| \leqq \varepsilon} |g''(X_i + t)| - \int_a^\infty \sup_{|t| \leqq \varepsilon} |g''(x+t)| f(x) dx \right| \geqq 1 \right\} \ .$$

Since $P\{M_n > \varepsilon\} = o(n^{-1})$, Lemma 1 follows from condition (vi) and Chebyshev's inequality.

Woodroofe [1] mentioned that condition (i) and

$$\int_0^\infty -g(x) f(x) dx < \infty \ ,$$

which is a weaker condition than (iv), imply all assumptions of Wald [4]. Thus we can make use of his results.

Lemma 2. *Let $\hat{\theta}_n$, $n \geqq 1$, be a sequence of m.l.e.'s. Suppose that conditions (i)–(iii) and (iv) hold. Then for every $\varepsilon > 0$ there exists $c \geqq 0$ such that*

$$P\{|\hat{\theta}_n| \geqq \varepsilon\} \leqq cn^{-1}$$

for all $n \geqq 1$.

Proof. Let M be a positive number chosen such that

$$E\left\{ \log \sup_{t < -\mathbf{M}} f(X-t) \right\} < E\{\log f(X)\} \ .$$

For every $t \in [-M, -\varepsilon]$ there exists an open neighborhood U_t of t such that

$$E\left\{ \log \sup_{u \in U_t} f(X-u) \right\} < E\{\log f(X)\} \ .$$

The existence of such a positive number M and that of such a U_t follow from Wald [4]. As $\{U_t : t \in [-M, -\varepsilon]\}$ covers the compact set $[-M, -\varepsilon]$, there exists a finite subcover of this set $[-M, -\varepsilon]$ determined by $t_j \in [-M, -\varepsilon], j = 1, \cdots, m$. For notational convenience, let $U_0 = (-\infty, -M)$ and $U_j = U_{t_j}, j = 1, \cdots, m$. If $|\hat{\theta}_n| \geqq \varepsilon$ and $M_n < \varepsilon$, then $-\infty < \hat{\theta}_n \leqq -\varepsilon$ and therefore $\hat{\theta}_n \in U_j$ for some $j \in \{0, 1, \cdots, m\}$, that is to say,

$$n^{-1} \sum_{i=1}^{n} \log \sup_{t \in \bar{U}_j} f(X_i - t) \geqq n^{-1} \sum_{i=1}^{n} \log f(X_i)$$

for some $j \in \{0, 1, \cdots, m\}$. Write

$$b_j = E\{\log f(X)\} - E\{\log \sup_{t \in \bar{U}_j} f(X-t)\} > 0, \quad j = 0, 1, \cdots, m$$

and let $2b = \min \{b_j; j = 0, 1, \cdots, m\} > 0$. Then

$$|n^{-1} \sum_{i=1}^{n} \log \sup_{t \in \bar{U}_j} f(X_i - t) - E\{\log \sup_{t \in \bar{U}_j} f(X-t)\}| < b, \quad j = 0, 1, \cdots, m$$

and

$$|n^{-1} \sum_{i=1}^{n} \log f(X_i) - E\{\log f(X)\}| < b$$

imply

$$n^{-1} \sum_{i=1}^{n} \log \sup_{t \in \bar{U}_j} f(X_i - t) < n^{-1} \sum_{i=1}^{n} \log f(X_i), \quad j = 0, 1, \cdots, m.$$

Hence we have

$$P\{|\hat{\theta}_n| \geqq \varepsilon\} \leqq P\{|\hat{\theta}_n| \geqq \varepsilon, M_n < \varepsilon\} + P\{M_n \geqq \varepsilon\}$$

$$\leqq \sum_{j=0}^{m} P\{|n^{-1} \sum_{i=1}^{n} \log \sup_{t \in \bar{U}_j} f(X_i - t) - E\{\log \sup_{t \in \bar{U}_j} f(X-t)\}| \geqq b\}$$

$$+ P\{|n^{-1} \sum_{i=1}^{n} \log f(X_i) - E\{\log f(X)\}| \geqq b\} + P\{M_n \geqq \varepsilon\}.$$

Now, by conditions (i)–(iii) and (iv), the assertion follows from Chebyshev's inequality.

For $i = 1, \cdots, n$ and $0 \leqq t \leqq (\log n)^{1/2}$, let

$$Z_{ni} = Z_{ni}(X_i, t) = Y_{ni} - E\{Y_{ni}\},$$

where

$$Y_{ni} = Y_{ni}(X_i, t) = g'(X_i + a_n^{-1}t), \quad \text{if } X_i > a_n^{-1},$$
$$= 0 \quad , \quad \text{if } X_i \leqq a_n^{-1}.$$

Here E denotes expectation. Moreover, let $b_n(t) = E\{Z_{n1}(X_1, t)\}^2$.

Lemma 3. *Let conditions (i)—(iii), (v) and (vi) be satisfied. Then there exists a constant c such that for all $x \in R$, $n \geqq 1$ and $0 \leqq t \leqq (\log n)^{1/2}$*

$$|P\{(nb_n(t))^{-1/2} \sum_{i=1}^{n} Z_{ni}(X_i, t) < x\} - \Phi(x)| \leqq c(\log n)^{-1}.$$

Proof. We shall first show that

(3.2) $$E\{Y_{n1}\} = -t\alpha a_n^{-1} \log a_n(1+t)^{-1} + O(a_n^{-1}(1+t)),$$

(3.3) $$E\{Y_{n1}^2\} = \alpha \log a_n(1+t)^{-1} + O(1),$$

(3.4) $E\{|Y_{n1}|^3\} = O(a_n(1+t)^{-1})$.

By condition (iii)', choose $a > 0$ and $c_0 \geqq 0$ such that

(3.5) $|f(x) - \alpha x| \leqq c_0 x^{1+r}$, $|g'(x) - x^{-1}| \leqq c_0 x^{r-1}$ and $|g''(x) + x^{-2}| \leqq c_0 x^{r-2}$

for $0 < x \leqq 2a$.   Next choose $\delta > 0$ such that conditions (v) and (vi) hold.   Then we may establish (3.2) as follows.   Since

$$g'(x + a_n^{-1}t) = g'(x) + \int_0^{a_n^{-1}t} g''(x+u)\,du$$

we have

$$E\{Y_{n1}\} = \int_{a_n^{-1}}^{\infty} g'(x)f(x)\,dx + \int_{a_n^{-1}}^{\infty} \left\{ \int_0^{a_n^{-1}t} g''(x+u)\,du \right\} f(x)\,dx$$
$$= I_1 + I_2\,,   \text{say.}$$

It is easily seen that

$$I_1 = -\int_0^{a_n^{-1}} g'(x)\,f(x)\,dx\,,$$

so that $I_1 = O(a_n^{-1})$ by (3.5).   Next we put

$$I_2 = \int_{a_n^{-1}}^{a} \left\{ \int_0^{a_n^{-1}t} g''(x+u)\,du \right\} f(x)\,dx + \int_a^{\infty} \left\{ \int_0^{a_n^{-1}t} g''(x+u)\,du \right\} f(x)\,dx$$
$$= I_{21} + I_{22}\,,   \text{say.}$$

By condition (vi), we have $I_{22} = O(a_n^{-1}t)$.   Moreover let

$$I_{21} = I_{211} + I_{212} + I_{213}\,,$$

where

$$I_{211} = -\int_{a_n^{-1}}^{a} \left\{ \int_0^{a_n^{-1}t} (x+u)^{-2}du \right\} \alpha x\,dx\,,$$

$$I_{212} = -\int_{a_n^{-1}}^{a} \left\{ \int_0^{a_n^{-1}t} (x+u)^{-2}du \right\} (f(x) - \alpha x)\,dx\,,$$

$$I_{213} = \int_{a_n^{-1}}^{a} \left\{ \int_0^{a_n^{-1}t} [g''(x+u) + (x+u)^{-2}]\,du \right\} f(x)\,dx\,.$$

By easy computation, (3.5) implies that

$$I_{211} = -t\alpha a_n^{-1} \log a_n(1+t)^{-1} + O(a_n^{-1}t)\,,$$
$$I_{212} = O(a_n^{-1}t)$$

and $$I_{213} = O(a_n^{-1}t)\,,$$

so that (3.2) is established.

To establish (3.3), let

$$E\{Y_{n1}^2\} = \int_{a_n^{-1}}^{a} \{g'(x+a_n^{-1}t)\}^2 f(x)\,dx + \int_{a}^{\infty} \{g'(x+a_n^{-1}t)\}^2 f(x)\,dx$$
$$= J_1 + J_2, \quad \text{say.}$$

Condition (v) implies that $J_2 = O(1)$. Divide J_1 into J_{11}, J_{12}, J_{13} and J_{14} as follows:

$$J_{11} = \int_{a_n^{-1}}^{a} (x+a_n^{-1}t)^{-2} \alpha x \, dx \, ,$$

$$J_{12} = \int_{a_n^{-1}}^{a} (x+a_n^{-1}t)^{-2} (f(x) - \alpha x)\,d\lambda \, ,$$

$$J_{13} = \int_{a_n^{-1}}^{a} 2(x+a_n^{-1}t)^{-1} \{g'(x+a_n^{-1}t) - (x+a_n^{-1}t)^{-1}\} f(x)\,dx \, ,$$

$$J_{14} = \int_{a_n^{-1}}^{a} \{g'(x+a_n^{-1}t) - (x+a_n^{-1}t)^{-1}\}^2 f(x)\,dx \, .$$

Then, by (3.5), we have

$$J_{11} = \alpha \log a_n(1+t)^{-1} + O(1) \, ,$$
$$J_{12} = O(1) \, ,$$
$$J_{13} = O(1)$$

and
$$J_{14} = O(1) \, ,$$

so that (3.3) is established.

Finally, we shall establish (3.4). Let

$$E\{|Y_{n1}|^3\} = \int_{a_n^{-1}}^{a} |g'(x+a_n^{-1}t)|^3 f(x)\,dx + \int_{a}^{\infty} |g'(x+a_n^{-1}t)|^3 f(x)\,dx$$
$$= K_1 + K_2, \quad \text{say.}$$

By condition (v), we have $K_2 = O(1)$. Also by (3.5) we have

$$K_1 \leqq \int_{a_n^{-1}}^{a} \{(1+(2a)^r c_0)\,(x+a_n^{-1}t)^{-1}\}^3 f(x)\,d(x)$$
$$= O(a_n(1+t)^{-1}) \, .$$

This implies (3.4).

From (3.2), (3.3) and (3.4), we have

(3.6)
$$E\{Z_{n1}^2\} = \alpha \log a_n(1+t)^{-1} + O(1) \, ,$$
$$E\{|Z_{n1}|^3\} = O(a_n(1+t)^{-1}) \, .$$

Now, the assertion of Lemma 3 follows from the Berry-Esseen theorem ([5], Theorem 12.4).

In the rest of this section, we shall study the conditional distribution of

$a_n^{-1} \sum_{i=1}^{n} g'(X_i - a_n^{-1}t)$ given $M_n > a_n^{-1}t$ for $0 < t \leq (\log n)^{1/2}$. The conditional ditribu-tion of $X_1, \cdots, X_n$, given $M_n > a_n^{-1}t$, is that of independent random variables with common density

$$f_n^*(x) = c_n f(x), \qquad x > a_n^{-1}t$$
$$= 0 \qquad , \qquad \text{otherwise}$$

where

$$c_n = [\int_{a_n^{-1}t}^{\infty} f(x) dx]^{-1}.$$

For $i = 1, \cdots, n$ and $0 < t \leq (\log n)^{1/2}$ let

$$Z_{ni}^* = Z_{ni}^*(X_i, t) = Y_{ni}^* - E^*\{Y_{ni}^*\}$$

where

$$Y_{ni}^* = Y_{ni}^*(X_i, t) = g'(X_i - a_n^{-1}t), \qquad \text{if} \quad X_i > a_n^{-1}(1+2t),$$
$$= 0 \qquad\qquad , \qquad \text{if} \quad X_i \leq a_n^{-1}(1+2t).$$

Here E^* denotes conditional expectation given $M_n > a_n^{-1}t$. It is easily seen that $c_n = 1 + O(n^{-1})$ for $0 < t \leq (\log n)^{1/2}$. Thus, in a similar way to Lemma 3, we obtain

$$E^*\{Y_{n1}^*\} = t\alpha\, a_n^{-1} \log a_n(1+t)^{-1} + O(a_n^{-1}(1+t)),$$
$$E^*\{Z_{n1}^{*2}\} = \alpha \log a_n(1+t)^{-1} + O(1)$$

and $\qquad E^*\{|Z_{n1}^*|^3\} = O(a_n(1+t)^{-1}),$

which lead to the following lemma.

Lemma 4. *Let conditions* (i)—(iii), (v) *and* (vi) *be satisfied. Then there exists a constant c such that for all $x \in R$, $n \geq 1$ and $0 < t \leq (\log n)^{1/2}$*

$$|P\{(nb_n^*(t))^{-1/2} \sum_{i=1}^{n} Z_{ni}^*(X_i, t) < x \mid M_n > a_n^{-1}t\} - \Phi(x)| \leq c(\log n)^{-1},$$

where $b_n^*(t) = E^*\{Z_{n1}^*(X_1, t)\}^2$.

4. Proof of Theorem. As the left sides of (2.1) and (2.2) are uniformly bounded for $\theta \in R$ and $t \in R$, it suffices to prove the assertion for all sufficiently large n. To simplify our notations we shall use n_0 as a generic constant instead of the phrase "for all sufficiently large n". In the same manner we shall use c as a generic constant to denote factors occurring in the bounds.

We shall use ideas related to Woodroofe [1]. It follows from Lemma 1 and Lemma 2 that

(4.1) $\qquad P\{a_n \hat{\theta}_n \leq -t\} = P\{a_n^{-1} \sum_{i=1}^{n} g'(X_i + a_n^{-1}t) \geq 0\} + O(n^{-1}),$

where $O(n^{-1})$ is uniform in $t \in [0, a_n \varepsilon)$. Here $\varepsilon > 0$ is chosen sufficiently small so that (3.1) of Lemma 1 holds. Similarly, it follows from Lemma 1 and Lemma 2 that

$$(4.2) \qquad P\{a_n \hat{\theta}_n > t\} = P\{a_n^{-1} \sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0,\ M_n > a_n^{-1}t\} + O(n^{-1})$$

$$= P\{a_n^{-1} \sum_{i=1}^n g'(X_i - a_n^{-1}t) < 0 \mid M_n > a_n^{-1}t\} P\{M_n > a_n^{-1}t\}$$

$$+ O(n^{-1}),$$

where $O(n^{-1})$ is uniform in $t > 0$.

We shall first show the validity of (2.1). By condition (iii)$'$

$$(4.3) \qquad | P\{a_n^{-1} \sum_{i=1}^n g'(X_i + a_n^{-1}t) \geqq 0\} - P\{a_n^{-1} \sum_{i=1}^n Y_{ni} \geqq 0\} |$$

$$\leqq \sum_{i=1}^n P\{X_i \leqq a_n^{-1}\}$$

$$\leqq c(\log n)^{-1}$$

for all $n \geqq n_0$ and $0 \leqq t \leqq (\log n)^{1/2}$. Since

$$P\{a_n^{-1} \sum_{i=1}^n Y_{ni} \geqq 0\} = P\{(nb_n(t))^{-1/2} \sum_{i=1}^n Z_{ni} \geqq x_n(t)\},$$

where

$$x_n(t) = -n^{1/2}(b_n(t))^{-1/2} E\{Y_{n1}\},$$

it follows from Lemma 3 that

$$(4.4) \qquad | P\{a_n^{-1} \sum_{i=1}^n Y_{ni} \geqq 0\} - \Phi(-x_n(t)) | \leqq c(\log n)^{-1}$$

for all $n \geqq 1$ and $0 \leqq t \leqq (\log n)^{1/2}$. According to (3.2) and (3.6)

$$-x_n(t) = (na_n^{-1} E\{Y_{n1}\})(na_n^{-2} b_n(t))^{-1/2}$$

$$= \{-t + 2t \log (1+t)(\log n)^{-1} + O((1+t)(\log n)^{-1})\}$$

$$\times \{1 - 2 \log (1+t)(\log n)^{-1} + O((\log n)^{-1})\}^{-1/2}$$

$$= -t + t \log (1+t)(\log n)^{-1} + O((1+t)(\log n)^{-1}).$$

Hence, for $n \geqq n_0$ and $0 \leqq t \leqq (\log n)^{1/2}$

$$(4.5) \quad | \Phi(-x_n(t)) - \Phi(-t) | \leqq \frac{1}{\sqrt{2\pi}} | t - x_n(t) | \max \left\{ \exp\left(-\frac{t^2}{2}\right), \exp\left(-\frac{x_n(t)^2}{2}\right) \right\}$$

$$\leqq c(\log n)^{-1}.$$

From (4.1), (4.3), (4.4) and (4.5), there exists a constant c such that

$$(4.6) \qquad | P\{a_n \hat{\theta}_n \leqq -t\} - \Phi(-t) | \leqq c(\log n)^{-1}$$

for all $n \geqq n_0$ and $0 \leqq t \leqq (\log n)^{1/2}$. For $t > (\log n)^{1/2}$ we have

$$|P\{a_n\hat{\theta}_n \leq -t\} - \Phi(-t)| \leq P\{a_n\hat{\theta}_n \leq -(\log n)^{1/2}\} + \Phi(-(\log n)^{1/2}).$$

Using (4.6) and Feller ([6], p. 166, Lemma 2), we obtain

$$(4.7) \qquad\qquad |P\{a_n\hat{\theta}_n \leq -t\} - \Phi(-t)| \leq c(\log n)^{-1}$$

for $n \geq n_0$ and $t > (\log n)^{1/2}$. Hence (4.6) and (4.7) imply (2.1).

We next show the validity of (2.2). By condition (iii)'

$$|P\{a_n^{-1}\sum_{i=1}^{n} g'(X_i - a_n^{-1}t) < 0 \,|\, M_n > a_n^{-1}t\} - P\{a_n^{-1}\sum_{i=1}^{n} Y_{ni}^* < 0 \,|\, M_n > a_n^{-1}t\}|$$

$$\leq \sum_{i=1}^{n} P\{X_i \leq a_n^{-1}(1+2t) \,|\, M_n > a_n^{-1}t\}$$

$$\leq cn\, a_n^{-2}(3t^2 + 4t + 1).$$

Hence, for every s, $0 < s < 1$, there exists a constant c such that for $n \geq n_0$ and $0 < t \leq (\log n)^{s/2}$

$$(4.8) \quad |P\{a_n^{-1}\sum_{i=1}^{n} g'(X_i - a_n^{-1}t) < 0 \,|\, M_n > a_n^{-1}t\} - P\{a_n^{-1}\sum_{i=1}^{n} Y_{ni}^* < 0 \,|\, M_n > a_n^{-1}t\}|$$

$$\leq c(\log n)^{s-1}.$$

Applying arguments similar to those used in (4.4) and (4.5), Lemma 4 implies

$$(4.9) \qquad |P\{a_n^{-1}\sum_{i=1}^{n} Y_{ni}^* < 0 \,|\, M_n > a_n^{-1}t\} - \{1 - \Phi(t)\}| \leq c(\log n)^{-1}$$

for $n \geq n_0$ and $0 < t \leq (\log n)^{1/2}$. By (4.8) and (4.9) we have

$$|P\{a_n^{-1}\sum_{i=1}^{n} g'(X_i - a_n^{-1}t) < 0 \,|\, M_n > a_n^{-1}t\} P\{M_n > a_n^{-1}t\} - \{1 - \Phi(t)\}|$$

$$\leq |P\{a_n^{-1}\sum_{i=1}^{n} g'(X_i - a_n^{-1}t) < 0 \,|\, M_n > a_n^{-1}t\} - \{1 - \Phi(t)\}| P\{M_n > a_n^{-1}t\}$$

$$+ \{1 - \Phi(t)\} P\{M_n \leq a_n^{-1}t\}$$

$$\leq \{1 - \Phi(t)\} P\{M_n \leq a_n^{-1}t\} + c(\log n)^{s-1}$$

for $n \geq n_0$ and $0 < t \leq (\log n)^{s/2}$. Using Feller ([6], p. 166, Lemma 2), we obtain

$$\{1 - \Phi(t)\} P\{M_n \leq a_n^{-1}t\} \leq ct \exp\left(-\frac{t^2}{2}\right)(\log n)^{-1}$$

$$\leq c(\log n)^{-1}$$

for $n \geq n_0$ and $t > 0$. Hence, from this and (4.2) it follows that for $n \geq n_0$ and $0 < t \leq (\log n)^{s/2}$

$$|P\{a_n\hat{\theta}_n > t\} - \{1 - \Phi(t)\}| \leq c(\log n)^{s-1},$$

from which (2.2) is shown by a similar argument used in (4.7). Thus we complete the proof of the theorem.

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