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Author(s)	Colombini, Ferruccio
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THE FEFFERMAN-PHONG INEQUALITY IN THE LOCALLY TEMPERATE WEYL CALCULUS

FERRUCCIO COLOMBINI, DANIELE DEL SANTO and CLAUDE ZUILY

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1. Introduction

The Fefferman-Phong inequality, which is a two derivatives Gårding type inequality, has been proved by its authors for pseudo-differential operators with symbols in the class $S_{1,0}^m$. It has been extended by HÖRMANDER [5], in his work on the Weyl calculus, to symbols in the general class $S(m, g)$ where g is a slowly varying and temperate metric satisfying the uncertainly principle. Further works on spectral theory and singularities for nonlinear hyperbolic equations showed the necessity to relax the temperacy condition on the metric and DENCKER [3], BONY-LERNER [1] introduced new classes allowing to deal with these applications. However the Fefferman-Phong inequality was not proved for these classes. In a recent work, COLOMBINI, DEL SANTO, ZUILY [2], we have also been led to consider non temperate metrics and the above mentioned inequality was required in the proof; it turned out that these metrics were locally temperate in the sense of DENCKER [3]. The purpose of this work is then to prove the Fefferman-Phong inequality for properly supported pseudo-differential operators with symbols in locally temperate classes. Unfortunately, because of the complexity of the quantification, this inequality is still not available in the general case of the Bony-Lerner classes.

2. Notations, statement of the result, examples

We first recall some definitions taken from HÖRMANDER [5] and DENCKER [3].

Let V be an n dimensional vector space and $W = V \oplus V'$ where V' is the dual of V . Elements in V will be denoted by x and those in W by w or (x, ξ) .

Let G be a metric on V , assumed to be slowly varying i.e.

$$(2.1) \quad \left\{ \begin{array}{l} \text{there exist constants } a_0 > 0, A_0 \geq 1 \text{ such that for } x, y \text{ in } V: \\ G_x(x-y) \leq a_0 \Rightarrow \frac{1}{A_0} \leq \frac{G_x}{G_y} \leq A_0. \end{array} \right.$$

Let g be a metric on W which is also slowly varying i.e.

$$(2.2) \quad \left\{ \begin{array}{l} \text{there exist constants } a_1 > 0, A_1 \geq 1 \text{ such that for } w, w' \text{ in } W: \\ g_w(w-w') \leq a_1 \Rightarrow \frac{1}{A_1} \leq \frac{g_w}{g_{w'}} \leq A_1. \end{array} \right.$$

We shall also assume that

$$(2.3) \quad \text{for every } (x, \xi) \text{ in } W, \quad G_x \leq g_{x, \xi}.$$

The space W is a symplectic space with the standard symplectic form

$$(2.4) \quad \sigma(w, w') = \langle y, \xi \rangle - \langle x, \eta \rangle \quad \text{if } w = (x, \xi), w' = (y, \eta) \in W.$$

The dual metric of g with respect to σ is then defined by

$$(2.5) \quad g_{w_0}^\sigma(w) = \sup_{w' \neq 0} \frac{|\sigma(w, w')|^2}{g_{w_0}(w')}.$$

The metric g will also be assumed to separate the uncertainly principle which reads

$$(2.6) \quad g_w \leq g_w^\sigma \quad \text{for every } w \text{ in } W.$$

We then define the function h on W by

$$(2.7) \quad h^2(w) = \sup_{w' \neq 0} \frac{g_w(w')}{g_w^\sigma(w')} \leq 1.$$

The metric g is said locally temperate if :

$$(2.8) \quad \left\{ \begin{array}{l} \text{there exist positive constants } a_2, A_2 \text{ and } N \in \mathbb{N} \text{ such that} \\ G_x(x-y) \leq a_2 \Rightarrow g_w \leq A_2 g_{w'} (1 + g_w^\sigma(w-w'))^N \\ \text{if } w = (x, \xi) \text{ and } w' = (y, \eta). \end{array} \right.$$

We introduce now the order functions. These are positive functions m on W for which one can find constants $b_j > 0, B_j \geq 1, j=0, 1$ and $M \in \mathbb{N}$ such that

$$(2.9) \quad g_w(w-w') \leq b_0 \Rightarrow \frac{1}{B_0} \leq \frac{m(w)}{m(w')} \leq B_0, \quad w, w' \in W,$$

$$(2.10) \quad \left\{ \begin{array}{l} G_x(x-y) \leq b_1 \Rightarrow m(w) \leq B_1 m(w') (1 + g_w^\sigma(w-w'))^M \\ \text{if } w = (x, \xi) \text{ and } w' = (y, \eta). \end{array} \right.$$

Given G, g, m as above we define the class of symbols $S(m, g)$ to be the set of C^∞ functions a on W such that

$$(2.11) \quad \left\{ \begin{array}{l} \text{for every } k \in N \text{ one can find } C_k > 0 \text{ such that for every} \\ w, w_1, \dots, w_k \text{ in } W \text{ we have} \\ |a^{(k)}(w)(w_1, \dots, w_k)| \leq C_k m(w) \prod_{i=1}^k (g_w(w_i))^{1/2}. \end{array} \right.$$

We define a metric \tilde{G} on $V \times V$ by

$$(2.12) \quad \tilde{G}_{x,y}(s, t) = G_x(s) + G_y(t)$$

and we consider the \tilde{G} distance of a point $(x, y) \in V \times V$ to the diagonal

$$(2.13) \quad D(x, y) = \inf_{x_0 \in V} \tilde{G}_{x_0, x_0}(x - x_0, y - x_0).$$

If $\varepsilon > 0$, we shall set $D_\varepsilon = \{(x, y) \in V \times V, D(x, y) \leq \varepsilon\}$.

Given $0 < \varepsilon' < \varepsilon$ one can construct χ such that

$$(2.14) \quad \chi \in S(1, \tilde{G}), \quad \text{supp } \chi \subset D_\varepsilon, \quad \chi = 1 \text{ on } D_{\varepsilon'}.$$

We shall call such a function properly supported.

Given $a \in \mathcal{S}(W)$ and χ properly supported we can define a pseudo-differential operator by the Weyl quantification

$$(2.15) \quad a_\chi^w u(x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi \rangle} \chi(x, y) a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

This formula can be extended to the case where $a \in S(m, g)$, as an operator sending $C_0^\infty(V)$ into $\mathcal{E}'(V)$ and $C^\infty(V)$ to $\mathcal{D}'(V)$. The symbolic calculus for this class of operators and the L^2 continuity, when m is bounded, have been achieved by DENCKER [3]. The purpose of this work is to prove that the FEFFERMAN-PHONG inequality [4], which has been proved by HÖRMANDER [5] when the metric is temperate, still holds for locally temperate metrics and properly supported operators. Thus the main result of this work is the following.

Theorem 2.1. *Let $a \in S(h^{-2}, g)$, $a \geq 0$ on W . Let χ be properly supported. Then one can find a positive constant C such that*

$$(a_\chi^w u, u) + C \|u\|_{L^2}^2 \geq 0, \quad \text{for } u \in C_0^\infty(V).$$

EXAMPLE 2.2. As we said before, locally temperate metrics occur in proving Carleman estimates with singular weights (see [2]). Let $V = \mathbf{R}_x^n \times \mathbf{R}_y$, $V' = \mathbf{R}_\xi^n \times \mathbf{R}_\eta$. Let $\theta(y)$ be a C^∞ function on \mathbf{R} such that

$$\theta(y) = e^y \text{ if } y \leq 0, \quad \theta(y) = 2 \text{ if } y \geq 1, \quad 0 \leq \theta \leq 2.$$

We set $\Phi^2(x, y; \xi, \eta) = 1 + \theta^2(y)|\xi|^2 + |\eta|^2$. For G we take the flat metric on V , $G_{x,y}(s, t) = |s|^2 + t^2$. Then the metric

$$g_{x,y;\xi,\eta} = dx^2 + dy^2 + \frac{d\xi^2 + d\eta^2}{\Phi^2(x, y; \xi, \eta)}$$

is slowly varying, locally temperate, satisfies the uncertainly principle and (2.3), but is not temperate in the sense of HÖRMANDER [5].

3. Proof of Theorem 2.1

The first step is to prove an analogue of Lemma 18.6.10 in [5].

Proposition 3.1. *Let G (resp. g) be one positive definite quadratic form on \mathbf{R}^n (resp. \mathbf{R}^{2n}). Let us assume*

$$(3.1) \quad G(t) \leq g(t, \tau), \quad (t, \tau) \in \mathbf{R}^{2n},$$

$$(3.2) \quad \sup_{(t, \tau)} \frac{g(t, \tau)}{g^\sigma(t, \tau)} = \lambda^2 \leq 1.$$

Let $a \in C^\infty(\mathbf{R}^{2n})$, $a \geq 0$, be such that for each $k \in \mathbf{N}$ one can find a constant $C_k > 0$ such that for X, T_1, \dots, T_k in \mathbf{R}^{2n}

$$(3.3) \quad |a^{(k)}(X)(T_1, \dots, T_k)| \leq C_k \lambda^{-2} \prod_{i=1}^k (g(T_i))^{1/2}.$$

Let $\chi \in \mathcal{S}(1, \tilde{G})$, $\text{supp } \chi \subset D_\varepsilon$, $\chi = 1$ on $D_{\varepsilon'}$, $\varepsilon' < \varepsilon$. Then there exists a constant $C > 0$ such that

$$(3.4) \quad (a_\chi^* u, u) + C \|u\|_{L^2}^2 \geq 0 \quad \text{for } u \in C_0^\infty(\mathbf{R}^n).$$

Here C is independent of g, G and depends on a only through a finite sum $\sum_{k=1}^{k_0} C_k$, where k_0 is independent of a .

Proof. Let us first assume $a \in \mathcal{S}(\mathbf{R}^{2n})$. Then we shall have $a_x^w = b^w$ if

$$\hat{a}\left(\frac{1}{2}(x+y), y-x\right)\chi(x, y) = \hat{b}\left(\frac{1}{2}(x+y), y-x\right)$$

where $\hat{\cdot}$ is the Fourier transform with respect to the second variable.

Taking the inverse Fourier transform we obtain

$$b(x, \xi) = \exp(-i\langle D_v, D_\eta \rangle) \left[\chi \left(x + \frac{t}{2}, x - \frac{t}{2} \right) a(x, \eta) \right] \Big|_{\eta=\xi}^{t=0}.$$

By Proposition 2.4 in [3] and Theorem 18.4.11 in [5], the mapping $a \mapsto b$ has a continuous extension to a weakly continuous linear mapping from $S(\lambda^{-2}, g)$ to $S(\lambda^{-2}, g)$. Moreover the remainder term

$$b(x, \xi) - \sum_{j=0}^N (-i\langle D_v, D_\eta \rangle)^j \left[\chi \left(x + \frac{t}{2}, x - \frac{t}{2} \right) a(x, \eta) \right] \Big|_{\eta=\xi}^{t=0} \frac{1}{j!}$$

is weakly continuous with value in $S(\lambda^{N-1}, g)$ and its semi-norms in this space depend on N and on the semi-norms of a in $S(\lambda^{-2}, g)$ of χ in $S(1, \tilde{G})$. Since $\chi = 1$ near the diagonal we get, if $N = 1 : b = a + r$ with $r \in S(1, g)$. Therefore $a_\chi^w = a^w + r^w$. Now, by Lemma 18.6.10 in [5] we have $(a^w u, u) \geq -C_0 \|u\|_{L^2}^2$ with C_0 independent of a and g . On the other hand, since $r \in S(1, g)$ we have by Theorem 18.6.3 in [5]: $|(r^w u, u)| \leq C_1 \|u\|_{L^2}^2$, where C_1 depends on a fixed number of semi-norms of r in $S(1, g)$, therefore on a fixed number of semi-norms of a in $S(\lambda^{-2}, g)$ and of χ in $S(1, \tilde{G})$. It follows that $(a_\chi^w u, u) \geq C \|u\|_{L^2}^2$ with C as claimed. \diamond

Now we return to general a, g, G and we localize the problem in the balls constructed in Lemma 18.4.4 of [5].

Let G, g be metrics on V and W , satisfying (2.1) to (2.8).

Let $\rho > 0$ be so small that, with a_j, A_j defined in (2.1) to (2.8),

$$(3.5) \quad \rho < a_1, \quad 32A_0^3 \rho \leq a_2, \quad 8A_0^4 \rho \leq a_0.$$

Let us set with $\rho_0 < \rho$,

$$B_v = \{w : g_{w_v}(w - w_v) \leq \rho_0\} \subset U_v = \{w : g_{w_v}(w - w_v) \leq \rho\} \subset U'_v \\ U'_v = \{w : g_{w_v}(w - w_v) \leq a_1\}.$$

Let $\varphi_v \in C_0^\infty(B_v)$ be real such that $\sum_v \varphi_v^2 = 1$ and $\theta_v \in C_0^\infty(U_v)$, $\theta_v \geq 0$, $\theta_v = 1$ on B_v , φ_v

and θ_v uniformly bounded in $S(1, g)$. If we set we get $a_v = \theta_v a$ we get $a = \sum_v \varphi_v a_v \varphi_v$

and the semi-norms of a_v in $S(h^{-2}, g)$ are uniformly bounded by those of a in $S(h^{-2}, g)$.

We also fix $\varepsilon > 0$ so small that

$$(3.6) \quad 16A_0^5 \varepsilon \leq a_2, \quad 8A_0^4 \varepsilon \leq a_0$$

and we take a properly supported χ with $\text{supp } \chi \subset D_\varepsilon$, $\chi = 1$ near the diagonal. We also assume that χ is real and $\chi(y, x) = \chi(x, y)$. Since $a_v \geq 0$, it follows from Proposition 3.1, that

$$(3.7) \quad (a_{v\chi}^w u, u) + C \|u\|_{L^2}^2 \geq 0, \quad u \in C_0^\infty,$$

where C is independent of G, g and v . Applying (3.7) to $\varphi_{v\chi}^w u$, instead of u , we get

$$(\varphi_{v\chi}^w a_{v\chi}^w \varphi_{v\chi}^w u, u) + C(\varphi_{v\chi}^w \varphi_{v\chi}^w u, u) \geq 0$$

since $(\varphi_{v\chi}^w)^* = \varphi_{v\chi}^w$, because χ is real and $\chi(x, y) = \chi(y, x)$.

Taking the sum with respect to v we get

$$(3.8) \quad \left(\sum_v \underbrace{\varphi_{v\chi}^w a_{v\chi}^w \varphi_{v\chi}^w u, u}_{(1)} \right) + C \left(\sum_v \underbrace{\varphi_{v\chi}^w \varphi_{v\chi}^w u, u}_{(2)} \right) \geq 0.$$

Lemma 3.2. *The operator $\sum_v \varphi_{v\chi}^w \varphi_{v\chi}^w$ is bounded on L^2 .*

Proof. As in the proof of the L^2 continuity in [5] we shall use the Cotlar-Knapp-Stein Lemma. Let us set $A_v = \varphi_{v\chi}^w \cdot \varphi_{v\chi}^w$. We have to show

$$(3.9) \quad \sup_v \sum_\mu \|A_v A_\mu^*\|^{1/2} \leq C, \quad \sup_v \sum_\mu \|A_v^* A_\mu\|^{1/2} \leq C.$$

The proof of course is completely symmetric for the two terms. Let us consider the first. We have (here the norm is the $L^2 - L^2$ operator norm), $\|A_v A_\mu^*\| \leq \|\varphi_{v\chi}^w\| \cdot \|\varphi_{v\chi}^w \varphi_{\mu\chi}^w\| \cdot \|\varphi_{\mu\chi}^w\|$.

Since $\varphi_v \in S(1, g)$ has uniformly bounded semi-norms in this space, $\varphi_{v\chi}^w$ is bounded in L^2 with uniformly bounded operator norm (see the Remark after Theorem 4.3 in [3]). Therefore the first part of (3.9) will be a consequence of the following estimate.

$$(3.10) \quad \left\{ \begin{array}{l} \text{For every } l \in N \text{ one can find } C_l > 0 \text{ such that for every } v \\ \|\varphi_{v\chi}^w \varphi_{\mu\chi}^w\| \leq C_l (1 + d_{v\mu})^{-l}. \\ \text{One can find } l_0 \in N \text{ such that} \\ \sup_v \sum_\mu (1 + d_{v\mu})^{-l_0} < +\infty. \end{array} \right.$$

The proof of 3.10 is close to that of Theorem 4.3 in [3] and we give it for sake of completeness and the convenience of the later use in the proof of Lemma 3.3. Let $\psi \in S(1, \tilde{G})$, $\psi = 1$ on $\{(x, y) : \exists z : \chi(x, z)\chi(z, y) \neq 0\}$.

Then by formula (4.22) in [3] we can write

$$(3.11) \quad \varphi_{v\chi}^w \varphi_{\mu\chi}^w = \theta_{v\mu\psi}^w$$

$$(3.12) \quad \begin{cases} \theta_{\nu\mu}(x, \xi) = \exp\left(\frac{i}{2}\sigma(D_z, D_\zeta; D_t, D_\tau)\right) [F_{x,\xi}(z, t, \zeta, \tau)] \Big|_{t=\tau=0}^{\zeta=z=0} \\ F_{x,\xi}(z, t, \zeta, \tau) = \chi(x+z+t, x+z-t)\chi(x+z+t, x-z+t) \\ \qquad \qquad \qquad \varphi_\nu(x+z, \xi+\zeta)\varphi_\mu(x+t, \xi+\tau). \end{cases}$$

We claim that, denoting by πU_ν the projection of U_ν on V :

$$(3.13) \quad \inf_{t \in \pi U_\nu} G_t(x-t) > A_0^2 \varepsilon \Rightarrow \text{supp } F_{x,\xi} = \emptyset.$$

Indeed let $(z, t, \zeta, \tau) \in \text{supp } F_{x,\xi}$. Then $x+z \in \pi U_\nu$, $x+t \in \pi U_\nu$, $(X, Y) \in \text{supp } \chi$, $(X, Z) \in \text{supp } \chi$ where $X = x+z+t$, $Y = x+z-t$, $Z = x-z+t$. It follows that $D(X, Y) \leq \varepsilon$, $D(X, Z) \leq \varepsilon$. By Lemma 2.2 in [3], since $\varepsilon \leq a_0$, we get $G_X(X-Y) \leq 4A_0\varepsilon$, $G_X(X-Z) \leq 4A_0\varepsilon$. It follows that $G_X(X - \frac{X+Y}{2}) = \frac{1}{4}G_X(X-Y) \leq A_0\varepsilon \leq a_0$ so by (2.1),

$$G_{x+z}(z) = \frac{1}{4}G_{\frac{X+Y}{2}}(X-Z) \leq \frac{A_0}{4}G_X(X-Z) \leq A_0^2\varepsilon.$$

Since $t = x+z \in \pi U_\nu$, this contradicts (3.13).

It follows that, on the support of $\theta_{\nu\mu}$ we have

$$(3.14) \quad \begin{cases} \inf_{t \in \pi U_\nu} G_t(x-t) \leq A_0^2\varepsilon \\ \inf_{z \in \pi U_\nu} G_z(x-z) \leq A_0^2\varepsilon. \end{cases}$$

We shall show that this implies

$$(3.15) \quad \begin{cases} \forall t \in \pi U_\nu \quad G_t(x-t) \leq 2A_0^4\varepsilon + 4A_0\rho & (\leq \min(a_0, a_2)) \\ \forall z \in \pi U_\mu \quad G_z(x-z) \leq 2A_0^4\varepsilon + 4A_0\rho & (\leq \min(a_0, a_2)). \end{cases}$$

Indeed let $t_0 \in \pi U_\nu$ be such that $G_{t_0}(x-t_0) \leq A_0^2\varepsilon$. Since t_0, t are in πU_ν we have $G_{t_0}(t-t_0) \leq g_{w_\nu}(w-w_\nu) \leq \rho$, $G_{t_0}(t_0-t_\nu) \leq \rho$. Since $\rho \leq a_0$ we obtain $G_t \leq A_0 G_{t_0} \leq A_0^2 G_{t_0}$. Therefore

$$G_t(x-t) \leq 2[G_t(x-t_0) + G_t(t_0-t)] \leq 2[A_0^4\varepsilon + 2A_0\rho].$$

◇

It follows from [5], formula (18.4.12) that, for every $k \in \mathbb{N}$ one can find $C_k > 0$, independent of ν such that

$$(3.16) \quad |\theta_{\nu\mu}(w)| \leq C_k \left(1 + \inf_{\substack{w' \in U_\nu \\ w'' \in U_\mu}} [g_{w'}^\sigma(w-w'') + [g_{w''}^\sigma(w-w')]] \right)^{-k}.$$

We would like to replace the right hand side of (3.16) by

$$\left(1 + \inf_{w' \in U_\nu} g_w^\sigma(w-w') + \inf_{w'' \in U_\mu} g_w^\sigma(w-w'')\right)^{-k}.$$

This will follow from

$$(3.17) \quad g_w^\sigma(w-w') + g_w^\sigma(w-w'') \leq C(1 + g_w^\sigma(w-w') + g_w^{\sigma'}(w-w'))^{n_0}$$

for some n_0 independent of w, w', w'' .

If we set $w=(x, \xi), w'=(y, \eta), w''=(z, \zeta)$, it follows from (3.15) that $G_y(x-y)$ and $G_z(x-z)$ are bounded by a_2 . It follows from (2.8) that

$$\begin{aligned} g_w^\sigma &\leq A_2 g_w^\sigma (1 + g_w^{\sigma'}(w-w'))^N \\ g_w^{\sigma'} &\leq A_2 g_w^\sigma (1 + g_w^{\sigma'}(w-w''))^N \end{aligned}$$

which implies

$$\begin{aligned} g_w^\sigma &\leq A_2 g_w^{\sigma'} (1 + g_w^{\sigma'}(w-w'))^N \\ g_w^{\sigma'} &\leq A_2 g_w^{\sigma'} (1 + g_w^{\sigma'}(w-w''))^N \end{aligned}$$

so

$$g_w^\sigma(w-w') + g_w^\sigma(w-w'') \leq A_2(2 + g_w^{\sigma'}(w-w') + g_w^{\sigma'}(w-w''))^{N+1}.$$

It follows that (3.17) will be a consequence of

$$(3.18) \quad g_w^\sigma(w-w') + g_w^{\sigma'}(w-w'') \leq C(1 + g_w^{\sigma'}(w-w'') + g_w^{\sigma'}(w-w'))^q$$

for some $q \in \mathbb{N}$.

Let us set $w_1 = w' + w'' - w$. Since $w'=(y, \eta) \in U_\nu, w''=(z, \zeta) \in U_\mu$ (3.13) implies that $G_y(x-y) \leq a_0, G_z(x-z) \leq a_0$. We deduce from (2.1) that $G_y(x-z) \leq A_0^2 G_z(x-z) \leq 2A_0^6 \varepsilon + 4A_0^3 \rho \leq a_2$. Since $w_1 - w' = w'' - w = (z-x, \zeta - \xi)$ and $w'=(y, \eta)$, (2.8) implies

$$\begin{aligned} g_w^\sigma &\leq A_2 g_{w_1}^\sigma (1 + g_w^{\sigma'}(w'' - w_1))^N \\ g_w^{\sigma'} &\leq A_2 g_{w_1}^\sigma (1 + g_w^{\sigma'}(w'' - w_1))^N. \end{aligned}$$

This implies

$$(3.19) \quad \begin{cases} g_{w_1}^\sigma(w-w'') \leq A_2(1 + g_w^{\sigma'}(w-w''))^{N+1} \\ g_{w_1}^\sigma(w-w') \leq A_2(1 + g_w^{\sigma'}(w-w'))^{N+1}. \end{cases}$$

By symmetry one gets

$$(3.20) \quad \begin{cases} g_w^\sigma \leq A_2 g_{w_1}^\sigma (1 + g_{w_1}^\sigma (w' - w_1))^N \\ g_{w''}^\sigma \leq A_2 g_{w_1}^\sigma (1 + g_{w_1}^\sigma (w'' - w_1))^N. \end{cases}$$

It follows from (3.19) and (3.20) that

$$g_w^\sigma (w - w') + g_{w''}^\sigma (w - w'') \leq C(A_2)(1 + g_w^\sigma (w - w'') + g_{w''}^\sigma (w - w'))^q$$

for some q . This is (3.18). Therefore (3.17) is proved and by (3.16) we get

$$(3.21) \quad |\theta_{v\mu}(x, \xi)| \leq C_l \left(1 + \inf_{w' \in U_\nu} g_w^\sigma (w - w') + \inf_{w'' \in U_\mu} g_w^\sigma (w - w'') \right)^{-l}.$$

Let $w'_0 \in U_\nu$ and $w''_0 \in U_\mu$ be such that

$$(3.22) \quad g_w^\sigma (w - w'_0) = \inf_{w' \in U_\nu} g_w^\sigma (w - w'), \quad g_w^\sigma (w - w''_0) = \inf_{w'' \in U_\mu} g_w^\sigma (w - w'').$$

Since $w'_0 \in U_\nu$ it follows from (3.15) that $G_{x'_0}(x - x'_0) \leq a_2$. Then (2.8) implies $g_{w'_0}^\sigma \leq A_2 g_w^\sigma (1 + g_w^\sigma (w - w'_0))^N$. By the slow variation of g in U_ν we get

$$\begin{aligned} g_{w'_0}^\sigma (w'_0 - w''_0) &\leq A_1 g_{w'_0}^\sigma (w'_0 - w''_0) \leq A_1 A_2 g_w^\sigma (w'_0 - w''_0) (1 + g_w^\sigma (w - w'_0))^N \\ &\leq 2A_1 A_2 [g_w^\sigma (w'_0 - w) + g_w^\sigma (w - w''_0)] (1 + g_w^\sigma (w - w'_0))^N \\ &\leq 2A_1 A_2 (1 + g_w^\sigma (w - w'_0) + g_w^\sigma (w - w''_0))^{N+1}. \end{aligned}$$

It follows from (3.21) and (3.22) that if we set

$$(3.23) \quad d_{v\mu} = \inf_{\substack{w' \in U_\nu \\ w'' \in U_\mu}} g_{w'_0}^\sigma (w' - w'')$$

we have for every $l \in \mathbb{N}$,

$$(3.24) \quad |\theta_{v\mu}(x, \xi)| \leq C_l (1 + d_{v\mu})^{-l}.$$

The same estimate is true for every semi-norm of $\theta_{v\mu}$ in $S(1, g)$. It follows from [3] (Remark after Theorem 4.3) that first part of (3.10) is valid. Let us now show the second part. We fix so we may assume that $g_{w'_0}(t) = |t|^2$. We have

$$\sum_{\mu=1}^{\mu_0} (1 + d_{v\mu})^{-l_0} = \sum_{k=1}^{k_0} \sum_{k-1 \leq d_{v\mu} < k} (1 + d_{v\mu})^{-l_0} \leq \sum_{k=1}^{k_0} k^{-l_0} \text{card } M_k$$

where $M_k = \{\mu : d_{v\mu} < k\}$. We shall show that $\text{card } M_k \leq Ck^q$ for some q , which will prove (3.10).

Now if $\mu \in M_k$ there exists $w'_\nu \in U_\nu, w''_\mu \in U_\mu$ such that $g_{w'_0}^\sigma (w'_\nu - w''_\mu) < k$. It follows from (2.3) that

$$(3.25) \quad g_{w'_0}^\sigma(w'_v - w''_\mu) \leq A_1 k.$$

Now, by (3.15), if $w \in \text{supp } \theta_{v_\mu}$, $w = (x, \xi)$ we have $G_{x'_0}^\sigma(x - x'_v) \leq a_0$ and $G_{x''_\mu}(x - x''_\mu) \leq a_0$. It follows that $G_x(x - x'_v) + G_x(x - x''_\mu) \leq 4A_0^5 \varepsilon + 8A_0^2 \rho$ so $G_x(x''_\mu - x'_v) \leq 8A_0^5 \varepsilon + 16A_0^2 \rho$. Therefore $G_{x'_0}(x'_v - x''_\mu) \leq 8A_0^6 \varepsilon + 16A_0^3 \rho \leq a_2$ by (3.5), (3.6). We deduce from (2.8) that $g_{w'_\mu}^\sigma(t) \leq C g_{w'_0}^\sigma(t) (1 + g_{w'_0}^\sigma(w'_v - w''_\mu))^N$. It follows from (2.6) and (3.25) that $g_{w'_\mu}^\sigma(t) \leq C(A_1) |t|^2 k^N$. Therefore $|t| \leq \delta k^{-N/2}$ implies $g_{w'_\mu}^\sigma(t) \leq C(A_1) \delta^2$ so:

$$|z - w''_\mu| \leq \delta k^{-N/2} \Rightarrow g_{w'_\mu}(z - w''_\mu) \leq C'(A_1) \delta^2.$$

Now $g_{w'_\mu}^{1/2}(z - w_\mu) \leq g_{w'_\mu}^{1/2}(z - w''_\mu) + g_{w'_\mu}^{1/2}(w_\mu - w''_\mu) \leq C'(A_1) \delta + \rho \leq a_1$ if δ is small enough. Therefore

$$(3.26) \quad |z - w''_\mu| \leq \delta k^{-N/2} \Rightarrow z \in U'_\mu.$$

Now, $|z - w'_v| \leq |z - w''_\mu| + |w''_\mu - w'_v| \leq \delta k^{-N/2} + g_{w'_0}^{1/2}(w''_\mu - w'_v) \leq \delta k^{-N/2} + k^{1/2}$, by (2.6). Then,

$$(3.27) \quad |z - w''_\mu| \leq \delta k^{-N/2} \Rightarrow |z - w'_v| \leq 2k^{1/2}.$$

Let us set $V_\mu = \{z : |z - w''_\mu| \leq \delta k^{-N/2}\}$. We deduce from (3.27)

$$(3.28) \quad \bigcup_{\mu \in M_k} V_\mu \subset \{z : |z - w'_v| \leq 2k^{1/2}\}.$$

Now there is a bound for the number of V_μ which can intersect (since it is true for U'_μ and $V_\mu \subset U'_\mu$). Therefore (3.28) implies

$$C_1 \sum_{\mu \in M_k} m(V_\mu) \leq m\left(\bigcup_{\mu \in M_k} V_\mu\right) \leq C_0 k^{n/2}.$$

It follows that $\text{card } M_k \cdot (\delta k^{-N/2})^n \leq C k^{n/2}$. This completes the proof of Lemma 3.2. ◇

We consider now the term (1) in (3.8). Using Theorem 3.3 in [3] we get

$$(3.29) \quad \varphi_{v_X}^w a_{v_X}^w \varphi_{v_X}^w = (\varphi_v^2 a_v)_\psi^w + r_{v_\psi}^w,$$

where $\psi = 1$ on the support of χ , $r_v \in S(1, g)$ and

$$(3.30) \quad |r_v^{(k)}(w)(w_1, \dots, w_k)| \leq C \prod_{i=1}^k g_w^{1/2}(w_i) \|a\|_{S(h-2, g)}.$$

Since $\left(\sum_v \varphi_v^2 a_v\right) = a_\psi^w$ we get

$$(a_\psi^w u, u) = \left(\sum_v \varphi_{v\chi}^w a_{v\chi}^w \varphi_{v\chi}^w u, u \right) - \left(\left(\sum_v r_v \right)^w u, u \right).$$

Lemma 3.3. *The operator $(\sum_v r_v)^w$ is bounded on L^2 .*

If we prove this Lemma, it will follow from (3.8) and Lemma 3.2, that $(a_\psi^w u, u) \geq -C\|u\|_{L^2}^2$. Since, by Corollary 2.5 in [3], we have $a_\psi^w - a_\chi^w = r_\chi^w$ with $r \in S(1, g)$, we shall have $(a_\chi^w u, u) \geq -C\|u\|_{L^2}^2$ for every properly supported χ , which proves Theorem 2.1.

Proof of Lemma 3.3. We shall estimate the semi-norms in $S(1, g)$ of the symbol of $r_{v\psi}^w$ more precisely than (3.30). We split the proof into two cases. Let $w = (x, \xi) \in W$ be fixed.

Case 1: $g_{w_v}(w - U_v) = \inf_{w' \in U_v} g_{w_v}(w - w') \leq \rho$.

We know by the construction of the balls U_v (see [5] Lemma 18.4.4) that there is at most N_0 (independent of w) such v . By (3.30) we get

$$\sum_{v \geq 1} |r_v^{(k)}(w)(w_1, \dots, w_k)| \leq N_0 \cdot C \prod_{i=1}^k g_{w_v}^{1/2}(w_i) \|a\|_{S(h^{-2}, g)}.$$

Case 2: $g_{w_v}(w - U_v) > \rho$.

Then $w \notin U_v$ and $\varphi_v^2 a_v(w) = 0$. Let us set $\varphi_{v\chi}^w a_{v\chi}^w \varphi_{v\chi}^w = d_{v\psi}^w$; then by (3.18) in [3] and (3.29) it is easy to see that $r_v(w) = d_v(w)$. We want to show that for every k, l in N ,

$$(3.31) \quad |d_v^{(k)}(w)(w_1, \dots, w_k)| \leq C_{kl} h^{-2}(w_v) \prod_{i=1}^k g_{w_v}^{1/2}(w_i) (1 + g_{w_v}^\sigma(w - U_v))^{-1/2},$$

where C_{kl} are independent of w, w_i and v .

This will follow essentially from the proof of Theorem 2.2.1 in [1]. Indeed, according to Bony-Lerner, a symbol c is said to be confined in U_v if for every integers k, l one can find constants C_{kl} such that for every X in U_v and every w_i in W we have

$$|c^{(k)}(X)(w_1, \dots, w_k)| \leq C_{kl} m(X) \prod_{i=1}^k g_{w_v}^{1/2}(w_i) (1 + g_{w_v}^\sigma(X - U_v))^{-1/2}.$$

We shall show that since a_v is supported in U_v and φ_v is confined in U_v then $a_{v\chi}^w \varphi_{v\chi}^w = b_v^w$, with b_v confined in U_v . Then since φ_v is supported in U_v , the same argument will show that d_v is confined in U_v , which proves (3.31).

According to (3.8) in [3] we have

$$(3.32) \quad b_v(X) = \pi^{-2n} \iint e^{2i\sigma(T-X, Z-X)} a_v(Z) \varphi_v(T) \chi(x+z-t, t+z-x) \cdot \chi(t+z-x, x+t-z) dZ dT,$$

where $X=(x, \xi), T=(t, \tau), Z=(z, \zeta)$.

Let $X_0=(x_0, \xi_0)$ be such that $g_{w_v}(X_0)=1, (g_{w_v}^\sigma(X))^{1/2} = \sigma(X, X_0)$. Then

$$\frac{1}{2} \langle X_0, D_Z \rangle (2i\sigma(T-X, Z-X)) = \sigma(T-X, X_0) = [g_{w_v}^\sigma(T-X)]^{1/2}.$$

It follows that

$$\left(1 + \frac{1}{2} \langle X_0, D_Z \rangle\right)^k e^{2i\sigma(T-X, Z-X)} = (1 + g_{w_v}^\sigma(T-X)^{1/2})^k e^{2i\sigma(T-X, Z-X)}.$$

Therefore

$$b_v(X) = \pi^{-2n} \iint e^{2i\sigma(T-X, Z-X)} (1 + g_{w_v}^\sigma(T-X)^{1/2})^{-k} \varphi_v(T) \left(1 + \frac{1}{2} \langle X_0, D_Z \rangle\right)^k [a_v(Z) \chi(x+z-t, t+z-x) \chi(t+z-x, x+t-z)] dZ dT.$$

Now since φ_v and a_v are confined in U_v and $\chi \in S(1, \tilde{G})$, the Leibniz formula will give

$$(3.33) \quad |b_v(X)| \leq C \|\varphi_v\|_{0,t} \|a_v\|_{k,N} \|\chi\|_{S(1, \tilde{G})}^2 \iint (1 + g_{w_v}^\sigma(T-X))^{-k/2} (1 + g_{w_v}^\sigma(T-U_v))^{-1/2} \cdot (1 + g_{w_v}^\sigma(T-U_v))^{-N} (G_{x+z-t}^{1/2}(x_0) + G_{t+z-x}^{1/2}(x_0) + G_{x+t-z}^{1/2}(x_0))^k dZ dT.$$

Now in the integral defining b_v , since a_v is supported in U_v , we get exactly as in (3.15)

$$(3.34) \quad \forall s \in \pi U_v \quad G_s(x-s) \leq 2A_0^4 \varepsilon + 4A_0 \rho \quad (\leq \min(a_0, a_2)).$$

It follows that on the support of the function inside the integral in (3.32) we have

$$\begin{cases} G_{x+z-t} \leq A_0 G_z \\ G_{t+z-x} \leq A_0 G_z \\ G_{x+t-z} \leq A_0 G_z. \end{cases}$$

Since $z \in \pi U_v$ (because $Z \in \text{supp } a_v \subset U_v$) we get $G_z \leq A_0^2 G_{x_v}$; by (2.3), $G_{x_v} \leq g_{w_v}$, therefore

$$G_{x+z-t}^{1/2}(x_0) + G_{t+z-x}^{1/2}(x_0) + G_{x+t-z}^{1/2}(x_0) \leq C(A_0) g_{w_v}(X_0) = C(A_0).$$

We deduce from (3.33),

$$(3.35) \quad |b_\nu(X)| \leq C \|\varphi_\nu\|_{0,l} \|a_\nu\|_{k,N} \|\chi\|_{\tilde{S}(1,\tilde{G})}^2 \iint (1 + g_{w_\nu}^\sigma(T-X))^{-k/2} (1 + g_{w_\nu}^\sigma(T-U_\nu))^{-l/2} \cdot (1 + g_{w_\nu}^\sigma(Z-U_\nu))^{-N} dZ dT.$$

Now

$$(3.36) \quad \begin{aligned} 1 + g_{w_\nu}^\sigma(X-U_\nu) &\leq C(1 + g_{w_\nu}^\sigma(X-T) + g_{w_\nu}^\sigma(T-U_\nu)). \\ 1 + g_{w_\nu}^\sigma(X-U_\nu) &\leq C(1 + g_{w_\nu}^\sigma(X-T))(1 + g_{w_\nu}^\sigma(T-U_\nu)). \end{aligned}$$

On the other hand for every w' in U_ν

$$1 + g_{w_\nu}^\sigma(Z-w_\nu) \leq 1 + 2g_{w_\nu}^\sigma(Z-w') + 2g_{w_\nu}^\sigma(w'-w_\nu) \leq 3 + 2g_{w_\nu}^\sigma(Z-w')$$

because $g \leq g^\sigma$ and the radius of U_ν is smaller than one. It follows that

$$(3.37) \quad (1 + g_{w_\nu}^\sigma(Z-U_\nu))^{-1} \leq C(1 + g_{w_\nu}^\sigma(Z-w_\nu))^{-1}.$$

Taking in (3.35) $N = 2n + 1$, $k = l + 2n + 1$, we deduce from (3.36) and (3.37)

$$|b_\nu(X)| \leq C \|\varphi_\nu\|_{0,l} \|a_\nu\|_{2n+1+l, 2n+1} (1 + g_{w_\nu}^\sigma(X-U_\nu))^{-l/2} \cdot \iint (1 + g_{w_\nu}^\sigma(X-T))^{-(n+1/2)} (1 + g_{w_\nu}^\sigma(Z-w_\nu))^{-(n+1/2)} dZ dT.$$

Since the product of the determinant of g_{w_ν} and $g_{w_\nu}^\sigma$ is equal to one and

$$\|\varphi_\nu\|_{0,l} \leq M, \quad \|a_\nu\|_{2n+1+l, 2n+1} \leq C_{l,n} \sup_{X \in U_\nu} h^{-2}(X),$$

we get our claim. We estimate the derivatives by the same method. This proves (3.31).

Now, from the definition (2.7) of h we have for every w' in U_ν ,

$$g_{w_\nu}^\sigma(w-w') \geq h^{-2}(w_\nu) g_{w_\nu}(w-w').$$

Since we are in case 2 it follows that

$$(3.38) \quad g_{w_\nu}^\sigma(w-U_\nu) \geq \rho h^{-2}(w_\nu).$$

Now, let $w'_0 = (x'_0, \xi'_0) \in U_\nu$ be such that

$$(3.39) \quad g_{w_\nu}^\sigma(w-U_\nu) = g_{w'_0}^\sigma(w-w'_0).$$

It follows from (3.34) that $G_{x'_0}(x-x'_0) \leq a_2$ and from (2.8)

$$g_{w'_0} \leq A_2 g_w (1 + g_{w'_0}^\sigma (w - w'_0))^N.$$

By the slow variation of g in U_ν we deduce

$$g_{w_0} \leq A_1 A_2 g_w (1 + A_1 g_{w_0}^\sigma (w - w'_0))^N.$$

From (3.31), (3.38) and (3.39) we get

$$(3.40) \quad |r_\nu^{(k)}(w)(w_1, \dots, w_k)| \leq C_{kIN} \prod_{i=1}^k g_w^{1/2}(w_i) (1 + g_{w_0}^\sigma (w - U_\nu))^{-\frac{1}{2} + \frac{N}{2} + 1}.$$

Assume $w \in U_\lambda$. Using the slow variation of g in U_λ and (3.34) we get

$$g_{w_\lambda}^\sigma \leq A_1 g_w^\sigma \leq A_1 A_2 g_{w'_0}^\sigma (1 + g_{w'_0}^\sigma (w - w'_0))^N$$

and therefore

$$g_{w_\lambda}^\sigma (w - w'_0) \leq A_1 A_2 (1 + A_1 g_{w_0}^\sigma (w - w'_0))^{N+1}.$$

$$(3.41) \quad 1 + \inf_{\substack{w' \in U_\nu \\ w'' \in U_\lambda}} g_{w_\lambda}^\sigma (w' - w'') \leq C(A_1, A_2) (1 + g_{w_0}^\sigma (w - U_\nu))^{N+1}.$$

Recall that we defined in (3.23) $d_{\lambda\nu}$ by

$$d_{\lambda\nu} = \inf_{\substack{w' \in U_\nu \\ w'' \in U_\lambda}} g_{w''}^\sigma (w' - w'').$$

By (3.40), (3.41) and the slow variation of g on U_λ we get for every $k, q \in \mathbb{N}$

$$(3.42) \quad |r_\nu^{(k)}(w)(w_1, \dots, w_k)| \leq C_{kq} \prod_{i=1}^k g_w^{1/2}(w_i) (1 + d_{\lambda\nu})^{-q}.$$

Using the second part of (3.10) we obtain that the semi-norms of Σr_ν are uniformly bounded in $S(1, g)$; it follows that the operator $(\Sigma r_\nu)_\psi^w$ is bounded on L^2 , by Theorem 4.3 in [3]. The proof is complete. \diamond

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F. Colombini
Dipartimento di Matematica
Università di Pisa
56127 Pisa, Italia
e-mail: colombin@dm.unipi.it

D. Del Santo
Dip. di Scienze Matematiche
Università di Trieste
Piazzale Europa 1
34127 Trieste, Italia
e-mail: delsant@univ.trieste.it

Claude Zuily
Université de Paris-Sud
Département de Mathématiques
91405 Orsay Cedex, France
e-mail: Claude.Zuily@math.u-psud.fr

