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Osaka University
ON MOTION OF AN ELASTIC WIRE
AND SINGULAR PERTURBATION
OF A 1-DIMENSIONAL PLATE EQUATION

NORIHITO KOISO

(Received March 23, 1999)

1. Introduction and preliminaries

Consider a springy circle wire in the euclidean space $\mathbb{R}^3$. We characterize such a wire as a closed curve $\gamma = \gamma(x)$ with unit line element and fixed length. For such a curve, its elastic energy is given by

$$E(\gamma) = \int_0^L |\gamma_{xx}|^2 \, dx.$$ 

Solutions of the corresponding Euler-Lagrange equation are called elastic curves. Closed elastic curves in the euclidean space are classified in [7]. We discuss on motion of a circle wire governed by the elastic energy.

We will see that the equation becomes an initial value problem for $\gamma = \gamma(x, t)$:

$$\begin{align*}
\gamma_t + \partial_x^4 \gamma + \mu \gamma_{tt} &= \partial_x \left\{ (w - 2|\gamma_{xx}|^2) \gamma_x \right\}, \\
-w_{xx} + |\gamma_{xx}|^2 w &= 2|\gamma_{xx}|^4 - |\partial_x^2 \gamma|^2 + |\gamma_{tx}|^2, \\
\gamma(x, 0) &= \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0.
\end{align*}$$

Here, $\mu$ is a constant which represents the resistance, and the ODE for $w$ corresponds to the constrained condition $(\gamma_{tx}, \gamma_{tx}) \equiv 0$ (i.e., $|\gamma_x| \equiv 1$.) When the resistance $\mu$ is very large, we can analyze the behavior of the solution replacing the time parameter $t$ to $\tau = \mu^{-1} t$. Then, (EW) becomes

$$\begin{align*}
\mu^{-2} \gamma_{\tau \tau} + \partial_x^4 \gamma + \gamma_{\tau} &= \partial_x \left\{ (w - 2|\gamma_{xx}|^2) \gamma_x \right\}, \\
-w_{xx} + |\gamma_{xx}|^2 w &= 2|\gamma_{xx}|^4 - |\partial_x^2 \gamma|^2 + \mu^{-2} |\gamma_{tx}|^2, \\
\gamma(x, 0) &= \gamma_0(x), \quad \gamma_{\tau}(x, 0) = \mu \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0.
\end{align*}$$

And, when $\mu \to \infty$, we get, omitting initial data $\gamma_{\tau}(x, 0)$,

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The equation (EP), treated in [4] and [5], has a unique all time solution for any initial data, and the solution converges to an elastic curve. In this paper, we will prove:

1) The equation (EW) has a unique short time solution for any initial data. (Corollary 3.13.)

2) If $\mu$ is large, then the solution of (EW) exists for long time, and converges to a solution of (EP) when $\mu \to \infty$. (Corollary 4.10.)

Note that in 2), the derivative $\gamma_t(x, 0) = \mu \gamma_1(x)$ diverges when $\mu \to \infty$.

If (EW) contained no 3rd derivatives and was not coupled with ODEs, i.e., if our equation was $\gamma_{tt} + \partial_x^3 \gamma + \mu \gamma_t = F(\gamma, \gamma_x, \gamma_{xx}, \gamma_t)$, it is standard to show the short time existence of solutions. (See [9] Section 11.7.) Being coupled is not main difficulty to solve the equation. We can overcome it by careful estimation similar to [4]. However, the difficulty due to the presence of 3rd derivatives is essential. We will overcome the difficulty using the new unknown variable $\xi := \gamma_x$. As we will see in Lemma 2.2, the equation for $\xi$ does not contain 3rd derivatives $\nabla_x \xi_x$. Owing to the lack of the term, we will be able to solve $(EW^\xi)$ by a usual method: perturb to a parabolic equation and show the solution of the parabolic equation converges to a solution of the original equation. This will be done in Section 3.

Remark 1.1. In this paper, we only treat curves in the 3-dimensional euclidean space $\mathbb{R}^3$. But, the result holds also on the case of any dimensional euclidean space, with no modification of proofs.

By similarity, we may assume that the length of the initial curve $\gamma_0$ is 1. From now on, a closed curve means a map from $S^1 \equiv \mathbb{R}/\mathbb{Z}$ into the euclidean space $\mathbb{R}^3$ or the unit sphere $S^2$. The inner product of vectors is denoted by $(\cdot, \cdot)$, and the norm is denoted by $|\cdot|$. We also use the covariant derivation $\nabla$ on $S^2$. For a tangential vector field $X(x)$ along a curve $\gamma(x)$ on $S^2$, the covariant derivative is defined by $\nabla_x X := (X'(x))^T$. The covariant differentiation is non-commutative, because the curvature tensor $R$ of $S^2$ is non-zero. For example, if $X(x, t)$ is a tangential vector field along a family $\gamma(x, t)$ of curves on $S^2$, we have

$$\nabla_x \nabla_t X - \nabla_t \nabla_x X = R(\gamma_x, \gamma_t) X = (\gamma_t, X) \gamma_x - (\gamma_x, X) \gamma_t.$$ 

For functions on $S^1$ and vector fields along a closed curve, we use $L_2$-inner product $(\cdot, \cdot)$ and $L_2$-norm $\|\cdot\|$. Sobolev $H^n$-norm is denoted by $\|\cdot\|_n$. For a tensor field along the closed curve on $S^2$, $\|\cdot\|_n$ is defined using covariant derivation. That is, $\|\zeta\|^2_n = \sum_{i=0}^n \|\nabla_x^i \zeta\|^2$. We also use $C^n$ norm $\|\cdot\|_{(n)}$. In particular, $\|\cdot\|_{(0)} = \max |\cdot|$.
2. The equations

To derive the equation of motion, we use Hamilton's principle. For a moving curve \( \gamma = \gamma(t, x) \), the velocity energy is given by \( \| \gamma_t \|^2 \) and the elastic energy is given by \( \| \gamma_{xx} \|^2 \). (By rescaling, we omit coefficients.) Therefore, the real motion is a stationary point of the integral

\[
L(\gamma) := \int_{t_1}^{t_2} \| \gamma_t \|^2 - \| \gamma_{xx} \|^2 \, dt.
\]

That is, the integral

\[
L' := \int_{t_1}^{t_2} (\gamma_t, \delta_t) - (\gamma_{xx}, \delta_{xx}) \, dt
\]

should vanish for all \( \delta = \delta(t, x) \) satisfying \( \delta(t_1, x) = \delta(t_2, x) = 0 \) and the constrained condition \((\gamma_x, \delta_x) \equiv 0\).

From integration by parts, we see

\[
L' = \int_{t_1}^{t_2} \gamma_{tt} + \partial_x^4 \gamma, \delta \, dt.
\]

On the other hand, the orthogonal complement of the space \( V := \{ \delta \mid (\gamma_x, \delta_x) \equiv 0 \} \) at each time \( t \) is \( V^\perp = \{ (u, \delta) \mid u = u(x) \} \). Therefore, \( \gamma \) is stationary if and only if \( \gamma_t \in V \) and \( \gamma_{tt} + \partial_x^4 \gamma = (u \gamma_x)_x \) for some function \( u = u(t, x) \).

REMARK 2.1. Many papers (e.g., [2], [3]) apply Hamilton's principle using \( \| \gamma_t \|^2 + \| \gamma_x \|^2 \) as the kinetic energy, and gets a wave equation. The wave equation is completely different from (EW). A linear version of our equation can be found, for example, in [1] p. 246.

This difference can be explained as follows. We characterize a planar thick wire of length \( L \), of radius \( R \) and of unit weight per length as a map \( u = u(x, y) : [0, L] \times [-R, R] \to \mathbb{R}^2 \) such that \( u(x, y) = \gamma(x) + yJ\gamma(x) \), where \( \gamma \) is a curve of unit line element and \( J \) is the \( \pi/2 \) rotation. When \( u \) moves, i.e. when we consider a family \( u = u(x, y, t) \) of such curves, the velocity energy becomes

\[
\frac{1}{2R} \int_0^L dx \int_{-R}^R |u_t(x, y)|^2 \, dy = \| \gamma_t \|^2 + \frac{1}{3} R^2 \| \gamma_{xt} \|^2.
\]

Hence, our wire is infinitely thin, while previous papers treat thick wires.

In this paper, we treat slightly more general equation, equation with resistance \( \mu \). That is,

\[
\gamma_{tt} + \mu \gamma_t + \partial_x^4 \gamma = (u \gamma_x)_x,
\]
coupled with an ODE for $u$, which is derived from the constrained condition: $|\gamma_x| \equiv 1$. From
\[
0 = \partial_x^2 |\gamma_x|^2 = 2(\gamma_{tt} x, \gamma_x) + 2|\gamma_{tx}|^2,
\]
the unknown $u$ satisfies
\[
(-\partial_x^2 \gamma + \partial_x^2 (u \gamma_x) - \mu \gamma_{tx}, \gamma_x) = -|\gamma_{tx}|^2.
\]
Using $|\gamma_x|^2 \equiv 1$, we can rewrite this to
\[
-u_{xx} + |\gamma_{xx}|^2 u = 2\partial_x^2 |\gamma_{xx}|^2 - |\partial_x^3 \gamma|^2 + |\gamma_{tx}|^2,
\]
and, putting $w := u + 2|\gamma_{xx}|^2$, we get (EW).

Since the principal part of (EW) is the operator of the plate equation:
\[
u_{tt} + \partial_x^4 u,
\]
we perturb it to a parabolic operator:
\[
u_{tt} - 2\varepsilon u_{txx} + (1 + \varepsilon^2)\partial_x^4 u
\]
\[
= (\partial_t - (\varepsilon + \sqrt{-1})\partial_x^2)(\partial_t - (\varepsilon - \sqrt{-1})\partial_x^2)u
\]
with $\varepsilon > 0$. It is possible to show that a perturbed equation of (EW)
\[
\begin{cases}
\gamma_{tt} - 2\varepsilon \gamma_{txx} + (1 + \varepsilon^2)\partial_x^2 \gamma + \mu \gamma_t = \partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\
-w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3 \gamma|^2 + |\gamma_{tx}|^2,
\end{cases}
\]
\[
\gamma(x, 0) = \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x), \quad (\gamma_0, \gamma_1) = 0
\]
has a short-time solution. However, we cannot get uniform estimate when $\varepsilon \to 0$, because
$\partial_x \{(w - 2|\gamma_{xx}|^2)\gamma_x\}$ contains the third derivative of $\gamma$. To overcome this difficulty,
we convert (EW) to an equation on $S^2$, and “remove” the third derivative.

We introduce a new unknown function $\xi$ by $\xi = \gamma_x$. The function $\xi$ is a family of
closed curves on $S^2$.

**Lemma 2.2.** The equation (EW) is equivalent to equation
\[
\begin{cases}
\nabla_t \xi + \nabla_x^3 \xi + \mu \xi_t = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x,
\end{cases}
\]
\[
- w_{xx} + |\xi_x|^2 w = |\xi_x|^4 - |\nabla_x \xi_x|^2 + |\xi_x|^4,
\]
\[
\xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x), \quad \int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0.
\]
and (EP) is equivalent to equation

\[\begin{aligned}
\dot{\xi}_x + \nabla^2_x \xi_x &= (w - |\xi_x|^2) \nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x, \\
-w_{xx} - |\xi_x|^2 w &= -|\nabla_x \xi_x|^2 + |\xi_x|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \int_0^1 \xi_0 \, dx = 0.
\end{aligned}\]

\text{(EP)}

Proof. It is straightforward to check the following decomposition:

\[\begin{aligned}
\dot{\xi}_x &= \nabla_x \xi_x - |\xi_x|^2 \xi_x, \\
\dot{\xi}_t &= \nabla_t \xi_t - |\xi_t|^2 \xi_t, \\
\partial_t^2 \xi_e &= \nabla_x \xi_x - |\xi_x|^2 \nabla_x \xi_x - \frac{5}{2} \partial_x |\xi_x|^2 \xi_x + (|\nabla_x \xi_x|^2 + |\xi_x|^4 - 2\partial_x^2 |\xi_x|^2) \xi_x.
\end{aligned}\]

Using these formulas, we see that the \(x\)-derivatives of (EW) imply (EW\(\xi\)). Conversely, (EW\(\xi\)) implies the equation

\[\dot{\xi}_t + \partial_t^4 \xi + \mu \xi_t = \partial_t^2 \{w - 2|\xi_x|^2)\xi_x\}.
\]

Under the assumption: \(\int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0\), we see that the closedness condition: \(\int_0^1 \xi \, dx \equiv 0\) is satisfied. Let \(\gamma\) be the solution of an ODE:

\[\begin{aligned}
\gamma_t + \mu \gamma_s &= -\partial_t^4 \xi + \partial_s \{w - 2|\xi_x|^2)\xi_x\}, \\
\gamma(x, 0) &= \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x).
\end{aligned}\]

Then

\[\gamma_{xxt} + \mu \gamma_{xxt} = -\partial_t^4 \xi + \partial_x^2 \{w - 2|\xi_x|^2)\xi_x\} = \xi_{tt} + \mu \xi_t
\]

and \((\gamma_x - \xi)_t + \mu (\gamma_x - \xi)_t \equiv 0\). Hence \(\gamma_x \equiv \xi\) and \(\gamma\) is a solution of (EW).

A similar calculation gives the equivalence of (EP) and (EP\(\xi\)). \qed

3. Short time existence

In this section, we fix \(\mu \in \mathbb{R}\).

To perturb (EW\(\xi\)), we introduce a function \(\rho(x, y)\). Since \(\xi_0\) is the derivative of a closed curve \(\gamma_0\) in the euclidean space, each component of \(\xi_0\) takes 0 at some \(x\). Therefore, by Wirtinger's inequality, we have \(\|\xi_{0x}\|^2 \geq \pi^2 \|\xi_0\|^2 \geq \pi^2\). (It is known in fact that \(\|\xi_{0x}\|^2 \geq 4\pi^2\).) Let \(\delta(r)\) be a \(C^\infty\) function on \(\mathbb{R}\) such that \(\delta(r) = 1\) on \(|r| \leq \pi^2/8\), \(\delta(r) = 0\) on \(\pi^2/4 \leq |r|\) and \(0 \leq \delta(r) \leq 1\) on \(\pi^2/8 \leq |r| \leq \pi^2/4\). We put

\[\rho(x, y) = \pi^2 + \delta(y^2 - |\xi_{0x}(x)|^2)(y^2 - \pi^2).
\]
Fix an interval $I$ such that $|\xi_0(x)|^2 \geq \pi^2/2$ for any $x \in I$. If $x \in I$ and $|y^2 - |\xi_0(x)|^2| \leq \pi^2/4$, then $\rho(x, y) \geq \min\{\pi^2, y^2\} \geq \pi^2/4$. And if $|y^2 - |\xi_0(x)|^2| \geq \pi^2/4$, then $\rho(x, y) = \pi^2$. Therefore, for any function $u(x)$,

$$\int_0^1 \rho(x, u(x)) \, dx \geq \frac{\pi^2}{4} \int_1^1 \, dx.$$  

**Remark 3.1.** Below, we use the function $\rho$ only to ensure $\rho \geq 0$ everywhere and $\int_0^1 \rho(x, u(x)) \, dx$ is bounded from below by a positive constant. Note that $\rho(x, y) := y$ satisfies this requirement if $\xi = \gamma_x$ for some closed curve $\gamma$ in the euclidean space.

**Proposition 3.2.** Let $\xi_0(x)$ be a $C^\infty$ closed curve on $S^2$ with $\|\xi_0\| \geq \pi$ and $\xi_1(x)$ a $C^\infty$ tangent vector field along $\xi_0$. Let $\rho$ be the function defined as above. Then, equation

$$\{ \begin{align*}
\partial_t \xi_t &- 2\varepsilon \nabla^2_\xi \xi_t + (1 + \varepsilon^2) \nabla^3_\xi \xi_t + \mu \xi_t \\
&= (w - |\xi_t|^2) \nabla_\xi \xi_t + 2w_\xi \xi_t - \frac{3}{2} \partial_\xi |\xi_t|^2 \xi_t, \\
-w_{xx} + \rho(x, |\xi_t|^2) w &= |\xi_t|^2 - |\nabla_\xi \xi_t|^2 + |\xi_t|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x)
\end{align*} \}$$

has a $C^\infty$ solution on some interval $0 \leq t < T$.

**Proof.** We can prove unique short-time existence of $(EW^\xi\varepsilon)$ by a similar method with that used in [4]. Here, we mention only two steps. One is an estimation of the ODE for $w$. Lemma 3.3 with the function $\rho$ ensures estimation of $w$ by $\xi$. Another, Lemma 3.4, is a crucial point to use the contraction principle. \[
\]

**Lemma 3.3** ([4] Lemma 4.1, Lemma 4.2). Let $a$ and $b$ be $L_1$-functions on $S^1$ such that $a \geq 0$ and $\|a\|_{L_1} > 0$. Then, the ODE for a function $w$ on $S^1$

$$-w'' + aw = b$$

has a unique solution $w$, and the solution $w$ is estimated as

$$\max|w| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1},$$

$$\max|w'| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1}.$$  

Moreover, there exists universal constants $C > 0$ and $N > 0$ depending on $n$ such that

$$\|w\|_{n+2} \leq C(1 + \|a\|_{L_1}^N) \|b\|_{n},$$

$$\|w\|_{(n+2)} \leq C(1 + \|a\|_{(0)}^N) \|b\|_{(n)}.$$
Lemma 3.4. We consider a linear PDE for $u$

\[
\begin{cases}
u_{tt} - 2\varepsilon u_{txx} + (1 + \varepsilon^2)\partial_x^4 u = f, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x).
\end{cases}
\]

If $f \in C^{2\alpha}$, $u_0 \in C^{4+2\alpha}_x$ and $u_1 \in C^{2+2\alpha}_x$, then there is a unique solution $u \in C^{4+2\alpha}$. Moreover, we have an estimation:

\[
\|u\|_{C^{4+2\alpha}} \leq C\{\|f\|_{C^{2\alpha}} + \|u_0\|_{C^{4+2\alpha}} + \|u_1\|_{C^{2+2\alpha}}\},
\]

where $\|\cdot\|_{C^{2\alpha}}$ means the Hölder norm for $x$-direction, and $\|\cdot\|_{C^{4+2\alpha}}$ means the weighted Hölder norm ($t$-derivatives are counted twice of $x$-derivatives.)

Proof. We decompose the equation to a parabolic equation as

\[
u_t - (\varepsilon + \sqrt{-1})u_{xx} = v, \quad v_t - (\varepsilon - \sqrt{-1})v_{xx} = f.
\]

Using the fundamental solution

\[
\Gamma(x, t) = \frac{1}{2\sqrt{\pi\varepsilon}\sqrt{-1}\sqrt{t}} \exp\left(-\frac{x^2}{4(\varepsilon \pm \sqrt{-1})t}\right)
\]

of the parabolic operator $\partial_t - (\varepsilon \pm \sqrt{-1})\partial_x^2$, we can estimate as

\[
\|u\|_{C^{4+2\alpha}} \leq C\{\|v\|_{C^{2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\} \\
\leq C\{\|f\|_{C^{2\alpha}} + \|v_0\|_{C^{4+2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\} \\
\leq C\{\|f\|_{C^{2\alpha}} + \|u_1\|_{C^{2+2\alpha}} + \|u_0\|_{C^{4+2\alpha}}\}.
\]

When we take the limit $\varepsilon \to 0$ in (EW$^{\xi\varepsilon}$), we should note that the term $\partial^3_x\xi_x$ is quasi-linear, and contains the third derivative of $\xi$. In fact, in local coordinate system,

\[
\nabla^3_x \xi_x = \{\partial^4_x \xi_x + 4\Gamma_q \cdot (\xi_x \partial^2_x \xi_x) \partial_{x^p} + \text{[lower order terms]}\}.
\]

However, when we integrate it by parts, we can treat it as though it contained no third derivatives.

Lemma 3.5. For any $K > 0$, there are $T > 0$ and $M > 0$ with the following property:

Let $\xi$ be a solution of (EW$^{\xi\varepsilon}$) with $\varepsilon \in [0, 1]$ on an interval $[0, t_1) \subset [0, T)$. If its initial value satisfies $\|\xi_1\|_1^2 + \|\xi_0\|_1^2 \leq K$, then $\|\xi_t\|_1^2 + \|\xi_x\|_1^2 \leq M$ holds on $0 \leq t < t_1$. 
Proof. Put
\[ f = (w - \rho(x, |\xi_x|^2))\nabla_x \xi_x + 2w_x\xi_x - \frac{3}{2}\partial_x|\xi_x|^2\xi_x. \]

We can estimate
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|\xi_t\|^2 + (1 + \epsilon^2)\|\nabla_x \xi_x\|^2 \right\} \\
= \langle \xi_t, \nabla_t \xi_t \rangle + (1 + \epsilon^2)\langle \nabla_x \xi_x, \nabla_t \nabla_x \xi_x \rangle \\
= \langle \xi_t, \nabla_t \xi_t + (1 + \epsilon^2)\nabla_x^2 \xi_x \rangle + (1 + \epsilon^2)\langle R(\xi_t, \xi_x)\xi_x, \nabla_x \xi_x \rangle \\
\leq \langle \xi_t, 2\epsilon \nabla_x^2 \xi_t + f \rangle - \mu\|\xi_t\|^2 + C\max|\xi_x|^2\|\xi_t\|\|\nabla_x \xi_x\| \\
\leq -2\epsilon\|\nabla_x \xi_x\|^2 + \langle \xi_t, f \rangle - \mu\|\xi_t\|^2 + C\|\xi_x\|^2\|\xi_t\|\|\nabla_x \xi_x\| \\
\leq (1 - \mu)\|\xi_t\|^2 + \|f\|^2 + C\|\xi_x\|^2\|\xi_t\|^2 + \|\nabla_x \xi_x\|^2,
\]

and,
\[
\frac{1}{2} \frac{d}{dt} \|\xi_x\|^2 = \langle \xi_x, \nabla_t \xi_x \rangle = -\langle \nabla_x \xi_x, \xi_t \rangle \leq \|\nabla_x \xi_x\|^2 + \|\xi_t\|^2.
\]

Here, by Lemma 3.3, \(\|f\| \leq C(1 + \|\xi_t\|^2 + \|\xi_x\|^2)^N\). Therefore, putting \(X(t) := 1 + \|\xi_t\|^2 + (1 + \epsilon^2)\|\xi_x\|^2\), we get
\[ X'(t) \leq C_1X(t)^N, \]
and, \(X(t)\) is bounded from above by a solution of the ODE: \(y'(t) = C_1y(t)^N\).

Remark 3.6. If we use original equation of \(\gamma\), which contains \(\partial_x^3\gamma\) in the right hand side, the term \(\langle \gamma_t, \partial_x^3\gamma \rangle\) appears in the estimation. Since we need the term \(-2\epsilon\|\gamma_x\|^2\) to cancel \(\langle \gamma_t, \partial_x^3\gamma \rangle\), we cannot get uniform estimate with respect to \(\epsilon\), and the following proof will fail.

Lemma 3.7. For any \(K > 0\) and \(n \geq 0\), there is \(M > 0\) with the following property:

Let \(\xi\) be a solution of (EW^ε) with \(\varepsilon \in [0, 1]\) on \([0, T)\). If its initial value satisfies \(\|\xi_0\|, \|\xi_0\|_{n+1} \leq K\), and if it satisfies \(\|\xi_t\|, \|\xi_t\|^2 \leq K\) on \(0 \leq t < T\), then \(\|\xi_t\|, \|\xi_t\|^2 \leq M\) holds on \(0 \leq t < T\).

Proof. The claim holds for \(n = 0\) by taking \(M = K\). We prove the claim by induction. Suppose that the claim holds for \(n\). In particular, we know bounds of \(\|\xi_t\|_n\),
Therefore, we have

\[
\| \nabla_x \nabla_x^{n+1} \xi_r - \nabla_x^{n+1} \xi_r \| = \left\| \sum_{i=0}^{n} \nabla_x^{i} (R(\xi_r, \xi_r) \nabla_x^{n-i} \xi_r) \right\|
\]

\[
\leq C \sum_{i+j \leq n} \| \nabla_x \xi_r \| \| \nabla_x \xi_r \| \leq C \sum_{i+j \leq n} \| \xi_r \|_i \| \xi_r \|_{j+1} \leq C \| \xi_r \|_{n+1},
\]

\[
\| \nabla_x \nabla_x^{n+2} \xi_r - \nabla_x^{n+3} \xi_r \| = \left\| \sum_{i=0}^{n+1} \nabla_x^{i} (R(\xi_r, \xi_r) \nabla_x^{n+1-i} \xi_r) \right\|
\]

\[
\leq C \left( \| \xi_r \| \| \nabla_x^{n+1} \xi_r \| + \sum_{i=0}^{n+1} \| \nabla_x \xi_r \| \right) \leq C \left( \| \xi_r \|_1 \| \xi_r \|_{n+1} + \| \xi_r \|_{n+1} \right)
\]

\[
\leq C \| \xi_r \|_{n+1},
\]

\[
\| w \|_{n+2} \leq C(1 + \rho(x, |\xi_r|^2) \| n \|_N \| \xi_r \|^2 - |\nabla_x \xi_r|^2 + |\xi_r|^4 \|_n
\]

\[
\leq C \left( \sum_{i+j \leq n} \| \xi_r \|_i \| \xi_r \|_{j+1} + \sum_{i+j \leq n, i \leq j} \| \nabla_x \xi_r \|_i \| \nabla_x \xi_r \|_{j+1} + 1 \right)
\]

\[
\leq C(\| \xi_r \|_{n+1} + \| \xi_r \|_1 \| \xi_r \|_{n+2} + 1) \leq C(\| \xi_r \|_{n+1} + \| \xi_r \|_{n+2} + 1).
\]

Put

\[
f := (w - |\xi_r|^2) \nabla_x \xi_r + 2w \xi_r - \frac{3}{2} \partial_x |\xi_r|^2 \xi_r.
\]

Then,

\[
\| f \|_{n+1} \leq C(1 + \| \xi_r \|_{n+2} + \| \xi_r \|_{n+2} \| \xi_r \|_1 + \| w \|_{n+2} \| \xi_r \|_1 + \| w \|_2 \| \xi_r \|_{n+1})
\]

\[
\leq C(1 + \| \xi_r \|_{n+2} + \| w \|_{n+2}) \leq C(1 + \| \xi_r \|_{n+2} + \| \xi_r \|_{n+1}).
\]

Using these, we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \nabla_x^{n+1} \xi_r \|^2 + (1 + e^2) \| \nabla_x^{n+2} \xi_r \|^2 \right\}
\]

\[
= \langle \nabla_x^{n+1} \xi_r, \nabla_x \nabla_x^{n+1} \xi_r \rangle + (1 + e^2) \langle \nabla_x^{n+2} \xi_r, \nabla_x \nabla_x^{n+2} \xi_r \rangle
\]

\[
\leq \langle \nabla_x^{n+1} \xi_r, \nabla_x^{n+1} \nabla_x \xi_r \rangle + (1 + e^2) \langle \nabla_x^{n+2} \xi_r, \nabla_x^{n+3} \xi_r \rangle
\]

\[
+ C(\| \xi_r \|_{n+1} + \| \xi_r \|_{n+2})(1 + \| \xi_r \|_{n+1})
\]

\[
\leq \langle \nabla_x^{n+1} \xi_r, \nabla_x^{n+1} \xi_r \rangle + (2e \nabla_x^{n+2} \xi_r - \mu \xi_r) + C(1 + \| \xi_r \|^2_{n+1} + \| \xi_r \|^2_{n+2})
\]

\[
\leq \langle \nabla_x^{n+1} \xi_r, 2e \nabla_x^{n+3} \xi_r \rangle + C(1 + \| \xi_r \|^2_{n+1} + \| \xi_r \|^2_{n+2})
\]

\[
\leq C(1 + \| \nabla_x^{n+1} \xi_r \|^2 + (1 + e^2) \| \nabla_x^{n+2} \xi_r \|^2).
\]
Lemma 3.8. For any smooth initial data \( \{ \xi_0, \xi_1 \} \), \( K > 0 \), \( T > 0 \) and \( m, n \geq 0 \), there is \( M > 0 \) with the following property:

Let \( \xi \) is a solution of \((\text{EW}^\varepsilon)\) with \( \varepsilon \in [0, 1] \) on \([0, T)\). If \( \|\xi_t\|, \|\xi_x\|_1 \leq K \) on \( 0 \leq t < T \), then \( \xi \) is smooth on \( S^1 \times [0, T) \), and the derivatives are bounded as \( \|\nabla^m_t \xi\| \leq M \).

Proof. By Lemma 3.7, the claim holds for \( m \leq 1 \). Suppose that the claim holds up to \( m \). In particular, we have \( C^\infty_x \) bounds of \( \xi \) and \( \nabla^m_t \xi \). Therefore, using

\[-(\partial^j_t w)_{xx} + \partial^j_t w = \partial^j_t f - \sum_{0 \leq i \leq j} \binom{j}{i} \partial^i_t \rho \partial^{j-i}_t w\]

for \( 0 \leq j \leq m - 1 \), we have \( C^\infty_x \) bounds of \( \partial^{m-1}_t w \). Since \( \nabla^{m+1}_t \xi \) is expressed as a polynomial of these lower derivatives, we get the result.

Proposition 3.9. The equation \((\text{EW}^\xi)\) has a short time solution for any smooth initial data.

Proof. We put \( K := \|\xi_1\|^2 + \|\xi_0\|_1^2 \) and take \( T > 0 \) in Lemma 3.5. Then, by Lemma 3.8, any solution has a priori estimate on \( 0 \leq t < T \).

Let \( [0, T_\varepsilon) \) be the maximal interval such that a solution exists for \( \varepsilon \). If \( T_\varepsilon < T \), then \( \xi \) is smoothly and uniformly bounded on \([0, T_\varepsilon)\), hence can be continued beyond \( T_\varepsilon \). This contradicts to the definition of \( T_\varepsilon \), therefore we see that \( T_\varepsilon \geq T \). We conclude that a solution \( \xi \) exists on the interval \([0, T)\) for each \( \varepsilon > 0 \), and these \( \xi \)'s have smooth uniform bounds on \( S^1 \times [0, T) \).

Therefore, taking a sequence \( \varepsilon_i \to 0 \), we get a solution of

\[
\begin{aligned}
\nabla_t \xi_1 + \nabla^2_x \xi_1 + \mu \xi_t &= (w - |\xi_x|^2) \nabla_x \xi_t + 2 w_x \xi_t - \frac{3}{2} \partial_x \xi_x |^2 \xi_t, \\
-w_{xx} + \rho(x, |\xi_x|^2) w &= |\xi_x|^2 - |\nabla_x \xi_t|^2 + |\xi_t|^4, \\
\xi(x, 0) &= \xi_0(0), \quad \xi_t(x, 0) = \xi_t(x).
\end{aligned}
\]

Since \( \rho(x, |\xi_x|^2) = |\xi_x|^2 \) when \( \xi \) is sufficiently close to \( \xi_0 \), we have a solution \( \xi \) of \((\text{EW}^\xi)\) on some time interval. Once we have a short time solution \( \xi \) of \((\text{EW}^\xi)\), we can estimate the solution as Lemma 3.8, and the solution \( \xi \) can be continued to the interval \([0, T)\).

Proposition 3.10. Let \( \xi \) and \( \tilde{\xi} \) be solutions of \((\text{EW}^\xi)\) on \([0, T)\). If \( \xi \) and \( \tilde{\xi} \) have same smooth initial data, then they identically coincide.

Proof. To express the difference of two solutions, we use local coordinates. We fix the initial value \( \{ \xi_0, \xi_1 \} \), and take a local coordinate \( U \) which contains the initial
value $\xi_0$. In $U$, (EW$^\xi$) is expressed as:

$$\begin{cases}
\xi_{tt}^p + 4 \gamma_q u_{\xi}^q \epsilon_{xi}^q \xi_{xi}^q + 4 \gamma_q u_{\xi}^q \epsilon_{xi}^q \xi_{xi}^q = F^p[\xi_{xx}, w_x, u], \\
-w_{xx} + g_{qr}(\xi) \xi_{xx}^q \xi_{xx}^r w = G[\xi_{xx}, \xi],
\end{cases}$$

where $F^p[\xi_{xx}, w_x, \xi]$ is a polynomial of $\xi_{xx}^q$, $\xi_{xx}^q$, $w_x$, $w_x$, $\xi$, functions of $\xi^q$, and $G[\xi_{xx}, \xi]$ is a polynomial of $\xi_{xx}^q$, $\xi_{xx}^q$, $\xi$, functions of $\xi^q$. (We only note highest derivatives.)

Let $\{\tilde{\xi}, \tilde{w}\}$ be another solution of (EW$^\xi$) on $[0, t_1)$ $(t_1 \leq T)$. Applying Lemma 3.5 and Lemma 3.8 with $\varepsilon = 0$, we have smooth bounds of $\xi$ and $\tilde{\xi}$. We put $\zeta := \tilde{\xi} - \xi$, $u := \tilde{w} - w$. Then, we see that

$$\zeta_{tt}^p + 4 \gamma_q u_{\xi}^q \epsilon_{xi}^q \xi_{xi}^q + 4 \gamma_q u_{\xi}^q \epsilon_{xi}^q \xi_{xi}^q \zeta$$

equals to a sum of terms containing at least one of $\zeta_{xx}$, $\zeta_{xx}$, $u$, $u_{xx}$, $\zeta_{tt}$ or the difference of the values of a function at $\tilde{\xi}$ and $\xi$. Similarly,

$$-w_{xx} + g_{qr}(\xi) \xi_{xx}^q \xi_{xx}^r \zeta$$

equals to a sum of terms containing at least one of $\zeta_{xx}$, $\zeta_{xx}$, $\zeta$, or the difference of the values of a function at $\tilde{\xi}$ and $\xi$.

Therefore, we can estimate $\zeta$ and $u$ linearly:

$$\|\zeta_{tt}^p + 4 \gamma_q u_{\xi}^q \epsilon_{xi}^q \xi_{xi}^q \zeta\| \leq C(\|\zeta\| + \|\zeta_x\| + \|\zeta_{xx}\| + \|u\| + \|u_x\| + \|u_{xx}\|),$$

$$\|-u_{xx} + g_{qr}(\xi) \xi_{xx}^q \zeta_{xx}^r u\| \leq C(\|\zeta\| + \|\zeta_x\| + \|\zeta_{xx}\| + \|\zeta_{tt}\|).$$

Regarding $\zeta$ as a vector field along $\xi$, these inequalities can be written using covariant derivation along $\xi$:

$$\|\nabla^2 \zeta + \nabla_\xi \zeta\| \leq C(\|\zeta\|_2 + \|\nabla \zeta\|),$$

$$\|\zeta\| + \|\zeta_x\| + \|\zeta_{xx}\| + \|\zeta_{tt}\|.$$
This proof applies at any time $t_0$ such that $\tilde{\xi}(t_0) = \xi(t_0)$. Therefore, the set $\{t \mid \tilde{\xi}(t) = \xi(t)\}$ is open and closed in $[0, T)$, hence agrees to $[0, T)$.

Combining Proposition 3.9 and Proposition 3.10, we get the following

**Theorem 3.11.** The equation $(EW^\xi)$ has a unique short time solution for any smooth initial data.

**Remark 3.12.** To show this theorem, we did not assume that $\mu \geq 0$. Hence the result is time-invertible. That is, a unique solution exists on some open time interval $(-T, T)$ containing $t = 0$.

**Corollary 3.13.** The equation $(EW)$ has a unique short time solution for any smooth initial data.

4. **Singular perturbation**

In this section, we assume that $\mu > 0$ and change the time variable $t$ of $(EW^\xi)$ to $\mu^{-1}t$.

$$(EW^{\xi\mu}) \quad \begin{cases} \mu^{-2}\nabla_t \xi_t + \nabla_t^2 \xi_t + \xi_t = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x, \\ -w_{xx} + |\xi_x|^2w = \mu^{-2}|\xi_x|^2 - |\nabla_x \xi_x|^2 + |\xi_x|^4, \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \mu \xi_t(x), \quad \int_0^1 \xi_0 dx = \int_0^1 \xi_1 dx = 0. \end{cases}$$

First, we show uniform existence and boundedness of solutions with respect to large $\mu$. Constants $T, M$ below are independent of $\mu$.

**Lemma 4.1.** For any $K > 0$, there are $T > 0$ and $M > 0$ with the following property:

If $\xi$ is a solution of $(EW^{\xi\mu})$ on an interval $[0, t_1) \subset [0, T)$ and if its initial value satisfies $\|\xi_0\|, \|\xi_1\| \leq K$, then $\|\xi\|_1, \mu^{-1}\|\xi\| \leq M$ holds on $0 \leq t < t_1$.

Proof. It is similar to the proof of Lemma 3.5. We put

$$f = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2} \partial_x |\xi_x|^2 \xi_x,$$

and we have

$$\frac{1}{2} \frac{d}{dt} \left\{ \mu^{-2}\|\xi_t\|^2 + \|\nabla_x \xi_x\|^2 \right\} + \|\xi_t\|^2 = \langle \xi_t, f \rangle + \langle \nabla_x \xi_x, R(\xi_t, \xi_x) \rangle \leq \left( \frac{1}{4} + \frac{1}{4} \right) \|\xi_t\|^2 + \|f\|^2 + C(\|\xi_x\|_1^2 \|\nabla_x \xi_x\|)^2.$$
Here, $\|f\|^2$ is bounded by a polynomial of $X := \mu^{-2}\|\xi_t\|^2 + \|\nabla_x \xi_t\|^2 + \|\xi_t\|^2$. Combining it with $d\|\xi_t\|^2/dt \leq \|\xi_t\|^2 + \|\nabla_x \xi_t\|^2$, we have a $\mu$-independent estimate of time derivative of $X$ by a polynomial of $X$. Therefore, there is a $\mu$-independent time $T > 0$ such that $\|\xi_t\| \leq C\mu$ and $\|\xi_t\|_1 \leq C$ on $[0, T)$.

**Lemma 4.2.** For any $K > 0$ and $n > 0$, there are $M > 0$ and $\mu > 0$ with the following property:

Let $\xi$ be a solution of $(EW^\xi\mu)$ on $[0, T)$ with $\mu \geq \mu_0$. If its initial value satisfies $\|\xi_0\|_{n+1}$, $\|\xi_1\|_n \leq K$ and if it satisfies $\|\xi_t\|_1$, $\mu^{-1}\|\xi_t\| \leq K$ on $[0, T)$, then it holds that $\|\xi_t\|_{n+1}$, $\|w\|_{n+1}$, $\mu^{-1}\|\xi_t\| \leq M$ on $[0, T)$.

**Proof.** It is similar to the proof of Lemma 3.7. Suppose that we have bounds: $\|\xi_t\|_{n+1}$, $\mu^{-1}\|\xi_t\|_n \leq M$. They imply that $\|\xi_t\|_n$, $\mu^{-1}\|\xi_t\|_{n+1} \leq C$, and,

$$
\|w\|_{n+2}, \|f\|_{n+1} \leq C(1 + \mu^{-1}\|\xi_t\|_{n+1} + \|\xi_t\|_{n+2}) \\
\leq C(1 + \mu^{-1}\|\nabla_x^n \xi_t\| + \|\nabla_x^{n+2} \xi_t\|).
$$

Using this, we have

$$
\frac{1}{2} \frac{d}{dt} \left( \mu^{-2}\|\nabla_x^{n+1} \xi_t\|^2 + \|\nabla_x^{n+2} \xi_t\|^2 \right) + \|\nabla_x^{n+1} \xi_t\|^2 \\
= \langle \nabla_x^{n+1} \xi_t, \mu^{-2}\nabla_x \nabla_x^{n+1} \xi_t \rangle + \langle \nabla_x^{n+2} \xi_t, \nabla_x \nabla_x^{n+2} \xi_t \rangle + \|\nabla_x^{n+1} \xi_t\|^2 \\
\leq \langle \nabla_x^{n+1} \xi_t, \mu^{-2}\nabla_x \nabla_x^{n+1} \xi_t \rangle + \langle \nabla_x^{n+2} \xi_t, \nabla_x \nabla_x^{n+2} \xi_t \rangle + \|\nabla_x^{n+1} \xi_t\|^2 \\
+ C\mu^{-2}\|\nabla_x^{n+1} \xi_t\| \cdot \mu \|\xi_t\|_{n+1} + C\|\xi_t\|_{n+2}^2 \|\xi_t\|_{n+1} \\
\leq \langle \nabla_x^{n+1} \xi_t, \nabla_x^{n+1} f \rangle + \left( C\mu^{-1} + \frac{1}{8} \right) (\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + C\|\xi_t\|_{n+2}^2 \\
\leq \left( C\mu^{-1} + \frac{1}{4} \right) (\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + C(1 + \|\nabla_x^{n+2} \xi_t\|^2).
$$

Assuming that $\mu \geq 4C_1$ and combining it with the first estimation:

$$
\frac{1}{2} \frac{d}{dt} \left( \mu^{-2}\|\xi_t\|^2 + \|\nabla_x \xi_t\|^2 \right) \leq -\frac{1}{2}\|\xi_t\|^2 + C,
$$

we can estimate

$$
X(t) := \mu^{-2}(\|\nabla_x^{n+1} \xi_t\|^2 + \|\xi_t\|^2) + (\|\nabla_x^{n+2} \xi_t\|^2 + \|\nabla_x \xi_t\|^2)
$$

by $X'(t) \leq C(1 + X(t))$. Hence we have $\|\xi_t\|_{n+2} \leq C$, $\|\xi_t\|_{n+1} \leq C\mu$. Substituting it to the estimate of $\|w\|_{n+2}$, we get $\|w\|_{n+2} \leq C$.

**Proposition 4.3.** For any initial data $\xi_0$ and $\xi_1$, there is $T > 0$ such that $(EW^\xi\mu)$ has a solution on $[0, T)$ for each $\mu > 0$. Moreover, for any $n \geq 0$, there are $\mu > 0$
and \( M > 0 \) such that the solution with \( \mu \geq \mu_0 \) satisfies \( \|\xi_x\|_n, \|w\|_n \leq M \) and \( \|\xi_t\|_n \leq M\mu \) on \([0, T)\).

Proof. Using Lemma 4.1 and Lemma 4.2, the proof is similar to that of Proposition 3.9.

Let \( \{\eta, v\} \) be a solution of the limiting equation \( (\mu \to \infty) \) of \((EW^\xi)\) omitting initial data \( \xi(x, 0) \).

\[
\begin{align*}
\eta_t + \nabla_\xi^2 \eta_t &= (v - |\eta_t|^2)\nabla_\xi \eta_t + 2v_\xi \eta_t - \frac{3}{2} \partial_\xi |\eta_t|^2 \eta_t, \\
-v_{xx} + |\eta_t|^2 v &= -(|\nabla_\xi \eta_t|^2 + |\eta_t|^4), \\
\eta(x, 0) &= \xi_0(0).
\end{align*}
\]

\((EP^0)\)

In [4] (Theorem 7.5), we know that the corresponding equation for closed curves in the euclidean space has a unique all time solution. Therefore, \((EP^0)\) has a unique all time solution, via Lemma 2.2.

We regard function \( \eta \) as the 0-th approximation of \( \xi \) for \( \mu \to \infty \). To compare \( \xi \) and \( \eta \), we divide the interval \([0, \infty)\) so that the image \( \xi(S^1 \times I) \) of each subinterval \( I \) is contained in a local coordinate \( U \) of \( S^2 \). For a solution \( \xi \) and an interval \([t_0, t_1) \subset I \) such that \( \xi(S^1 \times [t_0, t_1]) \) is contained in \( U \), we denote by \( \{\zeta, u\} \) the difference between \( \xi \) and \( \eta \) in the local coordinate, i.e., \( \zeta^p := \xi^p - \eta^p \), \( u := w - v \). We use the local expression of \((EW^\xi)\):

\[
\begin{align*}
\mu^{-2}(\zeta_t^p + \Gamma_q^p_r(\xi)\xi^q_\xi^r + \xi_\xi^p + 4\Gamma_q^p_r(\xi)\xi^q_\xi^r \xi_\xi^r + \xi_t^p = F^p[\xi_{xx}, w_x], \\
-w_{xx} + g_{qr}(\xi)\xi^q_\xi^r w = \mu^{-2}g_{qr}(\xi)\xi^q_\xi^r + G[\xi_{xx}], \\
\zeta(x, 0) = \zeta_0(0), \quad \xi_t(x, 0) = \mu \xi_1(x), \quad \int_0^1 \zeta_t^p dx = \int_0^1 \xi_1^p dx = 0,
\end{align*}
\]

where \( F^p[\xi_{xx}, w_x] \) are polynomials of \( \xi_x, \xi_{xx}, w, w_x \), functions of \( \xi \), and \( G[\xi_{xx}] \) is a polynomial of \( \xi_x, \xi_{xx} \), functions of \( \xi \). (We only note highest derivatives.) Since the local expression of \((EP^0)\) is given by the above equations substituting \( \mu^{-1} = 0 \), \( \{\zeta, u\} \) satisfies

\[
\begin{align*}
\mu^{-2}(\zeta_t^p + 2\Gamma_q^p_r(\eta)\eta^q_\eta^r \xi_t^p) + \xi_\xi^p + 4\Gamma_q^p_r(\eta)\eta^q_\eta^r \xi_\xi^r + \eta_t^p \xi_t^p = F^p[\eta_{xx}, \eta_x], \\
-w_{xx} + g_{qr}(\xi)\xi^q_\xi^r w = \mu^{-2}g_{qr}(\xi)\eta^q_\eta^r + G[\eta_{xx}], \\
\zeta(x, 0) = 0, \quad \xi(x, 0) = \mu \xi_1(x).
\end{align*}
\]
We regard $\zeta$ as a vector field along $\eta$. Then, we can rewrite the above expression as

$$(\text{EW}^\zeta)$$

$$\begin{align*}
\mu^{-2} \nabla^2 \zeta + \nabla^4 \zeta + \nabla_t \zeta \\
= L_1[\nabla^2 \zeta, u_x] + Q_1[\nabla^2 \zeta, u_x; \nabla^3 \zeta, u_x] - \mu^{-2}\{\nabla_t \eta_x + L_2[\zeta] + Q_2[\nabla_t \zeta; \nabla_t \zeta]\}, \\
- u_{xx} + |\xi_x|^2 u
\end{align*}$$

$$\begin{align*}
= \mu^{-2}|\eta_x|^2 + L_3[\nabla_t \zeta] + Q_3[\nabla_t \zeta; \nabla \zeta] + L_4[|\nabla^2 \zeta| + Q_4[\nabla^2 \zeta; \nabla^2 \zeta]]. \\
(|\xi_x|^2 = |\eta_x|^2 + L_5[\nabla_t \zeta] + Q_5[\nabla_t \zeta; \nabla \zeta]).
\end{align*}$$

$$\zeta(x, 0) = 0, \quad \nabla_t \zeta(x, 0) = \mu \xi_1(x),$$

where $L_i$ are linear, $|Q_i(\alpha; \beta)| \leq C|\alpha| |\beta|$. (We only note highest derivatives.)

To get estimate of $(\zeta, u)$, we need following

**Lemma 4.4** ([5] Lemma 1.5). For any $K_1$, $K_2 > 0$ and any $T > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:

If $\mu \geq \mu_0$ and $X(t)$, $Y(t)$ and $Z(t)$ are non-negative functions on $[0, T)$ such that

$$X(0) \leq K_1 \mu^{-2}, \quad |X'(0)| \leq K_1, \quad Y(0) \leq K_1, \quad Z(0) \leq K_1 \mu^2,$$

and that

$$\begin{align*}
\mu^{-2}X''(t) + X'(t) &\leq K_1(X(t) + \mu^{-2}Z(t) + \mu^{-2}) - K_2 Y(t), \\
Y'(t) + \mu^{-2}Z'(t) &\leq K_1(Y(t) + 1) - K_2 Z(t),
\end{align*}$$

on $[0, T)$, then they satisfy

$$X(t) < M \mu^{-2}, \quad Y(t) < M \quad \text{and} \quad Z(t) < M \mu^2$$

on $[0, T)$.

**Lemma 4.5.** For any $n \geq 0$ and any $K > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:

Let $\{\zeta, u\}$ be the solution of $(\text{EW}^\zeta)$ with $\mu \geq \mu_0$, defined on $[t_0, t_1) \subset [0, T)$. If $\|\zeta\|_n \leq K \mu^{-1}$ at $t = t_0$, then $\|\zeta\|_n \leq M \mu^{-1}$ holds on $[t_0, t_1)$.

Proof. Note that we have bounds of $\{\xi, u\}$ by Proposition 4.3. Therefore, we know $\|\zeta\|_n \leq C$, $\|\nabla \zeta\|_n \leq C \mu$ and $\|u\|_n \leq C$. We may assume that $\mu \geq \mu_0 \geq 1$. For

$$h := \mu^{-2}|\eta_x|^2 + L_3[\nabla \zeta] + Q_3[\nabla \zeta; \nabla \zeta] + L_4[\nabla^2 \zeta] + Q_4[\nabla^2 \zeta; \nabla^2 \zeta],$$
we have
\[ \|h\|_n \leq C\mu^{-2}(1 + \|\nabla_i \zeta\|_n + \|\nabla_i \zeta \|_1 \|\nabla_i \zeta\|_n) + \|\zeta\|_{n+2} + \|\zeta\|_3 \|\zeta\|_{n+2} \]
\[ \leq C(\mu^{-2} + \mu^{-1}\|\nabla_i \zeta\|_n + \|\zeta\|_{n+2}), \]
and, \( \|u\|_{n+2} \leq C\|h\|_n \leq C(\mu^{-2} + \mu^{-1}\|\nabla_i \zeta\|_n + \|\zeta\|_{n+2}) \). And, for
\[ f := L_1[\nabla^2_i \zeta, u, \zeta] + Q_1[\nabla^2, u, \zeta, \nabla^2_i \zeta, u, \zeta] - \mu^{-2}(\nabla_i \eta + L_2[\zeta] + Q_2[\nabla_i \zeta, \nabla_i \zeta]), \]
we have
\[ \|f\|_n \leq C(\|\zeta\|_{n+2} + \|u\|_{n+1} + \mu^{-2}(1 + \|\nabla_i \zeta\|_1 \|\nabla_i \zeta\|_n)) \]
\[ \leq C(\|\zeta\|_{n+2} + \mu^{-2} + \mu^{-1}\|\nabla_i \zeta\|_n). \]

Put \( X_n(t) := \|\nabla^2_i \zeta\| \) and \( Z_n(t) := \|\nabla^2_i \nabla_i \zeta\| \). Then, we see that
\[ (X_0^2)' = 2(\zeta, \nabla_i \zeta) \leq 2X_0 Z_0, \]
\[ (X_i^2)' = 2(\nabla_i \zeta, \nabla_i \nabla_i \zeta) \leq -2(\nabla_i \zeta, \nabla_i \nabla_i \zeta) + C\|\zeta\|_1 \|\zeta\|_n \]
\[ \leq 2X_2 Z_0 + C(X_0^2 + X_1^2), \]
\[ \mu^{-2}(Z_i^2)' + 2Z_i^2 + (X_{i+2}^2)' = 2(\nabla_i \nabla_i \zeta, \mu^{-2}\nabla_i \nabla_i \zeta + \nabla_i \nabla_i \zeta) + 2(\nabla^{i+2}_i, \nabla_i \nabla^{i+2}_i) \]
\[ \leq 2(\nabla_i \nabla_i \zeta, \nabla_i \zeta) + C\|\nabla_i \nabla_i \zeta\| \|\mu^{-2}\|\nabla_i \zeta\|_{i-1} + \|\zeta\|_{i-1}) + C\|\nabla^{i+2}_i\| \|\zeta\|_{i+1} \]
\[ \leq C Z_i (X_{i+2} + X_0 + \mu^{-2} + \mu^{-1}(Z_i + Z_0)) + C(X_{i+2}^2 + X_0^2). \]

Therefore,
\[ \mu^{-2}(\|\nabla_i \zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' + 2\|\nabla_i \zeta\|_n^2 \]
\[ \leq C\|\zeta\|_{n+2}^2 + C\mu^{-1}\|\nabla_i \zeta\|_n^2 + C\mu^{-2} + C \sum_{i=0}^n Z_i(X_{i+2} + X_0) \]
\[ \leq \frac{1}{2}\|\nabla_i \zeta\|_n^2 + C\|\zeta\|_{n+2}^2 + C_1\mu^{-1}\|\nabla_i \zeta\|_n^2 + C\mu^{-2}, \]
\[ \mu^{-2}(\|\nabla_i \zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' \leq C(\|\zeta\|_{n+2}^2 + \mu^{-2}) - \|\nabla_i \zeta\|_n^2 \]
if \( \mu \geq 2C_1 \).

We also have,
\[ \mu^{-2}(X_i^2)'' + (X_i^2)' + 2X_{i+2}^2 \]
\[ = 2\mu^{-2}\|\nabla_i \nabla_i \zeta\|^2 + 2(\nabla_i \zeta, \mu^{-2}\nabla_i \nabla_i \zeta + \nabla_i \nabla_i \zeta + \nabla^{i+2}_i) \]
\[ +C \mu^{-2} \| \zeta \|_{-1}^2 + C \| \nabla^2 \zeta \| \{ \mu^{-2} (\| \nabla \zeta \|_{i-1} + \| \zeta \|_{i-2}) + \| \zeta \|_{i-1} \} \]
\[ \leq 3 \mu^{-2} Z_i^2 + CX_i \{ X_{i+2} + X_0 + \mu^{-2} + \mu^{-1} (Z_i + Z_0) \} \]
\[ + C \mu^{-2} (X_i^2 + X_0^2) + CX_i \{ \mu^{-2} (Z_i + Z_0) + X_i + X_0 \} \]
\[ \leq X_{i+2}^2 + C \{ X_i^2 + X_0^2 + \mu^{-2} (Z_i^2 + Z_0^2) + \mu^{-4} \}, \]
\[ \mu^{-2} \{ \| \zeta \|_n^2 \}'' + (\| \zeta \|_n^2)' \leq C \{ \| \zeta \|_n^2 + \mu^{-2} \| \nabla \zeta \|_n^2 + \mu^{-4} \} - \| \zeta \|_{n+2}^2. \]

Setting \( X := \| \zeta \|_n^2, \quad Y := \| \nabla^2 \zeta \|^2 \) and \( Z := \| \nabla \zeta \|_n^2 \) in Lemma 4.4, we have \( \| \zeta \|_n \leq C \mu^{-1}. \) \( \square \)

**Lemma 4.6.** For any \( n, m > 0 \) and \( K > 0, \) there are \( M > 0 \) and \( \mu_0 > 0 \) with the following property:

Let \( \{ \zeta, u \} \) be the solution of (EW) with \( \mu \geq \mu_0, \) defined on \( [t_0, t_1] \subset [0, T). \) If \( \| \nabla^m \zeta \|_n \leq K \mu^{m-1} \) at \( t = t_0, \) then

\[ \| \nabla^m \zeta \|_n \leq M (\mu^{-1} + \mu^m e^{-\mu t/2}), \]
\[ \| \partial_t^m u \|_n \leq M (\mu^{-1} + \mu^m e^{-\mu t/2}) \]

hold on \( [t_0, t_1]. \)

Proof. We put \( V_j := \mu^{-1} + \mu^j e^{-\mu t/2}. \) Note the log-convexity:

\[ V_j^2 \leq V_{j-1} V_{j+1} \quad \text{and} \quad V_j V_k \leq V_0 V_{j+k} \leq (1 + \mu_0^{-1}) V_{j+k} \quad \text{for} \quad j, k \geq 0. \]

We know that \( \| \nabla \zeta \|_n \leq C \mu_0, \| u \|_n \leq C \) by Proposition 4.3, and \( \| \zeta \|_n \leq C \mu^{-1} \) by Lemma 4.5. In particular, \( \| \zeta \|_n \leq C V_{-1} \) holds. We prove the estimate of \( \partial_t^m u \) and the estimate of \( \nabla^{m+1} \zeta, \) assuming the estimate of \( \partial_t^j u \) and \( \nabla^j \zeta \) for \( j < m. \)

First, we estimate \( \partial_t^m u. \) Put

\[ h := \mu^{-2} \| \nabla \zeta \|^2 + L_3 [\nabla \zeta; \zeta] + Q_3 [\nabla \zeta; \zeta] + L_4 [\nabla^2 \zeta] + Q_4 [\nabla^2 \zeta; \nabla^2 \zeta]. \]

It is estimated as

\[ \| \partial_t^m h \|_n \leq C \{ \mu^{-2} (1 + \| \nabla^{m+1} \zeta \|_n) + V_{2m-1} \]
\[ + \| \nabla \zeta \|_n \| \nabla^{m+1} \zeta \|_n + V_3^* V_{2m-1} \} + V_{2m-1} \]
\[ \leq C \{ \mu^{-1} \| \nabla^{m+1} \zeta \|_n + V_{2m} \}, \]

\[ \| \zeta \|_n \]
where $V_3^*$ appears only if $m \geq 2$. Therefore, we have

$$\| \partial_t^m u \|_{(n+2)} \leq \| \partial_t^m h \|_{(n)} + C \sum_{j=1}^m \| \partial_t^j |\xi| \|_{(n)} \| \partial_t^{m-j} u \|_{(n)}$$

$$\leq C \{ \mu^{-1} \| \nabla_i^{m+1} \zeta \|_{(n)} + V_{2m} \} + C \sum_{j=1}^m (1 + V_{2j-1}) V_{2(m-j)}$$

$$\leq C \{ \mu^{-1} \| \nabla_i^{m+1} \zeta \|_{(n)} + V_{2m} \}.$$

Now, we estimate $\nabla_i^{m+1} \zeta$. Put

$$f := L_1[\nabla_i^2 \zeta, u_x] + Q_1[\nabla_i^2 \zeta, u_x; \nabla_i^2 \zeta, u_x] - \mu^{-2}(\nabla_i \eta + L_2[\zeta] + Q_2[\nabla_i \zeta; \nabla_i \zeta]).$$

Then,

$$\| \nabla_i^m f \|_{(n)} \leq C \{ V_{2m-1} + \| \partial_t^m u \|_{(n+1)} + \mu_{t}^{n+1} \| \partial_t^m u \|_{(n+1)}$$

$$\leq C \{ \mu^{-1} \| \nabla_i^{m+1} \zeta \|_{(n)} + V_{2m} \},$$

where $V_3^*$ appears only if $m \geq 2$. Therefore,

$$\| \nabla_i^m (\mu^{-2} \nabla_i^2 \zeta + \nabla_i \zeta) \|_{(n)} \leq \| \nabla_i^m \zeta \|_{(n+4)} + \| \nabla_i^m f \|_{(n)}$$

$$\leq C \{ \mu^{-1} \| \nabla_i^{m+1} \zeta \|_{(n)} + V_{2m} \}.$$ 

Thus,

$$\mu^{-2} \frac{\partial}{\partial t} |\nabla_i^m \nabla_i^{m+1} \zeta|^2 + 2|\nabla_i^m \nabla_i^{m+1} \zeta|^2$$

$$= 2(\nabla_i^m \nabla_i^{m+1} \zeta, \mu^{-2} \nabla_i^m \nabla_i^{m+1} \zeta + \nabla_i^m \nabla_i^{m+1} \zeta)$$

$$\leq 2(\nabla_i^m \nabla_i^{m+1} \zeta, \nabla_i^m (\mu^{-2} \nabla_i^{m+2} \zeta + \nabla_i^{m+1} \zeta))$$

$$+ C \mu^{-2} |\nabla_i^m \nabla_i^{m+1} \zeta| \| \nabla_i^{m+1} \zeta \|_{(n+1)}$$

$$\leq C |\nabla_i^m \nabla_i^{m+1} \zeta| \{ \mu^{-1} \| \nabla_i^{m+1} \zeta \|_{(n)} + V_{2m} \}.$$ 

From this, for $X(t) := \| \nabla_i^{m+1} \zeta \|_{(n)}^2$, we have

$$\mu^{-2} X'(t) + 2X(t) \leq C_1 \mu^{-1} X(t)^2 + CV_{2m} X(t) \leq \left( \frac{1}{2} + C_1 \mu^{-1} \right) X(t)^2 + CV_{2m}^2,$$

where $X(t) = \limsup_{\delta \to 0} (X(t + \delta) - X(t))/\delta$.

We set $\mu_0 \leq 2C_1$. Then,

$$\mu^{-2} X'(t) + X(t) \leq C_2 (\mu^{-2} + \mu^4 e^{-\mu t}),$$
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\[ X(t) \leq X(t_0)e^{-\mu^2 t} + C_2(\mu^{-2} + \mu^{2m+2}e^{-\mu^2 t}) \leq C(\mu^{-2} + \mu^{2m+2}e^{-\mu^2 t}) \]

that is, \(\|\nabla_{\ell+1}^m \zeta\|^2 \leq C V_{2m+1}\).

Substituting it to the estimate of \(\|\partial^m u\|_{(m+2)}\), we get the estimation of \(\partial^m u\). \(\blacksquare\)

**Proposition 4.7.** For any initial data \(\{\xi_0, \xi_1\}\), any interval \(I_0, t_1 \subset [0, T]\) and any local coordinate \(U\) of \(S^2\) such that the image \(\eta(S^1 \times [t_0, t_1])\) is contained in \(U\), there exists \(\mu_0 > 0\) with the following property:

If \(\xi\) is a solution of \((\text{EW}^{\mu})\) on \([0, T]\), then the image \(\xi(S^1 \times [t_0, t_1])\) is contained in \(U\). Moreover, \(\xi\) uniformly converges to \(\eta\) on \([0, T]\) when \(\mu \to \infty\).

**Proof.** We divide the interval \([0, T]\) so that the image \(\eta(S^1 \times I)\) of each subinterval \(I\) is included to a local coordinate \(U_I\). \(\blacksquare\)

Note that \(\zeta\) is defined only on each short time interval.

Starting from \(t = 0\) and applying this Lemma on each time interval where \(\{z_0, z_1\}\) is defined, we see that \(\|\zeta\|_n\) is small for large \(\mu\).

We sum up these results, and get the following

**Theorem 4.8.** For any non-negative integers \(m, n\) and any positive number \(T\), there are positive numbers \(\mu_0\) and \(M\) with the following properties:

For each \(\mu \geq \mu_0\), there exists a solution \(\xi\) of \((\text{EW}^{\mu})\) on \([0, T]\), and \(\xi\) uniformly converges to \(\eta\) when \(\mu \to \infty\). More precisely,

\[|\partial^m T^p (\xi^p - \eta^p)| \leq M(\mu^{-1} + \mu^{2m-1}e^{-\mu^2 T/2})\]

holds on each local coordinate.

**Remark 4.9.** In general, we cannot expect uniform estimation on the whole time \([0, \infty)\). The limit \(\eta(\infty)\) can be an unstable elastic curve, and in that case, \(\xi(\infty)\) and \(\eta(\infty)\) discontinuously depend on the initial data.

**Corollary 4.10.** For any positive number \(T\), there exists a unique solution \(\gamma\) of \((\text{EW}\gamma)\) on \([0, T]\) for sufficiently large \(\mu > 0\). Moreover, the solution converges to a solution \(\eta\) of \((\text{EP})\) when \(\mu \to \infty\).
References


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