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EVEN SETS OF (−4)-CURVES ON RATIONAL SURFACE

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Abstract

We study rational surfaces having an even set of disjoint (−4)-curves. The properties of the surface S obtained by considering the double cover branched on the even set are studied. It is shown, that contrarily to what happens for even sets of (−2)-curves, the number of curves in an even set of (−4)-curves is bounded (less or equal to 12). The surface S has always Kodaira dimension bigger or equal to zero and the cases of Kodaira dimension zero and one are completely characterized. Several examples of this situation are given.

1. Introduction

Let X be a smooth surface. A set of ν disjoint smooth rational curves $N_1, \ldots, N_\nu$ is called an *even set* if there exists $L \in \mathrm{Pic}(X)$ such that $2L \equiv N_1 + \cdots + N_\nu$. In this note we study even sets of curves $N_1, \ldots, N_\nu$ where each N_i is a (−4)-curve (i.e. a smooth rational curve with self-intersection −4) on rational surfaces. We prove that, contrarily to what happens for even sets of (−2)-curves (cf. [6]), the number of curves in an even set of (−4)-curves is bounded. More precisely we show that the maximal number of curves in such a set is 12.

Given an even set of smooth rational curves one can consider the double cover branched on these curves. For even sets of (−2)-curves on rational surfaces, such a double cover is again a rational surface (see [6]). In contrast again the double cover of a rational surface branched on an even set of (−4)-curves has always Kodaira dimension ≥ 0. In this paper we characterize completely the even sets of (−4)-curves on rational surfaces, such that the corresponding double cover has Kodaira dimension 0 or 1. More precisely we show that any even set of (−4)-curves on a rational surface, whose corresponding double cover has Kodaira dimension 0 or 1, are components of fibres of a not relatively minimal elliptic fibration. We give examples for all the possible numbers of the (−4)-curves when the Kodaira dimension is 0. We do not know any examples for which the Kodaira dimension of the double cover is 2 and we conjecture this should not occur.

NOTATION. We work over the complex numbers. All varieties are projective algebraic. All the notation we use is standard in algebraic geometry. We just recall the definition of the numerical invariants of a smooth surface X: the self-intersection

number K_X^2 of the canonical divisor K_X, the *geometric genus* $p_g(X) := h^0(K_X) = h^2(\mathcal{O}_X)$, the *irregularity* $q(X) := h^0(\Omega_X^1) = h^1(\mathcal{O}_X)$ and the *holomorphic Euler characteristic* $\chi(X) := 1 + p_g(X) - q(X)$.

A $(-r)$-*curve* on a surface X is a smooth irreducible rational curve with self-intersection $-r$. An *even set of* $(-r)$-*curves* is a disjoint union of $(-r)$-curves $C_1, \ldots, C_n$ such that the divisor $C_1 + \cdots + C_n$ is divisible by 2 in $\mathrm{Pic}(X)$.

We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$.

2. General facts

Throughout this section we make the following

ASSUMPTION 1. X is a smooth projective rational surface and $C_1, \ldots, C_n$ is an even set of disjoint (-4)-curves on X. We denote by L the divisor satisfying $C \equiv 2L$, where $C := C_1 + \cdots + C_n$.

From now on to the end of this chapter, we denote by X a surface in the conditions of Assumption 1.

REMARK 2. We can contract the curves C_i obtaining a rational surface with n quotient singularities of type $(1/4)(1, 1)$.

Proposition 2.1. *The divisor L satisfies the following*:
(i) $h^0(X, L) = 0$;
(ii) $K_X L + L^2 = 0$;
(iii) $(K_X + L)^2 = K_X^2 + n$;
(iv) $1 \leq h^0(X, 2K_X + C) \leq n$;
(v) $1 \leq h^0(X, K_X + L) \leq n$.

Proof. Assertion (i) is obvious, because $2L \equiv C$, $h^0(X, C) = 1$ and C is reduced. By assumption 1, we have $L^2 = -n$ and $K_X L = n$, this proves (ii) and (iii). Finally, by the Riemann–Roch theorem, one has $h^0(X, K_X + L) \geq 1$ and thus the left side of inequalities (iv) and (v). On the other hand by the long exact sequence obtained from the exact sequence:

$$0 \to \mathcal{O}_X(2K_X) \to \mathcal{O}_X(2K_X + C) \to \mathcal{O}_C((2K_X + C)|_C) \to 0,$$

one has $h^0(X, 2K_X + C) \leq n$ (and so also $h^0(X, K_X + L) \leq n$) because X is rational, $\mathcal{O}_C(2K_X + C) = \mathcal{O}_C$ and $h^0(C, \mathcal{O}_C) = n$. $\square$

REMARK 3. In what follows, we can assume that $K_X + L$ is nef. Otherwise, since $K_X + L$ is effective, there is an irreducible curve E such that $E^2 < 0$ and $E(K_X +$

$L) < 0$. Since each curve C_i satisfies $C_i(K_X + L) = 0$, E is not one of the curves C_i. So $EL \geq 0$ and thus $EK_X < 0$. Since E is irreducible and $E^2 < 0$, the only possibility is that E is a $(−1)$-curve disjoint from C and so we can contract it, without changing the initial assumptions.

REMARK 4. Nefness of $K_X + L$ implies that for each $(−1)$-curve θ, there exists at least one $(−4)$-curve C_i such that $\theta C_i > 0$.

Our next goal is to describe the double cover of X branched along $C = 2L$. Let

$$\pi : S \to X$$

be a double cover branched along C. Then S is a smooth surface and by the double cover formulas ([1]), we have
- $K_S = \pi^*(K_X + L)$;
- $\mathcal{X}(\mathcal{O}_S) = 2$.

REMARK 5. The surface S, having $\mathcal{X}(\mathcal{O}_S) = 2$ has Kodaira dimension ≥ 0. Since we are assuming that $K_X + L$ is nef and $K_S = \pi^*(K_X + L)$, also K_S is nef and thus S is minimal.

Lemma 2.2. *Let X be a rational surface with an even set C of $(−4)$-curves. Then $h^0(X, −2K_X) \leq 1$. Furthermore $h^0(X, −2K_X) \neq 0$ if and only if the double cover $S \to X$ is a K3 surface.*

Proof. Notice that $h^0(X, C) = 1$ and $h^0(X, 2K_X + C) \geq 1$ by Proposition 2.1. Thus we conclude that either $h^0(−2K_X) = 0$ or $h^0(X, −2K_X) = 1$ and $h^0(X, 2K_X + C) = 1$. Since $K_X C_i = 2$, if $h^0(−2K_X) = 1$, then each C_i is a component of $−2K_X$. So we can write $−2K_X = C + \Gamma$, where Γ is an effective divisor and letting $\Delta = 2K_X + C$ we obtain $\Delta + C + \Gamma \equiv C$. Hence $\Delta = \Gamma = 0$ namely $C = −2K_X$. Since X, being rational, has no 2-torsion, also $L = −K_X$ and so $K_S = \mathcal{O}_S$. Thus S having $p_g = 1$, $\chi = 2$ is a K3 surface. Conversely, if S is a K3 surface, $K_S = \mathcal{O}_S$ and the result follows. □

Next, we apply the above results to the following proposition.

Proposition 2.3. *Let X be a rational surface with an even set C of n $(−4)$-curves and such that $K_X + L$ is nef. Then $−n \leq K_X^2 \leq −1$. Furthermore*
(i) *if $K_X^2 = −n$, then*
  (i$_a$) *$\kappa(S) = 0 \Rightarrow S$ is a K3 surface, or*
  (i$_b$) *$\kappa(S) = 1 \Rightarrow S$ is an elliptic surface;*
(ii) *if $K_X^2 > −n$, then $K_S^2 \geq 2$ and S is a surface of general type.*

Proof. Since $K_X + L$ is effective and nef one has $(K_X + L)^2 \geq 0$. So by Proposition 2.1 (iii) $K_X^2 \geq -n$. As we have seen, one has $h^0(X, K_X + L) \geq 1$ and so since $h^0(X, L)$ must be 0 we conclude that $h^0(-K_X) = 0$, otherwise the map

$$H^0(X, -K_X) \otimes H^0(X, K_X + L) \to H^0(X, L)$$

would have nonzero image. So by the Riemann–Roch theorem necessarily $K_X^2 \leq -1$.

The rest of the proposition is clear, by the classification of minimal surfaces (see e.g. [2]) and $K_S^2 = 2(K_X^2 + n)$. $\square$

Finally we recall an important result due to Miyaoka.

Proposition 2.4 ([11]). *The number n of disjoint (-2)-curves on a surface W with K_W nef satisfies $3c_2(W) - K_W^2 \geq (9/2)n$.*

With this result we obtain:

Lemma 2.5. *If X is a rational surface with an even set C of n (-4)-curves, then*

$$n \leq 16.$$

Furthermore, if equality holds, then $K_S^2 = 0$.

Proof. Since $\mathcal{X}(\mathcal{O}_S) = 2$, one has $c_2(S) \leq 24$. On the other hand, for each curve C_i, $\pi^{-1}(C_i)$ is a (-2)-curve in S. Finally, applying Proposition 2.4 we obtain the result. $\square$

We will start by studying the case $K_X^2 = -n$, this is the K3 and the elliptic case.

3. The elliptic fibration

In this section we want to prove the following:

Proposition 3.1. *Let X be as in* Assumption 1 *and let S be the associated double cover of X. Then $\kappa(S) \leq 1$ if and only if X has an elliptic fibration containing the (-4)-curves.*

Before proving the above result, we will need various facts that we now establish. Suppose that X has an elliptic fibration

$$p' : X \to \mathbb{P}^1$$

with general fibre F' such that every (-4)-curve is contained in a fibre.

Since $F'C_i = 0$ and $F'C = 0$, $\mathcal{O}_{F'}(C) = \mathcal{O}_{F'}$ and so either $\mathcal{O}_{F'}(L) = \mathcal{O}_{F'}$ or $\mathcal{O}_{F'}(L) \neq \mathcal{O}_{F'}$.

In the first case for a general fibre F', $\pi^*(F')$ is disconnected. More precisely $\pi^*(F')$ is the union of two fibres of an elliptic fibration on S. In the second case $\pi^*(F')$ is connected and, by the Hurwitz formula, again elliptic.

So we have

Lemma 3.2. *With the above notation, $\pi^* F'$ gives an elliptic fibration on S and we have the following commutative diagram*:

$$
\begin{array}{ccc}
S & \xrightarrow{\ p\ } & B \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\
X & \xrightarrow{\ p'\ } & \mathbb{P}^1.
\end{array}
$$

Moreover,

Lemma 3.3. *In the above situation one of the following holds*
- $h^0(-2K_X) = 1$, $\kappa(S) = 0$ *and S is a K3 surface;*
- $h^0(-2K_X) = 0$, $\kappa(S) = 1$ *and S is an elliptic surface.*

Proof. The proof follows by Proposition 2.3 and Lemmas 2.2 and 3.2. $\qquad\square$

Having established the properties above we now examine the converse situation. Let X be as in Assumption 1 and S the double cover of X branched in C.

If $\kappa(S) = 0$, then S is a K3 surface, so $K_X + L \equiv 0$ and X is a Coble surface, this is, a nonsingular rational surface with empty anticanonical linear system $|-K_X|$ but nonempty bi-anticanonical system $|-2K_X|$. By the results in [7], in this case, there is a birational morphism $X \to \mathbb{P}^2$ such that the image of $C \in |-2K_X|$ in $\mathbb{P}^2$ is a member of $|-2K_{\mathbb{P}^2}|$, whence a plane sextic, that will be called a Coble sextic.

Lemma 3.4. *The classification in* ([7], Section 5) *yields*:
(i) *If $n = 1$, the image of the irreducible (-4)-curve C on $\mathbb{P}^2$ is an irreducible member of a pencil of sextics with nine distinct double base points and having an extra singular double point.*
(ii) *If $n > 1$, then the image of C is the union of two members (both singular) of a pencil of cubics.*

We are ready to prove Proposition 3.1

Proof of Proposition 3.1. If X has an elliptic fibration containing the (-4)-curves the result follows by Lemma 3.2.

Conversely, assume that $\kappa(S) \leq 1$. By Proposition 2.3, we have two possibilities: $\kappa(S) = 0$ or 1.

For $\kappa(S) = 0$, S is a K3 surface and applying Lemma 3.4 we obtain the result. In fact, if $n = 1$, then X is a rational elliptic surface with one multiple fibre of multiplicity 2 whilst, if $n > 1$, X is a rational elliptic surface without multiple fibres.

For $\kappa(S) = 1$, S is an elliptic surface and so there is a smooth curve B and a surjective morphism $p \colon S \to B$ whose generic fibre is a nonsingular elliptic curve F. It is well known that: $\mathcal{X}(\mathcal{O}_S) = 2$ implies

$$(3.1) \qquad\qquad K_S \sim 2g(B)F + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} F,$$

where $m_1 F_1, \ldots, m_r F_r$ are the multiple fibres of $p \colon S \to B$.

Note that $K_S \pi^* C_i = 2(K_X + L)C_i = 0$. Since the elliptic fibration of S must be invariant under the involution associated to the double cover $\pi \colon S \to X$, it induces a fibration of X, $p' \colon X \to \mathbb{P}^1$, whose general fibre we denote by F'. Since $F \pi^* C_i = 0$, also $F' C_i = 0$ and $F' C = 0$. Using the same reasoning as in Lemma 3.2, we see that also $p' \colon X \to \mathbb{P}^1$ is an elliptic fibration and we are done. $\qquad\square$

REMARK 6. Recall that a *Halphen pencil of index m* is an irreducible pencil of plane curves of degree $3m$ with 9 base points of multiplicity m (some of them may be infinitely near). By [3], the minimal resolution of a Halphen pencil of index m is a rational elliptic surface with a multiple fibre of multiplicity m.

Conversely, again by Theorem 5.6.1 in [3], if $f \colon X' \to \mathbb{P}^1$ is a rational elliptic surface with a multiple fibre of multiplicity m ($m = 1$ if it does not have multiple fibre), then there exists a birational morphism $\tau \colon X' \to \mathbb{P}^2$ such that the composition of rational maps $f \circ \tau^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is given by a Halphen pencil of index m.

So, if X is a surface as in Assumption 1, then $\kappa(S) \leq 1$ if and only if there is a Halphen pencil of index m in $\mathbb{P}^2$ corresponding to the elliptic fibration. In particular for $\kappa(S) = 0$, $m = 1$ or $m = 2$.

4. The K3 case

In this section we assume that $K_S \equiv \mathcal{O}_S$, thus one has $-K_X \equiv L$ and $-2K_X \equiv C$.

Notice that S is a smooth K3 surface with an involution σ such that $\sigma^* \omega = -\omega$ for a nonzero holomorphic 2-form. Zhang in [15] classified the quotients of K3 surfaces modulo involutions. In particular, with the approach top-down, he proved the following

Proposition 4.1 ([15]). *Assuming that X is a rational surface with an even set of n disjoint (-4)-curves. If the double cover S is a K3 surface, then we can conclude that*

$$1 \leq n \leq 10,$$

and n can take any value in this range (see examples below).

REMARK 7.   It is possible to give another proof of the previous proposition using similar arguments to those used in the following sections.

EXAMPLE 8.   Let us give an example of each possible case:

(i) For $n = 1$, as we have seen above, take a sextic curve in $\mathbb{P}^2$ with ten double points. After blowing them up, we obtain a rational surface with a (−4)-curve. See also ([15], Example 2.7).

(ii) For $n = 2$, in a pencil of cubics of $\mathbb{P}^2$, take two singular members with one node each, N_1 and N_2 respectively. Blowing up the base points, N_1 and N_2 we obtain the two (−4)-curves. This example, with another point of view, can be found in [10].

(iii) For $n = 3$, in a pencil of cubics of $\mathbb{P}^2$ take two singular members: one cubic with a node and a conic plus a line. Blowing up the base points and the singular points of these two singular members we obtain the result.

(iv) For $n = 4$, take a pencil of cubics and choose two singular cubics: three non-concurrent lines L_1, L_2, L_3 and one cubic, C_1, with one unique singular point, a double point N. Let

$$C = L_1 + L_2 + L_3 + C_1$$

with the following singular points:

$L_i \cap L_j = P_{ij}$ with $i, j = 1,2,3$, and $L_1 \cap C_1 = \{Q_1, Q_2, Q_3\}$, $L_2 \cap C_1 = \{R_1, R_2, R_3\}$, $L_3 \cap C_1 = \{S_1, S_2, S_3\}$ and N.

Blowing up the thirteen singular points of C we obtain $p\colon X \to \mathbb{P}^2$ with $\tilde{L}_i^2 = -4$, $\tilde{C}_1^2 = -4$ and rational.

(v) For $n = 5$, take a pencil of cubics and C is the sum of two singular members one conic plus a line, and three lines.

(vi) For $n = 6$, Let $L_1, \ldots, L_6$ be six nonconcurrent lines in $\mathbb{P}^2$ and $C = L_1 + \cdots + L_6 \sim 6H$. Then C has 15 singular points, we blow up each line in 5 different points $p\colon X \to \mathbb{P}^2$, then $\tilde{L}_i^2 = -4$, this is, take a pencil of cubics with two singular cubics $L_1 + L_2 + L_3$ and $L_4 + L_5 + L_6$.

(vii) For $n = 7$, in $\mathbb{F}_0$ denote by L_1 and L_2 the two rulings, then take seven effective divisors $R_1, R_2, R_3, M_1, M_2, M_3$ and S where $R_i \sim L_1$, $M_i \sim L_2$ with $i = 1, 2, 3$, and finally $S \sim L_1 + L_2$ without singular points. Blowing up the intersection points we obtain seven (−4)-curves and the double cover is a K3 surface. This is, in $\mathbb{P}^2$ take a pencil of cubics and the two singular cubics are one nonsingular conic C plus a line L_1 and three nonconcurrent lines L_2, L_3, L_4 with these intersection points $C \cap L_i = P_{i1}$ and P_{i2} such that $L_1 \cap L_2 = P_{11} = P_{21}$ and $L_3 \cap L_4 = P_{31} = P_{41}$, the other intersection points are all different.

(viii) For $n = 8$, in $\mathbb{F}_0$, with the same notation as above, take $R_1 \sim R_2 \sim R_3 \sim R_4 \sim L_1$ and $M_1 \sim M_2 \sim M_3 \sim M_4 \sim L_2$. As before, in $\mathbb{P}^2$ we take a pencil of cubics with these two singular members: $L_1 + L_2 + L_3$, three concurrent lines, write $L_1 \cap L_2 \cap L_3 = P$ and $L_3 + L_4 + L_5$ three concurrent lines as well, write $L_3 \cap L_4 \cap L_5 = Q$, such that $P \neq Q$ and the other intersection points are all different.

(ix) For $n = 9, 10$, in $\mathbb{P}^2$ take six lines $L_1, \ldots, L_6$ with this configuration: L_1, L_2 and L_3 with a common point P; L_1, L_4 and L_5 with a common point Q_1; L_2, L_4 and L_6 with a common point Q_2; L_3, L_5 and L_6 with a common point Q_3; finally $L_1 \cap L_6 = R_1$, $L_2 \cap L_5 = R_2$ and $L_3 \cap L_4 = R_3$, all different points. First of all, we blow up P, then $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ are fibres of $\mathbb{F}_1$ and denote by L the exceptional curve lying over the point P; now we blow up each fibre six times and we obtain ten (-4)-curves, two of them in each fibre, plus the strict transform of $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ and L.

We can obtain $n = 9$ in a similar way, see for instance Example 2.10 of [7].

5. The elliptic case

Now we examine more closely the case when S is an elliptic surface with $\kappa(S) = 1$. Let F be a general fibre of the elliptic fibration $p \colon S \to B$.

Since the Kodaira dimension of S is 1 there are effective nonzero n-canonical divisors. These are supported on the fibres of the elliptic pencil (see Equation (3.1)) and so the elliptic pencil is unique. Thus it is necessarily invariant under the involution associated to the double cover and we have a commutative diagram:

(5.1)
$$
\begin{array}{ccc}
S & \xrightarrow{\ p\ } & B \\
\pi \downarrow & & \downarrow \pi' \\
X & \xrightarrow{\ p'\ } & \mathbb{P}^1
\end{array}
$$

where $p' \colon X \to \mathbb{P}^1$ is also an elliptic fibration.

We want to prove:

Theorem 5.1. *In the above situation we have*:
(i) $1 \leq n \leq 12$;
(ii) *if F' is the general fibre of p', then $\pi^*(F') = F_1 + F_2$ is disconnected.*

The proof will be given throughout this section.

As we have seen in the proof of Proposition 3.1, we can write

$$
K_S \sim 2g(B)F + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} F,
$$

where $m_1 F_1, \ldots, m_r F_r$ are the multiple fibres of $p \colon S \to B$.

First of all we are going to describe the singular fibres of p' containing (-4)-curves.

Proposition 5.2. *There exists a birational morphism $\epsilon \colon X \to X'$, where X' is a relatively minimal elliptic surface with the following property*: *for $i = 1, \ldots, n$, the curve $C_i' := \epsilon(C_i)$ is an irreducible component of a fibre of type* II, III, IV *or* $m\mathrm{I}_r$ *with $1 \leq r \leq 9$ and $m \geq 1$.*

Proof. Notice first that, since $K_X^2 = -n$, X is not relatively minimal and so there exists a (-1)-curve θ_1 such that θ_1 is a component of one of the fibres F' of the elliptic fibration of X. Besides, by Remark 4, there exists a (-4)-curve C_1, with $\theta_1 C_1 = \alpha \geq 1$. As we have seen above C_1 is contained in some fibre F' and so also $(\theta_1 + C_1) \subset F'$. As a consequence, by Zariski's lemma

$$(\theta_1 + C_1)^2 = -1 + 2\alpha - 4 \leq 0$$

and so $\alpha \leq 2$.

If $\alpha = 2$, $2\theta_1 + C_1 \subseteq F'$ and $(2\theta_1 + C_1)^2 = 0$ mean that $m(2\theta_1 + C_1) = F'$ for some $m \in \mathbb{N}$, $m \geq 1$. Contracting θ_1, $\epsilon_1 \colon X \to X_1$, we obtain a fibre of type I_1, mI_1 or II in X_1 and $K_{X_1}^2 = -n + 1$.

If $\alpha = 1$, since $\theta_1 L \geq 1$, one has $\theta_1 C \geq 2$ and so θ_1 meets, at least, another (-4)-curve C_2. As in the precedent paragraph we conclude that $\theta_1 C_2 = 1$. Let $\epsilon_1 \colon X \to X_1$ be the blowing down of θ_1 and $\epsilon_1(C_i) = \tilde{C}_i$. Then $\tilde{C}_1$ and $\tilde{C}_2$ are curves with self-intersection -3 and $\tilde{C}_1 \tilde{C}_2 = 1$. As before, we have in this fibre a (-1)-curve $\tilde{\theta}_2$. Hence either $\tilde{\theta}_2$ comes from a (-2)-curve in S, or $\tilde{\theta}_2$ comes from a (-1)-curve in S.

If $\tilde{\theta}_2$ comes from a (-2)-curve θ_2, then it is easy to see that $4\theta_1 + 2\theta_2 + C_1 + C_2$ is a fibre. Contracting θ_1 and θ_2 we obtain a fibre of type III.

If $\tilde{\theta}_2$ comes from a (-1)-curve, there are three possibilities:

(i) $\theta_2 C_1 = \theta_2 C_2 = 1$. Then we have $(2\theta_1 + 2\theta_2 + C_1 + C_2)^2 = 0$ and so we have a fibre or a rational multiple of a fibre. Contracting θ_1 and θ_2, we obtain that an integer multiple of the image of $2\theta_1 + 2\theta_2 + C_1 + C_2$ is a fibre of type I_2.

(ii) $\theta_2 C_1 = \theta_2 C_2 = 0$. First of all, let us point out that every (-4)-curve C_i meets, at most, three (-1)-curves (possibly infinitely near). In fact if C_i meets four (-1)-curves, it is not very difficult to see that contracting these the image of C_i is a smooth rational curve with self-intersection 0. This is impossible because the fibres of an elliptic fibration have always $p_a = 1$. Thus, since $\tilde{C}_1^2 = -3$, there exists a (-2)-curve δ with $\delta C_1 = 1$ and $\delta \theta_2 = 1$. There are also two, and only two, (-4)-curves C_3 and C_4 intersecting θ_2, this is $\theta_2 C_3 = \theta_2 C_4 = 1$. Then we have $C_3 + C_4 + 4\theta_2 + 2\delta + C_1 + \cdots$ in the fibre, but as we have seen before $C_3 + C_4 + 4\theta_2 + 2\delta$ is a fibre, and we obtain a contradiction.

(iii) $\theta_2 C_2 = 0$ and $\theta_2 C_1 = 1$.

If there is another (-1)-curve θ_2' such that $\theta_2' C_1 = 1$, we have $\theta_2' C_2 = 0$; if not, $\theta_2' C_2 = 1$ and then $(2\theta_1 + 2\theta_2' + C_1 + C_2)^2 = 0$, but we have $2\theta_1 + 2\theta_2' + C_1 + C_2 + \theta_2 \subseteq f$, which is absurd. We have seen above that there exist C_3 and C_4, different (-4)-curves, such that $\theta_2 C_3 = 1$ and $\theta_2' C_4 = 1$. Then we have $4\theta_1 + 4\theta_2 + 4\theta_2' + 3C_1 + C_2 + C_3 + C_4 \subseteq f$ and $(4\theta_1 + 4\theta_2 + 4\theta_2' + 3C_1 + C_2 + C_3 + C_4)^2 = 0$, we get a fibre of type IV.

If there is not another (-1)-curve intersecting C_1 and since $\theta_2 C_2 = 0$, there exists another (-4)-curve C_3 with $\theta_2 C_3 = 1$. As before, blowing down θ_1 and θ_2, $\tilde{C}_3$ and $\tilde{C}_2$ are curves with self-intersection -3, so there exists another (-1)-curve θ_3. Since, as before, $\theta_3 C_3 = 1$, then either $\theta_3 C_2 = 1$ or $\theta_3 C_2 = 0$. If $\theta_3 C_2 = 1$ we have $2\theta_1 + 2\theta_2 +$

$2\theta_3 + C_1 + C_2 + C_3 \subseteq f$ and $(2\theta_1 + 2\theta_2 + 2\theta_3 + C_1 + C_2 + C_3)^2 = 0$, so we obtain a fibre (or a rational multiple of a fibre) of type I_3. In the other case, $\theta_3 C_2 = 0$, there is another (-4)-curve $C_4 \ldots$, repeating the same argument we obtain a fibre of type mI_r, $r \geq 4$ and $m \geq 1$.

In conclusion, since n is a finite number and $K_X^2 = -n$ there are n (-1)-curves in the fibres and after contracting them by $\epsilon \colon X \to X'$, one obtains a relatively minimal rational elliptic surface X' with these singular fibres.

By [8] a connected fibre on a relatively minimal rational elliptic surface has at most nine irreducible components, and so in particular for mI_r we have $1 \leq r \leq 9$. $\square$

Denote by F_j' the elliptic fibres in X containing (-4)-curves and by

$$\mathcal{J} = \{F_j', \ j = 1, \ldots, n'\}$$

the set of these fibres. Also denote by $\mathcal{J}' = \{\epsilon(F_j'), \ j = 1, \ldots, n'\}$ the image of $\mathcal{J}$ in X'.

Keeping this notation:

Corollary 5.3. *The number n of (-4)-curves in C is at most 12. In particular, if $n = 12$ the singular fibres of the elliptic fibration of X' are all in J'.*

Proof. Since X' is a relatively minimal rational elliptic surface we have that $c_2(X') = 12$ and by ([2], Lemma VI.4), we know that

$$c_2(X') = \sum_s \mathcal{X}_{\text{top}}(F_s'),$$

with F_s' the singular fibres.

Also, noticing that $\mathcal{X}_{\text{top}}(I_n) = n$, $\mathcal{X}_{\text{top}}(II) = 2$, $\mathcal{X}_{\text{top}}(III) = 3$ and $\mathcal{X}_{\text{top}}(IV) = 4$, the result follows by Proposition 5.2. $\square$

REMARK 9. The pull-back $\pi^*(F_j')$, where $F_j' \in \mathcal{J}$, will be one of following types:
- If $\epsilon(F_j')$ is of type mI_r, this is $F_j' = m\left(\sum_1^r C_i + 2\theta_i\right)$, then $\pi^*(F_j') = m\left(\sum_1^r 2\gamma_i + 2\hat{\theta}_i\right) = 2m\left(\sum_1^r \gamma_i + \hat{\theta}_i\right)$, therefore $\pi^*(F_j') = 2mI_{2r}$ with $1 \leq r \leq 9$ and $m \geq 1$.
- If $\epsilon(F_j')$ is of type II, then $\pi^*(F_j') = \pi^*(2\theta_1 + C_1) = 2(\hat{\theta}_1 + \gamma_1)$, therefore $\pi^*(F_j') = 2III$.
- If $\epsilon(F_j')$ is of type III, then $F_j' = 4\theta_1 + 2\theta_2 + C_1 + C_2$ with θ_2 a (-2)-curve such that $\theta_2 C = 0$, hence $\pi^*(F_j') = 2(2\hat{\theta}_1 + \theta_2^1 + \theta_2^2 + \gamma_1 + \gamma_2)$, so $\pi^*(F_j') = 2\tilde{D}_4$.
- Finally, if $\epsilon(F_j')$ is of type IV, then $\pi^*(F_j') = 2(2\hat{\theta}_1 + 2\hat{\theta}_2 + 2\hat{\theta}_3 + 3\gamma_1 + \gamma_2 + \gamma_3)$, so $\pi^*(F_j') = 2\tilde{E}_6$.

We want now to understand $\pi^*(F')$. So, we begin by supposing that $\pi^*(F') = F$. In this case $B = \mathbb{P}^1$ and we can consider the commutative diagram:

$$
\begin{array}{ccc}
S & \xrightarrow{\ p\ } & \mathbb{P}^1 \\
\ \downarrow{\scriptstyle \pi} & & \ \downarrow{\scriptstyle \pi'} \\
X & \xrightarrow{\ p'\ } & \mathbb{P}^1.
\end{array}
$$

Lemma 5.4. *If $\pi^*(F') = F$, every fibre $\epsilon(F'_j) \in \mathcal{J}'$ is a fibre of type $m\mathrm{I}_r$. In particular,*

$$
F'_j = m_j \left(\sum_1^{r_j} C_i + 2\theta_i \right),
$$

with n (−1)-curves θ_i, $1 \le r_j \le 9$ and $m_j \ge 1$.

Proof.   If $\pi^*(F') = F$, by Remark 9 the pull-back of any fibre containing a (−4)-curve is a double fibre of the elliptic fibration in S. Since for every multiple fibre mF_0 in a elliptic fibration F_0 cannot be simply-connected (cf. [1]), looking at the description of the fibres in Proposition 5.2 we obtain the statement.   □

It is well known (see [3]) that every relatively minimal rational elliptic surface has at most one multiple fibre.

We analyse the different possibilities for the multiple fibres to prove the next proposition:

Proposition 5.5. *The elliptic fibration F' of $p' \colon X \to \mathbb{P}^1$, Diagram (5.1), satisfies*

$$
\pi^*(F') = F_1 + F_2.
$$

Proof.   Under the assumption that $\pi^*(F') = F$, let mD be the unique multiple fibre in X', if it has any, otherwise let $m = 1$ and D be any fibre.

First of all assume that $mD \notin \mathcal{J}'$, then $m_j = 1$ in F'_j for all $j = 1, \ldots, n'$. The multiple fibres in S are $\pi^*(mD)$ of multiplicity m, and $\pi^*(F'_j)$ of multiplicity 2, $j = 1, \ldots, n'$. Thus, since $\mathcal{X}(\mathcal{O}_S) = 2$, and the elliptic fibration has base $\mathbb{P}^1$,

$$
K_S \equiv \frac{(m-1)}{m} F + \frac{n'}{2} F.
$$

On the other hand, since

$$
K_X \equiv -F' + \frac{(m-1)}{m} F' + \theta_1 + \cdots + \theta_n,
$$

by the double cover formulas we obtain

$$K_S \equiv -F + \frac{(m-1)}{m}F + \hat{\theta}_1 + \cdots + \hat{\theta}_n + \gamma_1 + \cdots + \gamma_n = \frac{(m-1)}{m}F + \left(\frac{n'}{2}-1\right)F,$$

a contradiction.

Now, assume that $m > 1$ and $mD \in \mathcal{J}'$. Then the multiple fibres in S are $\pi^*(mD)$ of multiplicity $2m$ and $\pi^*(F_j')$ of multiplicity 2, with $j = 1, \ldots, n'-1$. Thus

$$K_S \equiv \frac{(2m-1)}{2m}F + \frac{(n'-1)}{2}F.$$

As before, $K_X \equiv -F' + ((m-1)/m)F' + \theta_1 + \cdots + \theta_n$ and by the double cover formulas we obtain

$$K_S \equiv -F + \frac{(m-1)}{m}F + \hat{\theta}_1 + \cdots + \hat{\theta}_n + \gamma_1 + \cdots + \gamma_n$$
$$= \frac{(m-1)}{m}F + \left(\frac{(n'-1)}{2}-1\right)F + \frac{1}{2m}F,$$

since m is a natural number we have a contradiction again and the result follows. $\square$

So $\pi^*(F')$ is disconnected and there is a natural $2-1$ map $\pi'\colon B \to \mathbb{P}^1$. By using the Hurwitz formula we get

$$g := g(B) = \frac{\deg R}{2} - 1,$$

where R is the ramification divisor. Let us recall the commutative diagram (5.1)

$$\begin{array}{ccc} S & \xrightarrow{p} & B \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{p'} & \mathbb{P}^1. \end{array}$$

Then keeping the above notation:

Corollary 5.6. *One has*

$$n' \le \deg R \le n' + 1$$

Proof. Since S is branched along $C = C_1 + \cdots + C_n$, each $F_j' \in \mathcal{J}$ corresponds to a ramification point of π', so $n' \le \deg R$. Also, since $p'\colon X' \to \mathbb{P}^1$ has at most one multiple fibre, there is at most one more ramification point corresponding to a multiple fibre (necessarily of even multiplicity) of the elliptic fibration of X', hence the result. $\square$

Corollary 5.7. *The singular fibres of p coming from fibres $F'_j \in \mathcal{J}$ of X are of type $m\mathrm{I}_{2r}$ ($1 \leq r \leq 9$, $m \geq 1$), III, $\tilde{D}_4 D_4$ and $\tilde{E}_6 E_6$.*

Proof. Using Remark 9, notice that if $\epsilon(F'_j)$ is a fibre type $m\mathrm{I}_r$, then $\pi^*(F'_j) = 2m\left(\sum_1^r \gamma_i + \hat{\theta}_i\right)$. Therefore we obtain in S a fibre of type $m\mathrm{I}_{2r}$ with $1 \leq r \leq 9$ and $m \geq 1$. If $\epsilon(F'_j)$ is type II, III, or IV, then $\pi^*(F'_j) = 2F$, with F a fibre of type III, $\tilde{D}_4$ and $\tilde{E}_6$ respectively. $\square$

EXAMPLES. – We take a pencil of cubics with 12 nodal cubics, blowing up the base points and these 12 double points we will obtain a rational elliptic surface with $K_X^2 = -12$ and $n = 12$. The double cover S will be an elliptic surface. In a similar way, we can obtain examples for $n \geq 4$ even.
– Using the program Magma, we can find a pencil of sextics with 4 nodal sextics. Blowing up these double points and the base points, we get examples for $n = 2, 3$ or 4.

6. Some remarks on the case of general type

Suppose now that S is a surface of general type. From [12], $K_S^2 \leq 9\mathcal{X}(\mathcal{O}_S)$ and so $K_S^2 \leq 18$.

Recalling that $K_S^2 = 2(K_X^2 + n)$ and $-n < K_X^2 \leq -1$, so

(6.1) $\qquad 2 \leq K_S^2 \leq 2(n - 1), \quad 2 \leq h^0(2K_X + C) \leq n \quad \text{and} \quad n \geq 2.$

We do not know if this case can happen unlike in the previous cases we know examples. Below we give some properties for this situation. More precisely, throughout this section we will prove:

Proposition 6.1. *Suppose that S is a surface of general type. One of the following holds*:
- *if S is regular, then $2 \leq K_S^2 \leq 8$ and $2 \leq n \leq 9$;*
- *if S is irregular, then $q(S) = 1$. Also, $6 \leq K_S^2 \leq 10$ and $4 \leq n \leq 9$.*

Proof. We divide the proof into steps.
STEP 1: $h^0(2K_X + C) = h^0(2K_X + L) = K_X^2 + n + 1$.
By the Riemann–Roch theorem

$$h^0(2K_X + C) = K_X^2 + n + 1 + h^1(2K_X + C).$$

The projection formula $h^1(2K_S) = h^1(2K_X + C) + h^1(2K_X + L)$, together with $h^1(2K_S) = 0$, gives $h^1(2K_X + C) = 0$ and so $h^0(2K_X + C) = K_X^2 + n + 1$. Since $K_X + L$ is effective, nef and big, then $h^1(-K_X - L) = 0$ (see [13]) and thus $h^0(2K_X + L) = K_X^2 + n + 1$ as asserted.

STEP 2:   *The canonical divisor satisfies* $2 \leq K_S^2 \leq 10$ *and* $2 \leq n \leq 14$.

If $K_S^2 = 18$, by the Noether's formula one has $c_2(S) = 6$ and applying Miyaoka's formula (Proposition 2.4), we obtain $n = 0$: also for $K_S^2 = 16$ we obtain $n = 1$. Then $K_S^2 \leq 14$.

Similarly, if $K_S^2 = 14$, then $n \leq 3$. Then we have a contradiction by the inequalities (6.1). The same argument proves that $K_S^2 \neq 12$. Then $K_S^2 \leq 10$.

If $K_S^2 = 10$ then $n \leq 7$. From Proposition 2.3, one gets $K_X^2 \leq -1$, whence the only possibility is $6 \leq n \leq 7$. For $n = 7$ we have $K_X^2 = -2$ and for $n = 6$ we have $K_X^2 = -1$. In the same way, if $K_S^2 = 8$ then $5 \leq n \leq 8$, if $K_S^2 = 6$ then $4 \leq n \leq 10$, if $K_S^2 = 4$ then $3 \leq n \leq 12$, and finally if $K_S^2 = 2$, then $2 \leq n \leq 14$.

From now on, we are going to analyse separately the cases when S is regular and irregular.

STEP 3:   *If* S *is a regular surface of general type, then* $2 \leq K_S^2 \leq 8$ *and* $2 \leq n \leq 9$.

The hypothesis $q(S) = 0$ implies $b_2(S) = c_2(S) - 2$ and $p_g(S) = 1$, hence $b_2(S) = 22 - K_S^2$. Since $\pi \colon S \to X$ is an holomorphic map of degree 2, then

$$\pi^* \colon H^2(X, \mathbb{R}) \to H^2(S, \mathbb{R})$$

is an injective ring homomorphism. We have $b_2(X) = h^{1,1}(X)$ and

$$b_2(S) = h^{2,0}(S) + h^{1,1}(S) + h^{0,2}(S),$$

where $h^{2,0}(S) = h^{0,2}(S) = 1$. Since $h^{1,1}(X) \leq h^{1,1}(S)$, one has

$$b_2(X) \leq b_2(S) - 2.$$

Note that, since X is a rational surface, $b_2(X) = 10 - K_X^2$. If $K_S^2 = 2$, then $b_2(S) = 20$ and that implies $K_X^2 \geq -8$. In conclusion, $K_X^2 \geq -8$ and from $K_S^2 = 2(K_X^2 + n)$, $n \leq 9$.

Likewise, for $K_S^2 = 4$ one has $K_X^2 \geq -6$ and $n \leq 8$; for $K_S^2 = 6$ one has $K_X^2 \geq -4$ and $n \leq 7$; for $K_S^2 = 8$ one has $K_X^2 \geq -2$ and $n \leq 6$; finally $K_S^2 \neq 10$.

STEP 4:   *If* S *is an irregular surface of general type, then* S *is not of Albanese general type and the genus of a general fibre of the Albanese fibration is* > 2.

Suppose that $q(S) \geq 1$. Then $p_g(S) = q + 1 \geq 2$ and, since an irregular surface satisfies $K^2 \geq 2p_g$ by ([4]), $K_S^2 \geq 4$.

Since S is a double covering of a surface with $p_g(X) = q(X) = 0$ and $q(S) > 0$, by the de Franchis theorem ([5]) S is not of Albanese general type, and so we can consider the Albanese fibration

$$f \colon S \to B,$$

where $q(S)$ is the genus of B. We denote by g the genus of a general fibre of f and

write $q := q(S)$. We have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & B \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{\;f'\;} & \mathbb{P}^1 \end{array}$$

where π' is a $2:1$ map with $\deg R = 2q + 2$, where R is the ramification divisor.

By the appendix of [4]

$$K_S^2 \geq 8(g-1)(q-1).$$

Since, by Step 2, $K_S^2 \leq 10$, the only possibilities are $q = 1$, $g \geq 2$ and $\deg R = 4$, or $q = 2$, $g = 2$ and $\deg R = 6$. In this last case the slope inequality ([14])

$$\frac{4(g-1)}{g} \leq \frac{K_S^2 - 8(g-1)(q-1)}{\mathcal{X}(\mathcal{O}_X) - (g-1)(q-1)} \leq 12$$

yields $K_S^2 = 10$ and thus $6 \leq n \leq 7$.

Assume in either case that $g = 2$. Then f and f' are fibrations of genus 2 with $2q + 2$ fibres of f' corresponding to the ramification points of π'. Denote by $F_1, \ldots, F_{2q+2}$ those fibres of f' and let $\pi^*(F_i) = 2\tilde{F}_i$, where $\tilde{F}_i$ is a fibre of f. Since a fibration of genus 2 does not have multiple fibres all the fibres F_j, $j = 1, \ldots, 2q + 2$, have to contain some of the (−4)-curves and all its other components will appear with even multiplicity. So we can write $F_j = C_1 + \cdots + C_s + 2D$, where $C_1, \ldots, C_s$ are (−4)-curves in C appearing with odd multiplicity in F_j and D is an effective divisor. Since $K_X(C_1 + \cdots + C_s) = 2s$ and $K_X F_j = 2$, by the assumption $g = 2$, then $K_X D = 1 - s$. Now, $C_i F = 0$ implies $C_i D = 2$, and so from $DF = 0$, one obtains $2D^2 = -2s$, this is $D^2 = -s$. But then $K_X D + D^2 = 1 - 2s$, and this contradicts the adjunction formula. So we can conclude that $g > 2$.

STEP 5:   *If S is irregular, then $q(S) = 1$. Also, $6 \leq K_S^2 \leq 10$ and $4 \leq n \leq 9$.*

By the previous step $g > 2$, whence $q = 1$.

Notice that a surface with an Albanese fibration with $g \neq 2$ satisfies $K^2 \geq (8/3)\mathcal{X}(\mathcal{O})$ ([9]) and thus $K_S^2 \geq 6$.

Finally, as we have seen for the regular case, we have $h^{1,1}(X) \leq h^{1,1}(S)$ and since $q(S) = 1$, one has $b_2(S) = c_2(S) + 2$; also, $b_2(S) = h^{1,1}(S) + 4$, then $b_2(X) \leq c_2(S) - 2$. Applying this inequality for $K_S^2 = 6$, we obtain $n < 10$. □

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