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EXAMPLES OF NON-KÄHLER SYMPLECTIC MANIFOLDS

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1. Introduction

Symplectic manifolds are manifolds which admit a non-degenerate closed 2-form. It is well-known that the manifolds which admit a Kähler structure are sytplectic manifolds. In this paper, a symplectic manifold M is called *non-Kähler*, if M do not admit Kähler structures. There has been recent interest in examples of closed non-Kähler symplectic manifolds.

The first such example was constructed by Thurston [10]. This closed non-Kähler symplectic manifold was a total space of a flat torus bundle over a torus in [10] and was also a nil-manifold in Abbena [1] and Weinstein [12].

Other examples of closed non-Kähler symplectic manifolds have appeared in Cordero, Fernandez and Gray [2], Cordero, Fernandez and Leon [3], Mc Duff [8] and Watson [11]. With the exception of [8], all of these examples are nil-manofolds, which are a generalization of Thurston's example.

In this paper, we generalize the Thurston's example in another way and show that there is a new calss of closed non-Kähler symplectic manifolds. We prove that the total spaces of flat surface bundles over closed symplectic manifolds whose characteristic homomorphisms satisfy some conditions have natural symplectic structures but they are non-Kähler. To see that our symplectic manifolds are non-Kähler, we find non-zero Massey triple products, for it is well-known that all the Massey triple products on closed Kähler manifolds are zero.

We review some definitions in §2 and state our theorem in §3. As an application of the theorem, we construct the examples of closed non-Kähler symplectic manifolds in §4. We prove in §5 that our symplectic manifolds admit non-zero Massey triple products.

2. Preliminaries

We call that a closed manifold M is *non-Kähler symplectic manifold* if M is a symplectic manifold and do not admit Kähler structures. To prove non-existence of Kähler structures, we use the following:

Theorem. (see [4] p. 168) *All Massey triple products are zero on a com-*

pact Kähler manifold.

We review the definition of Massey triple products ([9]).

For a smooth closed manifold W, let $A^*(W)$ denote the de Rham complex and let $H^*_{DR}(W)$ be its homology. Let $A\in H^p_{DR}(W)$, $B\in H^q_{DR}(W)$, $C\in H^r_{DR}(W)$ be such that $A\cup B=0$ and $B\cup C=0$. Choose representatives a, b, c and $x, y\in A^*(W)$ such that $dx=a\wedge b$, $dy=b\wedge c$. Then $a\wedge y+(-1)^{p+1}x\wedge c$ is a closed $(p+q+r-1)$-form. Its class in $H^{p+q+r-1}_{DR}(W)$ defines a coset modulo $A\cdot H^{q+r-1}_{DR}(W)+C\cdot H^{p+q-1}_{DR}(W)$. This coset is called the *Massey triple product* of A, B and C, and is denoted by $\langle A, B, C\rangle$.

We define Dehn twist diffeomorphisms on surfaces.

Let Σ_g be an oriented closed surface of genus $g(\geq 1)$ and $\{a_1, \cdots, a_g; b_1, \cdots, b_g\}$ a symplectic system of oriented simple closed curves on Σ_g, namely $a_i\cap b_j=a_i\cap a_j=b_i\cap b_j=\phi$ for $i\neq j$ and a_i intersects b_i at one point with intersection number $+1$ for $i=1, \cdots, g$.

We denote by T^2 a torus with a coordinate $(\exp(\theta_1\sqrt{-1}), \exp(\theta_2\sqrt{-1}))$ and by a, b the closed curves such that $a(\theta)=(\exp(\theta\sqrt{-1}), 1)$, $b(\theta)=(1, \exp(\theta\sqrt{-1}))$, and define a neighbourhood U of $a\cup b$ by

$$U=\{(\exp(\theta_1\sqrt{-1}), \exp(\theta_2\sqrt{-1}))\in T^2;\ -3\varepsilon<\theta_1<3\varepsilon \text{ or } -3\varepsilon<\theta_2<3\varepsilon\}\ ,$$

where $\varepsilon(>0)$ is a fixed small number such that $3\varepsilon<\pi/2$.

Then we have a neighbourhood U_i of $a_i\cup b_i$, and a diffeomorphism f_i: $U_i\to U$ such that the images of a_i, b_i are a, b respectively.

We may assume that $U_1, \cdots, U_g$ are disjoint.

We identify U_i with U by f_i. Then *Dhen twist diffeomorphism* $T(a_i)$ (resp. $T(b_i)$) along a_i(resp. b_i) is a diffeomorphism on Σ_g whose support supp $T(a_i)$ (resp. supp $T(b_i)$) is contained in U_i and is defined on U_i by

$$\begin{aligned} &T(a_i)(\exp(\theta_1\sqrt{-1}), \exp(\theta_2\sqrt{-1}))\\ &\quad=(\exp((\theta_1+\gamma(\theta_2))\sqrt{-1}), \exp(\theta_2\sqrt{-1}));\\ &(\text{resp. } T(b_i)(\exp(\theta_i\sqrt{-1}), \exp(\theta_2\sqrt{-1})))\\ &\quad=(\exp(\theta_1\sqrt{-1}), \exp((\theta_2+\gamma(\theta_1))\sqrt{-1})))\ , \end{aligned}$$

where $\gamma=\gamma(\theta)$ is a smooth function on $\boldsymbol{R}$ satisfying the following conditions:

(1) $\gamma(\theta+2\pi)=\gamma(\theta)+2\pi$;

(2) $\gamma(\theta)=0$ for $\varepsilon-2\pi\leq\theta\leq-\varepsilon$, and $\gamma(\theta)=2\pi$ for $\varepsilon\leq\theta\leq 2\pi-\varepsilon$ where $\varepsilon(>0)$ is the fixed small number;

(3) γ is strictly increasing on $[-\varepsilon, \varepsilon]$.

Lemma. *For the fixed symplectic system $\{a_i, b_i; i=1, \cdots, g\}$, there exists a volume form v of Σ_g which is preserved by $T(a_1), \cdots, T(a_g)$ and $T(b_1), \cdots, T(b_g)$.*

Proof. Set $v_i=f_i^*(d\theta_1\wedge d\theta_2)$ for $i=1, \cdots, g$. Then the 2-form v_i is a volume form of U_i which is preserved by $T(a_i)$ and $T(b_i)$.

We set

$$U_\varepsilon = \{(\exp(\theta_1\sqrt{-1}), \exp(\theta_2\sqrt{-1})) \in T^2; \ -2\varepsilon < \theta_1 < 2\varepsilon \quad \text{or} \quad -2\varepsilon < \theta_2 < 2\varepsilon\}$$

and

$$U_{i\varepsilon} = f_i^{-1}(U_\varepsilon) \quad \text{for} \quad i = 1, \cdots, g\,.$$

Let v_0 be a volume form of the complement U_0 of the union of $\{U_{i\varepsilon}; i=1, \cdots, g\}$. Then, by use of the partition of unity for the covering $\{U_0, U_1, \cdots, U_g\}$ of Σ_g, we have a volume form v of Σ_g which coincide $\pm v_i$ on $U_{i\varepsilon}$ for $i=1, \cdots, g$ and v_0 on the complement of the uion of $\{U_i; i=1, \cdots, g\}$. Since the supports of $T(a_i)$ and $T(b_i)$ are contained in $U_{i\varepsilon}$ for $i=1, \cdots, g$, and v_i is preserved by $T(a_i)$ and $T(b_i)$, the volume form v is preserved by $T(a_1), \cdots, T(a_g)$ and $T(b_1), \cdots, T(b_g)$. Therefore we have Lemma.

3. Theorem

Let $\{a_1, \cdots, a_g; b_1, \cdots, b_g\}$ a symplectic system of oriented simple closed curves on Σ_g. By Lemma of §2, there exists a volume form v on Σ_g which is invariant under $T(c)$ for all $c=a_1, \cdots, a_g, b_1, \cdots, b_g$.

Now let (N, Ω) be a closed symplectic manifold admitting a homomorphism $\rho: \pi_1(N) \to \mathrm{Diff}(\Sigma_g)$ which satisfies the following condition:
(*) The image of ρ is generated by Dehn twist diffeomorphisms $T(c_1)$, $T(c_2)$, $\cdots$, $T(c_n)$ such that $\operatorname{supp} T(c_1) \cap \operatorname{supp} T(c_i) = \phi$ for $i \neq 1$, where $c_1, \cdots, c_n$ are elements of the symplectic system of oriented simple closed curves $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$.

Define a $\pi_1(N)$-action on $\tilde{N} \times \Sigma_g$ by

$$\Phi_g(\tilde{x}, z) = (\sigma(g)(\tilde{x}), \rho(g)(z)) \quad \text{for} \quad g \in \pi_1(N)\ ;$$

where $\pi: \tilde{N} \to N$ is the universal covering of N and $\sigma(g)$ is the covering transformation corresponding to $g (\in \pi_1(N))$.

We denote by M the quotient space of $\tilde{N} \times \Sigma_g$ by the above $\pi_1(N)$-action. Then M is the total space of the flat Σ_g-bundle over N whose characteristic homomorphism is ρ.

Let $p_1: \tilde{N} \times \Sigma_g \to \tilde{N}$ and $p_2: \tilde{N} \times \Sigma_g \to \Sigma_g$ be the projections. We have a closed 2-form $p_1^*(\pi^*\Omega) + p_2^*\, v$ on $\tilde{N} \times \Sigma_g$. Since this closed 2-form is invariant under the $\pi_1(N)$-action and non-degenerate, M has a natural symplectic structure ω which is the projection of $p_1^*(\pi^*\Omega) + p_2^*\, v$ down to M.

Theorem. *The above closed symplectic manifold (M, ω) is non-Kähler.*

By Theorem, we have a new class of closed non-Kähler symplectic manifolds. We construct the examples of such manifolds in §4.

In §5, we prove that the manifold M has a non-zero Massey triple product.

Then our theorem follows immediately from the well-known theorem in §2 that all the Massey triple products on closed Kähler manifolds vanish.

The fact that total spaces of Σ_g-bundles (not necessarily flat) adimt a symplectic structure is mentioned in Thurston [10] for $g(>1)$, and our construction appears essentially in Johnson [5].

4. Examples

Set $N=\Sigma_m(m\geq 1)$, Ω a symplectic structure on N. Let $\{\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m\}$ be the natural system of generators of $\pi_1(\Sigma_m)$ which have the relation such that $[\alpha_1, \beta_1]\cdot[\alpha_2, \beta_2]\cdots[\alpha_m, \beta_m]=1$, where $[\alpha_i, \beta_i]=\alpha_i\cdot\beta_i\cdot\alpha_i^{-1}\cdot\beta_i^{-1}$.

We define a homomorphism ρ_1: $\pi_1(N)\rightarrow\mathrm{Diff}(\Sigma_g)$ $(g\geq 2)$ by

$$\rho_1(\alpha) = T(a_1)\,, \quad \rho_1(\beta_1) = T(a_2)\ ;$$
$$\rho_1(c) = id, \quad \text{for} \quad c = \alpha_i,\ \beta_i\ (i\geq 2)\,,$$

where $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ is the symplectic system of oriented simple closed curves on Σ_g in §3. Then the homomorphism ρ_1 satisfies the condition (*) in §3. Therefore, by Theorem, we have a closed non-Kähler symplectic manifold (M_1, ω_1) of dimension 4.

The 4-dimensional closed manifold M_1 is also an example of almost complex manifold admitting no complex structure. To see this, it is sufficient to note the following three facts:

(1) A symplectic manifold admits always an almost complex structure (see for example [12]);

(2) The first Betti number $b_1(M_1)$ of M_1 is even (in fact, $b_1(M_1)=2m+2g-2$) and M_1 admits no Kähler structure (by our theorem);

(3) A compact complex surface with even first Betti number admits a Kähler structure (see K. Kodaira [7] Theorem 25).

Moreover, in §3, we reset $(N, \Omega)=(M_1, \omega_1)$ and define a homomorphism ρ_2: $\pi_1(M_1)\rightarrow\mathrm{Diff}(\Sigma_g)$ by $\rho_2=\rho_1\circ p_*$, where p: $M_1\rightarrow\Sigma_m$ is the projection of the flat Σ_g-bundle. Then, the homomorphism ρ_2 satisfies the condition (*). Therefore, by Theorem, we have a closed non-Kähler symplestic manifold (M_2, ω_2) of dimension 6.

Repeating this procedures, we have a closed non-Kähler symplectic manifold (M_n, ω_n) of dimension $(2n+2)$. Thus we have a new class of closed non-Kähler symplectic manifolds.

5. Massey products

We prove the following proposition:

Proposition. *The closed manifold M constructed in §3 has a non-zero Mas-*

sey triple product.

Our method to find a non-zero Massey triple product is motivated by Cordero, Fernandez and Gray [2], Cordero, Fernandez and Leon]3], Griffiths and Morgan [4] and Kawashima [6].

In this section, we fix a symplectic system $\{a_1, \cdots, a_g; b_1, \cdots, b_g\}$ of oriented simple colosed curves on Σ_g.

Let M be the symplectic manifold and ρ: $\pi_1(N) \to \mathrm{Diff}(\Sigma_g)$ be the homomorphism as in §3.

By the condition (*), we have a homomorphism

$$p\colon Im\ \rho \to \langle T(c_1)\rangle$$

defined by

$$p(T(c_1)) = T(c_1), p(T(c_i)) = id\ (i=2, \cdots, n)\ ;$$

where $\langle T(c_1)\rangle$ denotes the subgroup of Diff (Σ_g) generated by $T(c_1)$.

Lemma 1. *There exists an element φ of* Hom $(H_1(N, \boldsymbol{Z}), \boldsymbol{R})$ *such that $\varphi(x)=1$ (resp. 0) for $x=\Xi(g)$, if $(p\circ\rho)(g)=T(c_1)$ (resp. id), where Ξ: $\pi_1(N)\to H_1(N; \boldsymbol{Z})$ is the Hurewicz homomorphism.*

Proof. Since the image of $p\circ\rho$ is an abelian group, we have a well-defined homomorphism $\bar{\rho}\colon H_1(M; \boldsymbol{Z})\to\langle T(c_1)\rangle$ such that $\bar{\rho}\circ\Xi=p\circ\rho$. Let I: $\langle T(c_1)\rangle\to\boldsymbol{Z}$ be a natural isomorphism. Then, the desired homomorphism φ is defined by $\varphi=I\circ\bar{\rho}$. q.e.d.

Set ξ the closed 1-form on N whose De Rham cohomolgy class corresponds to the element φ of Lemma 1 under the isomorphisms $H^1_{MR}(N)\simeq H^1(N; \boldsymbol{R})\simeq$ Hom $(H_1(N; \boldsymbol{Z}), \boldsymbol{R})$ and F a smooth function on $\tilde{N}$ such that $\pi^*\xi=dF$, where $\pi\colon \tilde{N}\to N$ is the universal covering of N.

Lemma 2. *The above function F satisfies that $\sigma(g)^*F=F+1$ (resp. F), if $p\circ\rho(g)=T(c_1)$ (resp. id), where g is an element of $\pi_1(N)$.*

Proof. Let g be an element of $\pi_1(N)$ which is represented by an oriented closed curve c on N. We denote a lift of c on $\tilde{N}$ by $\tilde{c}$. Note that $\pi^*\xi=dF$, then we have

$$\sigma(g)^*F-F=\int_{\tilde{c}}\pi^*\xi=\int_{C}\xi=\varphi\circ\Xi(g)$$

Therefore, by Lemma 1, we have Lemma 2. q.e.d.

We need Lemma 2 and the following lemma to find a non-zero Massey triple product.

Lemma 3. *There exist closed 1-forms η, η' on Σ_g satisfying the following* *conditions*:

(1) $T(c_1)^*\eta=\eta,\ T(c_1)^*\eta'=\eta+\eta'$;

(2) η, η' *are invariant under* $T(c_i)$ *for* $i=2, \cdots, n$;

(3) $\int_{\Sigma_g}\eta\wedge\eta'\neq 0$;

(4) $\int_{c_1}\lambda\neq 0$ *for all closed* 1-*forms on* Σ_g *such that* $\int_{\Sigma_g}\eta\wedge\lambda\neq 0$.

Proof. We define closed 1-forms η_0, η_0' on a torus T^2 by

$$\eta_0=\frac{d\gamma}{d\theta}(\theta_2)\,d\theta_2\,,\quad \eta_0'=d\theta_1\,,$$

where $(\exp(\theta_1\sqrt{-1}), \exp(\theta_2\sqrt{-1}))$ is a coordinate of T^2 and $\gamma(\theta)$ is the smooth function which is used to define Dhen twist diffomorphisms in §2.

Then, the 1-forms satisfy the following conditions:

(4.1) (1) $T(a)^*\eta_0=\eta_0\,,\quad T(a)^*\eta_0'=\eta_0+\eta_0'$;

(2) $\int_{T^2}\eta_0\wedge\eta_0'\neq 0$;

(3) $\int_a \eta_0'\neq 0\,,\ \int_a\eta_0=0$;

where $T(a)$ is Dhen twist diffeomorphism along a and $\{a, b\}$ is the symplectic system of oriented simple closed curves of T^2 such that $a=a(\theta)=(\exp(\theta\sqrt{-1}, 1)$ and $b=b(\theta)=(1, \exp(\theta\sqrt{-1}))$.

We construct the desired closed 1-forms η, η' by use of η_0, η_0'.

We may assume that $c_1=a_1$.

Let f_1 be the diffeomorphism from U_1 to U as in §2. We set

$$\eta=f_1^*\eta_0\,,\quad \eta'=f_1^*\eta_0'\,.$$

Since $c_1=a_1$ and $f_1(a_1)=a$, we have (1) of Lemma 3. By the condition (*) in §3. the union of $\{\text{supp }T(c_i);\, i=2, \cdots, n\}$ is contained in Σ_g-U_1, we have (2) of Lemma 3. Moreover, by (4.1) (2), we have (3) of Lemma 3.

We prove (4) of Lemma 3. Let λ be a closed 1-form on Σ_g. Since the closed 1-forms η, η' satisfy (3) of Lemma 3 and their supports are contained in U_1, they form a basis of $H^1(U_1, \partial U_1; \boldsymbol{R})$. Hence the 1-form λ is cohomologous to a closed 1-form λ' such that $\lambda'=n\eta'+m\eta+\mu$, where the support of μ is contained in Σ_g-U_1 and $n, m\in\boldsymbol{R}$. We have

$$\int_{a_1}\lambda=\int_{a_1}\lambda'=\int_{a_1}n\eta'+m\eta$$
$$=\int_a n\eta_0'+m\eta_0=n\int_a\eta_0';$$

and

$$\int_{\Sigma_g} \eta\wedge\lambda = \int_{\Sigma_g} \eta\wedge\lambda' = n\int_{\Sigma_g} \eta\wedge\eta' = n\int_{T^2} \eta_0\wedge\eta_0' .$$

Therefore, if $\int_{\Sigma_g} \eta\wedge\lambda \neq 0$, n must be non-zero by (4.1) (2). Hence, by (4.1) (3), we have (4) of Lemma 3. q.e.d.

In the following, we may assume that $c_1 = a_1$.

Let p_1: $\tilde{N}\times\Sigma_g \to \tilde{N}$ and p_2: $\tilde{N}\times\Sigma_g \to \Sigma_g$ be the projections. We have a closed 1-form $p_2^*\eta$ on $\tilde{N}\times\Sigma_g$. By (1) and (2) of Lemma 3, $p_2^*\eta$ is invariant under the $\pi_1(N)$-action. Also the closed 1-form $p_1^*(\pi^*\xi)$ on $\tilde{N}\times\Sigma_g$ is invariant under the $\pi_1(N)$-action. Therefore, they dfinee the closed 1-forms $\hat{\eta}$, $\hat{\xi}$ on M which are the projections of the closed 1-forms $p_2^*\eta$, $p_1^*(\pi^*\xi)$ down to M. We denote their cohomology classes of $\hat{\eta}$, $\hat{\xi}$ by A, C respectively.

Then we have the following lemma which proves Proposition.

Lemma 4. *Massey triple product $\langle A, A, C\rangle$ is non-zero.*

Proof. We define a 1:form y on $\tilde{N}\times\Sigma_g$ by

$$y = p_1^* F\cdot p_2^*\eta - p_2^*\eta' .$$

Then we have

$$\Phi_g^* y = p_1^*(\sigma(g)^*F)\cdot p_2^*(\rho(g)^*\eta) - p_2^*(\rho(g)^*\eta') ,$$

and, by (2) of Lemma 3, we get

$$\Phi_g^* y = p_1^*(\sigma(g)^*F)\cdot p_2^*((p\circ\rho(g))^*\eta) - p_2^*((p\circ\rho(g))^*\eta') .$$

Therefore, by Lemma 2 and (1) of Lemma 3, the 1-form y is invariant under the $\pi_1(N)$-action. We denote by $\hat{y}$ the 1-form on M which is the projection of y down to M. Since

$$dy = p_1^*(dF)\wedge p_2^*\eta = p_1^*(\pi^*\xi)\wedge p_2^*\eta ,$$

we have

$$d\hat{y} = \hat{\xi}\wedge\hat{\eta} .$$

Therefore, by definition, we have

$$\langle A, A, C\rangle \equiv [-\hat{\eta}\wedge\hat{y}] \quad \text{mod.} \quad A\cdot H_{DR}^1(M) + C\cdot H_{DR}^1(M) .$$

We remark that $-p_2^*\eta\wedge y = p_2^*(\eta\wedge\eta')$ and note that $p_2^*(\eta\wedge\eta')$ is a closed 2-form on $\tilde{N}\times\Sigma_g$ which is invariant under the $\pi_1(N)$-action. Then the closed 2-form μ on M which is the projection of $p_2^*(\eta\wedge\eta')$ down to M satisfies

$$[-\hat{\eta}\wedge\hat{y}] = [\mu] \quad \text{in} \quad H_{DR}^2(M) .$$

Now, let $i\colon \Sigma_g \to M$ be the inclusion mapping of a fibre. Then, we have

$$i^* \mu = \eta \wedge \eta' .$$

Hence,

$$[\mu]\,(i_*([\Sigma_g])) = \int_{i(\Sigma_g)} \mu = \int_{\Sigma_g} i^* \mu = \int_{\Sigma_g} \eta \wedge \eta' .$$

Therefore, $[\mu]\,(i_*([\Sigma_g]))$ is non-zero by (3) of Lemma 3. That is, we have

(4.2) (1) $\langle A, A, C \rangle \equiv [\mu] \quad \text{mod.} \quad A \cdot H^1_{DR}(M) + C \cdot H^1_{DR}(M)$;

(2) $[\mu]\,(i_*([\Sigma_g])) \neq 0$.

In the following, we prove that $[\mu]$ does not belong to $A \cdot H^1_{DR}(M) + C \cdot H^1_{DR}(M)$.

First, we have

$$\hat{\xi} = p^* \xi ,$$

where $p\colon M \to N$ is the projection of the flat Σ_g-bundle. The cohomology class C is represented by $\hat{\xi}$. Therefore, if X is an element of $C \cdot H^1_{DR}(M)$, $i^* X = 0$ in $H^2_{DR}(\Sigma_g)$. That is, we have

$$X(i_*[\Sigma_g])) = 0 \quad \textit{for} \quad X \in C \cdot H^1_{DR}(M) . \tag{4.3}$$

Secondly, let X be an element of $A \cdot H^1_{DR}(M)$. There is a closed 1-form ζ such that $X = [\hat{\eta} \wedge \zeta]$ $(\in A \cdot H^1_{DR}(M))$. And we have

$$i^*(\hat{\eta} \wedge \zeta) = \eta \wedge i^* \zeta .$$

Hence,

$$X(i_*([\Sigma_g])) = \int_{i(\Sigma_g)} \hat{\eta} \wedge \zeta = \int_{\Sigma_g} i^*(\hat{\eta} \wedge \zeta) = \int_{\Sigma_g} \eta \wedge i^* \zeta .$$

Then, by (4) of Lemma 3, if $X(i_*([\Sigma_g]))$ is non-zero, $\int_{a_1} i^* \zeta$ must be non-zero. On the other hand, we assumed that $c_1 = a_1$. Then, the following property of $T(c_1)$ is well-known and is obtained easily by the definition:

$$T(c_1)_*([b_1]) = [a_1] + [b_1] \quad \text{in} \quad H_1(\Sigma_g; \boldsymbol{Z}) .$$

Since $i_* \circ T(c_1)_* = i_*\colon H_1(\Sigma_g; \boldsymbol{Z}) \to H_1(M; \boldsymbol{Z})$,

$$i_*([a_1]) = 0 \quad \text{in} \quad H_1(M; \boldsymbol{Z}) .$$

Hence, for all closed 1-form λ on M, we have

$$\int_{a_1} i^* \lambda = [\lambda]\,(i_*([a_1]) = 0 .$$

Consequently, we have

$$X(i_*([\Sigma_g])) = 0 \quad \textit{for} \quad X \in A \cdot H^1_{DR}(M). \tag{4.4}$$

Now, by (4.2) (2), (4.3) and (4.4), $[\mu]$ does not belong to $A \cdot H^1_{DR}(M) + C \cdot H^1_{DR}(M)$. Hence, by (4.2) (1), we have Lemma 4. q.e.d.

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