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<tr>
<td>Author(s)</td>
<td>Yamato, Kenji</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 27(2) P.431-P.439</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1990</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/12424">https://doi.org/10.18910/12424</a></td>
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<td>DOI</td>
<td>10.18910/12424</td>
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EXAMPLES OF NON-KÄHLER SYMPLECTIC MANIFOLDS

KENJI YAMATO

(Received June 12, 1989)

1. Introduction

Symplectic manifolds are manifolds which admit a non-degenerate closed 2-form. It is well-known that the manifolds which admit a Kähler structure are symplectic manifolds. In this paper, a symplectic manifold $M$ is called non-Kähler, if $M$ do not admit Kähler structures. There has been recent interest in examples of closed non-Kähler symplectic manifolds.

The first such example was constructed by Thurston [10]. This closed non-Kähler symplectic manifold was a total space of a flat torus bundle over a torus in [10] and was also a nil-manifold in Abbena [1] and Weinstein [12].

Other examples of closed non-Kähler symplectic manifolds have appeared in Cordero, Fernandez and Gray [2], Cordero, Fernandez and Leon [3], Mc Duff [8] and Watson [11]. With the exception of [8], all of these examples are nil-manifolds, which are a generalization of Thurston’s example.

In this paper, we generalize the Thurston’s example in another way and show that there is a new class of closed non-Kähler symplectic manifolds. We prove that the total spaces of flat surface bundles over closed symplectic manifolds whose characteristic homomorphisms satisfy some conditions have natural symplectic structures but they are non-Kähler. To see that our symplectic manifolds are non-Kähler, we find non-zero Massey triple products, for it is well-known that all the Massey triple products on closed Kähler manifolds are zero.

We review some definitions in §2 and state our theorem in §3. As an application of the theorem, we construct the examples of closed non-Kähler symplectic manifolds in §4. We prove in §5 that our symplectic manifolds admit non-zero Massey triple products.

2. Preliminaries

We call that a closed manifold $M$ is non-Kähler symplectic manifold if $M$ is a symplectic manifold and do not admit Kähler structures. To prove non-existence of Kähler structures, we use the following:

Theorem. (see [4] p. 168) All Massey triple products are zero on a com-
pact Kähler manifold.

We review the definition of Massey triple products ([9]).

For a smooth closed manifold $W$, let $A^*(W)$ denote the de Rham complex and let $H^k_{dR}(W)$ be its homology. Let $A \in H^p_{dR}(W)$, $B \in H^q_{dR}(W)$, $C \in H^r_{dR}(W)$ be such that $A \cup B = 0$ and $B \cup C = 0$. Choose representatives $a$, $b$, $c$ and $x$, $y$, $z$ in $A^*(W)$ such that $dx = a \wedge b$, $dy = b \wedge c$. Then $a \wedge y + (-1)^{p+1} x \wedge c$ is a closed $(p+q+r-1)$-form. Its class in $H^{p+q+r-1}_{dR}(W)$ defines a coset modulo $A \cdot H^{p+q-r-1}_{dR}(W) + C \cdot H^{p+q-r-1}_{dR}(W)$. This coset is called the Massey triple product of $A$, $B$ and $C$, and is denoted by $<A, B, C>$.

We define Dehn twist diffeomorphisms on surfaces.

Let $\Sigma_g$ be an oriented closed surface of genus $g(\geq 1)$ and $\{a_1, \ldots, a_g; b_1, \ldots, b_g\}$ a symplectic system of oriented simple closed curves on $\Sigma_g$, namely $a_i \cap b_j = a_i \cap b_i = b_i \cap b_j = \phi$ for $i \neq j$ and $a_i$ intersects $b_i$ at one point with intersection number $+1$ for $i = 1, \ldots, g$.

We denote by $T^2$ a torus with a coordinate $(\exp(\theta_1 \sqrt{-1}), \exp(\theta_2 \sqrt{-1}))$ and by $a$, $b$ the closed curves such that $a(\theta) = (\exp(\theta \sqrt{-1}), 1)$, $b(\theta) = (1, \exp(\theta \sqrt{-1}))$, and define a neighbourhood $U$ of $a \cup b$ by

$$U = \{(\exp(\theta_1 \sqrt{-1}), \exp(\theta_2 \sqrt{-1})) \in T^2 ; -3\varepsilon < \theta_1 < 3\varepsilon \text{ or } -3\varepsilon < \theta_2 < 3\varepsilon\},$$

where $\varepsilon(>0)$ is a fixed small number such that $3\varepsilon < \pi/2$.

Then we have a neighbourhood $U_i$ of $a_i \cup b_i$, and a diffeomorphism $f_i : U_i \rightarrow U$ such that the images of $a_i$, $b_i$ are $a$, $b$ respectively.

We may assume that $U_1, \ldots, U_g$ are disjoint.

We identify $U_i$ with $U$ by $f_i$. Then $D^n$ then twist diffeomorphism $T(a_i)$ (resp. $T(b_i)$) along $a_i$ (resp. $b_i$) is a diffeomorphism on $\Sigma_g$ whose support $\text{supp } T(a_i)$ (resp. $\text{supp } T(b_i)$) is contained in $U_i$ and is defined on $U_i$ by

$$T(a_i)(\exp(\theta_1 \sqrt{-1}), \exp(\theta_2 \sqrt{-1})) = (\exp((\theta_1 + \gamma(\theta))/\sqrt{-1}), \exp(\theta_2 \sqrt{-1})),$$

(resp. $T(b_i)(\exp(\theta_1 \sqrt{-1}), \exp(\theta_2 \sqrt{-1})) = (\exp(\theta_1 \sqrt{-1}), \exp((\theta_2 + \gamma(\theta))/\sqrt{-1})))$,

where $\gamma = \gamma(\theta)$ is a smooth function on $R$ satisfying the following conditions:

1. $\gamma(\theta + 2\pi) = \gamma(\theta) + 2\pi$;
2. $\gamma(\theta) = 0$ for $-2\pi \leq \theta \leq -\varepsilon$, and $\gamma(\theta) = 2\pi$ for $\varepsilon \leq \theta \leq 2\pi - \varepsilon$

where $\varepsilon(>0)$ is the fixed small number;
3. $\gamma$ is strictly increasing on $[-\varepsilon, \varepsilon]$.

**Lemma.** For the fixed symplectic system $\{a_i, b_i; i = 1, \ldots, g\}$, there exists a volume form $\nu$ of $\Sigma_g$ which is preserved by $T(a_1), \ldots, T(a_g)$ and $T(b_1), \ldots, T(b_g)$.

**Proof.** Set $\nu_i = f_i^*(d\theta_1 \wedge d\theta_2)$ for $i = 1, \ldots, g$. Then the 2-form $\nu_i$ is a volume form of $U_i$ which is preserved by $T(a_i)$ and $T(b_i)$.
We set
\[ U_e = \{ (\exp(\theta_1 \sqrt{-1}), \exp(\theta_2 \sqrt{-1})) \in T^2; -2\varepsilon < \theta_1 < 2\varepsilon \text{ or } -2\varepsilon < \theta_2 < 2\varepsilon \} \]
and
\[ U_{ie} = f_i^{-1}(U_e) \text{ for } i = 1, \ldots, g. \]

Let \( v_0 \) be a volume form of the complement \( U_0 \) of the union of \{\( U_{i1}; i=1, \ldots, g \}\}. Then, by use of the partition of unity for the covering \{\( U_0, U_1, \ldots, U_g \)\} of \( \Sigma_g \), we have a volume form \( v \) of \( \Sigma_g \) which coincide with \( v_i \) on \( U_{i1} \) for \( i=1, \ldots, g \) and \( v_0 \) on the complement of the union of \{\( U_{i1}; i=1, \ldots, g \}\}. Since the supports of \( T(a_i) \) and \( T(b_i) \) are contained in \( U_{i1} \) for \( i=1, \ldots, g \), and \( v_i \) is preserved by \( T(a_i) \) and \( T(b_i) \), the volume form \( v \) is preserved by \( T(a_i), \ldots, T(a_g) \) and \( T(b_i), \ldots, T(b_g) \). Therefore we have Lemma.

### 3. Theorem

Let \{\( a_1, \ldots, a_g; b_1, \ldots, b_g \)\} a symplectic system of oriented simple closed curves on \( \Sigma_g \). By Lemma of §2, there exists a volume form \( v \) on \( \Sigma_g \) which is invariant under \( T(c) \) for all \( c=a_1, \ldots, a_g, b_1, \ldots, b_g \).

Now let \( (N, \Omega) \) be a closed symplectic manifold admitting a homomorphism \( \rho: \pi_1(N) \to \text{Diff}(\Sigma_g) \) which satisfies the following condition:

(*) The image of \( \rho \) is generated by Dehn twist diffeomorphisms \( T(c_1), T(c_2), \ldots, T(c_n) \) such that \( \text{supp } T(c_i) \cap \text{supp } T(c_j) = \emptyset \) for \( i \neq j \), where \( c_1, \ldots, c_n \) are elements of the symplectic system of oriented simple closed curves \{\( a_1, \ldots, a_g; b_1, \ldots, b_g \)\}.

Define a \( \pi_1(N) \)-action on \( \bar{N} \times \Sigma_g \) by

\[ \Phi_g(\bar{x}, z) = (\sigma(g)(\bar{x}), \rho(g)(z)) \text{ for } g \in \pi_1(N); \]

where \( \pi: \bar{N} \to N \) is the universal covering of \( N \) and \( \sigma(g) \) is the covering transformation corresponding to \( g \in \pi_1(N) \).

We denote by \( M \) the quotient space of \( \bar{N} \times \Sigma_g \) by the above \( \pi_1(N) \)-action. Then \( M \) is the total space of the flat \( \Sigma_g \)-bundle over \( N \) whose characteristic homomorphism is \( \rho \).

Let \( p_1; \bar{N} \times \Sigma_g \to \bar{N} \) and \( p_2; \bar{N} \times \Sigma_g \to \Sigma_g \) be the projections. We have a closed 2-form \( p_1^*(\pi^*\Omega) + p_2^* v \) on \( \bar{N} \times \Sigma_g \). Since this closed 2-form is invariant under the \( \pi_1(N) \)-action and non-degenerate, \( M \) has a natural symplectic structure \( \omega \) which is the projection of \( p_1^*(\pi^*\Omega) + p_2^* v \) down to \( M \).

**Theorem.** The above closed symplectic manifold \( (M, \omega) \) is non-Kähler.

By Theorem, we have a new class of closed non-Kähler symplectic manifolds. We construct the examples of such manifolds in §4.

In §5, we prove that the manifold \( M \) has a non-zero Massey triple product.
Then our theorem follows immediately from the well-known theorem in § 2 that all the Massey triple products on closed Kähler manifolds vanish.

The fact that total spaces of \( \Sigma_g \)-bundles (not necessarily flat) admit a symplectic structure is mentioned in Thurston [10] for \( g(>1) \), and our construction appears essentially in Johnson [5].

4. Examples

Set \( N=\Sigma_m(m\geq 1) \), \( \Omega \) a symplectic structure on \( N \). Let \( \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\} \) be the natural system of generators of \( \pi_1(\Sigma_m) \) which have the relation such that \( [\alpha_1, \beta_1] \cdots [\alpha_m, \beta_m]=1 \), where \( [\alpha_i, \beta_i]=\alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1} \).

We define a homomorphism \( \rho_1: \pi_1(N) \to \text{Diff}(\Sigma_g) \) (\( g \geq 2 \)) by

\[
\rho_1(\alpha) = T(a_1), \quad \rho_1(\beta_1) = T(a_2);
\rho_1(c) = \text{id}, \quad \text{for} \quad c = \alpha_i, \beta_i (i \geq 2),
\]

where \( \{a_1, \ldots, a_g, b_1, \ldots, b_g\} \) is the symplectic system of oriented simple closed curves on \( \Sigma_g \) in § 3. Then the homomorphism \( \rho_1 \) satisfies the condition (*) in § 3. Therefore, by Theorem, we have a closed non-Kähler symplectic manifold \( (M_1, \omega_1) \) of dimension 4.

The 4-dimensional closed manifold \( M_1 \) is also an example of almost complex manifold admitting no complex structure. To see this, it is sufficient to note the following three facts:

1. A symplectic manifold admits always an almost complex structure (see for example [12]);
2. The first Betti number \( b_1(M_1) \) of \( M_1 \) is even (in fact, \( b_1(M_1)=2m+2g-2 \)) and \( M_1 \) admits no Kähler structure (by our theorem);

Moreover, in § 3, we reset \( (N, \Omega)=(M_1, \omega_1) \) and define a homomorphism \( \rho_2: \pi_1(M_1) \to \text{Diff}(\Sigma_g) \) by \( \rho_2=\rho_1 \circ p_* \), where \( p: M_1 \to \Sigma_m \) is the projection of the flat \( \Sigma_g \)-bundle. Then, the homomorphism \( \rho_2 \) satisfies the condition (*). Therefore, by Theorem, we have a closed non-Kähler symplectic manifold \( (M_2, \omega_2) \) of dimension 6.

Repeating this procedures, we have a closed non-Kähler symplectic manifold \( (M_n, \omega_n) \) of dimension \( (2n+2) \). Thus we have a new class of closed non-Kähler symplectic manifolds.

5. Massey products

We prove the following proposition:

**Proposition.** The closed manifold \( M \) constructed in § 3 has a non-zero Mas-
sey triple product.

Our method to find a non-zero Massey triple product is motivated by Cordero, Fernandez and Gray [2], Cordero, Fernandez and Leon [3], Griffiths and Morgan [4] and Kawashima [6].

In this section, we fix a symplectic system \( \{ a_1, \ldots, a_g; b_1, \ldots, b_g \} \) of oriented simple closed curves on \( \Sigma_g \).

Let \( M \) be the symplectic manifold and \( \rho: \pi_1(N) \to \text{Diff}(\Sigma_g) \) be the homomorphism as in §3.

By the condition (*), we have a homomorphism

\[
p: \text{Im} \rho \to \langle T(c_1) \rangle
\]
defined by

\[
p(T(c_1)) = T(c_1), \quad p(T(c_i)) = \text{id} \quad (i=2, \ldots, n);
\]
where \( \langle T(c_1) \rangle \) denotes the subgroup of \( \text{Diff}(\Sigma_g) \) generated by \( T(c_1) \).

**Lemma 1.** There exists an element \( \varphi \) of \( \text{Hom}(H_1(N, \mathbb{Z}), \mathbb{R}) \) such that \( \varphi(x) = 1 \) (resp. 0) for \( x = \Xi(g) \), if \( (p \circ p)(g) = T(c_1) \) (resp. id), where \( \Xi: \pi_1(N) \to H_1(N, \mathbb{Z}) \) is the Hurewicz homomorphism.

Proof. Since the image of \( p \circ p \) is an abelian group, we have a well-defined homomorphism \( p: H_1(M; \mathbb{Z}) \to \langle T(c_1) \rangle \) such that \( p \circ \Xi = p \circ \rho \). Let \( I: \langle T(c_1) \rangle \to \mathbb{Z} \) be a natural isomorphism. Then, the desired homomorphism \( \varphi \) is defined by \( \varphi = I \circ p \).

**Lemma 2.** The above function \( F \) satisfies that \( \sigma(g) * F = F + 1 \) (resp. \( F \)), if \( p \circ p(g) = T(c_1) \) (resp. id), where \( g \) is an element of \( \pi_1(N) \).

Proof. Let \( g \) be an element of \( \pi_1(N) \) which is represented by an oriented closed curve \( c \) on \( N \). We denote a lift of \( c \) on \( \bar{N} \) by \( \bar{c} \). Note that \( \pi^* \xi = dF \), then we have

\[
\sigma(g) * F = \int_{\bar{c}} \pi^* \xi = \int_{c} \xi = \varphi \circ \Xi (g)
\]

Therefore, by Lemma 1, we have Lemma 2. q.e.d.

We need Lemma 2 and the following lemma to find a non-zero Massey triple product.

**Lemma 3.** There exist closed 1-forms \( \eta, \eta' \) on \( \Sigma_g \) satisfying the following
conditions:

1. \( T(c_i) \eta = \eta, \ T(c_i) \eta' = \eta + \eta' \);
2. \( \eta, \eta' \) are invariant under \( T(c_i) \) for \( i = 2, \ldots, n \);
3. \( \int_{\Sigma} \eta \wedge \eta' = 0 \);
4. \( \int_{\Sigma} \lambda = 0 \) for all closed 1-forms on \( \Sigma_g \) such that \( \int_{\Sigma_g} \eta \wedge \lambda = 0 \).

Proof. We define closed 1-forms \( \eta_0, \eta'_0 \) on a torus \( T^2 \) by

\[
\eta_0 = \frac{d\gamma}{d\theta} (\theta_2) d\theta_2, \quad \eta'_0 = d\theta_1,
\]

where \( (\exp(\theta_1/\sqrt{-1}), \exp(\theta_2/\sqrt{-1})) \) is a coordinate of \( T^2 \) and \( \gamma(\theta) \) is the smooth function which is used to define Dhen twist diffeomorphisms in §2.

Then, the 1-forms satisfy the following conditions:

(4.1) \( T(a) \eta_0 = \eta_0, \ T(a) \eta'_0 = \eta_0 + \eta'_0 \);
(2) \( \int_{\Sigma} \eta_0 \wedge \eta'_0 = 0 \);
(3) \( \int_{\Sigma} \eta'_0 = 0, \int_{\Sigma} \eta_0 = 0 \);

where \( T(a) \) is Dhen twist diffeomorphism along \( a \) and \( \{a, b\} \) is the symplectic system of oriented simple closed curves of \( T^2 \) such that \( a = a(\theta) = (\exp(\theta_1/\sqrt{-1}), 1) \) and \( b = b(\theta) = (1, \exp(\theta_2/\sqrt{-1})) \).

We construct the desired closed 1-forms \( \eta, \eta' \) by use of \( \eta_0, \eta'_0 \).
We may assume that \( c_1 = a_1 \).
Let \( f_i \) be the diffeomorphism from \( U_i \) to \( U \) as in §2. We set

\[
\eta = f_1^* \eta_0, \quad \eta' = f_1^* \eta'_0.
\]

Since \( c_i = a_i \) and \( f_i(a_i) = a \), we have (1) of Lemma 3. By the condition (*) in §3, the union of \( \{\text{supp } T(c_i); i = 2, \ldots, n\} \) is contained in \( \Sigma_g - U_1 \), we have (2) of Lemma 3. Moreover, by (4.1) (2), we have (3) of Lemma 3.

We prove (4) of Lemma 3. Let \( \lambda \) be a closed 1-form on \( \Sigma_g \). Since the closed 1-forms \( \eta, \eta' \) satisfy (3) of Lemma 3 and their supports are contained in \( U_1 \), they form a basis of \( H^1(U_1, \partial U_1; \mathbb{R}) \). Hence the 1-form \( \lambda \) is cohomologous to a closed 1-form \( \lambda' \) such that \( \lambda' = n\eta' + m\eta + \mu \), where the support of \( \mu \) is contained in \( \Sigma_g - U_1 \) and \( n, m \in \mathbb{R} \). We have

\[
\int_{\Sigma_g} \lambda = \int_{\Sigma_g} \lambda' = \int_{\Sigma_g} n\eta' + m\eta = \int_{\Sigma_g} n\eta'_0 + m\eta_0 = n \int_{\Sigma_g} \eta'_0;
\]
and
\[ \int_{\mathcal{E}} \eta \wedge \lambda = \int_{\mathcal{E}} \eta' \wedge \lambda' = n \int_{\mathcal{E}} \eta \wedge \eta' = n \int_{\mathcal{E}} \eta_0 \wedge \eta_0'. \]

Therefore, if \( \int_{\mathcal{E}} \eta \wedge \lambda \neq 0 \), \( n \) must be non-zero by (4.1) (2). Hence, by (4.1) (3), we have (4) of Lemma 3.

In the following, we may assume that \( c_1 = a_1 \).

Let \( p_1: \bar{N} \times \Sigma_g \to \bar{N} \) and \( p_2: \bar{N} \times \Sigma_g \to \Sigma_g \) be the projections. We have a closed 1-form \( p_1^* \eta \) on \( \bar{N} \times \Sigma_g \). By (1) and (2) of Lemma 3, \( p_1^* \eta \) is invariant under the \( \pi_1(N) \)-action. Also the closed 1-form \( p_1^* (\pi_1^* \xi) \) on \( \bar{N} \times \Sigma_g \) is invariant under the \( \pi_1(N) \)-action. Therefore, they define the closed 1-forms \( \hat{\eta}, \hat{\xi} \) on \( M \) which are the projections of the closed 1-forms \( p_1^* \eta, p_1^* (\pi_1^* \xi) \) down to \( M \). We denote their cohomology classes by \( A, C \) respectively.

Then we have the following lemma which proves Proposition.

**Lemma 4.** *Massey triple product \( \langle A, A, C \rangle \) is non-zero.*

Proof. We define a 1-form \( y \) on \( \bar{N} \times \Sigma_g \) by
\[ y = p_1^* F \cdot p_1^* \eta - p_1^* \eta'. \]

Then we have
\[ \Phi^*_1 y = p_1^* (\rho (g)^* F) \cdot p_1^* (\rho (g)^* \eta) - p_1^* (\rho (g)^* \eta'), \]
and, by (2) of Lemma 3, we get
\[ \Phi^*_1 y = p_1^* (\rho (g)^* F) \cdot p_1^* ((\pi \circ \rho (g))^* \eta) - p_1^* ((\pi \circ \rho (g))^* \eta'). \]

Therefore, by Lemma 2 and (1) of Lemma 3, the 1-form \( y \) is invariant under the \( \pi_1(N) \)-action. We denote by \( \hat{y} \) the 1-form on \( M \) which is the projection of \( y \) down to \( M \). Since
\[ dy = p_1^* (dF) \wedge p_1^* \eta = p_1^* (\pi_1^* \xi) \wedge p_1^* \eta, \]
we have
\[ d\hat{y} = \hat{\xi} \wedge \hat{\eta}. \]

Therefore, by definition, we have
\[ \langle A, A, C \rangle \equiv [-\hat{\eta} \wedge \hat{y}] \quad \text{mod.} \quad A \cdot H_{br}(M) + C \cdot H_{br}(M). \]

We remark that \( -p_1^* \eta \wedge y = p_1^* (\eta \wedge \eta') \) and note that \( p_1^* (\eta \wedge \eta') \) is a closed 2-form on \( \bar{N} \times \Sigma_g \) which is invariant under the \( \pi_1(N) \)-action. Then the closed 2-form \( \mu \) on \( M \) which is the projection of \( p_1^* (\eta \wedge \eta') \) down to \( M \) satisfies
\[ [-\hat{\eta} \wedge \hat{y}] = [\mu] \quad \text{in} \quad H^2_{br}(M). \]
Now, let \( i: \Sigma_g \to M \) be the inclusion mapping of a fibre. Then, we have
\[
i^* \mu = \eta \wedge \eta'.
\]
Hence,
\[
[i_*([\Sigma_g])] = \int_{i(\Sigma_g)} \mu = \int_{\Sigma_g} i^* \mu = \int_{\Sigma_g} \eta \wedge \eta'.
\]
Therefore, \([i_*([\Sigma_g])]\) is non-zero by (3) of Lemma 3. That is, we have
\[
(4.2) \quad \langle A, A, C \rangle \equiv [\mu] \mod. A \cdot H_{br}(M) + C \cdot H_{br}(M);
\]
\[
(2) \quad [\mu] (i_*([\Sigma_g])) \neq 0.
\]
In the following, we prove that \([\mu] \) does not belong to \( A \cdot H_{br}(M) + C \cdot H_{br}(M) \).

First, we have
\[
\xi = p^* \xi,
\]
where \( p: M \to N \) is the projection of the flat \( \Sigma_g \)-bundle. The cohomology class \( C \) is represented by \( \xi \). Therefore, if \( X \) is an element of \( C \cdot H_{br}(M) \), \( i^* X = 0 \) in \( H_2 \). That is, we have
\[
(4.3) \quad X(i_*([\Sigma_g])) = 0 \quad \text{for} \quad X \in C \cdot H_{br}(M).
\]
Secondly, let \( X \) be an element of \( A \cdot H_{br}(M) \). There is a closed 1-form \( \zeta \) such that \( X = [\hat{\eta} \wedge \zeta] \in A \cdot H_{br}(M) \). And we have
\[
i^* (\hat{\eta} \wedge \zeta) = \eta \wedge i^* \zeta.
\]
Hence,
\[
X(i_*([\Sigma_g])) = \int_{i(\Sigma_g)} \hat{\eta} \wedge \zeta = \int_{\Sigma_g} i^* (\hat{\eta} \wedge \zeta) = \int_{\Sigma_g} \eta \wedge i^* \zeta.
\]
Then, by (4) of Lemma 3, if \( X(i_*([\Sigma_g])) \) is non-zero, \( \int_{\Sigma_g} i^* \zeta \) must be non-zero. On the other hand, we assumed that \( c_i = a_i \). Then, the following property of \( T(c_i) \) is well-known and is obtained easily by the definition:
\[
T(c_i)([b_i]) = [a_i] + [b_i] \quad \text{in} \quad H_1(\Sigma_g; Z).
\]
Since \( i_* T(c_i) = i_*: H_1(\Sigma_g; Z) \to H_1(M; Z) \),
\[
i_*([a_i]) = 0 \quad \text{in} \quad H_1(M; Z).
\]
Hence, for all closed 1-form \( \lambda \) on \( M \), we have
\[
\int_{a_i} i^* \lambda = [\lambda](i_*([a_i])) = 0.
\]
Consequently, we have

\[ (4.4) \quad X(\iota_*(\Sigma_g)) = 0 \quad \text{for} \quad X \in A \cdot H^1_{dR}(M). \]

Now, by (4.2) (2), (4.3) and (4.4), \([\mu]\) does not belong to \(A \cdot H^1_{dR}(M) + C \cdot H^1_{dR}(M)\). Hence, by (4.2) (1), we have Lemma 4. 

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References


