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Osaka University
CERTAIN INVARIANT SUBRINGS ARE GORENSTEIN I

KEICHI WATANABE

(Received April 20, 1973)

Introduction

Let \( R = k[X_1, \ldots, X_n] \) be a polynomial ring over a field \( k \) and \( G \) be a finite subgroup of \( GL(n, K) \). We assume that \( |G| \), the order of \( G \), is not zero in \( k \). Then \( G \) acts on 1-forms of \( R \) and thus \( G \) can be considered as an automorphism group of \( R \). We want to investigate the invariant subring \( R^G \). We have two theorems concerning \( R^G \) already.

Theorem ([4], Théorème 1) \( R^G \) is again a polynomial ring if and only it \( G \) is generated by pseudo-reflections. (We call \( g \in G \) a pseudo-reflection if rank \( (g - I)^{1} \leq 1 \), where \( I \) is the unit matrix).

Theorem ([2], Proposition 13) \( R^G \) is a Macaulay ring.

After these theorems, we ask:

“When is \( R^G \) a Gorenstein ring?”

We prove in this paper the following theorems.

Theorem 1. If \( G \subset SL(n, k) \), then \( R^G \) is a Gorenstein ring.

We apply this theorem to the case of regular local rings. If \( (R, m) \) is a regular local ring and if \( G \) is a finite subgroup of Aut \( (R) \), then \( G \) acts linearly on \( m/m^2 \). Thus we have the canonical homorphism \( \chi: G \to GL(m/m^2) \). We also assume that \( |G| \) is a unit in \( R \). Then applying Theorem 1, we get the following theorem.

Theorem 3. If \( \chi(G) \subset SL(m/m^2) \), then \( R^G \) is Gorenstein.

To reduce the case of regular local rings to the case of polynomial rings, we use the following theorem.

Theorem 4. Let \( (A, m) \) be a local ring. (We assume always the Noetherian property.) We suppose that \( A \) has a filtration \( F = (F_i)_{i \geq 0} \) satisfying the following conditions.

(i) \( F_0 = A \) and \( F_i = m_i \).

(ii) \( (F_i)_{i \geq 0} \) defines the same topology as the \( m \)-adic topology on \( A \). We put \( R = Gr(A) = \bigoplus_{i \geq 0} F_i/F_{i+1} \) the associated graded algebra and \( M = R_+ = \bigoplus_{i \geq 1} F_i/F_{i+1} \) the
canonical maximal ideal of $R$. Then,

1. Preliminaries

The contents of this section can be found elsewhere. But for the convenience of the readers, I put the proofs. As for the definition and the fundamental properties of Gorenstein rings, see [1].

In this section, $R$ is a Noetherian ring and $G$ is a finite group acting on $R$. We assume that $|G|$, the order of $G$, is a unit in $R$. We denote by $R^G$ the invariant subring of $R$ by $G$ and by $\rho$ the Reynolds operator $R \to R^G$ defined by $\rho(r) = \frac{1}{|G|} \sum_{g \in G} g(r)$ for $r \in R$.

**Lemma 1.** If $f_1, \ldots, f_s$ are elements in $R^G$ which form an $R$-regular sequence, then they form also an $R^G$-regular sequence and $R^G/(f_1, \ldots, f_s) \cong (R/(f_1, \ldots, f_s))^G$.

**Proof.** It suffices to show the latter part. Let's put $a = (f_1, \ldots, f_s)R$. If $h \in R$ and $h-g(h) \in a$ for all $g \in G$, $h^{-1}(h) \in a$ and $\rho(h) \in R^G$ obtaining that $R^G/(f_1, \ldots, f_s)R^G \to (R/(f_1, \ldots, f_s))^G$ is surjective. Since injectivity is clear, we are done.

**Lemma 2.** If $R$ is Macaulay, then $R^G$ is Macaulay.

**Proof.** If $(f_1, \ldots, f_s)$ is a parameter system of $R^G$, it is also a parameter system for $R$. Since $R$ is Macaulay, $(f_1, \ldots, f_s)$ forms an $R$-regular sequence and by Lemma 1, it forms an $R^G$-regular sequence. So $R^G$ is Macaulay.

**Lemma 3.** If $(A, m)$ is an Artinian local ring, the following conditions are equivalent.

(a) $A$ is Gorenstein.

(b) $\text{length}_A(0; m) = 1$.

(c) There exists an element $z$ in $A$, $z \neq 0$, such that for every $x \neq 0$ in $A$ there exists an element $y$ in $A$ satisfying $xy = z$.

**Proof.** (a)$\Rightarrow$(b) is almost the definition itself. (b)$\Rightarrow$(c) is straightforward.

**Lemma 4.** Let $(A, m)$ be an Artinian local Gorenstein ring, $G$ a finite group acting on $A$. We assume that $|G|$ is a unit in $A$ and we denote by $z$ an element in $A$ satisfying the condition (c) of Lemma 3. If $z$ is invariant under $G$, then $A^G$ is Gorenstein.

**Proof.** We check the condition (c) of Lemma 3 for $A^G$. Take $x \neq 0$ in $A^G$. By assumption, there exists $y$ in $A$ satisfying $xy = z$. Then $\rho(y) = z$ and $\rho(y)$
Lemma 5. Let \( A \) be a ring which contains a field \( k \) and let \( k' \) be an extension field of \( k \). If a group \( G \) acts on \( A \) and \( G \) acts trivially on \( k \), we can extend the action of \( G \) to \( A' = A \otimes_k k' \) naturally. Then \((A')^G = A^G \otimes_k k'\). Thus \((A')^G\) is faithfully flat over \( A^G \) and if \( (A)^0 \) is Gorenstein, \( A^G \) is Gorenstein.

Proof. We write elements of \( A' \) in the form \( x' = \sum x_i c_i \) where \( x_i \in A \), \( c_i \in k' \) and \( c_i \)'s are linearly independent over \( k \). For any \( g \in G \), \( g(x') = \sum g(x_i) \otimes c_i \) and if \( x \) is \( G \)-invariant, all \( x_i \)'s are \( G \)-invariant. Thus we have \((A')^G = A^G \otimes_k k'\) and so \((A')^G\) is faithfully flat over \( A^G \). The latter part holds by [5], Theorem 1'.

2. The case when \( G \) is cyclic

In this section, we use the following notations. 
\( R = k[X_1, \ldots, X_n] \), the polynomial ring over a field \( k \).
\( G \) is finite cyclic subgroup of \( GL(n, k) \). We assume that \((\text{ch}(k), |G|) = 1\).
\( g \) is a generator of \( G \). We put \( |G| = m \) and we denote by \( \varepsilon \) a primitive \( m \)-th root of unity. We write \( e_i = \varepsilon^{a_i} \).

Lemma 6. If \( \det(g) = 1 \), then \( \mathcal{O} \) is Gorenstein.

Proof. \( X_1^m, \ldots, X_n^m \) are in \( R^G \) and by Lemma 1, we have \( \mathcal{O}/(X_1^m, \ldots, X_n^m)\mathcal{O} \cong (R/(X_1^m, \ldots, X_n^m)R)^G \). \( A = R/(X_1^m, \ldots, X_n^m)R \) is an Artinian local ring. As \( A \) is a complete intersection, \( A \) is Gorenstein. In \( A \), \( z = (X_1 \cdots X_n)^{m-1} \) satisfies the condition of Lemma 3 (c). If \( \det(g) = 1 \), \( z \in A^G \) and by Lemma 4, \( A^G \) is Gorenstein. Thus \( \mathcal{O} \) is Gorenstein.

Before proving the converse of Lemma 6, we need to fix some terminology.

Definition 1. \( m \) and \( a_i \) are as in the beginning of this section. We put \( I = \{(r_1, \ldots, r_n) | r_i \)'s are integers and \( 0 \leq r_i < m \) for \( i = 1, \ldots, n \} \)
\( J = \{(r_1, \ldots, r_n) \in I | \sum_{i=1}^n r_ia_i \equiv 0 \pmod{m}\} \).

We define an order in \( I \) and \( J \). Namely, \((r_1, \ldots, r_n) \geq (s_1, \ldots, s_n) \) if \( r_i \geq s_i \) for \( i = 1, \ldots, n \). We call an element of \( J \) minimal if it is minimal among the elements of \( J \) which are not \((0, \ldots, 0)\).

Recall that, if \( (A, m) \) is an \( n \)-dimensional local Macaulay ring, the 'type' of \( A \) is defined by the number \([\text{Ext}_A^n(A/m, A)/A/m]\). To say that \( A \) is Gorenstein
is equality to say that $A$ is Macaulay and $\text{type}(A) = 1$. We denote by $\text{emb}(A)$ the embedding dimension of $A$. $\text{emb}(A) = [m/m^2: A/m]$.

**Lemma 7.** If the number of minimal element of $J$ is $E$ and the number of maximal element of $J$ is $r$, then $\text{emb}(\mathcal{O}/(X^n_1, \ldots, X^n_n)) = E$ and $\text{type}(\mathcal{O}) = r$.

Proof. $X^n_1 \cdots X^n_n \equiv 0 \pmod{(X^n_1, \ldots, X^n_n)} \Rightarrow (r_1, \ldots, r_n) \in I$, and $X^n_1 \cdots X^n_n \in R^G \Rightarrow (r_1, \ldots, r_n) \in J$, and $\text{type}(\mathcal{O}) = \text{type}(\mathcal{O}/(X^n_1, \ldots, X^n_n))$. From these facts, the conclusion is immediate.

**Definition 2.** We call an element $g$ of $GL(n, k)$ a pseudo-reflection if the order of $g$ is finite and $\text{rank}(g-I_n) = 1$. (Where $I_n$ denotes the unit matrix).

**Proposition 1.** If $R^G$ is Gorenstein and if $G$ does not contain any pseudo-reflections other than the unity, then $G \subset SL(n, k)$.

Proof. It is clear that $(m, a_1, \ldots, a_n) = 1$. Since type $(\mathcal{O}) = 1$, $J$ must have unique maximal element $(r_1, \ldots, r_n)$. It is sufficient to prove that $(r_1, \ldots, r_n) = (m-1, \ldots, m-1)$. If this is not the case, we may assume that $r_1 < m-1$. Since $(r_1, \ldots, r_n)$ is the unique maximal element of $J$, for any $s_i$, $0 \leq s_i \leq m-1$ $(i=2, \ldots, n)$, $(m-1, s_2, \ldots, s_n) \in J$. If $(a_1, \ldots, a_n, m) = 1$, this can not happen and so $d = (a_2, \ldots, a_n, m) > 1$. Then if we put $m' = m/d$, $g^{m'} \neq 1$ and $g^{m'}$ is a pseudo-reflection. This contradicts the hypothesis that $G$ does not contain any pseudo-reflections other than the unity.

**Example 1.** If $\xi$ is a primitive 6-th root of unity and if we put $g = \begin{bmatrix} \xi & \xi^2 \\ \xi^2 & \xi \end{bmatrix}$, $R^G$ is Gorenstein but $\det(g) \neq 1$. This is due to the fact that $g^3 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is a pseudo-reflection. If we put $H = \{1, g^2\}$, $R^G = (R^H)^G/H$, $R^H = k[X^2, Y, Z]$. The action of $g = g \mod H$ on $k[X^2, Y, Z]$ is represented by $\begin{bmatrix} \xi^2 & \xi^2 \\ \xi^2 & \xi^2 \end{bmatrix}$ and $\det(g) = 1$.

More generally (we don’t suppose that $G$ is cyclic), let $H$ be the subgroup of $G$ generated by all its pseudo-reflections. Then $H$ is a normal subgroup of $G$ and $R^H$ is again a polynomial ring over $k$ (Serre [4], Théorème 1). Thus the hypothesis “$G$ does not contain any pseudo-reflections” is quite natural.

3. $R^G$ is Gorenstein at the origin

**Theorem 1a.** If a finite group $G \subset SL(n, k)$ acts on $R = k[X_1, \ldots, X_n]$ naturally and if $(|G|, \text{ch}(k)) = 1$, then $R^G$ is Gorenstein ‘at the origin’. Namely, if we put $n = R^G \cap (X_1, \ldots, X_n)$ and $\mathcal{O} = (R^G)_n$, then $\mathcal{O}$ is Gorenstein.

Proof. We take a parameter system $(f_1, \ldots, f_n)$ of $\mathcal{O}$ as follows;
1. Each $f_i$ is homogenous of the same degree $m$.
2. $m$ is a multiple of $|G|$.

We put $A = R/(f_1, \ldots, f_n)R$ and we want to apply Lemma 4. For this purpose we notice the following fact.

**Lemma 8.** Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring. We assume that $A_0 = k$ is a filed and that each $A_i$ is a finite dimensional vector space over $k$. If $f$ is a homogenous element of $A$ which is not a zero-divisor of $A$, then $\dim_k(A/fA)_n$ depends only on $A$, $n$ and $\deg(f)$.

**Proof.** If $\deg(f) = d$, $\dim_k(A/fA)_n = \dim_k(R/(X^n, \ldots, X^n))_d$. If we take $z \in A$ satisfying the condition of Lemma 3 (c) (A is Gorenstein), $\deg(z) = n(m-1)$. Then we take an element $g \in G$ and assume that $g$ is in a diagonal form. We put $H$ the cyclic subgroup of $G$ generated by $g$. Applying Lemma 8 to $R^H$, $\dim_k(R^H/(X^n, \ldots, X^n))_d = \dim_k(A^H)_d = \dim_k(R^H/(f_1, \ldots, f_n))_d$. As we have $(X^n, \ldots, X^n)_{m-1} \in R^H (g$ is in a diagonal form and $\det(g) = 1)$, $\dim_k(A^H)_{m-1} = 1$. As $\dim_k A_{m-1} = 1$, $z$ is invariant under $H$. As $g$ is arbitrary, $z \in A^G$. By Lemma 4, $A^G = \mathcal{O}(f_1, \ldots, f_n) \mathcal{O}$ is Gorenstein. Thus $\mathcal{O}$ is Gorenstein.

4. $R^G$ is globally Gorenstein

**Theorem 1.** If a finite subgroup $G$ of $SL(n, k)$ acts naturally on $R = k[X_1, \ldots, X_n]$ and if $(|G|, \text{ch}(k)) = 1$, then $R^G$ is Gorenstein.

**Proof.** By Lemma 5, we may assume that $k$ is algebraically closed. If we take a maximal ideal $n'$ of $R^G$, we can write $n' = (X_1-a_1, \ldots, X_n-a_n)R \cap R^G (a_1, \ldots, a_n \in k)$. We put $H = \{g \in G \mid g(a_1, \ldots, a_n) = (a_1, \ldots, a_n)\}$. We consider the diagram $R^G \rightarrow R^H \rightarrow R$. Then it is known that $R^G \rightarrow R^H$ is étale in a neighbourhood of $n'$ (Raynaud [3], P. 103, Th. 1). Thus $(R^G)_{n'} \rightarrow (R^H)_q$ is flat (where $q = (X_1-a_1, \ldots, X_n-a_n) \cap R^H$). If $(R^H)_q$ is Gorenstein, then $(R^G)_{n'}$ is Gorenstein ([5], Theorem 1). But by the coordinate transformation $(X_1, \ldots, X_n) \rightarrow (X_1-a_1, \ldots, X_n-a_n)$, $H$ can be regarded as a subgroup of $SL(n, k)$ and $q = (X_1, \ldots, X_n) \cap R^H$. By theorem 1a, $(R^H)_q$ is Gorenstein and we are done.

**Question 1.** Is the converse of Theorem 1 true? Let $G$ be a finite subgroup of $GL(n, k)$ and let us assume that $(|G|, \text{ch}(k)) = 1$ and that $G$ contains no pseudo-reflections other than the unity. If $R^G$ is Gorenstein, then $G \subset SL(n, k)$?

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1) Added in proof. The statement in Question 1 has been proved by the author. The proof will appear in [6].
Question 2. Is the following statement true? Let \( A = \bigoplus_{i \geq 0} A_i \) be a Noetherian graded ring with \( A_0 \) a field. We put \( M = A_+ = \bigoplus_{i \geq 1} A_i \). If \( A_M \) is Gorenstein, is \( A \) globally Gorenstein?

5. Base extensions

**Theorem 2.** Let \( A \) be a Noetherian ring and \( G \) be a finite subgroup of \( SL(n, A) \). We assume that \( |G| \) is a unit in \( A \). Then \( G \) acts naturally on \( R = A[X_1, \cdots, X_n] \). Then \( R^G \) is Gorenstein if and only if \( A \) is Gorenstein.

**Lemma 9.** Under the assumptions of Theorem 2, \( R^G \) is faithfully flat over \( A \).

Proof of Lemma 9. (i) If \( a \) is an ideal of \( A \), then \( a(R^G) = (aR)^G \). (If \( \sum a_i f_i \in (aR)^G \) with \( a_i \in a \) and \( f_i \in R \), \( \sum a_i f_i = \rho(\sum a_i f_i) = \sum a_i \rho(f_i) \) and we have \( (aR)^G \subset aR^G \). The converse inclusion is clear).

(ii) As \( R \) is \( A \)-flat, \( (aR)^G \cong (a \otimes_A R)^G \).

(iii) \( (a \otimes_A R)^G \cong a \otimes_A R^G \) (The isomorphisms is given by \( \sum a_i \otimes f_i \rightarrow \sum a_i \otimes \rho(f_i) \)). By (i), (ii), (iii), \( aR^G \cong a \otimes_A R^G \) and \( R^G/aR^G \cong (R/aR)^G \). Thus \( R^G \) is faithfully flat over \( A \).

Proof of Theorem 2. The fiber of the map \( f: \text{Spec}(R^G) \rightarrow \text{Spec}(A) \) at \( p \in \text{Spec}(A) \) is the Spec of \( R^G \otimes_A k(p) \cong (k(p)[X_1, \cdots, X_n])^G \) which is Gorenstein by Theorem 1. Thus \( f \) is a Gorenstein morphism in the sense of [5], Definition (1.7). The conclusion follows from [5], Theorem 1'.

**Remark.** In Lemma 9, the assumption "\( |G| \) is a unit in \( A \)" is essential. For example, let \( A = k[e] \), \( k \) be a field of characteristic 2, \( e^2 = 0 \), \( G = \langle g \rangle \), \( g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). If we put \( a = eA \), then \( eX_i \in R^G \) and \( e \otimes eX_i \neq 0 \) in \( a \otimes_A R^G \), while \( e.eX_i = 0 \). Thus \( a \otimes_A R^G \rightarrow aR^G \) is not injective and \( R^G \) is not flat over \( A \).

6. A theorem on the associated graded algebra of a local ring

**Theorem 3.** If \( (A, m) \) is a Noetherian local ring and \( (F_n)_{n \geq 0} \) be a filtration on \( A \) satisfying the two conditions.

1. \( F_0 = A \) and \( F_1 = m \).
2. \( (F_n)_{n \geq 0} \) defines the same topology as the \( m \)-adic topology on \( A \).

We put \( R = \text{Gr}(A) = \bigoplus_{i \geq 0} F_i/F_{i+1} \) the associated graded algebra and \( M = R_+ = \bigoplus_{i \geq 1} F_i/F_{i+1} \) the canonical maximal ideal of \( R \). Then,

(i) if \( R_M \) is Macaulay, then \( A \) is Macaulay.

(ii) if \( R_M \) is Gorenstein, then \( A \) is Gorenstein.

Proof. The proof follows immediately from the two lemmas below.
**Lemma 10.** Let \( f_1, \ldots, f_s \) be homogenous elements of \( R \) which make an \( R \)-sequence. If \( x_1, \ldots, x_s \) are elements of \( A \) with \( \text{In}(x_i) = f_i (i = 1, \ldots, s) \), then \((x_1, \ldots, x_s)\) form an \( A \)-regular sequence and \( \text{Gr}'(A/(x_1, \ldots, x_s)) = R/(f_1, \ldots, f_s) \). (If \( x \in A, x \in F_n \) and \( x \equiv F_{n+1} \), then \( \text{In}(x) = x \mod F_{n+1} \in \text{Gr}'(A) \). The filtration of \( A/(x_1, \ldots, x_s) \) is the one induced from \((F_n)\).)

**Proof.** We note the fact that if \( x, y \in A \) and \( \text{In}(x) \text{In}(y) \neq 0 \), then \( \text{In}(xy) = \text{In}(x) \text{In}(y) \).

**Case 1.** \( s = 1 \) (we omit the subscript 1).

If \( y \in A \) and \( \text{In}(y) \neq 0 \), by assumption \( \text{In}(x) \text{In}(y) \neq 0 \). Thus \( \text{In}(xy) = \text{In}(x) \text{In}(y) \neq 0 \) and \( xy \neq 0 \). On the other hand, \( \text{Gr}'(A/xA) \approx R/\text{Gr}'(xA) \) where \( \text{Gr}'(xA) \) is the homogenous ideal of \( R \) generated by \( \text{In}(x) \), \( z \in xA \). But if \( z = xy \in xA \), then \( \text{In}(z) = \text{In}(x) \text{In}(y) \) and so \( \text{In}(z) \in fR \). Thus we have \( \text{Gr}'(A/xA) \approx R/fR \).

**Case 2.** General case.

We assume that the assumption is true for \( s = i \) and prove for \( s = i + 1 \). As \( f_{i+1} \) is not a zero-divisor on \( \text{Gr}'(A/(x_1, \ldots, x_i)) \approx R/(f_1, \ldots, f_i) \), Case 1 applies.

**Lemma 11.** If \((A, m)\) is an Artinian local ring, \((F_n)\) is a filtration on \( A \) which satisfies the conditions of Theorem 3 and if \( R = \text{Gr}'(A) \) is Gorenstein, then \( A \) is Gorenstein.

**Proof.** We use Lemma 3. Let \( h \) be a homogenous element of \( R \) which satisfies the condition of Lemma 3(c) for \( R(M : M \) is a homogenous ideal of \( R) \). Then if \( z \in A \) be such that \( \text{In}(z) = h \), then for any \( x \in A, x \neq 0 \), there exists an element \( f \in R \) such that \( \text{In}(x)f = h \). If we take \( y \in A \) such as \( \text{In}(y) = f \) and if \( \deg(h) = m, \text{In}(y) \text{In}(x) = h \) and \( xy \equiv z \mod F_{m+1} \). But as \( F_{m+1} = 0, xy = z \) and \( z \) satisfies the condition (c) of Lemma 3 for \( A \).

**7. The case of regular local rings**

The statement of Theorem 4 was indicated to me by Professor M. Miya- nishi with an outline of a proof. I wish to express my deep gratitude to him.

**Theorem 4.** Let \((R, m)\) be a regular local ring of dimension \( n \) and \( G \) be a finite subgroup of \( \text{Aut}(R) \) satisfying the following conditions.

1. \( |G| \) is a unit in \( R \).
2. The automorphisms of \( k = R/m \) induced by the elements of \( G \) are identities.
3. If we denote \( \lambda : G \rightarrow GL(m/m^2) \) the canonical homomorphism, then \( \lambda(G) \subset SL(m/m^2) \).

Then \( S = R^G \) is Gorenstein.

The proof is divided into several steps. First we need a lemma.
Lemma 12. ([2], Proposition 10) Let $R$ be a commutative ring and $G$ be a finite group acting on $R$. We assume that $|G|$ is a unit in $R$ and we put $S=R^G$. Then if $a$ is an ideal of $S$, then $aR \cap S=a$.

Proof. If $\sum a_ir_i \in S$, $a_i \in a$, $r_i \in R$, then $\sum a_ir_i=\rho(\sum a_ir_i)=\sum a_i\rho(r_i) \in a$. Thus we get the inclusion $\subset$ and the converse is trivial.

We return to the proof of Theorem 4. From Lemma 12, we get

(1) $S$ is a Noetherian local ring.

Proof. Since $R$ is integral over $S$, $S$ is local and by Lemma 12, $S$ is Noetherian.

We put,

$A=\text{Gr}_{m}(R)\approx k[X_1, \cdots, X_n]$.

$G$ acts naturally on $A$. We denote by $n$ the maximal ideal of $S$ and we put $F_n=S \cap m^n$. $(F_n)_{n \geq 0}$ defines a filtration on $S$. We denote by $B$ the graded ring associated to this filtration. Then we have;

(2) $B \approx A^G$.

Proof. If $f=\ln(x) \in A_n$ is invariant under $G$, then $x-\rho(x) \in m^{n+1}$ and $\rho(x) \in F_n$. Thus $A^G \subset B$. The converse implication is trivial.

(3) The filtration $(F_n)$ defines on $S$ the same topology as $n$-adic topology.

Proof. If suffices to say that for any integer $t \geq 0$, there exists an integer $t'$ such that $S \cap m^{t'} \subset n^t$. But as $nR$ is $m$-primary, for some $s$, $m^s \subset nR$. Then, by Lemma 12, $m^s \cap S \subset (nR)^t \cap S=n^t$.

By (2), (3), Theorem 1 and Theorem 3, Theorem 4 is proved.

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References


Added in proof; Question 2 in section 4 was solved affirmatively by Y. Aoyama, S. Goto, J. Matijevic and R.C. Cowsik independently and in more general forms.