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LINKS OF ORIENTED DIVIDES AND FIBRATIONS IN LINK EXTERIORS

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1. Introduction

A divide P is the image of a generic immersion of a finite number of copies of the unit interval or the unit circle into the unit disk. Let (x_1, x_2) be a point on P and $T_{(x_1,x_2)}(P)$ the set of the tangent vectors to P at (x_1, x_2). We define the link of the divide P by the set

$$\{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 \mid (x_1, x_2) \in P, (u_1, u_2) \in T_{(x_1,x_2)}(P), x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1\} \subset S^3.$$

In the 1970's, the divide appeared as immersed curves on the real 2-plane, for the purpose of studying real morsifications of complex plane curve singularities, in the works of N. A'Campo [1], [2] and S.M. Gusein-Zade [11], [12], [13]. Recently A'Campo introduced the divide on the unit disk, defined the links of such divides, and showed that if a divide has the same configuration as the real morsification of an isolated plane curve singularity then the link of the divide is ambient isotopic to the link of the singularity [3]. Also, he proved in [4] that if the divide is connected then the link exterior has a fibration over S^1, and visualized the core curves of the right Dehn twists of the geometrical monodromy of the fibration on the figure of the divide.

In the present paper we mainly discuss the following oriented divide.

DEFINITION 1.1. An *oriented divide* is the image of a generic immersion of a finite number of copies of the oriented unit circle into the unit disk.

Consequently, the link of the oriented divide C is given by the set

$$L(C) := \{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 \mid (x_1, x_2) \in C, (u_1, u_2) \in \vec{T}_{(x_1,x_2)}(C), x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1\}$$

where $\vec{T}_{(x_1,x_2)}(C)$ is the set of the vectors tangent to C at (x_1, x_2) which are in the same direction as the assigned orientation. The construction of the link of an oriented divide is a natural extension of the link construction of a divide. From any divide we

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can obtain an equivalent oriented divide by using a simple "doubling" method. This will be described in Section 4.

Let D_L be a regular link projection of an oriented link L into the unit disk. So D_L is also oriented. Let $c(D_L)$ be the number of double points of D_L, and $r_0(D_L)$ the number of embedded circle components of D_L. Here, by the definition, an embedded circle component is a simple closed curve in the unit disk, and hence it corresponds to a trivial, unlinked component of L.

Theorem 1.2. *For any regular link projection D_L of an oriented link L, there is an oriented divide C, which can be obtained from D_L by attaching a finite number of small loops* (*see* Fig. 1), *such that the link $L(C)$ is ambient isotopic to L. Moreover, the number of necessary small loops is at most $3c(D_L) + r_0(D_L)$.*

The existence of oriented divides corresponding to any link L is already known in the context of Legendrian knots [8] (see Remark 2.6).

It is known by J. Alexander that every closed, orientable 3-manifold has a fibred link [6]. For every link exterior in S^3 we can prove the existence of a fibred link with stronger properties by combining the fibration theorem of connected divides with Theorem 1.2.

Theorem 1.3. *Let $E(L)$ be the exterior of an oriented link L in S^3. Then there is a knot A in $E(L)$, which is trivial in S^3, and a π-rotation map $R_\pi \colon S^3 \to S^3$ around A such that*

- *$R_\pi(L)$ is contained in $E(L)$, and*
- *the complement of $R_\pi(L)$ in $E(L)$ is fibred, its monodromy is the product of right Dehn twists, and the number of right Dehn twists is equal to the first Betti number of the fibre.*

In the proof, the fibration will be realized as that of a connected divide obtained from a regular link projection by attaching small loops. Therefore the geometry of the fibration is determined by the immersed curves of such a divide.

By Theorem 1.2, for any regular link projection D_L, a corresponding oriented divide can be constructed by attaching appropriate small loops to D_L. For each D_L we define the integer $d(D_L)$ to be the minimal number of small loops, over all possible choices, required to induce the link L. The genus of the fibre of the corresponding fibration in Theorem 1.3 can be calculated from the immersed curves of D_L with small loops, and is given by $c(D_L)+d(D_L)-r(L)+1$. Here $r(L)$ is the number of link components of L, which is equal to the number of immersed circles of the corresponding oriented divide. So now denote by $g_{\min}(L)$ the minimal value of $c(D_L)+d(D_L)-r(L)+1$ over all possible regular link projections. The integer $g_{\min}(L)$ is clearly a link invariant and, from the inequality $d(D_L) \leq 3c(D_L) + r_0(D_L)$ in Theorem 1.2, it is bounded above by $4c(D_L) + r_0(D_L) - r(L) + 1$.

Now we state the fact that there are infinitely many fibrations satisfying the properties in Theorem 1.3.

Theorem 1.4. *Let $E(L)$ be the exterior of a link L in S^3 and $g_{\min}(L)$ the invariant described above. Then for any integer $g \geq g_{\min}(L)$ there is a fibred link L' of the 3-manifold $E(L)$ which possesses the properties outlined in* Theorem 1.3 *and whose fibre is a genus g surface.*

In this paper, we will present methods for constructing oriented divides from link diagrams and vice-versa. A construction of a link diagram of the link of a non-oriented divide was presented by M. Hirasawa [14] independently of our work. He also showed in [14] an algorithm for drawing a Seifert fibre surface of the link. A construction of the link of a special class of divide, the *slalom curves of rooted planar trees*, was presented by C.V. Quach Hongler and C. Weber [16], [17].

This paper is organised as follows: In Section 2 we outline the geometric preliminaries, prove Theorem 1.2 and present an example which shows the method for constructing an oriented divide from a link diagram. In Section 3 we will prove Theorems 1.3 and 1.4. In Section 4 we introduce methods for drawing the link diagram of an oriented divide by using an inverse algorithm of the proof of Theorem 1.2. This method can also be applied to a non-oriented divide by doubling it.

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2. Geometric preliminaries and the proof of Theorem 1.2

In this paper all our discussions take place in the smooth category. Hence any mention of isotopy will implicitly refer to ambient isotopy.

It is more convenient for us to consider the following half plane model of a (oriented) divide. A half plane model for a divide is the image of a generic immersion of a finite number of copies of the unit interval $[0, 1]$ and the unit circle into the closure $\bar{H}$ of the half plane $H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\}$. By generic we mean

- the image has neither self-tangent points nor triple points;
- the immersion of each interval is relative to the boundary $\partial\bar{H}$;
- the image of each interval intersects $\partial\bar{H}$ transversely;
- the image of each circle does not intersect $\partial\bar{H}$.

We call it the *half plane divide* and each image of the unit interval or the unit circle a *component* of the divide.

For a parametrized curve $a := \{(x_1(t), x_2(t)) \mid 0 \leq t \leq 1\}$ in H, we define its embedding ϕ into $\mathbb{R}^3$ by

$$\phi(a) = \left\{ \left(\frac{\sqrt{x_{1,i}(t)}\, x'_{1,i}(t)}{l(t)}, \frac{\sqrt{x_{1,i}(t)}\, x'_{2,i}(t)}{l(t)}, x_{2,i}(t) \right) \right\} \subset \mathbb{R}^3, \tag{2.1}$$

where $\left(x'_1(t), x'_2(t)\right)$ are the derivatives of $\left(x_1(t), x_2(t)\right)$ by t, and $l(t) = \left|\left(x'_{1,i}(t), x'_{2,i}(t)\right)\right|$. Note that $\phi(a)$ depends on the parametrization assigned to a.

Now let P be a half plane divide, and denote its interval components by $I_1, \ldots, I_s$ and its circle components by $C_1, \ldots, C_r$. We give a parametrization $I_i^+ := \{\left(x_{1,i}(t), x_{2,i}(t)\right) \mid 0 \leq t \leq 1\}$ to each interval component I_i such that the points at $t = 0$ and 1 correspond to the endpoints of I_i, and also define the reverse parametrization $I_i^-(t)$ by $I_i^- := \{\left(x_{1,i}(1-t), x_{2,i}(1-t)\right) \mid 0 \leq t \leq 1\}$. Then we define the link of the (non-oriented) interval component I_i by the union $\phi(I_i^+) \cup \phi(I_i^-)$, and denote it by $\phi(I_i)$.

The link of each circle component C_j is defined using the same method as follows: first give a parametrization $C_j^+ := \{\left(x_{1,j}(t), x_{2,j}(t)\right) \mid 0 \leq t \leq 1\}$ to C_j such that the points at t_0 and t_1 correspond to the same point in C_j, and also define the reverse parametrization $C_j^- := \{\left(x_{1,j}(1-t), x_{2,j}(1-t)\right) \mid 0 \leq t \leq 1\}$. Then we define the link of the (non-oriented) circle component C_j by the disjoint union $\phi(C_j^+) \sqcup \phi(C_j^-)$, and denote it by $\phi(C_j)$.

The link of the half plane divide P is then defined as the disjoint union of these links, i.e. $\sqcup_i \phi(I_i) \sqcup_j \phi(C_j)$. We call it the *link of the half plane divide P* and denote it by $\phi(P)$.

The preceding definition for the link of a half-plane divide can be derived from A'Campo's original definition using an appropriate orientation-preserving complex transformation from the unit disk to the half-plane (see [I]). Note that the link of a half plane divide is isotopic, after the compactification $S^3 = \mathbb{R}^3 \cup \{\infty\}$, to that of the unit disk model. Sometimes we regard the link of the half plane model, lying in $\mathbb{R}^3$, as the link in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ induced by the compactification.

Now we assume each component of the half plane divide is an immersed circle, and construct a *half-plane oriented divide C* by assigning an orientation to each component. Let $C_1, \ldots, C_r$ be the circle components of C. For each C_i, choose a parametrization C_i^+ or C_i^- according to the assigned orientation, and denote the parametrized circle component also by C_i. Then we call the image $\sqcup_i \phi(C_i)$ the *link of the half-plane oriented divide C* and denote it by $\phi(C)$. Note that $\phi(*)$ represents the link of a half-plane divide or oriented divide depending on $*$. We call the vector

$$\left(\frac{\sqrt{x_{1,i}(t)}\, x'_{1,i}(t)}{l(t)}, \frac{\sqrt{x_{2,i}(t)}\, x'_{2,i}(t)}{l(t)} \right)$$

a *speed vector* of C at $\left(x_{1,i}(t), x_{2,i}(t)\right)$.

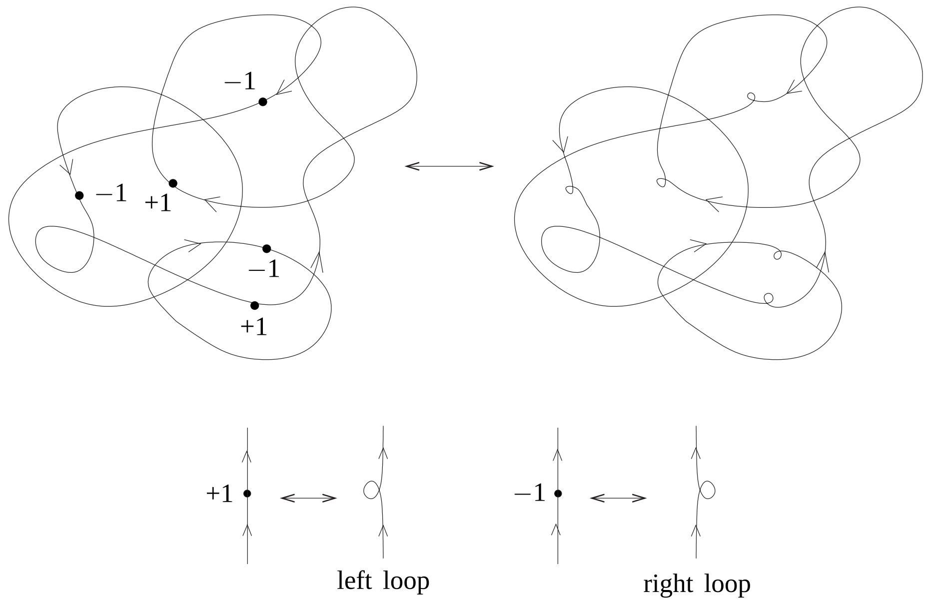

Fig. 1. An oriented divide obtained from D_L by attaching the divisor.

For brevity we will usually drop the adjective *half-plane*, and thus talk of *oriented divides* and *links of oriented divides* without any risk of confusion.

Let L be an oriented link in $H \times \mathbb{R}$ ($\subset \mathbb{R}^3$) and $D_L := p(L)$ a regular link projection of L, where $p\colon H \times \mathbb{R} \to H$ is the canonical projection map.

DEFINITION 2.1. A *divisor* on a regular link projection D_L is a finite set of points on $D_L \setminus \{$*double points*$\}$, each equipped with a +1 or -1.

For each point with +1 or -1 we apply a small left or right "loop" relative to the orientation as shown in Fig. 1. Then the regular link projection D_L with loops attached according to the divisor satisfies the conditions of the oriented divide. We call this the *oriented divide obtained from D_L by attaching the divisor*.

We will now rewrite Theorem 1.2 in terms of divisors, and then present a proof.

Theorem 2.2 (cf. Theorem 1.2). *For any oriented link L and its regular projection D_L, there is a divisor on D_L such that the* (*oriented*) *link of the oriented divide obtained from D_L by attaching the divisor is isotopic to L. Moreover, the divisor can be selected in such a way that the number of points in the divisor is at most $3c(D_L) + r_0(D_L)$, where $c(D_L)$ is the number of double points and $r_0(D_L)$ the number of embedded circle components of D_L.*

Before proving the theorem we will prepare some terms and notations which we will need in the proof, and also in the rest of the paper.

Let C be an oriented divide. By the definiton, the link $\phi(C)$ of C is contained in $\left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R}$, which is homeomorphic to the trivial circle bundle $H \times S^1$. Here we explicitly define the homeomorphism by

$$(y_1, y_2, y_3) \mapsto \left(y_1^2 + y_2^2, y_3, \arg(y_1, y_2)\right)$$

and denote it by Ψ. The image $\Psi\left(\phi(C)\right)$ of the link $\phi(C)$ in $\left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R} \subset \mathbb{R}^3$ constitutes a smooth 1-manifold in $H \times S^1$ such that $q\left(\Psi(\phi(C))\right)$ coincides with the immersed curves of the oriented divide C, where $q \colon H \times S^1 \to H$ is the canonical projection map. In other words, the image $\Psi\left(\phi(C)\right)$ is a smooth 1-manifold in $C \times S^1$ (here C means the immersed curves of the oriented divide C). Note that the third component of each point a in $\Psi\left(\phi(C)\right)$ coincides with the argument of the speed vector of C at $q(a)$.

We remark that, by identifying $H \times S^1$ with the set $\left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R} \subset \mathbb{R}^3$ by Ψ, any isotopy move of a smooth 1-manifold in $H \times S^1$ always induces an isotopy move of the corresponding link in $\mathbb{R}^3$.

We mainly deal with the following smooth 1-manifolds in $C \times S^1 \subset H \times S^1$. Let $\hat{C}$ be the disjoint union of oriented unit circles and define the immersed curve of an oriented divide C as the image $h(\hat{C})$ of $\hat{C}$ by a fixed map $h \colon \hat{C} \to H$, i.e. $C = h(\hat{C})$. Then consider an embedding map $\tilde{h} \colon \hat{C} \to H \times S^1$ satisfying $q\tilde{h} = h$, and set $s := \tilde{h}(\hat{C}) \subset H \times S^1$. We call such a smooth 1-manifold s a *lifted* 1-*manifold over* C. Two lifted 1-manifolds s_0 and s_1 over C are said to be *vertical isotopic* if there is a continuous family s_t, $t \in [0,1]$, of lifted 1-manifolds. Note that if there is a vertical isotopy between s_0 and s_1, then they are ambient isotopic in $H \times S^1$ and also, by the map Ψ^{-1}, in $\left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R} \subset \mathbb{R}^3$.

Since a lifted 1-manifold s over C lies in $H \times S^1$, each point $a := (x, \theta) \in s$ is a pair consisting of a point $x \in C \subset H$ and a unitary vector in the θ direction. By the definition of s, if $x \in C$ is a regular point of the immersed curve C then there is only one point a such that $q(a) = x$, while if it is a double point of C then there are two points a_1 and a_2 such that $q(a_1) = q(a_2) = x$. In the latter case, the two points $a_i := (x, \theta_i)$, $i = 1, 2$, correspond to two strands of C at the double point x and the unitary vectors θ_1 and θ_2 lie in mutually different directions. This smooth assignment of unitary vectors, tangential to H, to the immersed curve C is called a *vector field on* C. Each vector field on C corresponds to a lifted 1-manifold over C. In particular, the smooth 1-manifold $\Psi\left(\phi(C)\right)$ corresponds to a vector field which is everywhere in the same direction as the speed vectors of C.

Finally we define a relative winding number of a vector field for each edge of C. An *edge* e of the immersed curve C is the closure of a component of the set $C \setminus \{double\ points\}$. Suppose e has a parametrization with a parameter $t \in [0,1]$, and let V be a vector field on e. Then the parametrization of e induces a parametrization

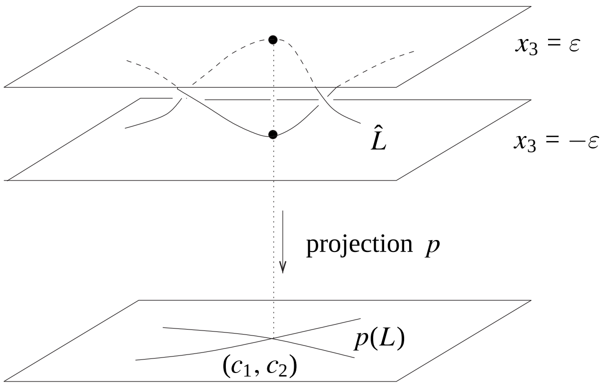

Fig. 2. The link in the thick plane.

$\big(x(t), \theta(t)\big) \in H \times S^1$ of V. Let $\alpha(t)$ be the speed vector of e at $x(t)$. Then the *relative winding number of* V *on* e is the rotation degree of $\theta(t) - \arg\big(\alpha(t)\big)$ when t runs from 0 to 1. Note that it is a real number and that the counterclockwise rotation is positive. In particular, if the vector field V is tangential to C at both endpoints of the edge e, the relative winding number of V on e is an integer.

Proof of Theorem 2.2. In this proof we always measure the arguments of tangent vectors on H in the counterclockwise direction relative to a fixed direction θ_0. We choose θ_0 so that the arguments of the tangent vectors to the oriented link projection $D_L := p(L)$ at the crossing points are not equal to θ_0. Now we deform L smoothly to $\hat{L}$ in $H \times \mathbb{R}$ so that $\hat{L}$ satisfies the following:

- $p(L) = p(\hat{L})$;
- $\hat{L} \subset \{(x_1, x_2, x_3) \in H \times \mathbb{R} \mid -\varepsilon \leq x_3 \leq \varepsilon\}$ for some sufficiently small $\varepsilon > 0$;
- At each double point (c_1, c_2) of $p(\hat{L})$, $\hat{L}$ passes through $(c_1, c_2, -\varepsilon)$ and (c_1, c_2, ε).

These conditions are presented pictorially in Fig. 2, and we will refer to the link $\hat{L}$ as the *link in the thick plane*. We define the embedding Ψ' of $H \times [-\varepsilon, \varepsilon]$ into the trivial circle bundle $H \times S^1$ by $(x_1, x_2, x_3) \in H \times [-\varepsilon, \varepsilon] \mapsto (x_1, x_2, \theta_0 + \pi - x_3) \in H \times S^1$ (here a minus sign is introduced in front of the x_3 term so that the embedding is oriented correctly). Note that the thick plane $H \times [-\varepsilon, \varepsilon]$ is naturally embedded in $\big(\mathbb{R}^2 \setminus \{(0,0)\}\big) \times \mathbb{R} \subset \mathbb{R}^3$ by the composite map $\Psi^{-1} \circ \Psi' \colon H \times [-\varepsilon, \varepsilon] \to \big(\mathbb{R}^2 \setminus \{(0,0)\}\big) \times \mathbb{R}$.

Let $s := \Psi'(\hat{L})$ and $s' := \Psi\big(\phi(D_L)\big)$. We will construct an isotopy s_t from $s_0 := s$ to s_1, where s_1 is a smooth 1-manifold in $H \times S^1$ which coincides with s' everywhere outside a particular open set $E \subset H \times S^1$.

By the construction of the link $\hat{L}$ in the thick plane, for every point (x_1, x_2) on D_L the unitary vector $v(x_1, x_2)$ at (x_1, x_2) satisfies $\theta_0 + \pi - \varepsilon \leq \arg\big(v(x_1, x_2)\big) \leq \theta_0 + \pi + \varepsilon$, and for each double point of D_L the unitary vectors satisfy $\arg\big(v(x_1, x_2)\big) = \theta_0 + \pi \pm \varepsilon$. By the definition of the embedding $\Psi' \colon H \times [-\varepsilon, \varepsilon] \to H \times S^1$, on each double point of D_L the overstrand of $\hat{L}$ corresponds to the argument $\theta_0 + \pi - \varepsilon$ and

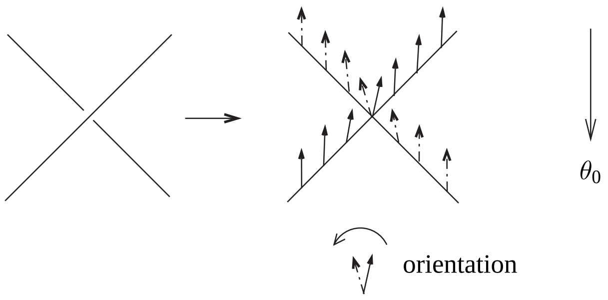

Fig. 3. The embedded link s_0 $\bigl(= \Psi'(\hat{L})\bigr)$.

the understrand of $\hat{L}$ corresponds to the argument $\theta_0 + \pi + \varepsilon$ (see Fig. 3).

Let $\{p_j\}$ be the set of double points of D_L, and $v_{j,1,0}$, $v_{j,2,0}$ the two unitary vectors on each double point p_j, i.e. so that $(p_j, v_{j,1,0})$, $(p_j, v_{j,2,0})$ are the points in s. We assume that $(p_j, v_{j,1,0})$ corresponds to the overstrand, and $(p_j, v_{j,2,0})$ corresponds to the understrand, of $\hat{L}$ at the double point. Let $U_{p_j} \subset H$ be a small neighbourhood of the double point p_j.

We construct the vertical isotopy s_t in two parts, first from s_0 to $s_{1/2}$, then from $s_{1/2}$ to s_1. Let $v_{j,1,t}$ and $v_{j,2,t}$ be the two unitary vectors based at the double point p_j, for $t \in [0, 1]$, which correspond to the points in s_t. In the first stage of vertical isotopy, s_t is the identity in $(H \setminus \sqcup_j U_{p_j}) \times S^1$, while within each U_{p_j} we assume the following.
(1) At $t = 1/2$, the unitary vector $v_{j,1,t}$ (resp. $v_{j,2,t}$) must lie tangential to the branch corresponding to the overstrand (resp. understrand) of $\hat{L}$ and in the same direction as the orientation assigned to D_L;
(2) The two unitary vectors $v_{j,1,t}$, $v_{j,2,t}$ at the double point p_j must not pass through each other, as t progresses from 0 to $1/2$;
(3) One of $v_{j,1,t}$ and $v_{j,2,t}$ does not pass through the argument θ_0. The other may pass through it at most once.
That such a vertical isotopy $s_0 \sim s_{1/2}$ exists is clear.

On each open edge $\overset{\circ}{e}_k$ select a disjoint union $\imath_k$ of $|w_k|$ open intervals, where w_k is the relative winding number of $s_{1/2}$ on the edge e_k. Define $E := (\sqcup_k \imath_k) \times S^1$. Then there exists a vertical isotopy $s_{1/2} \sim s_1$ such that s_1 coincides with s' outside E, and the relative winding number of s_1 on each component of $\imath_k$ is equal to $\mathrm{sign}(w_k)$. The existence is clear by definition of the relative winding number.

Now for the edge e_k, select a point on each component of $\imath_k$, and assign each of these points with the value $\mathrm{sign}(w_k)$. The set of these signed points constitutes a divisor, and we let C be the oriented divide obtained from D_L by attaching this divisor. Then it is clear that $\Psi\bigl(\phi(C)\bigr)$ is isotopic to s in $H \times S^1$. After the natural embedding $\Psi^{-1}\colon H \times S^1 \to \bigl(\mathbb{R}^2 \setminus \{(0,0)\}\bigr) \times \mathbb{R} \subset \mathbb{R}^3$, we can conclude that they are ambient isotopic in $\mathbb{R}^3$.

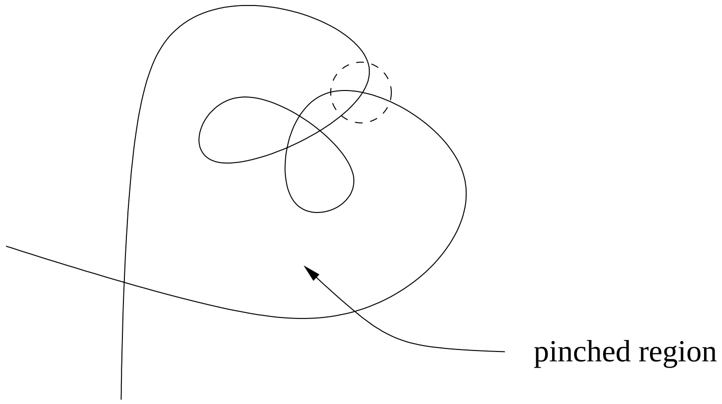

Fig. 4. An example of a pinched region (the dashed circle highlights the position of the 'pinch').

It remains to show that the divisor can be selected so that the number of points in the divisor is at most $3c(D_L) + r_0(D_L)$. The oriented projection D_L consists of a finite number of connected components. If a connected component Γ_i consists of a single embedded circle then the number of necessary points in the divisor for this component is 1, since the relative winding number for this component will always be ± 1. This coincides with the desired upperbound $3c(\Gamma_i) + r_0(\Gamma_i)$ since $c(\Gamma_i) = 0$ and $r_0(\Gamma_i) = 1$. Therefore the proof is completed by establishing the fact that the number of necessary points of the divisor for each connected, non-trivial component Γ_j is at most $3c(\Gamma_j)$.

We first prove the aforementioned assertion for a connected, non-trivial component which does not have pinched regions. A *pinched region* is defined to be a component of $H \setminus D_L$ such that the complement of its closure is not connected (see Fig. 4).

Let Γ be a connected, non-trivial, oriented link projection without pinched regions, and let $v(\Gamma)$ be the number of points at which the argument of the speed vector is θ_0. Since we can assume that θ_0 is in general position with respect to the speed vectors, it follows that $v(\Gamma)$ is finite. We then consider a triangulation on H with respect to Γ which we now describe.

Since there are no pinched regions, any region bounded by Γ is an n-gon with $n \geq 2$. For each n-gon with $n \geq 3$, we place a point in its interior and connect this point with each vertex of the n-gon by a simple edge. For each 2-gon we connect the two vertices by a simple edge in the 2-gon. Let Γ' be the subset of Γ obtained from it by deleting the interiors of the edges of all 2-gons. Actually, to ensure that we really do get a triangulation, we must make the additional assumption that none of the 2-gons of Γ share a common edge, since this would give a 2-gon in our supposed 'triangulation'. This pathological case occurs if and only if one of the components of Γ is an embedded circle intersecting exactly twice with one edge of another component.

We ignore such embedded circles and make appropriate adjustments later. With this extra assumption, the union of Γ' and the simple edges described above constitutes a triangulation T in H. Now each vertex of T is connected by at least three edges, so by using Fary's theorem [9] we can modify the triangulation so that every edge is straight. Then, by rotation if necessary, we can assume that no edge of the triangulation lies in the θ_0 direction.

Now the regular link projection Γ can be drawn as a polygon on the graph of T, except for the edges of 2-gons of Γ. Each 2-gon Δ also can be drawn along the simple edge e, which connects the two vertices b_1, b_2 of Δ, so that the edges of Δ are parallel and close to e in $\Delta \setminus (W_{b_1} \cup W_{b_2})$, where W_{b_i} is a small neighbourhood of b_i. Then we obtain a regular link projection $\tilde{\Gamma}$ from the above polygon along T by rounding the corners (note that this can be done without introducing any inflexion points). The strands of $\tilde{\Gamma}$ are straight except possibly in small neighbourhoods $W_{\tilde{p}_j}$ of double points $\tilde{p}_j$ of $\tilde{\Gamma}$. In each $W_{\tilde{p}_j}$ we may have at most two points at which the arguments of the speed vectors are θ_0 or $\theta_0 + \pi$. So if $v(\tilde{\Gamma}) > c(\tilde{\Gamma})$, then a π-rotation of $\tilde{\Gamma}$ yields a new position in which $v(\tilde{\Gamma}) \leq c(\tilde{\Gamma})$ holds.

If we relax the additional assumption concerning an embedded circle intersecting a single edge at exactly two points, the result $v(\tilde{\Gamma}) \leq c(\tilde{\Gamma})$ also follows from the fact that any such embedded circle can be drawn with only one tangent in the θ_0 direction (though we must ensure that the circle is sufficiently far away from any double points).

Now we assume $\tilde{\Gamma}$ lies in such a good position and construct a required divisor for $\tilde{\Gamma}$. Recall that $U_{\tilde{p}_j}$ is a small neighbourhood of the double point $\tilde{p}_j$ of $\tilde{\Gamma}$. We can assume $U_{\tilde{p}_j}$ is sufficiently small such that it is included in $W_{\tilde{p}_j}$, and we also assume that in $U_{\tilde{p}_j}$ the immersed curve $\tilde{\Gamma}$ consists of two crossed straight lines. So now $\tilde{\Gamma}$ is straight outside the annuli $\sqcup_j (W_{\tilde{p}_j} \setminus U_{\tilde{p}_j})$ and, in the annuli, all points on $\tilde{\Gamma}$ at which the speed vectors of $\tilde{\Gamma}$ lie in the θ_0 direction are included. We denote these points by v_i.

First perform the vertical isotopy $s_0 \sim s_{1/2}$. The family s_t, $t \in [0, 1/2]$, of vector fields is fixed on $\tilde{\Gamma} \setminus (\sqcup_j U_{\tilde{p}_j})$ during the isotopy and, by assumption (3), in each $U_{\tilde{p}_j}$ the vector field $s_{1/2}$ does not lie in the θ_0 direction on one of the crossed straight line, while it may lie in that direction at most at two points on the other line. We denote these two points by q_j and q'_j if they exist. Let $U'_{\tilde{p}_j} \subset U_{\tilde{p}_j}$ be a small neighbourhood of the boundary of $U_{\tilde{p}_j}$ such that q_j, $q'_j \in U_{\tilde{p}_j} \setminus U'_{\tilde{p}_j}$.

Next we perform the following vertical isotopy $s_{1/2} \sim s_{3/4}$. The family s_t, $t \in [1/2, 3/4]$, of vector fields is fixed on $\tilde{\Gamma} \setminus (\sqcup_j U_{\tilde{p}_j})$ and all q_i and q'_j during the isotopy. In each $U_{\tilde{p}_j}$ the isotopy is performed, without passing through the θ_0 direction at any point, and such that, in each $U_{\tilde{p}_j} \setminus U'_{\tilde{p}_j}$, $s_{3/4}$ lies in the same direction as the orientation of $\tilde{\Gamma}$ except in small neighbourhoods of q_j and q'_j included in $U_{\tilde{p}_j} \setminus U'_{\tilde{p}_j}$. We remark that $s_{3/4}$ does not lie in the θ_0 direction in each $U'_{\tilde{p}_j}$ and, on its inside boundary, $s_{3/4}$ lies in the same direction as the orientation of $\tilde{\Gamma}$, while on its outside boundary it lies in the $\theta_0 + \pi$ direction.

Finally we perform the vertical isotopy $s_{3/4} \sim s_1$ described next. Put $Z := \tilde{\Gamma} \setminus \big((\sqcup_j(U_{\tilde{p}_j} \setminus U'_{\tilde{p}_j})) \cup (\sqcup_i v_i)\big)$. For each connected component Z_k of Z, $s_{3/4}$ lies in the $\theta_0 + \pi$ direction on $Z_k \setminus (\sqcup_j U'_{\tilde{p}_j})$ and, in $Z_k \cap (\sqcup_j U'_{\tilde{p}_j})$, it lies in the position remarked above. Moreover the speed vectors of $\tilde{\Gamma}$ on Z_k never lie in the θ_0 direction. Therefore there exists a vertical isotopy $s_{3/4} \sim s_1$ that is fixed on $\tilde{\Gamma} \cap (\sqcup_j U_{\tilde{p}_j} \setminus U'_{\tilde{p}_j})$ and all v_i during the isotopy, and such that s_1 lies in the same direction as the orientation of $\tilde{\Gamma}$ except for small neighbourhoods of q_j, q'_j and v_i.

In each of the small neighbourhoods of the q_j, q'_j and v_i, the vector field s_1 rotates once. This corresponds to a twist in $H \times S^1$ and hence to a necessary signed point of the divisor. Since the pair q_j, q'_j appears in each $U_{\tilde{p}_j}$ at most once, and $v(\tilde{\Gamma})$ is the number of v_i's, the necessary number of signed points of the divisor is at most $v(\tilde{\Gamma}) + 2c(\tilde{\Gamma})$. Thus, because we already have the inequality $v(\tilde{\Gamma}) \leq c(\tilde{\Gamma})$, the number is bounded above by $3c(\tilde{\Gamma})$.

The divisor on $\tilde{\Gamma}$ induces a divisor on Γ by an isotopy deformation from $\tilde{\Gamma}$ to Γ. Hence the result is also true for Γ, i.e. the necessary number of signed points on Γ is at most $3c(\Gamma)$, which proves our assertion for link projections without pinched regions.

Finally we deal with the case of a link projection Γ with pinched regions. Let $\{r_k\}$ be the set of double points which represent the pinches, and let R_k be a small neighbourhood of r_k for each k. Then in each R_k perform a smoothing which agrees with the assigned orientation. This yields a disjoint union $\Lambda = (\sqcup_i T_i) \sqcup (\sqcup_j N_j)$ of connected oriented link projections without pinched regions, where the T_i are trivial components and the N_j are connected non-trivial components. We note that

$$\#\{T_i\} + \#\{N_j\} = \#\{r_k\} + 1, \tag{2.2}$$

where $\#S$ is the number of elements of the set S. By an isotopy deformation of Γ, we assume that, for each R_k, $\Lambda \cap R_k$ consists of two parallel straight lines with the same orientation and they are sufficiently closed. Since none of the T_i and N_j have any pinched regions, there is a vertical isotopy $s'_0 \sim s'_1$ for Λ such that the vector field s'_1 is in the same direction as the orientation of Λ except for a finite number of small neighbourhoods. In such a small neighbourhood, the vector field s'_1 has a twist in $H \times S^1$. By the conclusion arrived at in the 'no pinched region' case, on each T_i there is only one such a twist, while the number of twists on each N_j is at most $3c(N_j)$.

Now we perform the following vertical isotopy $s_0 \sim s_1$ for the original link projection Γ with pinched regions. In $\Gamma \setminus (\sqcup_k R_k)$, which coincides with $\Lambda \setminus (\sqcup_k R_k)$, the isotopy is completely the same as $s'_0 \sim s'_1$ applied to Λ above. We remark that since $\Lambda \cap R_k$ consists of two straight lines, the family s_t, $t \in [0, 3/4]$, is fixed on the four points of the boundary of $\Lambda \cap R_k$ and, during $s_{3/4} \sim s_1$, they isotope continuously and in unison. In each R_k there are two cases depending on the orientations of s_0 and Γ at the double point r_k. Let a and b be the two unitary vectors of s_0 at r_k such that $\theta_0 + \pi - \varepsilon = \theta_b < \theta_a = \theta_0 + \pi + \varepsilon$, where θ_a and θ_b are the arguments of a and b

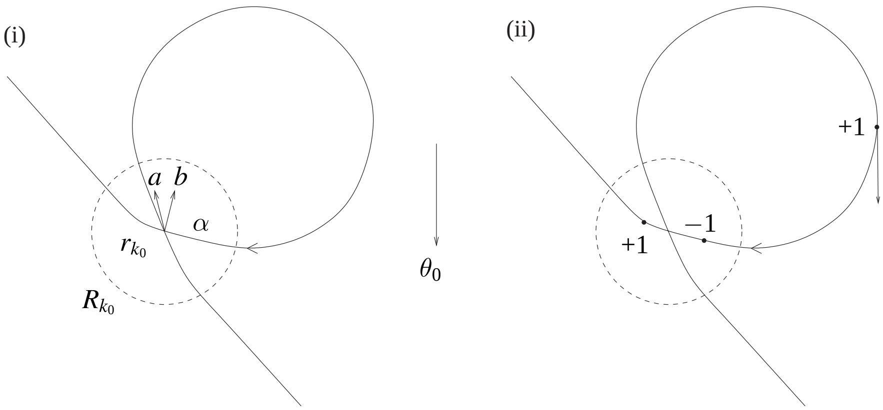

Fig. 5. Possible configuration for the divisor on R_{k_0}.

respectively. Let α and β be the crossed curves of Γ in R_k such that the arguments θ_α and θ_β of their speed vectors at r_k satisfy $\theta_0 < \theta_\beta < \theta_\alpha < \theta_0 + 2\pi$. Then the two cases are

(1) a corresponds to the curve α, or

(2) b corresponds to the curve α.

In case (1) the vertical isotopy $s_0 \sim s_{1/2}$ in R_k can be defined so that the unitary vectors do not pass through the θ_0 direction during the isotopy. In case (2) we perform a vertical isotopy such that a passes through the θ_0 direction once during the isotopy. The subsequent isotopy $s_{1/2} \sim s_1$ is defined to be same as that performed in small neighbourhoods of the double points of Γ which are not in $\{r_k\}$.

Now we count the number of twists of s_1 in $H \times S^1$. We have seen that on each T_i there is only one such a twist while the number of twists on each N_j are at most $3c(N_j)$. In each R_k, in case (1) there is no twist, while in case (2) there are two twists. Therefore there are in total $\#\{T_i\} + \sum_j 3c(N_j) + 2\hat{r}$ twists, where $\hat{r}$ is the number of case (2) double points in $\{r_k\}$.

If $\{N_j\}$ is not empty then, by (2.2), we have $\#\{T_i\} \leq \#\{r_k\}$, and so the number of twists is bounded above by $\#\{T_i\}+\sum_j 3c(N_j)+2\hat{r} \leq \sum_j 3c(N_j)+3\#\{r_k\} \leq 3c(\Gamma)$ as required. If $\{N_j\}$ is empty and at least one of the r_k is a case (1) double point, then $\#\{T_i\} = \#\{r_k\}+1$ and $\hat{r} < \#\{r_k\}$. So, in this case, the number of twists is also bounded above by $\#\{T_i\} + 2\hat{r} \leq 3\#\{r_k\} = 3c(\Gamma)$. If $\{N_j\}$ is empty and every r_k is a case (2) double point then $\#\{T_i\} = \#\{r_k\} + 1$ and $\hat{r} = \#\{r_k\}$, so the number of twists becomes $\#\{T_i\}+2\hat{r} = 3\#\{r_k\}+1 = 3c(\Gamma)+1$. The unwanted 1 can be eliminated from this bound as follows. First observe that in such a case there exists at least one innermost circle T_{i_0} in $\sqcup_i T_i$, and there exists only one double point r_{k_0} whose neighbourhood R_{k_0} connects T_{i_0} to another component of Λ. Since every r_k is a case (2) double point, r_{k_0} is as shown in Fig. 5 (i). Hence the signed points of the divisor corresponding to

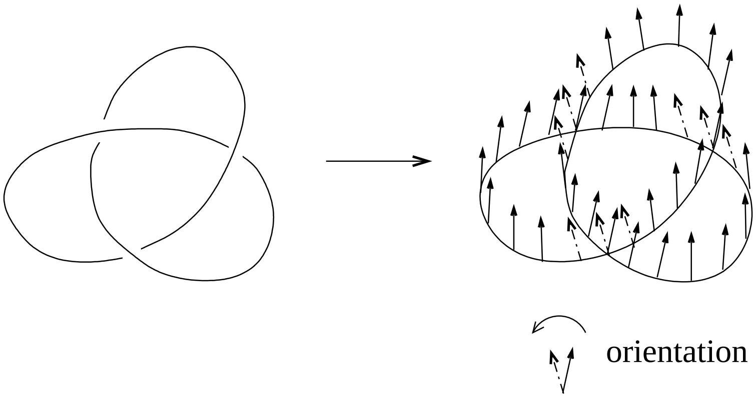

Fig. 6. A link diagram of the $(2, 3)$ torus knot and the embedded link in the trivial circle bundle.

the twists are +1 and -1 as shown in Fig. 5 (ii). However there is also a signed point on the circle T_{i_0} corresponding to the point at which the speed vector is in the θ_0 direction, and this sign is +1. This +1 and the -1 in R_{k_0} can be cancelled by a vertical isotopy. Thus we can eliminate the extra 1 from the upperbound. $\square$

REMARK 2.3. The inequality $v(\tilde{C}) \leq c(\tilde{C})$ in the above proof can also be shown using a result of Andreev [7], which states that there exists a unique circle packing which is dual to the above triangulation. The circle packing algorithm, used to complete this method of proof, was constructed for regular link projections by K. Stephenson, and is used for drawing knots in Knotscape. See “Help” in Knotscape [19].

We now present an example of the construction of an oriented divide from a link diagram. Let L be the $(2, 3)$ torus knot as shown on the left side of Fig. 6. The link $\hat{L}$ in the thick plane is realized in the trivial circle bundle $H \times S^1$ as shown on the right.

Next we assign an orientation to the regular link projection D_L of L and apply smooth rotations of the unitary vectors at each double point so that the unitary vectors coincide with the speed vectors of D_L. Then for each arc of $D_L \backslash \{double\ points\}$ we count the relative winding number and place this number at a point on the arc (Fig. 7).

The set consisting of two points, each with the sign +1, is a divisor of the regular knot projection D_L of the $(2, 3)$ torus knot. By attaching a positive loop to each point we obtain an oriented divide C whose link $\phi(C)$ is the $(2, 3)$ torus knot. This final step is shown in Fig. 8.

Before continuing the example we will briefly discuss an important fact concerning points of a divisor which are attached to “outside” arcs of an oriented divide.

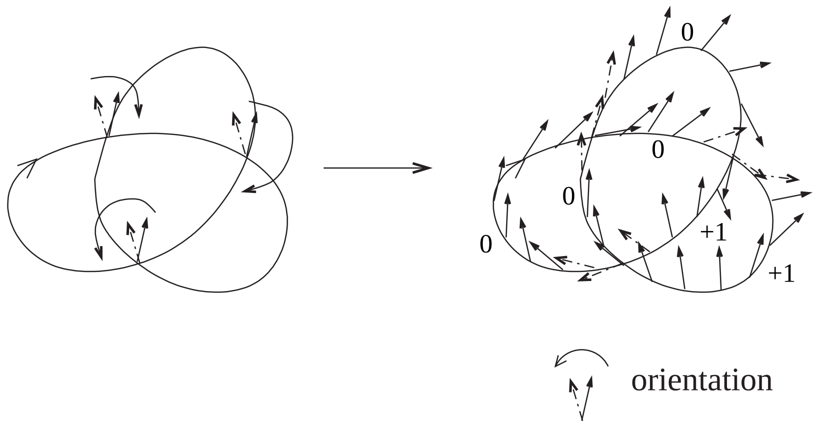

Fig. 7. Rotations of the unitary vectors and the relative winding numbers.

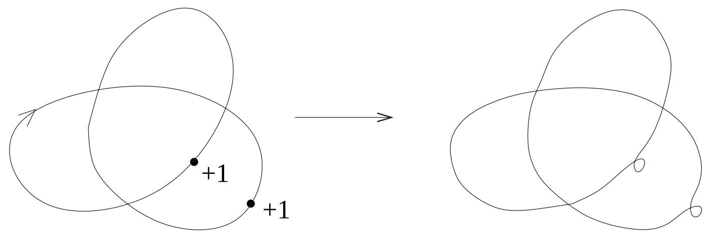

Fig. 8. A divisor and the corresponding oriented divide.

DEFINITION 2.4. Let C be an oriented divide and let $\{a_i\}$ be the set of arcs of $C \setminus \{double\ points\}$. If we can connect the arc a_i and the boundary $\partial\overline{H}$ by a path in $H \setminus C$, we say a_i is an *outside arc of* C. Otherwise it is an *inside arc*.

Lemma 2.5. *Let C be an oriented divide, and construct from it a new oriented divide C' by adding a loop to one of the outside arcs. Then the link $\phi(C)$ is isotopic to the link $\phi(C')$.*

This means that we can ignore any element of a divisor which appears on an outside arc of a regular link projection D_L of a link L.

Proof. The loop on an outside arc of C' constitutes a twist in $\Psi^{-1}(H \times S^1) \subset \mathbb{R}^3$ around the axis $\{(0,0)\} \times \mathbb{R}$, and the corresponding strand of $\phi(C')$ does not wind around any other strand. So we can remove the twist and isotope the link to $\phi(C)$. $\square$

Now we continue the example. Let C be the oriented divide with the divisor as

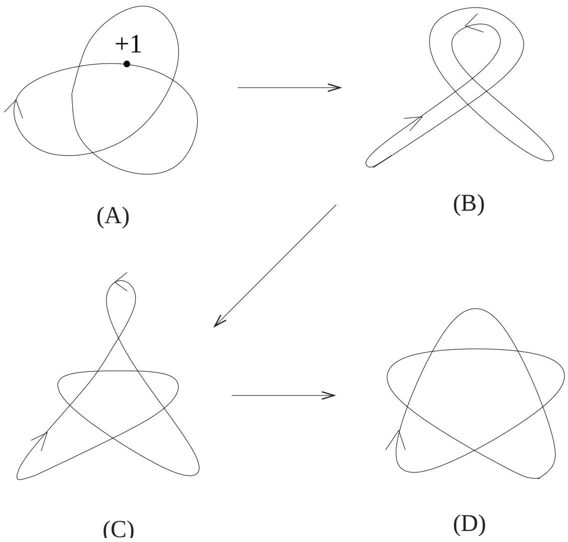

Fig. 9. An isotopic deformation of the oriented divide of the $(2, 3)$ torus knot.

shown in Fig. 9 (A). By Lemma 2.5 the link of the oriented divide C with this divisor is also the $(2, 3)$ torus knot. By attaching a left-handed loop at the point with the sign $+1$, the figure is deformed to (B) in the figure. This is just the oriented divide derived from the doubled curve of the divide of the $(2, 3)$ torus knot (see Section 4 below). Since the orientations of the loop and the outside arc parallel to the loop are opposite, the loop can be passed through the outside arc. Also the double point of the loop passes through the outside arc since they do not intersect in the trivial circle bundle. This allows our next deformation from (B) to (C). By Lemma 2.5 the loop on the outside arc of (C) can be ignored and is hence equivalent to (D).

REMARK 2.6. Instead of the link of an oriented divide C, we can construct another link by using the co-orientation on C, that is, the set

$$\{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 \mid (x_1, x_2) \in C, (u_1, u_2) \in \overrightarrow{T'}_{(x_1,x_2)}(C), x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1\} \subset S^3$$

is a link in S^3, where $\overrightarrow{T'}_{(x_1,x_2)}(C)$ is the set of normal vectors in the left direction with respect to the orientation on C. This is same as the construction of Legendrian knots via fronts of generically immersed curves. In the context of Legendrian knots it is already known that any oriented knot has such a corresponding immersed circle (see [8]).

3. Proofs of Theorems 1.3 and 1.4

In this section we will prove Theorems 1.3 and 1.4. We first define the rotation map $R_\theta\colon H \times S^1 \to H \times S^1$ of the trivial circle bundle $H \times S^1$ by $(x, \varphi) \mapsto (x, \varphi + \theta)$, for $\theta \in \mathbb{R}$. Then R_θ extends to a rotation map $\tilde{R}_\theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ by the embedding $\Psi^{-1}(H \times S^1) \subset \mathbb{R}^3$. If two links L and L' in $\mathbb{R}^3$ satisfy $\tilde{R}_\pi(L) = L'$ we say they *lie in mutually symmetric positions.*

Now let C be an oriented divide, defined by immersed circles each with a chosen orientation, and let C^{-1} be the oriented divide defined by the same immersion but with opposite orientations assigned to each component.

Proposition 3.1. *Let C be an oriented divide and let C^{-1} be the oppositely oriented divide of C. Then the links $\phi(C)$ and $\phi(C^{-1})$ lie in mutually symmetric positions. In particular, they are isotopic.*

Proof. Since $\phi(C^{-1}) = \tilde{R}_\pi\big(\phi(C)\big)$, $\phi(C)$ and $\phi(C^{-1})$ are isotopic and lie in mutually symmetric positions. □

Proof of Theorem 1.3. Let C be an oriented divide obtained from an oriented, connected, regular link projection D_L, of the link L, with a divisor such that the link $\phi(C)$ is isotopic to L. The existence of such a divisor is guaranteed by Theorem 1.2. Let P be the non-oriented half plane divide defined by the same immersed circles as the oriented divide C. Since the union of the immersed circles is connected, the link $\phi(P)$ of P is a fibred link in the compactified manifold $S^3 = \mathbb{R}^3 \cup \{\infty\}$ due to A'Campo's theorem [4], and furthermore, its monodromy is the product of right Dehn twists and the number of these right Dehn twists is equal to the first Betti number of the fibre. Then the assertion follows from these facts and Proposition 3.1. □

REMARK 3.2. The monodromy of the fibration in Theorem 1.3 can be described using a Dynkin diagram [1], [2], [11], [12], [13]. For this reason we emphasize that the monodromy has more explicit properties than stated in the theorem.

The oriented divide can be deformed while preserving the isotopy type of its link, as shown, for example, in Fig. 9. However, the fibred links in the link exterior $E(L)$ depend on the deformed oriented divides. So we can construct infinitely many distinct fibrations by deforming the oriented divide. This is the point of Theorem 1.4.

Proof of Theorem 1.4. By the definition of $g_{\min}(L)$ there is an oriented divide C obtained from a regular link projection D_L by attaching a divisor such that the link $\phi(C)$ is isotopic to L and $c(D_L) + d(D_L) - r(L) + 1 = g_{\min}(L)$. Now we regard the immersed circles of C in H as the divide P with both orientations. Then, after the compactification $S^3 = \mathbb{R}^3 \cup \{\infty\}$, the genus of the fibre of the fibration in $S^3 \setminus \phi(P)$ is

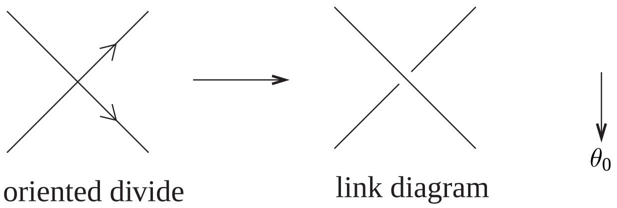

Fig. 10. “Over and under” rule.

$c(D_L) + d(D_L) - r(L) + 1 = g_{\min}(L)$. Since $S^3 \setminus \phi(P) = S^3 \setminus \left(\phi(C) \sqcup \phi(C^{-1})\right) = E(L) \setminus \phi(C^{-1})$, the link $L' = \phi(C^{-1})$ is a fibred link in $E(L)$ whose fibre is a genus $g_{\min}(L)$ surface. Next we attach n loops on an outside arc of the oriented divide C and denote the new oriented divide by C_n. The link $\phi(C_n)$ of C_n and the link $\phi(C)$ of C are isotopic in $\mathbb{R}^3$, by Lemma 2.5, and also in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Hence the link exterior $S^3 \setminus \phi(C_n)$ of the link $\phi(C_n)$ is homeomorphic to $E(L)$. We regard the immersed circles of C_n in H as the divide P_n with both orientations. Then the genus of the fibre of the fibration in $S^3 \setminus \phi(P_n)$ is $g_{\min}(L) + n$. Since $S^3 \setminus \phi(P_n) = S^3 \setminus \left(\phi(C_n) \sqcup \phi(C_n^{-1})\right) = E(L) \setminus \phi(C_n^{-1})$, the link $L'_n = \phi(C_n^{-1})$ is a fibred link in $E(L)$ whose fibre is a genus $g_{\min}(L) + n$ surface. $\square$

4. A link diagram of an oriented divide

Let C be an oriented divide on the half plane H and let $\phi(C)$ be the link of C in $\mathbb{R}^3$. In this section we construct a method for obtaining a link diagram of $\phi(C)$ from the oriented divide C. As seen in Section 2, the link $\phi(C)$ is naturally embedded in the trivial circle bundle $H \times S^1$ on H.

Now we regard the trivial circle bundle as the direct product of points in the half plane H with the unitary vectors based at each point in H. We assign a coordinate to the unitary vectors in S^1 which measures argument in the counterclockwise direction. For a technical reason we fix an argument θ_0 on S^1 such that the oriented divide C satisfies the following:

- At each double point of C the arguments of the unitary tangent vectors of the two branches are neither θ_0 nor $\theta_0 + \pi$;
- Every point on C whose argument of the unitary tangent vector is θ_0 or $\theta_0 + \pi$ is not an inflexion point of C.

DEFINITION 4.1 (“Over and under” rule). Let a, b be the branches of a double point of C and let $\theta_a, \theta_b \in (\theta_0, \theta_0 + 2\pi)$ be the arguments of their speed vectors respectively. If $\theta_a < \theta_b$ we define the branch a as “over” and b as “under”. Otherwise we define the branch a as “under” and b as “over” (see Fig. 10).

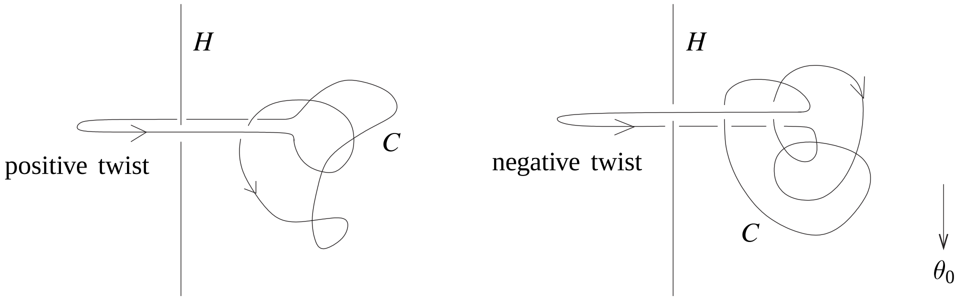

Fig. 11. Winding rule.

DEFINITION 4.2 (Winding rule). Let $r: H \times S^1 \to S^1$ be the canonical projection and let p be a point on an inside arc of C at which the argument of the unitary speed vector of C is θ_0. Then, as shown in Fig. 11 (where H is the half-plane),

- if the orientation of $r\big(\phi(C)\big)$ at p is positive, replace the small segment of the arc on which p lies by a positive twist;
- if the orientation of $r\big(\phi(C)\big)$ at p is negative, replace the small segment of the arc on which p lies by a negative twist.

Note that the orientation of the circle of the trivial circle bundle is in the direction from the "over" position to the "under" position of the plane of the regular link projection by the definition of the embedding of the thick plane into the trivial circle bundle.

Theorem 4.3. *Let C be an oriented divide. Then a link diagram of $\phi(C)$ is obtained by modifying C according to the "over and under" rule and the winding rule.*

Proof. Let p be a point on an arc a_i of $C \setminus \{double\ points\}$ at which the argument of the unitary speed vector of C is θ_0. We deform the arc a_i smoothly as shown in Fig. 12.

It will be noted that the deformed arc does not have a point at which the argument of the speed vector is θ_0 except for a single point on the small left or right loop. This loop corresponds to a positive or negative twist around the circle of the trivial circle bundle. Let C' be the deformed oriented divide with a divisor consisting of these small loops. The argument of the speed vector at all points on the oriented divide C' is contained in the interval $[\theta_0 + R, \theta_0 + 2\pi - R]$ for some $0 < R < \pi$ except for small neighborhoods of the loops. Now we deform the arguments of the speed vectors corresponding to the link $\phi(C')$ by the map $\Theta\colon [\theta_0 + R, \theta_0 + 2\pi - R] \to [\theta_0 + \pi - \varepsilon, \theta_0 + \pi + \varepsilon]$, defined by $\Theta\colon \theta \mapsto \big(\varepsilon/(\pi - R)\big)\big(\theta - (\theta_0 + \pi)\big) + (\theta_0 + \pi)$, while preserving the twists of the loops. Then we obtain a deformed link in the thick plane which has several loops.

By temporarily ignoring the loops, it is clear that the "over and under" rule cor-

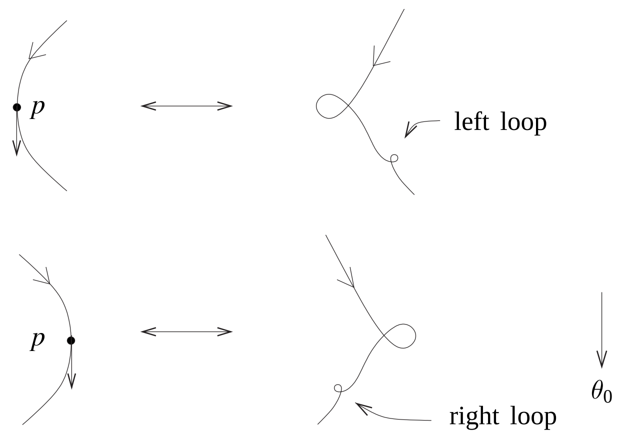

Fig. 12. Deformations of an arc whose speed vectors passing through the argument θ_0.

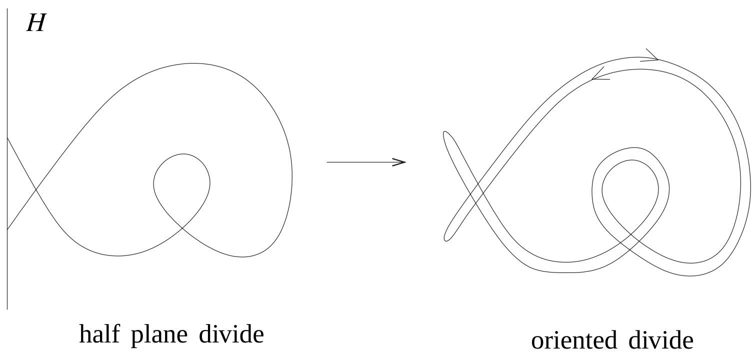

Fig. 13. Doubling of a divide.

rectly decodes the crossing information at each double point of the projection. The required link diagram materializes by noting that, since each of the small loops of the deformed oriented divide represents a twist around the circle of the trivial circle bundle, the information represented by these loops is correctly decoded by the winding rule. □

For any link $\phi(P)$ of a half plane divide P we can obtain an oriented divide C whose link $\phi(C)$ is isotopic to $\phi(P)$ by using the "doubling" method shown in Fig. 13.

From now on we fix the orientation of the oriented divide C induced by doubling a half plane divide P by always following the left hand side of the doubled curve. We assume that for any double point of the divide P the arguments of the tangents to

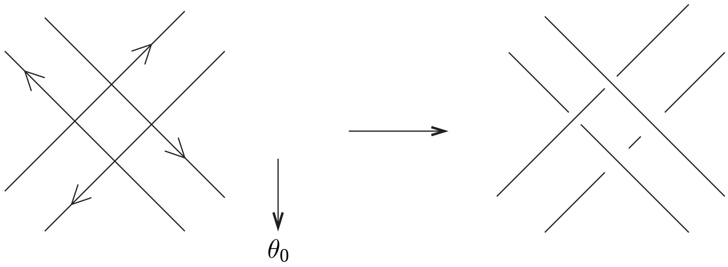

Fig. 14. “Over and under” rule for divides.

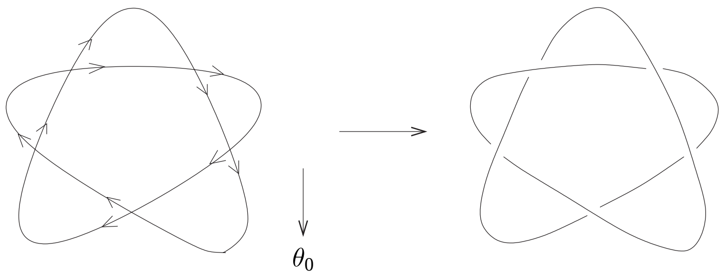

Fig. 15. A link diagram constructed from an oriented divide representation of the (2,3) torus knot.

the two branches are $\theta_0 + \pi/4$ and $\theta_0 + (3/4)\pi$ for a fixed argument θ_0 in S^1.

DEFINITION 4.4 (“Over and under” rule for divides). For each “#” crossing of the oriented divide C obtained by doubling a half plane divide P, we define the over and under crossings of the strands as shown in Fig. 14.

Since the “over and under” rule for divides is equivalent to four applications of the “over and under” rule for oriented divides, we can easily draw a link diagram of the link of a divide P using Theorem 4.3:

Corollary 4.5. *Let P be a half plane divide and let C be the oriented divide obtained by doubling P. Then a link diagram of the link $\phi(P)$ is constructed by modifying C by the “over and under” rule for divides and the winding rule.*

Now we show a few examples of link diagrams of the links of oriented and non-oriented divides. Let C be the oriented divide shown on the left in Fig. 15. The “over and under” rule applied to C produces the knot shown on the right. Note that the divide C does not contain an arc which requires application of the winding rule.

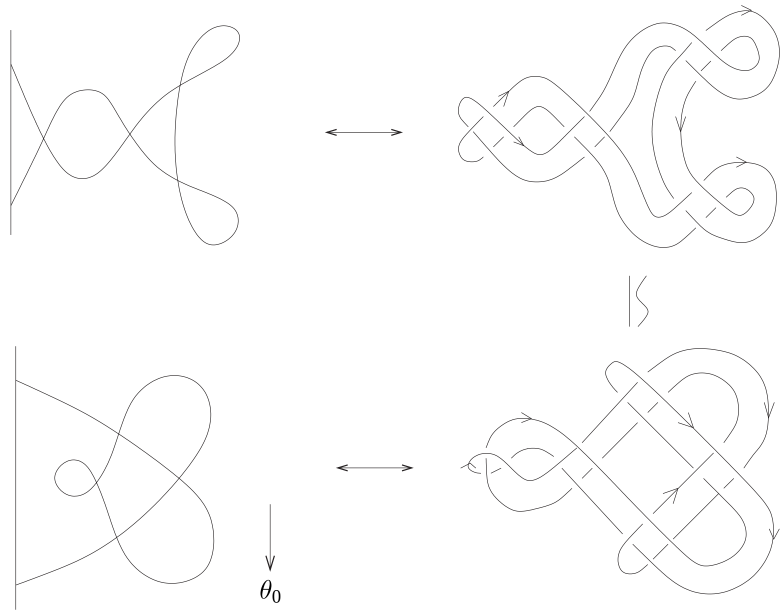

Fig. 16. Two divide representations of the 10_{139} knot and their link diagrams.

The result is clearly the $(2,3)$ torus knot. We have already seen this fact in Section 2 (cf. Fig. 9 (D))

The knot 10_{139} in the knot table in [18] has two different divide representations, which were found by A'Campo. Link diagrams of the divides are as shown in Fig. 16. The two links are isotopic, but the divides are distinct, i.e. they are not transversely isotopic as divides. Recently all divides up to 4 crossings have been listed by the authors in [10]. In this list only two divide knots were found to have more than one divide representation (up to divide equivalence).

To conclude the present paper we consider the class of divide called the *slalom curves of rooted planar trees*. A rooted planar tree is an embedded tree B in the half plane $\bar{H}$ such that $B \cap \partial \bar{H}$ consists of one point, which is a terminal vertex of B. For a rooted planar tree B there exists a divide $P_B \subset \bar{H}$ with the following properties:

- The double points of P_B lie in the interior of the edges of B, such that the local branches are transversal to the edge of B.
- Each connected component bounded by P_B contains exactly one vertex of B.
- The only intersection points of P_B with B are the double points of P_B.

The original slalom curve was defined not on $\bar{H}$ but on the unit disk (see [5]). The link $\phi(P_B)$ of P_B is a knot because P_B consists of just one immersed interval.

Theorem 4.6. *Let P_B be a slalom curve of a rooted planar tree B. By an isotopic deformation of B we assume that the tree B grows in the positive direction of*

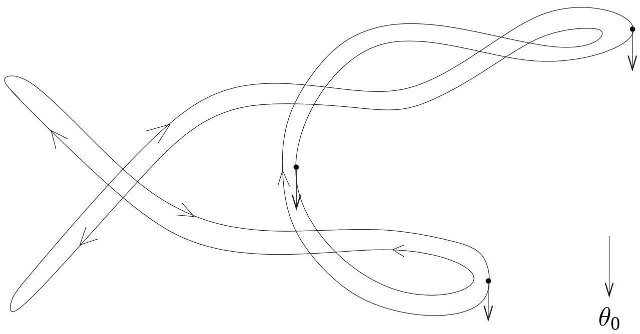

Fig. 17. A doubled curve of a slalom divide.

the x_1 coordinate of H. Then we can obtain a link diagram of the knot of P_B by applying only the "over and under" rule for divides to the oriented divide obtained by doubling P_B.

Proof. We assume the fixed argument θ_0 on S^1 is pointing down, i.e. in the negative direction of the x_2 coordinate of H. Then we can deform the slalom curve P_B into a good position, as shown in Fig. 17, so that all arcs which would be modified by the winding rule are outside arcs, and hence their windings can be ignored by Lemma 2.5. $\square$

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