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1-GENUS 1-BRIDGE SPLITTINGS FOR KNOTS

Dedicated to Professor Yukio Matsumoto on his 60th birthday

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1. Introduction

A knot K in the 3-sphere S^3 is called a 1-genus 1-bridge knot if a torus H splits S^3 into two solid tori each of which K intersects in a trivial arc. Such splittings for 2-bridge knots are studied in [25]. We show that, if K is neither a 2-bridge knot nor a satellite knot, and if K has two non-isotopic 1-genus 1-bridge splitting tori H_1, H_2, then we can move them to be interesect each other in two essential circles, and either (1) K is obtained from a component of a 2-bridge link by twisting along the other component, or (2) we can move H_1 and H_2 further to meet in a torus with two holes each circle of which c_i bounds a disc in $H_i - K$ and a twice punctured disc in $H_j - K$ for $(i, j) = (1, 2)$ and $(2, 1)$. We consider also knots in lens spaces.

We recall the precise definition of g-genus n-bridge splittings of links. Let W be a 3-manifold with non-empty boundary ∂W, and $T = \{t_1, \ldots, t_n\}$ a set of disjoint arcs properly embedded in W, that is, $t_i \cap W = \partial t_i$ for every $1 \leq i \leq n$. We say T is *trivial* in (W, T) if there is a set $\{D_1, \ldots, D_n\}$ of disjoint discs embedded in W so that $D_i \cap (\cup T) = \partial D_i \cap t_i = t_i$ and so that $D_i \cap \partial W$ is the arc $\mathrm{cl}(\partial D_i - t_i)$. We call D_i a *cancelling disc* of t_i. When W is a ball and T is trivial, the pair (W, T) is called a *trivial n-string tangle*.

Let M be a closed orientable 3-manifold, and L a link in M. Let H be a genus g Heegaard splitting surface of M, that is, H divides M into two handlebodies W_1 and W_2 of genus g. Suppose that H is transverse to L. Then we say H is a *g-genus n-bridge splitting* of (M, L) if L intersects W_i in a trivial set of n arcs for $i = 1$ and 2. A link L is called a g-genus n-bridge link if it admits a g-genus n-bridge splitting. A link in S^3 is simply called an *n-bridge link* if it has a 0-genus n-bridge splitting. For studies on positive genus n-bridge splittings, see [6], [22], [25], [17], [18], [19], [20], [15], and [24].

H. Rubinstein and M. Scharlemann showed in [31] that two strongly irreducible Heegaard splitting surfaces are isotoped to intersect each other in essential loops. They applied this result to isotope two Heegaard splitting surfaces of genus two to intersect each other beautifully ([32]). In [25] T. Kobayashi and O. Saeki generalized the

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above result, and made two g-genus n-bridge splitting surfaces to intersect each other in "K-essential" loops as below.

We need to recall some terminologies. In general, let X be a 3-manifold, and T a properly embedded 1-manifold in X, that is, $T \cap \partial X = \partial T$. Let F be a 2-manifold properly embedded in X. Suppose that F is transverse to T. In particular $\partial F \cap T = \emptyset$. We say F is *T-compressible* if there is a disc D embedded in X such that D is disjoint from T, that $D \cap F = \partial D$ and that ∂D does not bound a disc disjoint from T on F. Such D is called a *T-compressing disc* of F. We call F is *T-incompressible* if it is not T-compressible. Note that these definitions are different from those in [25].

Let H be a 2-manifold properly embedded in X so that H is transverse to T. The 2-manifold H is said to be *meridionally compressible* in (X, T) if there is a disc D embedded in X such that $\operatorname{int} D$ intersects T transversely in a single point, that $D \cap H = D\partial \cap (H - T) = \partial D$ and that ∂D in H does not bound a disc whose interior intersects T transversely in a single point. Such D is called a *meridionally compressing disc* of H. We call H is *meridionally incompressible* if it is not meridionally compressible. We define a T-compressible 2-submanifold and a meridionally compressible 2-submanifold of ∂X similarly.

Assume that either P is a 2-manifold properly embedded in X such that P is transverse to T, or P is a 2-submanifold of ∂X with $\partial P \cap T = \emptyset$. A simple loop l on P is said to be *T-essential* if it is disjoint from T and if it does not bound a disc which intersects T transversely in zero or one point.

Let M be a closed orientable 3-manifold, and L a link in M. Let H be a g-genus n-bridge splitting surface of (M, L), and W_1, W_2 the handlebodies obtained by cutting M along H. We say that H is *weakly K-reducible* if W_1 and W_2 contain K-compressing or meridionally compressing discs D_1 and D_2 of H respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. We call H *strongly K-irreducible* if it is not weakly K-reducible. Note that these definitions coincide with those in [25] but do not coincide with those in [17].

Theorem 1.1 (Proposition 6.19 in [25]). *Suppose that (M, L) is not the pair of the 3-sphere and the trivial knot and that M has a double cover branched along L. Let H_i be a strongly K-irreducible g_i-genus n_i-bridge splitting of (M, L) for $i = 1$ and 2. Then we can isotope H_1 and H_2 in (M, L) so that they intersect each other in a non-empty union of disjoint simple loops which are L-essential both in H_1 and in H_2.*

It is well-known that it is easy to isotope the splitting surfaces to be disjoint from each other. Hence the condition "non-empty" in the conclusion is very important. This result has been applied to studies of splittings of 2-bridge knots in [25] and [24]. In particular, every 1-genus 1-bridge splitting of a 2-bridge knot is weakly K-reducible, and hence is isotopic to a torus obtained by performing a tubing operation on the

2-bridge splitting 2-sphere along a single string of one of the two trivial 2-string tangles (Theorem 8.2 in [25]).

In this paper, we study on 1-genus 1-bridge splittings of 1-genus 1-bridge knots in the 3-sphere or a lens space, where a lens space is a genus one 3-manifold except $S^2 \times S^1$.

The class of 1-genus 1-bridge knots (1-bridge torus knots, or $(1, 1)$-knots for short) contains all torus knots and 2-bridge knots ((1.6) in [28]), and is important in light of Heegaard splitting theory ([23], Theorem 4 in [37], [21]) and Dehn surgery theory ([1], [7], [8], [33], [39], [40], [41]). See also [5], [3], [4], [9], [11], [13], [14], [16], [26], [27], [29], [35], [34].

Let M be the 3-sphere or a lens space. A knot in M is called the *trivial knot* if it bounds an embedded disc in M. A knot K in M is called a *core knot* if its exterior $M - \operatorname{int} N(K)$ is a solid torus. As we will see in Section 3, (M, K) has a weakly K-reducible 1-genus 1-bridge splitting if and only if K is the trivial knot, a core knot, a 2-bridge knot in S^3 or a connected sum of a core knot and a 2-bridge knot. A knot in M is called a *torus knot* if it can be isotoped onto a circle in a Heegaard splitting torus of M.

Let V be a solid torus, and l, m essential loops on ∂V. The loop m is of the *meridional slope* if it bounds a disc in V. The loop l is of a *longitudinal slope* if it is isotopic to a loop l' on ∂V such that l' intersects m transversely in a single point.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting. We say that the splitting has a *satellite diagram* if there is an essential circle l on the torus H such that t_1 and t_2 have cancelling discs C_1 and C_2 disjoint from l. We call the set of arcs $\partial C_1 \cap H$ and $\partial C_2 \cap H$ a satellite diagram, and l the *slope* of it. We say that the slope of the satellite diagram is *meridional* (resp. *longitudinal*) if it is meridional (resp. *longitudinal*) on ∂V_1 or ∂V_2. When the slope is meridional, K is clearly trivial. When the slope is longitudinal on ∂V_i, K can be obtained from a component of a 2-bridge link by a Dehn surgery on the other component, as is essentially shown in [28]. (In fact, K has a 1-bridge diagram on the annulus $A = \operatorname{cl}(\partial V_i - N(l))$, and an adequate Dehn surgery on a core of the other solid torus V_j deforms A to a flat annulus in S^3.) When $M = S^3$, the Dehn surgery is the same operation as a twisting. A knot with 1-genus 1-bridge splitting is a non-trivial non-core torus knot or a satellite knot if and only if the splitting has a satellite diagram of non-meridional and non-longitudinal slope. See Theorem 3 in [27], [28] and Theorem III in [14].

Theorem 1.2. *Let M be the S^3 or a lens space, and K a knot in M. Let H_1 and H_2 be* 1*-genus* 1*-bridge splitting tori of (M, K) such that they intersect each other in non-empty disjoint union of loops which are K-essential on both H_1 and H_2. Then one of the four conditions* (1)–(4) *below holds.*

(1) *H_1 and H_2 are isotopic in (M, K).*

(2) *One of the splittings is weakly K-reducible.*

(3) *One of the splittings has a satellite diagram of non-meridional and non-longitudinal slope. Moreover, after an adequate isotopy of* H_1 *and* H_2 *in* (M, K), *a loop of* $H_1 \cap H_2$ *gives the slope of the satellite diagram.*
(4) *We can isotope* H_1 *and* H_2 *in* (M, K) *so that they intersect each other in one or two loops which are* K*-essential on both* H_1 *and* H_2.

In general, let X be a 3-manifold, and T a 1-manifold properly embedded in X. Let F be a 2-manifold embedded in X so that F is transverse to T. Let γ be a subarc of T such that $\gamma \cap F = \partial\gamma$. We take a small tubular neighbourhood of γ, say $N(\gamma) \cong \gamma \times D^2$, so that $N(\gamma) \cap F = \partial\gamma \times D^2$. A *tubing operation* on F along γ is the operation deforming F into the 2-manifold $(F - (\partial\gamma \times D^2)) \cup (\gamma \times \partial D^2)$.

Theorem 1.3. *In case of the conclusion* (4) *of* Theorem 1.2, *one of the four conditions* (a)–(d) *below holds after an adequate isotopy of* H_1 *and* H_2 *in* (M, K).
(a) *One of the conclusions* (1)–(3) *of* Theorem 1.2 *holds.*
(b) (M, K) *is a sum of a trivial* 2*-string tangle and a pair of a once punctured lens space* X *and two strings* $S = s_1 \cup s_2$ *properly embedded in* X *such that the exterior* $E_i = \mathrm{cl}(X - N(s_i))$ *is homeomorphic to a solid torus and the other string* s_j *is trivial in* E_i *for* $(i, j) = (1, 2)$ *and* $(2, 1)$. *Moreover,* H_i *is obtained from* ∂X *by performing a tubing operation along* s_i, *for* $i = 1$ *and* 2.
(c) *One of the splittings has a satellite diagram of a longitudinal slope, two splitting tori intersect each other in precisely two loops which are essential on both* H_1 *and* H_2, *and one of them gives the slope of the satellite diagram.*
(d) *There are a solid torus* V *embedded in* M, *and two disjoint discs* D_1 *and* D_2 *on* ∂V *as below. The exterior* $\mathrm{cl}(M - V)$ *is also a solid torus.* K *intersects* V *in two arcs. There are two disjoint balls* B_1 *and* B_2 *in* $\mathrm{cl}(M - V)$ *such that* $V \cap B_i = D_i$, *that* K *intersects* B_i *in a trivial arc and that* K *intersects the solid torus* $V_i = V \cup B_i$ *in a trivial arc for* $i = 1$ *and* 2. *Moreover,* $H_i = \partial V_i$ *for* $i = 1$ *and* 2.

We will obtain the conclusion (c) precisely in Lemmas 5.8 and 7.5. We will obtain the conclusion (d) precisely in the end of Section 7.

K. Morimoto showed in Theorem 3 in [27] that 1-genus 1-bridge splitting torus of a torus knot is unique. H.J. Song and K.H. Ko showed in [35] that the pretzel knot $P(-2, 3, 7)$ has at least two non-isotopic 1-genus 1-bridge splitting tori.

The author expects that the situation of the conclusion (c) gives many examples of mutually non-isotopic 1-genus 1-bridge splitting tori. In case (c), K has a 1-bridge diagram on the annulus A obtained from the splitting torus H_i by cutting along the circle of slope. Note that the core circle of A forms a core knot in M. There are cancelling discs C_1, C_2 which form the satellite diagram composed of the arcs $(\partial C_i \cap H_i) \subset A$, $i = 1, 2$. Then $S_k = \partial N(A \cup C_i)$ is a 1-genus 1-bridge splitting torus having a satellite diagram of longitudinal slope for $k = 1$ and 2. When are S_1 and S_2 isotopic?

We give an example of case (b) in Section 13 such that the two strings s_1 and s_2 are not parallel in X.

Problem 1.4. Is there a knot which admits two non-isotopic 1-genus 1-bridge splittings located as described in (d)?

The author is wondering whether the conclusion (d) occurs for all the 1-genus 1-bridge knots or only for a special subclass of them. It may be possible that the splitting tori can be isotoped to intersect each other more beautifully in case (d).

The next corollary is on 1-genus 1-bridge knots in S^3. There is a double covering of M branched along a knot K if M is the 3-sphere (or a (p,q)-lens space with p odd). Hence we can apply Theorem 1.1 to 1-genus 1-bridge splittings, and obtain the result below from Theorems 1.2 and 1.3.

Corollary 1.5. *Let H_i be a* 1*-genus* 1*-bridge splitting of a knot K in S^3 for i =* 1 *and* 2*. Suppose that H_1 and H_2 are not isotopic in (S^3, K) and that K is not a* 2*-bridge knot nor a satellite knot. Then either*
(1) *K is obtained from a component of a* 2*-bridge link by twisting along the other component, or*
(2) *the conclusion* (d) *of* Theorem 1.3 *holds.*

We can classify all the knot types of 2-bridge knots from the uniquness of the isotopy classes of 2-bridge splitting spheres. The author expects that all the knot types of 1-genus 1-bridge knots are classified after studies in the course of this paper in the future.

This paper is made up of 13 sections and 4 appendixes. In Section 2, we prepare preliminary lemmas. In particular, we see in Remark 2.6 that every 1-genus 1-bridge splitting has infinitely many 1-bridge diagrams on the torus, while it has a unique "Heegaard diagram". t-incompressible and t-∂-incompressible surfaces in (V,t) are studied in Lemma 2.10, where V is a solid torus, and t a trivial arc in V. In Section 3, we consider weakly K-reducible 1-genus 1-bridge splittings. In Sections 4–12, we give a proof of Theorems 1.2 and 1.3. In Section 4, we consider the "general" case where $H_1 \cap H_2$ contains three parallel loops on $H_1 - K$ or $H_2 - K$. In Section 5, we consider the case $H_1 \cap H_2$ is a single loop. We consider the case $|H_1 \cap H_2| = 2$ in Sections 6–9, the case $|H_1 \cap H_2| = 3$ in Section 10 and the case $|H_1 \cap H_2| = 4$ in Section 11. In Section 12, we show that the conclusion "one of the splittings has a semi-satellite diagram of non-meridional and non-longitudinal slope" of Lemma 6.3 implies that either H_1 or H_2 has a satellite diagram, or K is a torus knot. In Section 13, we give an example of the conclusion (b) of Theorem 1.3 such that s_1 and s_2 are not parallel in X.

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2. Preliminaries

The next lemma implies that a trivial arc in a solid torus does not have a "local knot".

Lemma 2.1. *Let M be an irreducible 3-manifold and t a trivial arc in M. Any 2-sphere S embedded in the interior of M and intersecting t transversely in two points bounds a trivial 1-string tangle (B, t'), where $t' = t \cap B$.*

Proof. Let C be a cancelling disc of the trivial arc t in M. Since M is irreducible, S bounds a ball, say B, in M. A standard innermost loop argument shows that we can isotope C so that C intersects S in a single arc connecting the two points $t \cap S$. This arc cuts off from C a cancelling disc of the arc $t \cap B$ in the ball B. □

In the rest of this section, V denotes a solid torus, and t a trivial arc in V. We study "essential" surfaces in (V, t).

Lemma 2.2. *Let D be a t-compressing disc of ∂V. Then either D is a meridian disc disjoint from t, or a peripheral disc which cuts off a ball B containing t from V. In the latter case, we can take a cancelling disc C of t with $C \cap D = \emptyset$.*

Let Q be a meridionally compressing disc of ∂V in (V, t). Then Q is a meridian disc of V.

Proof. When ∂D is essential on ∂V (ignoring ∂t), D is a meridian disc disjoint from t. When ∂D is inessential on ∂V, D is a peripheral disc cutting off a ball, say B, from V. If B were disjoint from the arc t, then D would not be a t-compressing disc. Hence B contains t entirely. By Lemma 2.1, t is trivial in B. We can take a cancelling disc C of t in the ball B so that the arc $\partial C \cap \partial B$ is disjoint from D.

Suppose that ∂Q bounds a disc Q' on ∂V. Since Q is a meridionally compressing disc, Q' contains zero or two points of ∂t. Hence the 2-sphere $Q \cup Q'$ intersects t in one or three points, which contradicts that V is irreducible. □

Lemma 2.3. *Let C be a cancelling disc of t in V, and D a meridian disc of V with $D \cap t = \emptyset$. Then we can isotope D in (V, t) to be disjoint from C.*

Proof. We isotope D in (V, t) so that ∂D intersects the arc $\partial C - t$ transversely in minimum number of points and that D intersects C transversely. A standard innermost loop argument allows us to isotope D in (V, t) to cancel all the intersection loops of $D \cap C$. Suppose for a contradiction that $D \cap C$ contains one or more arcs. Let γ be an arc of $D \cap C$ such that γ is outermost on C away from t, that is, γ cobounds a disc Q_C on C with a subarc of $\partial C - t$ such that $Q_C \cap D = \gamma$. Let A be an annulus obtained by cutting ∂V along ∂D, and l_1 and l_2 the boundary loops of A. Then the arc $\gamma' = \partial Q_C \cap A$ has its endpoints in the same component of ∂A, say l_1, and γ' and a subarc of l_1 cobound a disc Q_A on A. If Q_A contained no endpoint of t, then we could isotope D along Q_A to decrease $|\partial D \cap (\partial C - t)|$. Hence Q_A contains the two endpoints ∂t. $\partial C \cap A$ contains an arc with its both endpoints in l_2 since $|l_1 \cap (\partial C - t)| = |l_2 \cap (\partial C - t)|$. Near an outermost such arc, we can isotope D along the outermost disc, to decrease $|\partial D \cap (\partial C - t)|$. This is again a contradiction. $\square$

Lemma 2.4. *Let D_i be a meridian disc of V with $D_i \cap t = \emptyset$ for $i = 1$ and 2. Then D_1 and D_2 are isotopic in (V, t).*

Proof. Let C be a cancelling disc of t in V. Lemma 2.3 allows us to isotope D_i in (V, t) so that D_i is disjoint from C for $i = 1$ and 2. After an adequate small isotopy of D_1 in (V, t), D_1 and D_2 intersect each other transversely. Since D_1 and D_2 are meridian discs of V, ∂D_1 and ∂D_2 are isotopic in ∂V (ignoring ∂t). If $\partial D_1 \cap \partial D_2 \neq \emptyset$, then $\partial D_1 \cup \partial D_2$ has two bigons on H, where a bigon is an open disc component of $H - (\partial D_1 \cup \partial D_2)$ incident to a single arc of $\partial D_1 - \partial D_2$ and a single arc of $\partial D_2 - \partial D_1$. One of the bigons does not contain the arc $\partial C \cap H$, and hence is disjoint from ∂t. We can isotope D_1 along the bigon, to decrease $|\partial D_1 \cap \partial D_2|$. Repeating such operations as above, we can isotope D_1 in (V, t) so that $\partial D_1 \cap \partial D_2 = \emptyset$. A standard innermost loop argument shows that we can isotope D_1 to be disjoint from D_2. D_1 and D_2 together divide V into two balls, one of which is disjoint from t. Hence D_1 and D_2 are isotopic in (V, t). $\square$

The next lemma implies that there are infinitely many isotopy classes of cancelling discs of a trivial arc in a solid torus.

Lemma 2.5. *Let D be a meridian disc of V with $D \cap t = \emptyset$. Let α be an arc on ∂V with $\partial \alpha = \partial t$ and $\alpha \cap \partial D = \emptyset$. Then there is a cancelling disc C of t in V with $\partial C \cap \partial V = \alpha$ and $C \cap D = \emptyset$.*

Proof. Let B be the ball obtained by cutting V along D. By Lemma 2.3 B contains a cancelling disc C of t. We can isotope C in B near the arc $\partial C \cap \partial B$ so that $\partial C \cap \partial B = \alpha$. $\square$

REMARK 2.6. The above lemma implies that every homeomorphism class of 1-genus 1-bridge splitting has infinitely many isotopy classes of 1-bridge diagrams on the torus up to homeomorphism, while it has a unique *Heegaard diagram* up to homeomorphism as below. Let H be an abstract torus, and p, q two distinct points on H. Let l_1 and l_2 be two essential loops on H with $(l_1 \cup l_2) \cap (p \cup q) = \emptyset$. From such a system, we can form a 1-genus 1-bridge knot by attaching a solid torus V_1 containing a trivial arc t_1 on the front side of H so that l_1 bounds a disc disjoint from t_1 in V_1, and attaching (V_2, t_2) similarly on the reverse side. By Lemma 2.4, such a diagram is unique under the condition "$|l_1 \cap l_2|$ is minimal up to isotopy on $H - (p \cup q)$". Theorem B in [13] and Lemma 2.2 together imply that such a Heegaard diagram represents the trivial knot if and only if p and q are in the same component of $H - (l_1 \cup l_2)$.

A standard innermost loop and outermost arc argument as in the proofs of the above lemmas shows the next lemma. We omit the proof.

Lemma 2.7. *Let D be a meridian disc of V with $D \cap t = \emptyset$. Let Q be a meridionally compressing disc of ∂V in (V, t). Then Q can be isotoped in (V, t) to be disjoint from the disc D. Moreover, after such an isotopy we can take a cancelling disc C of t so that C is disjoint from D and intersects Q in a single arc connecting the point $t \cap Q$ and a point in the arc $\partial C - t$. Hence we obtain a trivial* 2*-string tangle by cutting (V, t) along Q.*

The next lemma implies that there are infinitely many isotopy classes of meridionally compressing discs of ∂V in (V, t).

Lemma 2.8. *Let D be a meridian disc of V with $D \cap t = \emptyset$. Let A be the annulus obtained by cutting ∂V along ∂D. Let l be an essential loop on* $\operatorname{int} A$ *such that l separates the two points ∂t on A. Then l bounds a meridionally compressing disc of ∂V in (V, t).*

Proof. The loop l divides A into two anuuli, say A_1 and A_2. We push the interior of the disc $A_i \cup D$ slightly into $\operatorname{int} V$, to obtain a meridionally compressing disc as desired. □

In general, let X be a compact 3-manifold, and T a 1-manifold properly embedded in X. Let F be a 2-manifold such that either F is properly embedded in X transversely to T, or F is a subsurface of ∂X with $\partial F \cap T = \emptyset$. We say that F is *T-∂-compressible* if there is a disc D embedded in X such that (1) $D \cap T = \emptyset$, (2) $D \cap (F \cup \partial X) = \partial D$, (3) $\partial D \cap F = \alpha$ is an essential arc in $F - T$, (that is, α does not cobound a disc with a subarc of ∂F on $F - T$) and (4) $\partial D \cap (\partial X - \operatorname{int} F) = \beta$ is an essential arc in the surface obtained by cutting $(\partial X - \operatorname{int} F) - T$ along ∂F. We call such a disc D a *T-∂-compressing disc* of F. If F is not T-∂-compressible, then it is *T-∂-incompressible*.

REMARK 2.9. In the usual definition, the above condition (4) is omitted. But, this definition of T-∂-compressibility is equivalent to the usual one for T-incompressible surfaces.

Lemma 2.10. *Let F be a 2-manifold properly embedded in (V, t) such that F is transverse to t. Suppose that F is t-incompressible and t-∂-incompressible. Then F is a union of several surfaces of types* (1)–(6) *below.*
(1) *A 2-sphere disjoint from t.*
(2) *A 2-sphere intersecting t transversely in two points.*
(3) *A meridian disc of V disjoint from t.*
(4) *A meridian disc of V intersecting t transversely in a single point.*
(5) *A peripheral disc disjoint from t.*
(6) *A peripheral disc intersecting t transversely in a single point.*

Proof.
STEP 1. Let C be a cancelling disc of t in V. Suppose first that F is disjoint from C. Let V' be the solid torus obtained by cutting V along C, and C' be the disc composed of the two copies of C in $\partial V'$. Then F is contained in V' and is disjoint from C'. Since F is incompressible and ∂-incompressible also in V', F is of type (1) or (3) or (5).

STEP 2. We can assume that F intersects C. A standard innermost loop argument allows us to isotope F so that $F \cap C$ consists of arcs only since F is t-incompressible.

STEP 3. Suppose that $F \cap C$ contains an arc component which has both endpoints in the arc $\partial C \cap \partial V$. Let α be an outermost one away from t among such arcs, and C_1 the outermost disc of α. Note that $C_1 \cap t = \emptyset$. We perform a t-∂-compressing operation on F along C_1, to obtain another 2-manifold F_1. Since F is t-incompressible and t-∂-incompressible, Remark 2.9 implies that α cuts off a disc, say Q, from F such that $Q \cap t = \emptyset$. F is obtained from F_1 by taking a band sum of the disc $Q \cup C_1$ disjoint from t and another component. Note that F_1 is t-incompressible and t-∂-incompressible in (V, t). To show that F is a union of surfaces of types (1)–(6), it is enough to show that F_1 is so. Hence we can assume that $F \cap C$ does not contain such an arc.

STEP 4. If $F \cap C$ contains an arc component which has both endpoints in t. Then let β be an outermost one among such arcs, and C_2 the outermost disc. Note that $\partial C_2 - \beta \subset t$. Let N' be a regular neighbourhood of C_2 in the 3-manifold obtained by cutting V along F. Then N' intersects F in a disc R which forms a regular neighbourhood of the arc β in F. Let R_1 be the disc $\mathrm{cl}(\partial N' - R)$. Note that $R_1 \cap F = \partial R_1$. Since F is t-incompressible, the loop ∂R_1 bounds a disc R_2 disjoint from t in F. Thus $R \cup R_2$ forms a 2-sphere intersecting t in two points. Let F_2 be a 2-manifold or an emptyset obtained by discarding this 2-sphere from F. F is a union of surfaces of types (1)–(6) if F_2 is a union of such surfaces or an emptyset. Hence we can assume that $F \cap C$ does not contain such an arc.

STEP 5. Thus $F \cap C$ contains an arc component which has an endpoint in t and the other endpoint in the arc $\partial C \cap \partial V$. Let γ be an outermost one among such arcs. That is, γ cuts off a disc C_3 from C such that $C_3 \cap F = \gamma$ and that C_3 is cobounded by γ, a subarc of t and a subarc of $\partial C \cap \partial V$. Let N'' be a regular neighbourhood of C_3 in the 3-manifold obtained by cutting V along F. Then N'' intersects F in a disc R' which forms a regular neighbourhood of γ in F. Let R'_1 be the disc $\mathrm{cl}((\partial N'' \cap \mathrm{int}\, V) - R')$. Since F is t-incompressible and t-∂-incompressible, Remark 2.9 implies that the arc $F \cap \partial R'_1$ cuts off a disc R'_2 disjoint from t from F. Thus $R' \cup R'_2$ forms a disc intersecting t in a single point. Let F_3 be a 2-manifold or an emptyset obtained by discarding this disc from F. It is easy to see that F is a union of surfaces of types (1)–(6) if F_3 is a union of such surfaces or an emptyset. Hence we complete the proof by an induction on the number of the arcs $F \cap C$. □

3. Weakly K-reducible splittings

We will show that a 1-genus 1-bridge splitting is weakly K-reducible if and only if the knot K is the trivial knot, core knots, 2-bridge knots or composite knots of a core knot and a 2-bridge knot.

Throughout this section, let M denote the 3-sphere or a lens space, and K a knot in M with a 1-genus 1-bridge splitting $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$.

In this paper, we call H is *K-reducible* if there are K-compressing discs D_1 and D_2 of H in V_1 and V_2 respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. A K-reducible 1-genus 1-bridge splitting is weakly K-reducible. If there are a cancelling disc C_1 of t_1 and a meridian disc D_2 of V_2 disjoint from t such that $\partial C_1 \cap D_2 = \emptyset$, then H is K-reducible.

Lemma 3.1. *If the splitting H is K-reducible, then K is the trivial knot.*

Proof. Let D_1, D_2 be discs as in the above definition of K-reducibility. By Lemma 2.2, D_i is either a meridian disc disjoint from the arc t_i, or a peripheral disc cutting off from V_i a ball B_i with $t_i \subset B_i$. Since $\partial D_1 \cap \partial D_2 = \emptyset$, at least one of D_1 and D_2 is a peripheral disc. (Otherwise, $M \cong S^2 \times S^1$.)

First we suppose that both are peripheral discs. For $i = 1$ and 2, set $Q_i = B_i \cap H$, which is a disc containing ∂t. By Lemma 2.1, in B_i we can take a cancelling disc C_i of t_i with $C_i \cap \partial B_i \subset Q_1$. We can isotope the disc C_1 near the arc $\partial C_1 \cap (Q_1 \cup Q_2)$ so that $\partial C_1 \cap H = \partial C_2 \cap H$. Thus K is the trivial knot.

Suppose that one of D_1 and D_2, say D_1, is a meridian disc. Then D_2 is peripheral, and cuts off a ball B_2 from V_2. We perform a K-compressing operation on a copy of the once punctured torus $H' = \mathrm{cl}(H - B_2)$ along D_1, to obtain a peripheral disc disjoint from t_1 in V_1. Then K is the trivial knot as shown in the previous paragraph. □

Lemma 3.2. *If the splitting H is weakly K-reducible, then K is the trivial knot, a 2-bridge knot in S^3, a core knot in a lens space or a connected sum of a core knot in a lens space and a 2-bridge knot in S^3.*

Proof. Let D_1 and D_2 be discs as in the definition of weakly K-reducibility. We can assume that one of them, say D_1, is a meridionally compressing disc by Lemma 3.1. Then D_1 is a meridian disc of V_1 by Lemma 2.2. Since $M \not\cong S^2 \times S^1$, D_2 is a peripheral K-compressing disc, and cuts off a ball B containing t_2 from V_2. The pair (B, t_2) is a trivial 1-string tangle by Lemma 2.2. Let N be a small regular neighbourhood of D_1 in V_1 such that t_1 intersects N in a single short trivial arc t. By Lemma 2.7, $(\mathrm{cl}(V_1 - N), \mathrm{cl}(t_1 - t)))$ is a trivial 2-string tangle. Set $X_1 = N \cup \mathrm{cl}(V_2 - B)$, $X_2 = \mathrm{cl}(V_1 - N) \cup B$ and $s_i = K \cap X_i$. Then X_1 is a ball if M is the 3-sphere, and is a once punctured lens space if M is a lens space. The exterior of s_1 in X_1 is homeomorphic to the solid torus $\mathrm{cl}(V_2 - B)$. Since (X_2, s_2) is the sum of a trivial 1-string tangle and a trivial 2-string tangle along the disc $B \cap H$, we obtain the trivial knot or a 2-bridge knot in the 3-sphere if we attach a trivial 1-string tangle to (X_2, s_2). □

Conversely, we consider weakly K-reducibility for splittings of such knots.

Lemma 3.3 (Theorem B in [13]). *If K is a trivial knot, then H is K-reducible.*

Katura Miyazaki told us a very easy proof of the above lemma using the handle addition theorem. We omit it. Similar argument gives an easier proof of the weakly K-reducibility for core knots as below.

Lemma 3.4 (Essentially Theorem C in [13], 6.2 Lemma in [21]). *Suppose that H is not K-reducible. Then K is a core knot if and only if there are a cancelling disc C_i of t_i in V_i and a meridian disc R_j of V_j with $R_j \cap t_j = \emptyset$ such that the arc $\partial C_i \cap H$ intersects ∂R_j transversely in a single point for $(i, j) = (1, 2)$ or $(2, 1)$.*

REMARK 3.5. We can easily see weakly K-reducibility by isotoping R_j near ∂R_i along a subarc of $\partial C_i \cap H$.

Proof. "If" part follows from Lemma 2.5. We consider the "only if" part. Suppose that K is a non-trivial core knot. Let D_i be a meridian disc of V_i with $D_i \cap t_i = \emptyset$ for $i = 1$ and 2. We take regular neighbourhoods $N(D_2)$ in V_2 and $N(t_1)$ in V_1 with $N(D_2) \cap N(t_1) = \emptyset$. Set $E(t_1) = \mathrm{cl}(V_1 - N(t_1))$ and $X = E(t_1) \cup N(D_2)$. Then X is homeomorphic to the exterior of the knot K in M. We take an essential loop m on the annulus $\partial N(t_1) - H$. D_1 gives a compressing disc of the surface $\partial E(t_1) - m$ in $E(t_1)$. The torus ∂X has a compressing disc since K is a core knot. By the generalized handle addition theorem (Theorem 1 (a) in [38]), either $\partial E(t_1) - (\partial D_2 \cup m)$

has a compressing disc in $E(t_1)$, or $\partial E(t_1) - \partial D_2$ has a compressing disc D in $E(t_1)$ such that $\partial D \cap m \neq \emptyset$. In the former case, H is K-reducible, which is a contradiction. We consider the latter case. Let C_1 be a cancelling disc of t_1 in $E(t_1)$, that is, $|\partial C_1 \cap m| = 1$. We take C_1 so that $C_1 \cap D$ consists of arcs only and $|C_1 \cap D|$ is minimal over all the cancelling discs of t_1. Let A be the annulus obtained from H by cutting along ∂D_2.

When the arcs $\partial C_1 \cap A$ separate the two points ∂t_1, we will obtain a contradiction as below. Since $\partial D \cap m \neq \emptyset$, between every two adjacent points of $\partial D \cap m$ on ∂D, there is an intersection point of $\partial D \cap \partial C_1$. Hence there is an outermost arc α of $C_1 \cap D$ on D such that α cuts off an outermost disc Q from D with $|\partial Q \cap m| = 0$ or 1. The arc α divides C_1 into two discs C and C' where $|\partial C \cap m| = 1$. When $|\partial Q \cap m| = 1$, the cancelling disc $Q \cup C'$ intersects D in less number of arcs than C_1 does, which is a contradiction. When $|\partial Q \cap m| = 0$, we obtain a contradiction again, considering $C \cup Q$.

When $\partial C_1 \cap A$ does not separate ∂t_1, there is an arc, say α, on A such that α is contained in A, connects the two points ∂t_1 and is disjoint from $\mathrm{int}(\partial C_1 \cap H)$. We can take a cancelling disc C_2 of t_2 in (V_2, t_2) so that $\partial C_2 = \alpha$ by Lemma 2.5. Then $\partial C_2 \cap \partial C_1 = K \cap H$, and K has a 1-bridge diagram with no crossings on H.

The loop $K' = (\partial C_1 \cup \partial C_2) \cap H$ is of non-meridional slope of the solid tori V_1 and V_2. Otherwise, K would be the trivial knot. If K' is of non-longitudinal slope of V_1 and V_2, then the exterior of K is a Seifert fibred space over a disc with two exceptional points, which contradicts it is a solid torus. Hence K' is of a longitudinal slope of V_1 or V_2, say V_1 and we can take a meridian disc R_1 of V_1 such that R_1 intersects K' in a single point and that the intersection point is contained in ∂C_2. A standard innermost loop and outermost arc argument allows us to isotope $\mathrm{int}\, R_1$ so that R_1 is disjoint form the cancelling disc C_1. $\square$

Weakly K-reducibility of 1-genus 1-bridge splittings is shown in Theorem 8.2 in [25] for 2-bridge knots. For composite knots, it is essentially shown in Theorem 1.6 in [6]. See also Theorem II in [14].

4. General case

We begin to prove Theorems 1.2 and 1.3. This proof is completed at the end of Section 13. Let M be the 3-sphere or a lens space, and K a knot in M. Let H_i be a torus giving a 1-genus 1-bridge splitting $(M, K) = (V_{i1}, t_{i1}) \cup_{H_i} (V_{i2}, t_{i2})$ for $i = 1$ and 2. We assume that H_1 and H_2 intersect transversely in non-empty collection of loops which are K-essential on both H_1 and H_2. If a loop l of $H_1 \cap H_2$ is inessential on one of H_1 and H_2, say on H_1, then l bounds a disc intersecting K transversely in two points on H_1. Each of H_1 and H_2 contains zero or even number of essential loops of $H_1 \cap H_2$ since the splitting tori are separating in M.

The goal of this section is the next proposition.

Proposition 4.1. *Suppose that the loops $H_1 \cap H_2$ contains three parallel loops on $H_1 - K$ or $H_2 - K$, say $H_2 - K$. Then one of the following three conditions holds.*
(1) *We can isotope H_1 and H_2 in (M, K) so that they intersect in non-empty collection of smaller number of loops which are K-essential on both H_1 and H_2. Moreover, we can decrease the number of the intersection loops by two or more.*
(2) *The splitting H_1 is K-reducible.*
(3) *The splitting H_1 has a satellite diagram of non-meridional and non-longitudinal slope. Moreover, a loop of $H_1 \cap H_2$ gives the slope of the satellite diagram.*

To prove this proposition, we need the next three lemmas. We use the basic lemmas in Appendix A.

In general, let X be a 3-manifold, and T a 1-manifold properly embedded in X. Let F_1 and F_2 be 2-manifolds embedded in X so that $F_1 \cap F_2 = \partial F_1 = \partial F_2$. We say F_1 and F_2 are *T-parallel* if the 2-manifold $F_1 \cup F_2$ bounds a submanifold M of X such that the triple $(M, F_1, T \cap M)$ is homeomorphic to the triple $(F_1 \times [0, 1], F_1 \times \{0\}, P \times [0, 1])$, where P is a union of finite number of points in $\operatorname{int} F_1$.

Lemma 4.2. *Suppose that one of V_{11} and V_{12}, say V_{11} contains a component S of $H_2 \cap V_{11}$ such that S is K-parallel to a subsurface S' of ∂V_{11}. Suppose that $H_1 \cap H_2$ consists of larger number of loops than $|\partial S|$. Then we can isotope H_1 and H_2 in (M, K) so that H_1 and H_2 intersect in non-empty collection of smaller number of loops which are K-essential on both H_1 and H_2. Moreover, if S is an annulus disjoint from K, then we can decrease the number of intersection loops by two or more.*

Proof. Suppose that $\operatorname{int} S'$ is disjoint from H_2. Then we isotope H_2 near S slightly beyond S' along the parallelism, to cancel the intersection loops ∂S. Since $|H_1 \cap H_2| > |\partial S|$ before this isotopy, $H_1 \cap H_2 \neq \emptyset$ after the isotopy. If S is an annulus, then we have decreased $|H_1 \cap H_2|$ by two.

Suppose $(\operatorname{int} S') \cap H_2 \neq \emptyset$. Then we isotope S' very closely to S along the parallelism to cancel the intersection curves $(\operatorname{int} S') \cap H_2$. The loops ∂S remain to be intersection loops of $H_1 \cap H_2$. We consider the case where S is an annulus disjoint from K. If $\operatorname{int} S'$ intersects H_2 in two or more loops, then we have decreased the number of the intersection loops by two or more. Suppose for a contradiction that $\operatorname{int} S'$ intersects H_2 in a single loop c. Since c is K-essential in H_1, it is essential in the annulus S', and bounds a surface in the parallelism between S and S'. This is a contradiction since c generates the homology group of the solid torus of parallelism. □

Recall that we consider the general case where $H_1 \cap H_2$ contains three parallel loops l_1, l_2, l_3, appearing in this order, on $H_2 - K$. (They may be essential or inessential on H_2.) Let A_1 and A_2 be the annuli on $H_2 - K$ between l_1 and l_2 and between l_2 and l_3 respectively. We can assume, without loss of generality, that A_i is contained

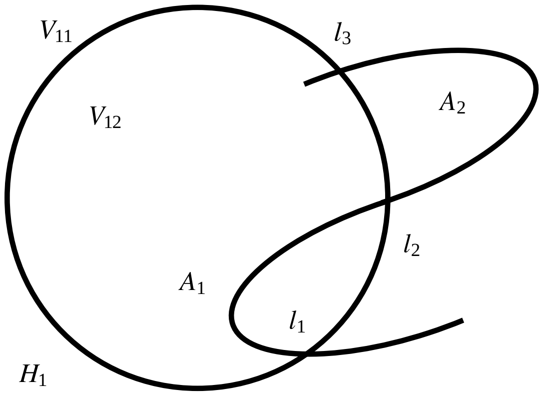

Fig. 4.1.

in V_{1i} for $i = 1$ and 2. See Fig. 4.1.

Lemma 4.3. *Suppose that at least one of A_1 and A_2, say A_1, has boundary loops which are inessential in H_1. Then one of the two conditions below holds.*
(1) *A_i is K-parallel in (V_{1i}, t_{1i}) to an annulus on H_1 for $i = 1$ or 2.*
(2) *K is the trivial knot.*

Proof. We apply Lemma A.3 in Appendix A to A_1 in (V_{11}, t_{11}). (1) of Lemma A.3 implies (1) of this lemma. Hence we can assume that (2) of Lemma A.3 holds. Then there is a cancelling disc, say C_1, of t_{11} such that $\partial C_1 \cap H_1$ is contained in the disc, say Q, bounded by the outermost loop among l_1 and l_2 on H_1.

Since l_2 is inessential on H_1, by Lemmas A.2 and A.3, either A_2 is K-parallel in (V_{12}, t_{12}) to an annulus in H_1, or there is a cancelling disc C_2 of t_{12} in (V_{12}, t_{12}) with $\partial C_2 \cap H_1 \subset Q$. In the former case, we obtain the conclusion (1). In the latter case, the knot K has a 1-bridge diagram on the disc Q, and hence is the trivial knot. □

Lemma 4.4. *Suppose that one component of ∂A_1 is essential and the other is inessential on ∂V_{11}. Then one of the two conditions below holds.*
(1) *A_2 is K-parallel in (V_{12}, t_{12}) to an annulus in H_1.*
(2) *The splitting H_1 is K-reducible.*

Proof. We assume that the conclusion (1) does not occur to show that (2) occurs. Then, applying Lemmas A.1, A.2 and A.3 to A_2 in (V_{12}, t_{12}), there is a cancelling disc C_2 of t_{12} with $C_2 \cap \partial A_2 = \emptyset$. By Lemma A.2 a component of ∂A_1 bounds a meridian disc D disjoint from K in V_{11}. If ∂A_2 contains ∂D, then H_1 is K-reducible since $C_2 \cap \partial D = \emptyset$. If a component l of ∂A_2 is inessential on H_1, then the arc $\partial C_2 \cap H_1$ is contained in the disc bounded by l on H_1, and hence ∂C_2 is disjoint from ∂D. Thus

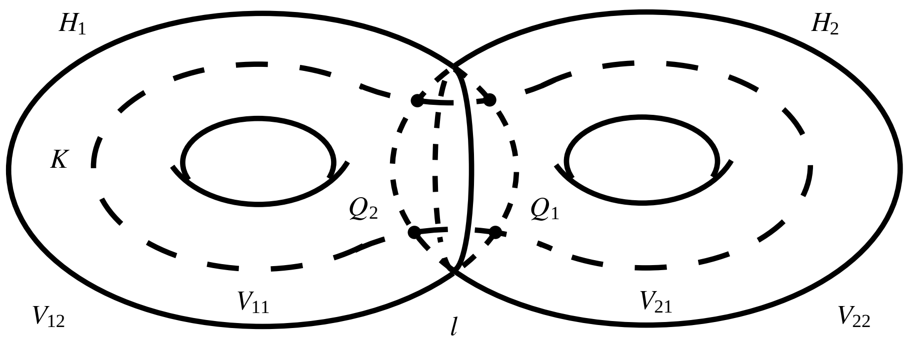

Fig. 5.1.

H_1 is K-reducible. $\square$

Proof. We prove Proposition 4.1. (1) of Lemma 4.3 and (1) of Lemma 4.4 imply (1) of this proposition together with Lemma 4.2 since $H_1 \cap H_2$ contains three or more loops. (2) of Lemma 4.3 implies (2) of this proposition by Theorem B in [13]. (2) of Lemma 4.4 is contained in (2) of this proposition.

Hence we can assume that the loops of ∂A_1 and ∂A_2 are essential in H_1. If at least one of A_1 and A_2, say A_1, is K-parallel in (V_{11}, t_{11}) to an annulus in H_1, then we obtain the conclusion (1) by Lemma 4.2 since $H_1 \cap H_2$ consists of three or more loops. Then we can assume that the annuli are not K-parallel into H_1, and that ∂A_i is of non-longitudinal slope of V_{1i} for $i = 1$ and 2 by Lemma A.1. Hence there is a cancelling disc C_i of t_i in (V_{1i}, t_{1i}) with $\partial C_i \cap \partial A_i = \emptyset$ for $i = 1$ and 2. l_1, l_2 and l_3 together divide H_1 into three annuli, one of which, say R, contains the two points $K \cap H_1$. We can isotope C_1 and C_2 near their boundary so that the arcs $\partial C_1 \cap H_1$ and $\partial C_2 \cap H_1$ are contained in R. This implies that H_1 admits a satellite diagram of a non-longitudinal slope. If the slope of the satellite diagram is meridional, then K is the trivial knot, and H_1 is K-reducible by Lemma 3.3. This completes the proof of the proposition. $\square$

5. When $|H_1 \cap H_2| = 1$

In this section, we study the case where H_1 and H_2 intersect each other in a single loop l. We are going to use the basic lemmas in Appendix B. Since 1 is odd, l is inessential and K-essential in both H_1 and H_2. l bounds a disc, say Q_i, intersecting K in two points in H_i for $i = 1$ and 2. See Fig. 5.1.

For $(i, j) = (1, 2)$ and $(2, 1)$, Q_i is contained in V_{j1} or V_{j2}, say V_{j1}, and forms a 2-sphere bounding a 3-ball B together with Q_j in the solid torus V_{j1}. K intersects B in two subarcs each of which connects Q_i and Q_j. Set $V'_{j1} = \mathrm{cl}(V_{j1} - B)$ and $t'_{j1} = t_{j1} \cap V'_{j1}$ for $j = 1$ and 2. Note that $t'_{j1} = t_{i2}$ for $(i, j) = (1, 2)$ and $(2, 1)$.

By Lemma 2.10, Q_i is K-compressible or K-∂-compressible in (V_{j1}, t_{j1}). Hence it is sufficient to consider the four cases (1), (2), (3)(a), (3)(b) below.
(1) Q_i is K-compressible in (V_{j1}, t_{j1}) for $(i, j) = (1, 2)$ and $(2, 1)$.
(2) Q_1 or Q_2, say Q_1 is K-incompressible in (V_{21}, t_{21}), K-∂-incompressible in (V'_{21}, t'_{21}) and K-∂-compressible in $(B, K \cap B)$.
(3) Q_1 or Q_2, say Q_1 is K-incompressible in (V_{21}, t_{21}) and is K-∂-compressible in (V'_{21}, t'_{21}). There are two subcases.
(a) Q_2 is K-compressible in (V_{11}, t_{11}).
(b) Q_2 is K-incompressible in (V_{11}, t_{11}) and K-∂-compressible in (V'_{11}, t'_{11}).

We do not need to consider the subcase where Q_2 is K-incompressible in (V_{11}, t_{11}), K-∂-incompressible in (V'_{11}, t'_{11}) and K-∂-compressible in $(B, K \cap B)$, since this case is contained in the case (2).

Lemma 5.1. *In case* (1) H_2 *is* K*-reducible.*

Proof. Since Q_1 is K-compressible in (V_{21}, t_{21}), ∂Q_2 bounds a K-compressing disc D of H_2 in (V_{21}, t_{21}) by Lemma B.1 in Appendix B. Let D' be a K-compressing disc of Q_2 in (V_{11}, t_{11}). By Lemma B.1(1), D' is contained in $V'_{11} = V_{11} \cap V_{22}$. Then D and D' show that H_2 is K-reducible. $\square$

Lemma 5.2. *In case* (3)(a) K *is isotopic to a core of* V_{21} *in* M.

Proof. Since Q_2 is K-compressible in (V_{11}, t_{11}), by Lemma B.1 (2), there is a cancelling disc C of t_{11} in (V_{11}, t_{11}) with $C \cap H_1 \subset Q_1$. Let D be a K-∂-compressing disc D of Q_1 in (V'_{21}, t'_{21}). By Lemma B.3 (1) D is a meridian disc of V'_{21}, and by Lemma B.3 (4) there is a cancelling disc P of t'_{21} in (V'_{21}, t'_{21}) such that P is disjoint from D and that $P \cap Q_1$ consists of two arcs each of which contains a point of $\partial t'_{21}$. Since K intersects Q_1 in precisely two points, we can take C and P so that $\partial C \cap \partial D$ is a single point and that $\partial C \cap \partial P = K \cap Q_1$. Hence K is isotopic to the circle $(\partial C \cup \partial P) \cap \partial V'_{21}$, and is isotopic to a core of V_{21}. $\square$

Lemma 5.3. *In case* (3)(b) H_j *is isotopic to the torus obtained by performing a tubing operation on* ∂B *along* t_{i2} *for* $(i, j) = (1, 2)$ *and* $(2, 1)$. *Moreover,* $(B, K \cap B)$ *is a trivial* 2*-string tangle, and its complementary tangle* (X, S) *with* $X = \mathrm{cl}(M - B)$ *and* $S = K \cap X$ *is as below.*
(i) X *is a ball or a once punctured lens space and* S *is a disjoint union of two arcs* s_1 *and* s_2,
(ii) $E_i = \mathrm{cl}(X - N(s_i))$ *is a solid torus for* $i = 1$ *and* 2 *and*
(iii) s_j *is trivial in* E_i *for* $(i, j) = (1, 2)$ *and* $(2, 1)$.
In particular, K *is the trivial knot or a* 2*-bridge knot when* $M = S^3$.

Proof. For $(i, j) = (1, 2)$ and $(2, 1)$, we consider the arguments below. Let D_j be a K-∂-compressing disc of Q_i in (V'_{j1}, t'_{j1}). The arc $D_j \cap Q_i$ separates the two points $K \cap Q_i$ in Q_i. Let $N(D_j)$ be a regular neighbourhood of D_j in V'_{j1} and set $B' = B \cup N(D_1) \cup N(D_2)$. Then B' is a ball isotopic to B in (M, K). Set $X = \mathrm{cl}(M - B')$, $s_j = t'_{j1} = t_{i2}$ and $S = s_1 \cup s_2$. The ball $N_j = \mathrm{cl}(V'_{j1} - N(D_j))$ forms a regular neighbourhood of s_j in X by Lemma B.3. Hence H_j is isotopic in (M, K) to the torus obtained by performing a tubing operation on ∂B along t_{i2}. Since $V_{j1} \cup B'$ is a solid torus isotopic to V_{j1} in (M, K), the exterior $E_j = \mathrm{cl}(X - N_j)$ of s_j is isotopic to the solid torus V_{j2}.

We can take a cancelling disc C_j of t_{j2} in (V_{j2}, t_{j2}) with $C_j \cap D_i = \emptyset$. Then C_j is also a cancelling disc of $s_i = t_{j2}$ in $E_j = \mathrm{cl}(X - N_j)$. Since the tangle $(B', K \cap B')$ is isotopic to $(B, K \cap B)$ in (M, K), and since $(B, K \cap B)$ is a trivial 2-string tangle by Lemma B.2 (3), $(B', K \cap B')$ is a trivial 2-string tangle.

When $M = S^3$, X is a ball, $\mathrm{cl}(X - N_1)$ and $\mathrm{cl}(X - N_2)$ are solid tori, and $\mathrm{cl}(X - N_1 \cup N_2)$ is a handlebody because s_1 is trivial in E_2. Hence (X, S) is a trivial 2-string tangle by Theorem 1 in [10]. Thus K is the trivial knot or a 2-bridge knot. □

Lemma 5.4. *Assume that $(B, K \cap B)$ is a trivial 2-string tangle. Suppose that the once punctured torus $H_2 \cap V_{12}$ or $H_1 \cap V_{22}$, say $H_2 \cap V_{12}$ is compressible in $V_{12} \cap V_{22}$. Then $M = S^3$ and K is the trivial knot or a 2-bridge knot.*

Proof. Let D be a compressing disc of $H_2 \cap V_{12}$ in $V_{12} \cap V_{22}$. Note that $V_{12} \cap V_{22}$ is disjoint from K. We perform a compressing operation on a copy of $H_2 \cap V_{12}$ along D. Then we obtain a disc D' (and possibly a torus component) with $\partial D' = \partial Q_i$. The 2-sphere $D' \cup Q_i$ bounds a ball W_i in V_{i2} for $i = 1$ and 2. Moreover, t_{i2} is trivial in W_i by Lemma 2.1 for $i = 1$ and 2. Then $(W_1 \cup_{D'} W_2, t_{12} \cup t_{22})$ is a trivial 2-string tangle, and hence K is the trivial knot or a 2-bridge knot. □

Lemma 5.5. *In case (2) either K is the trivial knot or a 2-bridge knot in S^3, or we can isotope H_1 in (M, K) so that*

(i) *$H_1 \cap H_2$ consists of two essential loops on both H_1 and H_2,*

(ii) *$H_1 \cap H_2$ divide H_i into two annuli one of which, say A_i, intersects K in two points for $i = 1$ and 2 and*

(iii) *there is a parallelism $(P, K \cap P)$ of A_1 and A_2 with $(\mathrm{int}\, P) \cap (H_1 \cup H_2) = \emptyset$.*

Proof. In case (2), $(B, K \cap B)$ gives a parallelism between Q_1 and Q_2 by Lemma B.4. Suppose first that $H_2 \cap V_{12}$ has a K-compressing disc D in (V_{12}, t_{12}). Then by Lemma B.2 (4) D is contained in $V_{12} \cap V_{22}$. By Lemma B.2 (3), $(B, K \cap B)$ is a trivial 2-string tangle, and hence K is the trivial knot or a 2-bridge knot in S^3 by Lemma 5.4.

Hence we can assume that $H_2 \cap V_{12}$ is K-incompressible in (V_{12}, t_{12}). Then $H_2 \cap V_{12}$ has a K-∂-compressing disc R in (V_{12}, t_{12}) by Lemma 2.10. If R is contained

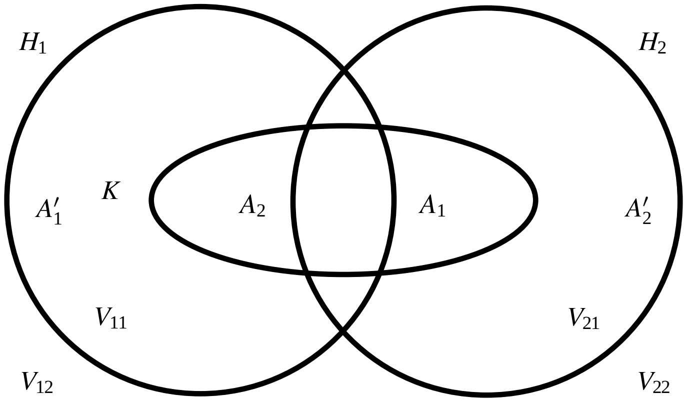

Fig. 5.2.

in V_{21}', then $R \cap Q_1$ is an arc essential in $Q_1 - K$ by the unusual definition of K-∂-compressibility. Hence R is also a K-∂-compressing disc of Q_1, which contradicts that we are now considering case (2). Then R is contained in $V_{12} \cap V_{22}$. The arc $R \cap (H_1 \cap V_{22})$ is essential in $H_1 \cap V_{22}$ by the definition of K-∂-compressibility. We isotope H_2 along R, and obtain the desired conclusion. $\square$

In the rest of this section, we consider the latter half of the conclusion of Lemma 5.5, that is, we assume that the conditions (i), (ii) and (iii) are satisfied. See Fig. 5.2. Set $A_i' = \mathrm{cl}(H_i - A_i)$ for $i = 1$ and 2. Let V_{i1} be the solid torus bounded by H_i and containing the annulus A_j for $(i, j) = (1, 2)$ and $(2, 1)$. Let V_{i2} be the other solid torus bounded by H_i in M. Set $t_{ij} = K \cap V_{ij}$. Let V_{i1}' be the solid torus $V_{i1} - P$, and let $t_{i1}' = K \cap V_{i1}'$ for $i = 1$ and 2. In (V_{j2}, t_{j2}) the annulus A_i' is K-compressible or K-∂-compressible by Lemma 2.10. Hence there are five cases (A), (B), (C), (D)(i), (D)(ii) below.

(A) One of A_1' and A_2', say A_1' has a K-compressing disc in $V_{12} \cap V_{22}$.

(B) A_1' and A_2' are K-compressible in (V_{11}', t_{11}') and in (V_{21}', t_{21}') respectively.

(C) One of A_1' and A_2', say A_1' has a K-∂-compressing disc in $V_{12} \cap V_{22}$.

(D) One of A_1' and A_2', say A_1' is K-∂-compressible in (V_{11}', t_{11}'). There are two subcases.

(i) A_2' is K-compressible in (V_{21}', t_{21}').

(ii) A_2' is K-∂-compressible in (V_{21}', t_{21}').

Lemma 5.6. *In case* (A), K *is the trivial knot.*

Proof. A K-compressing operation on a copy of A_1' in $V_{12} \cap V_{22}$ yields meridian discs D_1 and D_2 of V_{12} and V_{22}. By Lemma A.1, there is a cancelling disc C_i of t_{i2} disjoint from D_1 and D_2 in (V_{i2}, t_{i2}) for $i = 1$ and 2. Then $\partial C_i \cap H_i \subset A_i$. Since A_1

and A_2 are K-parallel in $(P, K \cap P)$, we can extend C_2 to a cancelling disc C'_2 of t_{11} in (V_{11}, t_{11}) with $\partial C_1 \cap H_1 \subset A_1$. Thus K admits a 1-bridge diagram on A_1. (Note that C'_2 may intersect A'_1.) Since ∂A_1 is meridional on ∂V_{12}, A_1 is an unknotted and untwisted annlus, and K is the trivial knot. □

Lemma 5.7. *In case* (C), *H_1 and H_2 are isotopic in* (M, K).

Proof. Let D be a K-∂-compressing disc of A'_1 with $D \subset V_{12} \cap V_{22}$. The arc $\partial D \cap A'_1$ connects two distinct components of $\partial A'_1$. Hence the arc $\partial D \cap A'_2$ is also essential in A'_2. If we perform a K-∂-compressing operation on A'_1 along D, then we obtain a disc whose boundary bounds a disc on A'_2. These discs together form a 2-sphere bounding a ball in V_{12}. Thus $V_{12} \cap V_{22}$ gives a K-parallelism between A'_1 and A'_2 in (M, K). Since $(P, K \cap P)$ gives K-parallelism between A_1 and A_2, H_1 and H_2 are isotopic in (M, K). □

Lemma 5.8. *In case* (D)(ii), *there is an annulus A embedded in M satisfying the following two conditions* (a) *and* (b).
(a) *A core loop of A forms a core knot in M.*
(b) *K has a* 1*-bridge diagram on A. That is, K intersects A transversely in two points and is divided into two subarcs t_1 and t_2, and there is an embedded disc C_i with $K \cap C_i = t_i \subset \partial C_i$ $\mathrm{cl}(\partial C_i - t_i) = C_i \cap \mathrm{int}\, A$ and $C_1 \cap C_2 \subset \mathrm{int}\, A$ for $(i, j) = (1, 2)$ and* $(2, 1)$.
Moreover, let R_i be the annulus obtained by isotoping A fixing ∂A along C_i slightly beyond t_i. Then H_i is isotopic to the torus $A \cup R_i$ in (M, K) for $i = 1$ and 2, *after changing suffix numbers if necessary.*

Proof. For $(i, j) = (1, 2)$ and $(2, 1)$, we consider the argument below. Since the annulus A'_i is disjoint from the knot K, performing K-∂-compressing operation on a copy of A'_i, we obtain a disc Q_i such that ∂Q_i bounds a disc Q'_i on A_j. The 2-sphere $Q_i \cup Q'_i$ bounds a ball B_i in V_{j2}. Hence A'_i and A_j are parallel in M (ignoring K). A standard innermost loop and outermost arc argument allows us to take a cancelling disc C_{j2} of t_{j2} in (V_{j2}, t_{j2}) with $C_{j2} \cap Q_i = \emptyset$. Since A_1 and A_2 are K-parallel in $(P, K \cap P)$, we have the desired conclusion. □

Lemma 5.9. *In case* (B), *we have a contradiction.*

Proof. A K-compressing operation on A'_1 yields meridian discs D_1 and D_2 of V_{22}. Since A'_2 has a K-compressing disc in V'_{21}, $M \cong S^2 \times S^1$, which contradicts our assumption. □

Lemma 5.10. *In case* (D)(i), *H_2 is K-reducible.*

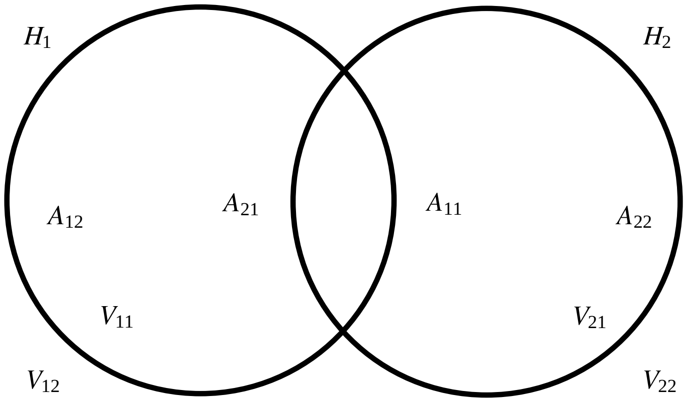

Fig. 6.1.

Proof. Performing a K-∂-compressing operation on A'_1, we obtain a K-compressig disc of A_2, which is peripheral in V_{22} (ignoring t_{22}). This disc and a K-compressing disc of A'_2 in V_{21} show that H_2 is K-reducible. □

6. When $H_1 \cap H_2$ consists of two essential loops (I)

In this and the next sections, we consider the case where $H_1 \cap H_2$ consists of two loops which are essential on both H_1 and H_2 (ignoring the points $K \cap H_i$).

See Fig. 6.1. The loops $H_1 \cap H_2$ divide H_i into two annuli A_{i1} and A_{i2} for $i = 1$ and 2. Let V_{i1} be the solid torus bounded by H_i in M with $A_{j1} \subset V_{i1}$ for $(i, j) = (1, 2)$ and $(2, 1)$. Let V_{i2} be the other solid torus bounded by H_i. Set $t_{ij} = K \cap V_{ij}$.

Lemma 6.1. *A_{11}, A_{12}, A_{21}, A_{22} are parallel to an annulus in the boundary of V_{21}, V_{22}, V_{11}, V_{12} respectively (ignoring K). Hence these solid tori are divided the anuuli into two solid tori.*

Proof. In this proof we ignore K. We consider A_{11}. The same argument will do for the other annuli.

We can assume that ∂A_{11} are of meridional slope of V_{21}. (Otherwise, the conclusion is a well-known fact.) Then A_{11} is compressible in V_{21}. Moreover, we can assume, without loss of generality, that A_{11} has a compressing disc Q in $V_{21} \cap V_{12}$. Then, compressing A_{11}, we obtain a compressing disc R of A_{22} in $V_{21} \cap V_{12}$. ∂Q is of meridional slope of V_{12}, and hence it is of non-meridional slope of V_{11}, otherwise, $M \cong S^1 \times S^2$. Hence A_{21} is parallel to one of A_{11} and A_{12} in V_{11}. When A_{21} is parallel to A_{11}, we are done. Hence we can assume that A_{21} is parallel to A_{12}. Then we can isotope H_1 into $\operatorname{int} V_{21}$ so that it is disjoint from the meridian disc R of V_{21}. Let S be the 2-sphere obtained by compressing H_2 along R. S bounds a ball containing H_1

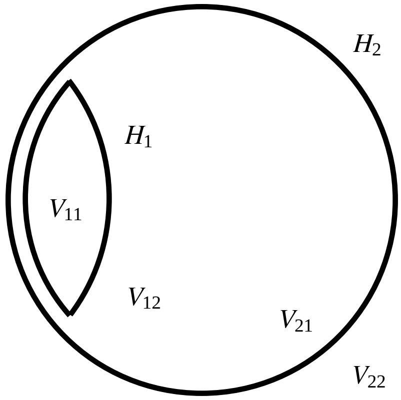

Fig. 6.2.

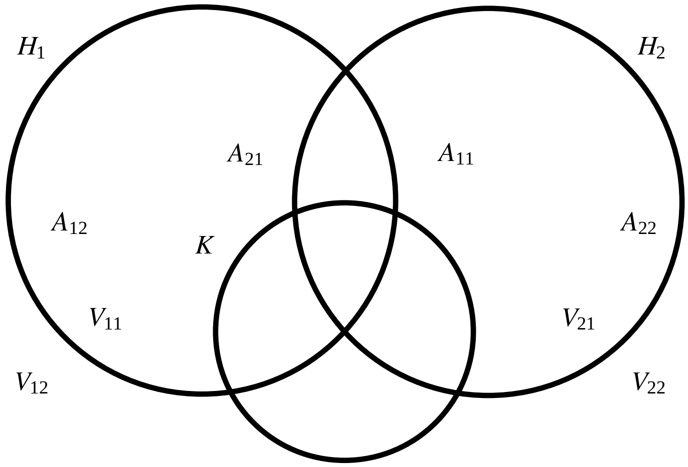

Fig. 6.3.

in V_{21}, and on the other side bounds a ball in V_{12}. See Fig. 6.2. Hence $M = S^3$, and ∂A_{21} is of longitudinal slope of V_{11} before the isotopy of H_1. Hence A_{21} is parallel to A_{11} in V_{11}. $\square$

We can assume, without loss of generality, that $|A_{i1} \cap K| \geq |A_{i2} \cap K|$. There are two cases: in case (I) $|A_{11} \cap K| = |A_{21} \cap K| = |A_{12} \cap K| = |A_{22} \cap K| = 1$ (See Fig. 6.3), and in case (II) $|A_{11} \cap K| = |A_{21} \cap K| = 2$ and $A_{12} \cap K = A_{22} \cap K = \emptyset$. We consider case (I) in this section, and case (II) in the next section. In this section, we will use lemmas in Appendix C.

When A_{ik} has a K-∂-compressing disc D in (V_{jk}, t_{jk}) for $(i, j) = (1, 2)$ or $(2, 1)$

and $k \in \{1, 2\}$, we say that D is *essential* if the arc $\partial D \cap A_{ik}$ is essential on A_{ik}, and *inessential* otherwise, following the definition in Appendix C.

There are many cases as below.

(1) Precisely one of the four solid tori $V_{11} \cap V_{21}$, $V_{11} \cap V_{22}$, $V_{12} \cap V_{21}$ and $V_{12} \cap V_{22}$ contains an essential K-∂-compressing disc (Lemma 6.6).

(2) Precisely two adjacent solid tori do (Lemma 6.5).

(3) Two non-adjacent solid tori do (Lemma 6.4).

(4) None of the solid tori does. There are 2 subcases.

(a) At least one of A_{11}, A_{12}, A_{21}, A_{22} is K-compressible (Lemma 6.2).

(b) All of the annuli are K-incompressible, and hence have inessential K-∂-compressing discs (Lemma 6.3).

Lemma 6.2. *Suppose that A_{ik} is K-compressible in (V_{jk}, t_{jk}) and A_{il} does not have an essential K-∂-compressing disc in (V_{jl}, t_{jl}) for $(i, j) = (1, 2)$ or $(2, 1)$ and $(k, l) = (1, 2)$ or $(2, 1)$. Then H_j is weakly K-reducible.*

Proof. Suppose, without loss of generality, that the preliminary conditions hold for $(i, j) = (1, 2)$ and $(k, l) = (1, 2)$. Let D be a K-compressing disc of A_{11}. We can assume, without loss of generality, that D is contained in $V_{11} \cap V_{21}$. Then A_{21} has a K-compressing disc in $V_{11} \cap V_{21}$ by Lemma C.1. Moreover, $\partial A_{11} = \partial A_{21}$ is meridional on V_{21} and V_{11}.

First, suppose that A_{12} is K-compressible in (V_{22}, t_{22}). Then ∂A_{12} is of meridional slope of V_{22}, and $M \cong S^2 \times S^1$, which is a contradiction.

Secondly, suppose that A_{12} is K-incompressible in (V_{22}, t_{22}). Then A_{12} has a K-∂-compressing disc D' by Lemma 2.10. By the preliminary condition, the arc $\partial D' \cap A_{12}$ is inessential in A_{12}. By Lemma C.3, we can take a cancelling disc C of t_{22} in (V_{22}, t_{22}) so that ∂C is disjoint from a component of ∂A_{12}. ((1) of Lemma C.3 contradicts our assumption in this lemma.) On the other hand, by performing a K-compressing operation on A_{11} along D, we obtain meridian discs, each of which intersects K at most one point. Hence H_2 is weakly K-reducible. $\square$

We introduce the notion of “semi-satellite diagrams”. In general, let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a 1-genus 1-bridge splitting. We say that H has a *semi-satellite diagram* if there is a pair of disjoint simple loops l_1 and l_2 on H such that l_1 and l_2 are essential on H and that t_i has a cancelling disc C_i disjoint from l_i in V_i for $i = 1$ and 2. A satellite diagram is a semi-satellite diagram. We call l_1 and l_2 the *slopes* of the semi-satellite diagram. We say that l_1 and l_2 are *meridional* (resp. *longitudinal*) if they are meridional (resp. longitudinal) on ∂V_1 or ∂V_2.

The next lemma immediately follows from Lemma C.3.

Lemma 6.3. *Suppose that A_{11}, A_{12}, A_{21} and A_{22} are K-incompressible in (V_{21}, t_{21}), (V_{22}, t_{22}), (V_{11}, t_{11}) and (V_{12}, t_{12}) respectively. Suppose that two non-adjacent solid tori among $V_{11} \cap V_{21}$, $V_{11} \cap V_{22}$, $V_{12} \cap V_{21}$, $V_{12} \cap V_{22}$ contain inessential K-∂-compressing discs of A_{11}, A_{12}, A_{21}, A_{22}, and that these solid tori do not contain an essential K-∂-compressing disc. Then H_i has a semi-satellite diagram of non-longitudinal and non-meridional slope for $i = 1$ and 2. Moreover, the other splitting torus is a union of two annuli as in* (2) *of* Lemma C.3.

In Section 12, we will study what the conclusion of Lemma 6.3 implies.

The next lemma immediately follows from Lemma C.2.

Lemma 6.4. *Suppose that two non-adjacent solid tori among $V_{11} \cap V_{21}$, $V_{11} \cap V_{22}$, $V_{12} \cap V_{21}$, $V_{12} \cap V_{22}$, say $V_{11} \cap V_{21}$ and $V_{12} \cap V_{22}$ contain essential K-∂-compressing discs of A_{11}, A_{12}, A_{21}, A_{22}. Then H_1 and H_2 are isotopic in (M, K).*

Lemma 6.5. *Suppose that only two adjacent solid tori among $V_{11} \cap V_{21}$, $V_{11} \cap V_{22}$, $V_{12} \cap V_{21}$, $V_{12} \cap V_{22}$, say $V_{12} \cap V_{22}$ and $V_{11} \cap V_{22}$ contain essential K-∂-compressing discs of A_{12}, A_{21}, A_{22}. Then either*

(1) *one of H_1 and H_2 is K-reducible, or*

(2) *we can isotope H_1 and H_2 so that they are in the situation of the latter half of* Section 5 (*which is considered in* Lemmas 5.6–5.10).

Proof. Suppose that A_{11} is K-compressible in (V_{21}, t_{21}). Then a K-compressing operation on A_{11} yields a meridian disc, say Q, disjoint from K and bounded by a loop of ∂A_{11}. Since A_{21} and A_{22} are parallel in (V_{22}, t_{22}), we can take a cancelling disc C of t_{22} in (V_{22}, t_{22}) with $\partial C \cap \partial Q = \emptyset$. Thus H_2 is K-reducible.

Hence we can assume that A_{11} is K-∂-compressible by Lemma 2.10. Then one of $V_{11} \cap V_{21}$ and $V_{12} \cap V_{21}$, say $V_{11} \cap V_{21}$ contains an inessential K-∂-compressing disc D of A_{11}. Then D cuts off a ball B intersecting K in a single arc, say t, from $V_{11} \cap V_{21}$, and we can take a cancelling disc Δ of t in B so that each of $\partial\Delta \cap \partial V_{11}$ and $\partial\Delta \cap \partial V_{21}$ is a single arc. See Fig. 6.4. We isotope K near t along Δ slightly beyond the arc $\partial\Delta - t$. See Fig. 6.5. Then K is in the same situation as in the latter half of Section 5, since $(V_{12} \cap V_{22}, K \cap V_{12} \cap V_{22})$ gives a K-parallelism between A_{12} and A_{22} before the isotopy. Note that this isotopy does not change the isotopy classes of H_1 and H_2 in (M, K). □

Lemma 6.6. *Suppose that precisely one solid torus among $V_{11} \cap V_{21}$, $V_{11} \cap V_{22}$, $V_{12} \cap V_{21}$, $V_{12} \cap V_{22}$, say $V_{11} \cap V_{21}$ contains an essential K-∂-compressing disc of A_{11}, A_{21}. Then one of the following three conditions holds.*

(1) *One of H_1 and H_2 is weakly K-reducible.*

(2) *We can isotope H_1 and H_2 so that they are in the situation of the latter half of* Section 5.

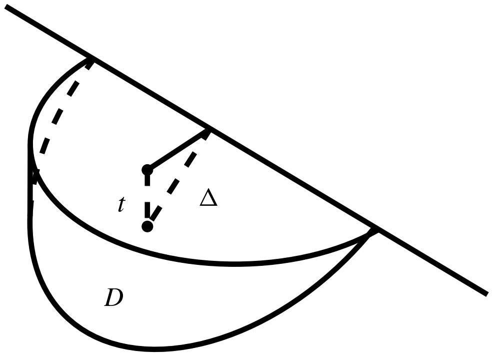

Fig. 6.4.

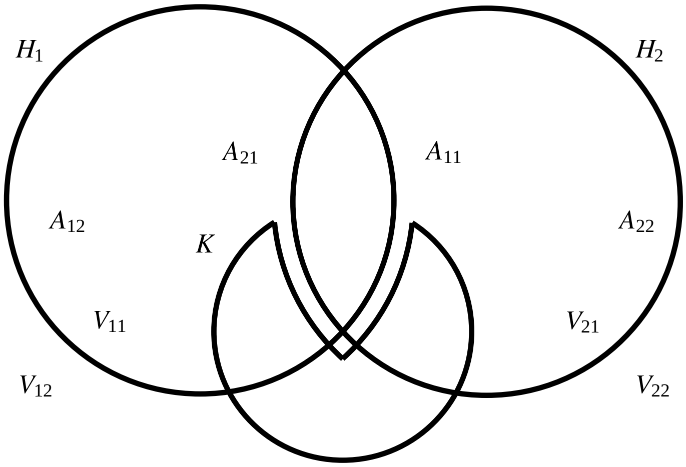

Fig. 6.5.

(3) *The conclusion of* Lemma 6.3 *holds.*

Proof. By Lemma C.2, A_{11} and A_{21} are K-parallel in $(V_{11} \cap V_{21}, K \cap V_{11} \cap V_{21})$. Suppose that one of A_{11} and A_{21}, say A_{11} is K-compressible in (V_{21}, t_{21}). Then H_2 is weakly K-reducible by Lemma 6.2 since A_{12} does not have an essential K-∂-compressing disc by the assumption of this lemma. Hence we can assume that A_{11} and A_{21} are K-incompressible.

Suppose that one of A_{12} and A_{22}, say A_{12} has a K-compressing disc D in (V_{22}, t_{22}). If D is contained in $V_{11} \cap V_{22}$, then A_{21} is also K-compressible by Lemma C.1, which contradicts our assumption. Hence D is contained in $V_{12} \cap V_{22}$. By Lemma C.1, $t = K \cap (V_{12} \cap V_{22})$ has a cancelling disc C' in $V_{12} \cap V_{22}$ such that each of $\partial C' \cap A_{12}$ and $\partial C' \cap A_{22}$ is a single arc. We isotope K near t along C' slightly

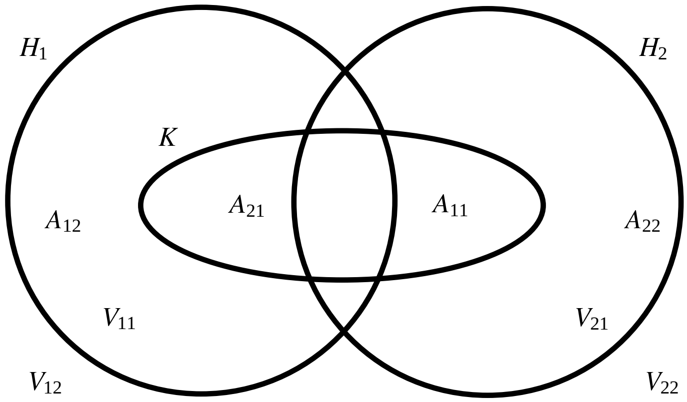

Fig. 7.1.

beyond the arc $\partial C' - t$. Then K is in the situation in the latter half of Section 5.

Hence we can assume that A_{12} and A_{22} are K-incompressible, and hence K-∂-compressible by Lemma 2.10. Suppose that $V_{12} \cap V_{22}$ contains an inessential K-∂-compressing disc D of A_{12} and A_{22}. Then D cuts off a ball B from $V_{12} \cap V_{22}$, and we can take a cancelling disc C of the arc $K \cap B$ so that $\partial C \cap A_{i2}$ is an arc for $i = 1$ and 2. Then we isotope K along C to be in the situation of latter half of Section 5.

Hence we can suppose that $V_{11} \cap V_{22}$ and $V_{12} \cap V_{21}$ contain inessential K-∂-compressing discs of A_{12} and A_{22} respectively. This is the situation of Lemma 6.3. $\square$

7. When $H_1 \cap H_2$ consists of two essential loops (II)

In this section, we will use Lemma D.1 in Appendix D. Let H_i, V_{ij}, A_{ij}, t_{ij} be as in Section 6. In this section, we consider case (II) where $|A_{11} \cap K| = |A_{21} \cap K| = 2$ and $A_{12} \cap K = A_{22} \cap K = \emptyset$. See Fig. 7.1.

$K \cap (V_{11} \cap V_{21})$ consists of two arcs, say t_1 and t_2, each of which connects A_{11} and A_{21}. By Lemma 2.10, A_{j1} is K-compressible or K-∂-compressible in (V_{i1}, t_{i1}) for $(i, j) = (1, 2)$ and $(2, 1)$.

Suppose that A_{j1} is K-∂-compressible in $(V_{i1} \cap V_{j2}, t_{j2})$. Any K-∂-compressing disc of A_{j1} intersects A_{i2} in an essential arc by the unusual definition of K-∂-compressibility, and hence it is also a K-∂-compressing disc of A_{i2}. K-∂-compressing a copy of A_{i2}, we obtain a K-compressing disc of A_{j1} in $(V_{i1} \cap V_{j2}, t_{j2})$.

Hence A_{j1} is K-compressible in (V_{i1}, t_{i1}) or K-∂-compressible in $(V_{11} \cap V_{21}, t_1 \cup t_2)$. In the latter case, a K-∂-compressing disc of A_{j1} is also a K-∂-compressing disc of A_{i1} in $(V_{11} \cap V_{21}, t_1 \cup t_2)$. Thus it is sufficient to consider the following two cases.

(1) A_{11} and A_{21} are K-∂-compressible in $(V_{11} \cap V_{21}, t_1 \cup t_2)$.

(2) A_{11} and A_{21} are K-∂-incompressible in $(V_{11} \cap V_{21}, t_1 \cup t_2)$, and A_{j1} is

K-compressible in (V_{i1}, t_{i1}) for $(i, j) = (1, 2)$ and $(2, 1)$.

Lemma 7.1. *In case* (1), *one of the three conditions* (a)–(c) *below holds.*
(a) *We can isotope H_1 and H_2 so that H_1 and H_2 intersect each other in a single inessential and K-essential loop.* (*We have already considered this situation in* Section 5.)
(b) *We can isotope H_1 and H_2 so that $H_1 \cap H_2$ divide H_i into two annuli each of which intersects K in a single point for $i = 1$ and* 2. (*We have already considered this situation in* Section 6.)
(c) *One of H_1 and H_2 is weakly K-reducible.*

Proof. Let D be a K-∂-compressing disc of A_{11} and A_{21} in $V_{11} \cap V_{21}$. Then the arc $\partial D \cap A_{i1}$ is either essential in both A_{11} and A_{21}, or inessential in them.

In the former case, we isotope H_1 along D. Then H_1 and H_2 intersect each other in a single inessential and K-essential loop. This is the conclusion (a).

In the latter case, let D_i be the disc cut off from A_{i1} by the arc $\partial D \cap A_{i1}$ for $i = 1$ and 2. Then $|K \cap D_1| = |K \cap D_2|$. This number of the intersection points is determined by the number of the arcs contained in the ball B bounded by the 2-sphere $D \cup D_1 \cup D_2$ in $V_{11} \cap V_{21}$.

When $|K \cap D_1| = |K \cap D_2| = 1$, $K \cap B$ is a single trivial arc. Hence we can isotope H_1 along B so that H_1 and H_2 are in position as in case (I). This is the conclusion (b).

When $|K \cap D_1| = |K \cap D_2| = 2$, we isotope H_1 along D. Then $H_1 \cap H_2$ is a union of a single inessential and K-essential loop and two essential loops on both H_1 and H_2. We apply Lemma 10.1 in Section 10 to this situation. The conclusion (1) of Lemma 10.1 implies that we can isotope H_1 and H_2 so that they intersect each other in a single inessential and K-inessential loop. This is the conclusion (a) of this lemma. The conclusion (3) of Lemma 10.1 is the conclusion (c) of this lemma. The conclusion (2) of Lemma 10.1 implies that the discs D_1 and D_2 are K-parallel after the isotopy along D. Before the isotopy, B contains a K-∂-compressing disc of A_{11} and A_{21} as we considered in the previous paragraph. Hence we obtain the conclusion (b). □

In the rest of this section, we consider case (2) just before Lemma 7.1. Let D_j be a K-compressing disc of A_{j1} in (V_{i1}, t_{i1}) for $(i, j) = (1, 2)$ and $(2, 1)$. We say that D_j is *essential* if $\partial D_j \cap A_{j1}$ is essential on A_{j1}, and otherwise it is *inessential*, following the definition in Appendix D.

We consider 4 cases below.
(1) Both D_1 and D_2 are essential (Lemma 7.2).
(2) One of D_1 and D_2 is essential, and the other is inessential (Lemma 7.4).
(3) Both D_1 and D_2 are inessential. There are two subcases.

(a) A_{11} or A_{21}, say A_{11} does not have a K-∂-compressing disc in $(V_{12} \cap V_{21}, t_{12})$ (Lemma 7.5).
(b) A_{j1} has a K-∂-compressing disc in $(V_{j2} \cap V_{i1}, t_{j2})$ for $(i, j) = (1, 2)$ and $(2, 1)$ (right after Lemma 7.5).

Lemma 7.2. *If D_j is essential for $j = 1$ and 2, then one of H_1 and H_2 is K-reducible or weakly K-reducible.*

Proof. Suppose first that one of D_1 and D_2, say D_1 is not contained in $V_{11} \cap V_{21}$. Then D_1 is contained in $V_{12} \cap V_{21}$. Similar argument as in the proof of Lemma A.1 (1) shows that D_2 is not contained in $V_{11} \cap V_{21}$, and hence is contained in $V_{11} \cap V_{22}$. We perform a K-compressing operation on A_{11} along D_1 and obtain a meridian disc D_1' of V_{21}. Since D_2 is a meridian disc of V_{22}, D_2 and D_1' shows that $M \cong S^2 \times S^1$, which contradicts our assumption.

Secondly, we assume that both of D_1 and D_2 are contained in $V_{11} \cap V_{21}$. A_{12} is K-compressible or K-∂-compressible in (V_{22}, t_{22}) by Lemma 2.10.

If A_{12} has a K-compressing disc, then K-compressing A_{12}, we obtain a meridian disc of V_{22}. Since D_2 is a meridian disc of V_{21}, $M \cong S^2 \times S^1$. This is again a contradiction.

Hence A_{12} has a K-∂-compressing disc D in (V_{22}, t_{22}). First, we consider the case where D is contained in $V_{11} \cap V_{22}$. K-∂-compressing A_{12} along D, we obtain a peripheral disc D' in V_{22}. D' is a K-compressing disc of A_{21}. We perform a K-compressing operation on A_{11} along D_1, to obtain a meridian disc Q of V_{21} such that Q intersects K in at most one point. Since $\partial Q \subset \partial A_{11}$ is disjoint from $\partial D'$, H_2 is weakly K-reducible.

Secondly, we consider the case where D is contained in $V_{12} \cap V_{22}$.

Since $V_{12} \cap V_{22}$ is disjoint from K, $V_{12} \cap V_{22}$ gives K-parallelism between A_{12} and A_{22}. Hence we can isotope H_1 near A_{12} in (M, K) so that H_1 is contained in V_{21} and that H_1 is disjoint from the K-compressing disc D_2 of A_{21}. Since $D_2 \subset V_{11} \cap V_{21}$, ∂D_2 is essential on A_{21}, and D_2 is a meridian disc of V_{21}. Note that $M = S^3$ because H_1 is contained in the ball obtained by cutting V_{21} along D_2. Then we can apply Theorem 7.3 below, which is an extension of Lemma 4.5 in [31]. Hence H_2 is weakly K-reducible. □

Theorem 7.3 ([24]). *Let M be a closed orientable 3-manifold, and L a link in M. Assume that M has a double cover branched along L. Let H_i be a g_i-genus n_i-bridge splitting of (M, L) for $i = 1$ and 2, and W a handlebody of genus g bounded by H_2 in M. Suppose that H_1 is contained in $\operatorname{int} W$, and that there is an L-compressing or meridionally compressing disc D of H_2 in $(W, L \cap W)$ with $D \cap H_1 = \emptyset$. Then either $M = S^3$ and L is the trivial knot, or H_2 is weakly L-reducible.*

Lemma 7.4. *Suppose that one of D_1 and D_2, say D_1 is essential, and that the other disc D_2 is inessential. Then H_2 is weakly K-reducible.*

Proof. K-compressing A_{11} along D_1, we obtain a meridian disc Q of V_{21} such that Q intersects K in at most one point.

Since ∂D_2 is inessential in A_{21} and the arcs $K \cap V_{11} \cap V_{21}$ connects A_{11} and A_{21}, D_2 is contained in (V_{22}, t_{22}). Hence H_2 is weakly K-reducible because $\partial Q \cap \partial D_2 = \emptyset$. $\square$

Lemma 7.5. *Suppose that D_1 and D_2 are inessential. If A_{11} or A_{21}, say A_{11} does not have a K-∂-compressing disc in $(V_{12} \cap V_{21}, t_{12})$, then H_2 has a satellite diagram on A_{21}.*

Proof. D_1 and D_2 are inessential. Then, by Lemma D.1, there is a cancelling disc C_1 of t_{21} in (V_{21}, t_{21}) with $\partial C_1 \cap H_2 \subset A_{21}$. There is a cancelling disc C_2 of t_{22} in (V_{22}, t_{22}) with $C_2 \cap D_2 = \emptyset$. Note that $\partial C_2 \cap H_2 \subset A_{21}$. Then C_1 and C_2 together give a 1-bridge diagram on A_{21}. $\square$

REMARK 7.6. Since A_{11} is K-∂-incompressible in $(V_{12} \cap V_{21}, t_{12})$, the loops $H_1 \cap H_2$ are of non-longitudinal slopes of V_{21} and V_{12}. But they may be of longitudinal slopes of V_{22} and V_{11}.

Thus we can assume that A_{j1} has an inessential K-compressing disc D_j in (V_{i1}, t_{i1}), and that A_{j1} has a K-∂-compressing disc in $(V_{j2} \cap V_{i1}, t_{j2})$ for $(i, j) = (1, 2)$ and $(2, 1)$. Isotoping H_1 and H_2 near A_{11} and A_{21} along these K-∂-compressing discs, we obtain the conclusion (d) of Theorem 1.3.

8. When $H_1 \cap H_2$ consists of two inessential loops

In this section, we consider the case where $H_1 \cap H_2$ consists of two loops which are inessential and K-essential on both H_1 and H_2. For $i = 1$ and 2, H_i contains two loops, say l_{i1} and l_{i2}, one of which, say l_{i1}, bounds a disc Q_i intersecting K in two points and disjoint from l_{i2}. Let R_i be the annulus cobounded by l_{i1} and l_{i2} on H_i, and H_i' the once punctured torus cut off by l_{i2} from H_i for $i = 1$ and 2. Note that R_i and H_i are disjoint from K for $i = 1$ and 2. Let V_{i1} be the solid torus bounded by H_i in M such that V_{i1} contains Q_j and H_j' for $(i, j) = (1, 2)$ and $(2, 1)$. Let V_{i2} be the other solid torus bounded by H_i in M with $R_j \subset V_{i2}$. Set $t_{ij} = K \cap V_{ij}$.

If $l_{11} = l_{22}$ and $l_{12} = l_{21}$, then $H_1' \cup R_1 \cup H_2'$ forms a closed surface of genus two which is disjoint from K and separates Q_1 and Q_2. See Fig. 8.1, which is schematic. This contradicts that both Q_1 and Q_2 intersect K and K is a knot rather than a link. Hence $l_{11} = l_{21}$ and $l_{12} = l_{22}$. See Fig. 8.2, which is also schematic. Let B be the ball component of $V_{12} \cap V_{22}$ bounded by $Q_1 \cup Q_2$.

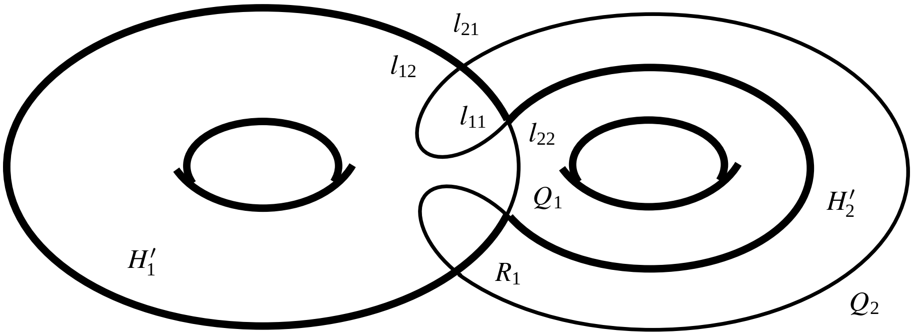

Fig. 8.1.

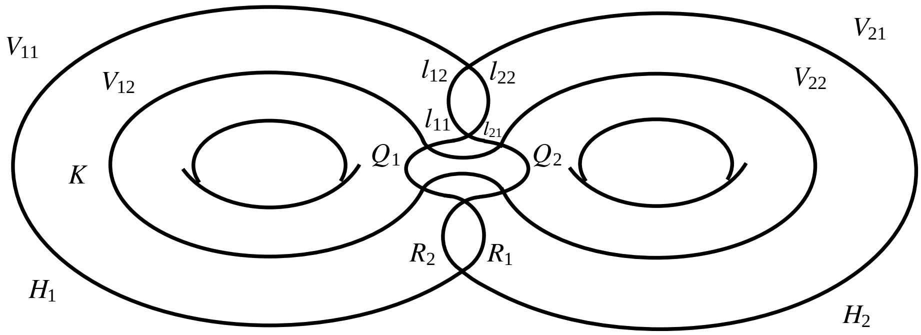

Fig. 8.2.

By Lemma 2.10, $Q_i \cup H_i'$ is K-compressible or K-∂-compressible in (V_{j1}, t_{j1}) for $(i, j) = (1, 2)$ and $(2, 1)$. Hence one of the following four conditions (i)–(iv) holds.
(i) H_i' has a K-compressing disc D in (V_{j1}, t_{j1}) with $D \cap Q_i = \emptyset$.
(ii) H_i' has a K-∂-compressing disc D in (V_{j1}, t_{j1}) with $D \cap Q_i = \emptyset$.
(iii) Q_i has a K-compressing disc D in (V_{j1}, t_{j1}) with $D \cap H_i' = \emptyset$.
(iv) Q_i has a K-∂-compressing disc D in (V_{j1}, t_{j1}) with $D \cap H_i' = \emptyset$.

Lemma 8.1. *Suppose that the condition* (iv) *holds for* $(i, j) = (1, 2)$ *or* $(2, 1)$. *Then* Q_1 *and* Q_2 *are* K*-parallel in* $(B, K \cap B)$, *and we can isotope* Q_1 *along* B *slightly beyond* Q_2 *to make* H_1 *and* H_2 *intersect in a single inessential and* K*-essential loop as in* Section 5.

Proof. Suppose, without loss of generality, that the condition (iv) holds for $i = (1, 2)$. Then there is a K-∂-compressing disc D of Q_1 in (V_{21}, t_{21}) with $D \cap H_1' = \emptyset$. If the arc $\partial D \cap H_2$ is contained in R_2, then it is an inessential arc in R_2 because it has both endpoints in l_{21}. This contradicts the unusual definition of K-∂-compressing disc. Hence the arc $\partial D \cap H_2$ is contained in Q_2, and Q_1 and Q_2 are K-parallel in $(B, K \cap B)$ by Lemma B.4. $\square$

Lemma 8.2. *Suppose that the condition* (ii) *holds for* $i = 1$ *or* 2. *Then one of the two conditions below holds.*
(1) *We can isotope* H_1 *and* H_2 *so that they intersect each other in a single loop which is inessential and* K*-essential on both* H_1 *and* H_2 *as in* Section 5.
(2) *One of* H_1 *and* H_2 *is weakly* K*-reducible.*

Proof. Assume that (ii) holds for $i = 1$. By the unusual definition of K-∂-compressing disc, the arc $\partial D \cap H_2$ is contained in H_2'. Hence D is also a K-∂-compressing disc of H_2' in (V_{11}, t_{11}). We isotope H_1 along the K-∂-compressing disc D, and make H_1 and H_2 intersect in a single inessential loop and two essential loops both in H_1 and H_2. We apply Lemma 10.1. The conclusion (1) of Lemma 10.1 implies that we can isotope one of the parallel annuli along the parallelism, and obtain the conclusion (1) of this lemma. The conclusion (2) of Lemma 10.1 implies that Q_1 and Q_2 are K-parallel before the isotopy along D. Then we can isotope Q_1 along the parallelism slightly beyond Q_2, to cancel the intersection $l_{11} = l_{21}$. Hence we obtain the conclusion (1) of this lemma again. The conclusion (3) of Lemma 10.1 is the conclusion (2) of this lemma. $\square$

Lemma 8.3. *Suppose that the condition* (i) *holds for* $(i, j) = (1, 2)$ *or* $(2, 1)$. *Then* Q_i *has a* K*-compressing disc in* (V_{j1}, t_{j1}), (*which may intersect* H_i').

Proof. Suppose that the condition (i) holds for $(i, j) = (1, 2)$. K-compressing H_1' along D, we obtain a disc D' with $D' \cap (K \cup Q_1) = \emptyset$ and $\partial D' = \partial H_1'$. Then the disc $R_2 \cup D'$ forms a K-compressing disc of Q_1. $\square$

By Lemmas 8.1, 8.2 and 8.3, we can assume that Q_i is K-compressible in (V_{j1}, t_{j1}) for $(i, j) = (1, 2)$ and $(2, 1)$.

Lemma 8.4. *Suppose that* Q_i *is* K*-compressible in* (V_{j1}, t_{j1}) *for* $(i, j) = (1, 2)$ *and* $(2, 1)$. *Then* K *is the trivial knot.*

Proof. Applying Lemma B.1 (2) to Q_1 in (V_{21}, t_{21}), we obtain a cancelling disc C_1 of t_{21} in V_{21} with $\partial C_1 \cap H_2 \subset Q_2$. Applying Lemma B.1 (3) to Q_2 in (V_{11}, t_{11}), we obtain a cancelling disc C_2 of t_{22} in $\mathrm{cl}(V_{11} - B)$ with $\partial C_2 \cap H_2 \subset Q_2$. Note that the interior of C_1 and C_2 may intersect each other in $V_{11} \cap V_{21}$. However, a standard innermost loop argument allows us to retake these discs so that their interiors are disjoint form each other. Then K has a 1-bridge diagram on the disc Q_2, and hence is trivial. $\square$

9. Two essential and inessential loops

In this section, we consider the case where $H_1 \cap H_2$ are consists of two loops both of which are inessential in one of H_1 and H_2, say H_1, and both essential in the other (ignoring the points $K \cap H_1$ and $K \cap H_2$). See Fig. 9.1.

Then a component of $H_1 \cap H_2$, say l_1, bounds a disc, say Q, in H_1 with $Q \cap H_2 = l_1$ and $|K \cap Q| = 2$. The other loop, say l_2, cobounds an annulus, say R, with l_1 in $H_1 - K$. Let H_1' be the once punctured torus bounded by l_2 in $H_1 - K$. Let V_{21} be the solid torus bounded by H_2 in M with $Q \cup H_1' \subset V_{21}$. The other solid torus bounded by H_2 is denoted by V_{22}. Then $R \subset V_{22}$. l_1 and l_2 together divide H_2 into two annuli, say A_1 and A_2. Since R is separating in V_{22} and is disjoint from K, one of A_1 and A_2, say A_1 intersects K in two points, and the other disjoint from K. Let V_{11} be the solid torus bounded by H_1 in M with $A_1 \subset V_{11}$. The other solid torus bounded by H_1 is denoted by V_{12}. Then $A_2 \subset V_{12}$. Set $t_{ij} = K \cap V_{ij}$ for $i,\ j \in \{1, 2\}$.

Lemma 9.1. *$M = S^3$, l_1 and l_2 are longitudinal loops of V_{22} and R is K-parallel to A_2 in (V_{22}, t_{22}).*

Proof. We ignore K in this paragraph. Since ∂Q is essential in H_2, it is of the meridional slope of V_{21}. Hence l_1 and l_2 are not of the meridional slope of V_{22}. Otherwise, $M \cong S^2 \times S^1$, which contradicts our assumption. Thus R is parallel to one of A_1 and A_2 in V_{22}. Hence we can isotope H_1 into $\operatorname{int} V_{21}$ so that H_1 is disjoint from a parallel copy of the meridian disc Q of V_{21}. Performing a K-compressing operation on H_2 along Q, we obtain a 2-sphere S in V_{21}, where S bounds a ball which contains H_1. This sphere S bounds another ball on the other side in a solid torus bounded by H_1. Hence $M = S^3$.

Since before the isotopy, l_1 and l_2 are of the meridional slope of V_{21}, they are of a longitudinal slope of V_{22}. Then R is parallel to A_1 and A_2 in V_{22} ignoring t_{22}. Because $V_{12} \cap V_{22}$ does not intersect K, A_{12} is K-parallel to A_2 in (V_{22}, t_{22}). □

By Lemma 2.10, $Q \cup H_1'$ is K-compressible or K-∂-compressible in (V_{21}, t_{21}). Hence we have the four cases below.

(1) Q has a K-compressing disc D in (V_{21}, t_{21}) with $D \cap H_1' = \emptyset$.
(2) Q has a K-∂-compressing disc D in (V_{21}, t_{21}) with $D \cap H_1' = \emptyset$.
(3) H_1' has a K-compressing disc D in (V_{21}, t_{21}) with $D \cap Q = \emptyset$.
(4) H_1' has a K-∂-compressing disc D in (V_{21}, t_{21}) with $D \cap Q = \emptyset$.

Lemma 9.2. *Suppose that A_1 has a K-compressing or a meridionally compressing disc P in $(V_{11} \cap V_{21},\ K \cap (V_{11} \cap V_{21}))$. Then H_2 is weakly K-reducible.*

Proof. Since R and A_2 are K-parallel in (V_{22}, t_{22}), we can isotope H_1 into $\operatorname{int} V_{21}$ so that H_1 is disjoint from P. Since $M \cong S^3$ by Lemma 9.1, M has a dou-

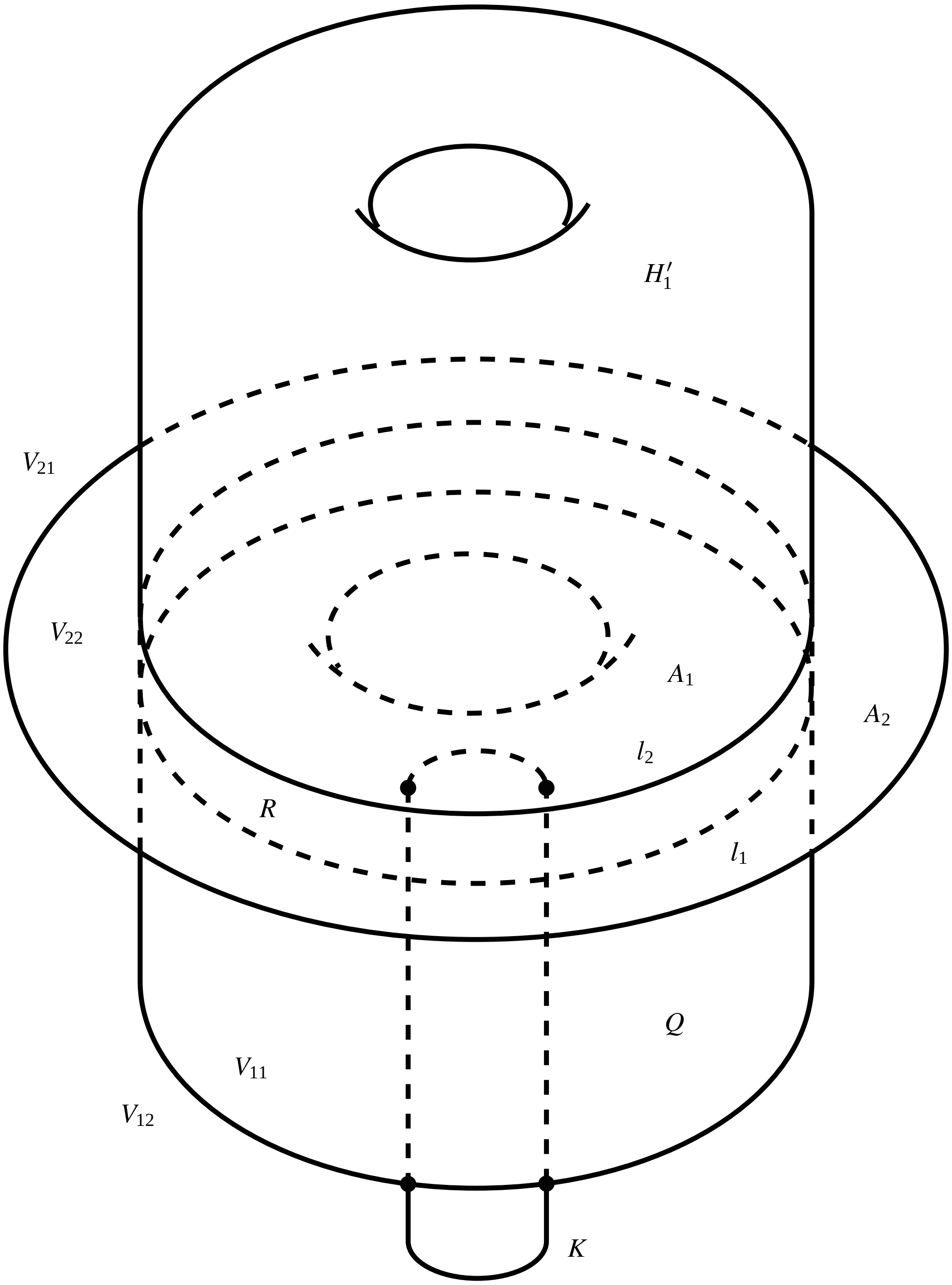

Fig. 9.1.

ble cover branched along K. Hence we can apply Theorem 7.3, to conclude that H_2 is weakly K-reducible. □

Lemma 9.3. *In case* (2) H_2 *is weakly* K*-reducible.*

Proof. Q has a K-∂-compressing disc D in (V_{21}, t_{21}) with $D \cap H_1' = \emptyset$. Since A_2 is disjoint from K, the arc $\partial D \cap H_2$ is contained in A_1, and D in $V_{11} \cap V_{21}$. The arc $\partial D \cap Q$ divides Q into two discs, each of which intersects K in a single point, and one of which, say Q', forms a meridian disc, say P, of V_{21} together with D. P is a meridionally compressing disc of A_1. Thus we conclude that H_2 is weakly K-reducible by Lemma 9.2. □

Lemma 9.4. *In case* (4) H_2 *is weakly* K*-reducible.*

Proof. H_1' has a K-∂-compressing disc D in (V_{21}, t_{21}) with $D \cap Q = \emptyset$. The arc $\partial D \cap H_2$ is essential in $A_1 - K$, and D is contained in $V_{11} \cap V_{21}$. Let O be the annulus obtained by K-∂-compressing a copy of H_1' along D. Note that O is contained in $V_{11} \cap V_{21}$. A component of ∂O is essential on H_2, and the other one is inessential and bounds a disc, say D', on A_1.

Suppose first that D' intersects K in a single point. Then the disc $O \cup D'$ gives a meridionally compressing disc of A_1. Hence H_2 is weakly K-reducible by Lemma 9.2.

Secondly, we suppose that D' intersects K in two points. We isotope H_1 in (M, K) along D slightly beyond the arc $\partial D \cap H_2$. Then H_1 and H_2 intersect each other in three K-essential loops, and $H_1 \cap H_2$ contains a loop which is essential on H_i and inessential on H_j for $(i, j) = (1, 2)$ and $(2, 1)$. Then by Lemma 10.3, H_1 and H_2 are weakly K-reducible. □

Lemma 9.5. *In case* (1) H_2 *is weakly* K*-reducible.*

Proof. In case (1), by K-compressing Q, we obtain a meridian disc Q' of V_{21} with $K \cap Q' = \emptyset$. A_1 is K-compressible or K-∂-compressible in (V_{11}, t_{11}) by Lemma 2.10.

Suppose that A_1 has a K-compressing disc. If it is contained in $V_{11} \cap V_{22}$, then, together with Q', it shows that H_2 is weakly K-reducible. If it is in $V_{11} \cap V_{21}$, then H_2 is weakly K-reducible by Lemma 9.2.

Hence we can assume that A_1 has a K-∂-compressing disc, say D_1, in (V_{11}, t_{11}). If D_1 is contained in V_{21}, then it is also a K-∂-compressing disc of Q or H_1'. Hence, by Lemmas 9.3 and 9.4, H_2 is weakly K-reducible. Thus D_1 is contained in $V_{11} \cap V_{22}$, and it is also a K-∂-compressing disc of R. By performing a K-∂-compressing operation on a copy of R, we obtain a peripheral K-compressing disc of A_2. This disc and Q' show that H_2 is K-reducible. □

Lemma 9.6. *In case* (3) H_2 *is weakly* K*-reducible.*

Proof. K-compressing H_1', we obtain a meridian disc E of V_{21} with $E \cap K = \emptyset$. If the K-compressing disc D is contained in $V_{11} \cap V_{21}$, then E is also in $V_{11} \cap V_{21}$, and H_2 is weakly K-reducible by Lemma 9.2. If E is contained in $V_{12} \cap V_{21}$, then the disc $E' = E \cup A_2$ forms a K-compressing disc of Q in (V_{21}, t_{21}), and H_2 is weakly K-reducible by Lemma 9.5. $\square$

10. When $|H_1 \cap H_2| = 3$

We consider the case where $|H_1 \cap H_2| = 3$ in this section. By Proposition 4.1, we can assume that $H_i - K$ does not contain three parallel loops of $H_1 \cap H_2$. Then H_i contains two essential loops and a single inessential loop of $H_1 \cap H_2$ for $i = 1$ and 2. Let Q_i be the disc bounded by the inessential intersection loop on H_i. Note that Q_i intersects K transversely in two points for $i = 1$ and 2. Let A_i be the annulus cut off from H_i by the two essential intersection loops such that $A_i \cap Q_i = \emptyset$ for $i = 1$ and 2. Set $H_i' = \mathrm{cl}(H_i - (A_i \cup Q_i))$, the 2-sphere with three holes, for $i = 1$ and 2. Let V_{i1} be the solid torus bounded by H_i in M with $Q_j \cup A_j \subset V_{i1}$ for $(i, j) = (1, 2)$ and $(2, 1)$. The other solid torus bounded by H_i is denoted by V_{i2}. Set $t_{ij} = K \cap V_{ij}$ for i, $j \in \{1, 2\}$.

Lemma 10.1. *Suppose that among the components of* $H_1 \cap H_2$ *essential loops on* H_1 *are essential also on* H_2*, and the inessential loop on* H_1 *is inessential also on* H_2*. See* Fig. 10.1*, which is schematic. Then one of the three conditions below holds.*
(1) A_1 *and* A_2 *are* K*-parallel, and the interior of the parallelism intersect neither* H_1 *nor* H_2*.*
(2) Q_1 *and* Q_2 *are* K*-parallel, and the interior of the parallelism intersect neither* H_1 *nor* H_2*.*
(3) *One of* H_1 *and* H_2 *is weakly* K*-reducible.*
In cases (1) *and* (2)*, we can isotope* H_1 *and* H_2 *in* (M, K) *so that* H_1 *and* H_2 *intersect in smaller number of non-empty collection of loops which are* K*-essential on both* H_1 *and* H_2*.*

Proof.
CLAIM 10.2. For each pair of $(i, j) = (1, 2)$ and $(2, 1)$, one of the three conditions below holds.
(i) The conclusion (1) or (2) of the lemma holds.
(ii) Q_j has a K-compressing disc in (V_{i1}, t_{i1}). (The compressing disc may intersect A_j.)
(iii) There is a meridian disc, say R_j, of V_{i1} with $\partial R_j \subset H_i'$, $R_j \cap (Q_j \cup A_j) = \emptyset$ and $|K \cap R_j| = 1$.

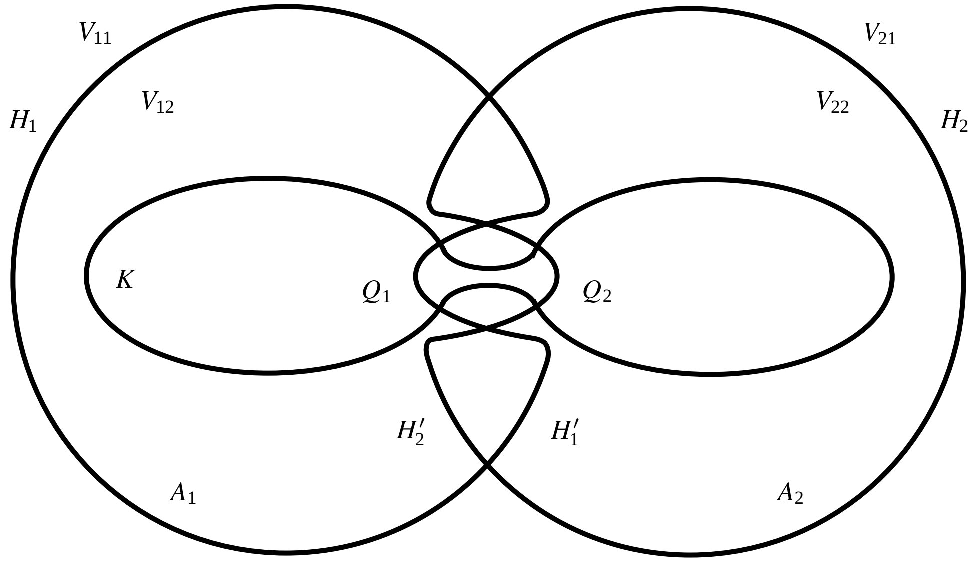

Fig. 10.1.

Proof. By Lemma 2.10, $Q_j \cup A_j$ is K-compressible or K-∂-compressible in (V_{i1}, t_{i1}).

Suppose first that $Q_j \cup A_j$ has a K-∂-compressing disc D. We assume first that D is incident to A_j. When $\partial D \subset A_1 \cup A_2$, A_1 and A_2 are K-parallel. This is the conclusion (1) of this lemma. Hence we can assume that $\partial D \subset A_j \cup H_i'$. K-∂-compressing a copy of A_j, we obtain a peripheral disc, say D', in V_{i1} such that $\partial D'$ and ∂Q_j cobounds an annulus on H_i'. The union of D' and this annulus forms a K-compressing disc of Q_j. This is the conclusion (ii). Hence we can assume that D is incident to Q_i. When $\partial D \subset Q_1 \cup Q_2$, Q_1 and Q_2 are K-parallel by Lemma B.4. This is the conclusion (2) of this lemma. When $\partial D \subset Q_j \cup H_i'$, by K-∂-compressing a copy of Q_j, we obtain two meridian discs intersecting K in a single point. After an adequate small isotopy, this meridian discs are disjoint from Q_j and A_j. This is the conclusion (iii).

Suppose that $Q_j \cup A_j$ has a K-compressing disc P. If P is incident to Q_j, then we are done. If P is incident to A_j, then, by K-compressing a copy of A_j, we obtain two meridian discs disjoint from K in V_{i1}. By K-compressing a copy of $H_i' \cup A_i$ along such a meridian disc, we obtain a K-compressing disc of Q_j in (V_{i1}, t_{i1}). This is the conclusion (ii). Thus Claim has proven. □

In case (ii), performing a K-compressing operation on a copy of Q_j, we obtain a peripheral disc S_j disjoint from K such that $\partial S_j = \partial Q_j$. Note that the disc S_j may intersect the annulus A_j. In case (iii), ∂R_j is parallel to a component of $\partial A_i = \partial A_j$ on H_i', and the loops ∂A_j are of meridional slope of the solid tori V_{i1} and V_{j2}.

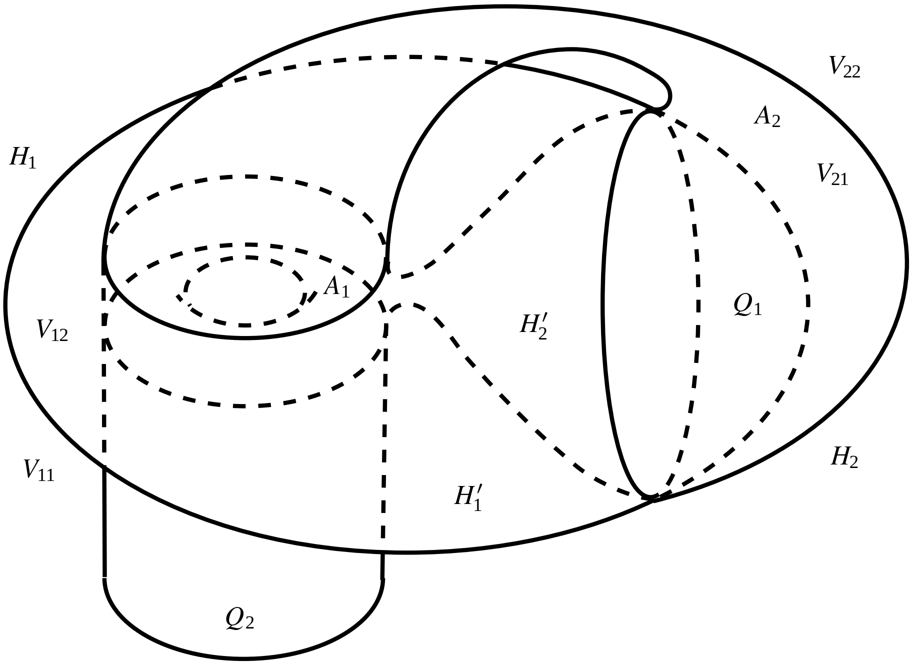

Fig. 10.2.

If (iii) holds for $(i, j) = (1, 2)$ and $(2, 1)$, then $M \cong S^2 \times S^1$, which contradicts our assumption. If (ii) holds, say for $(i, j) = (1, 2)$, and (iii) holds for $(i, j) = (2, 1)$, then S_2 and R_2 show that H_1 is weakly K-reducible.

Suppose that (ii) holds for $(i, j) = (1, 2)$ and $(2, 1)$. Since Q_1 is K-compressible in (V_{21}, t_{21}), Lemma B.1 (2) implies that t_{21} has a cancelling disc C_1 in V_{21} with $\partial C_1 \cap H_2 \subset Q_2$. Let B be the ball bounded by $Q_1 \cup Q_2$ in V_{11}. Since Q_2 is K-compressible in (V_{11}, t_{11}), Lemma B.1 (3) implies that t_{22} has a cancelling disc C_2 in $V_{11}' = \mathrm{cl}(V_{11} - B)$ with $\partial C_1 \cap \partial V_{11}' \subset Q_2$. C_1 and C_2 may intersect each other in loops in their interior. A standard innermost loop argument allows us to retake C_1 and C_2 so that their interiors are disjoint from each other and so that they give a 1-bridge diagram of K on Q_2. Hence K is the trivial knot, and Theorem B in [13] implies that H_2 is weakly K-reducible. □

Lemma 10.3. *Suppose that the inessential loop of $H_1 \cap H_2$ on H_i is essential on H_j for $(i, j) = (1, 2)$ and $(2, 1)$. See* Fig. 10.2. *Then H_i is K-reducible for $i = 1$ and 2.*

Proof.

CLAIM 10.4. For $(i, j) = (1, 2)$ and $(2, 1)$, Q_j has a K-compressing disc D_j with $D_j \cap A_j = \emptyset$ in (V_{i1}, t_{i1}).

Proof. Note that Q_j is a meridian disc of V_{i1} and that a component of ∂A_j, say l_1 is essential on H_i and the other, say l_2 is inessential on H_i. By Lemma A.2, there is a K-compressing disc D of A_j bounded by l_1. We take D so that $D \cap Q_j$ consists of minimal number of loops. Then a standard innermost loop argument shows that the intersection loops are essential on $Q_j - K$. If D is disjoint from Q_j, then the disc $D \cup A_i$ forms a K-compressing disc of Q_j. If D does intersect Q_j, then an innermost loop on D bounds a K-compressing disc of Q_j as desired. Thus Claim has proven. $\square$

For $(i, j) = (1, 2)$ and $(2, 1)$, the K-compressing disc D_j of Q_j is contained in $V_{i1} \cap V_{j2}$ since $V_{11} \cap V_{21}$ contains two subarcs of K connecting Q_1 and Q_2. By K-compressing Q_j along D_j, we obtain a meridian disc P_j of V_{i1} with $P_j \cap K = \emptyset$. Since D_i is a K-compressing disc of H_i in V_{i2}, and since ∂D_i and ∂P_j are disjoint, H_i is K-reducible. $\square$

11. When $|H_1 \cap H_2| = 4$

We consider in this section the case where $|H_1 \cap H_2| = 4$. By Proposition 4.1 we can assume for $i = 1$ and 2 that H_i does not contain 3 loops of $H_1 \cap H_2$ which are parallel in $H_i - K$. Then for each of $i = 1$ and 2, either

(I) H_i contains two essential intersection loops parallel in $H_i - K$ and two inessential intersection loops parallel in $H_i - K$, or

(II) H_i contains two families of two parallel essential intersection loops in $H_i - K$.

In case (I) the two points $K \cap H_i$ are contained in the disc component of $H_i - H_j$, and in case (II) the two points $K \cap H_i$ are contained in distinct and non-adjacent annulus components of $H_i - H_j$. In both cases, for $(i, j) = (1, 2)$ and $(2, 1)$, $H_i - H_j$ has two annulus components, say A_{i1} and A_{i2}, disjoint from K. They are contained in the same solid torus, say V_{j1}, bounded by H_j in M. Let V_{j2} be the other solid torus bounded by H_j in M, and set $t_{jk} = K \cap V_{jk}$ for j, $k \in \{1, 2\}$.

Lemma 11.1. *For $(i, j) = (1, 2)$ and $(2, 1)$, one of the two conditions below holds.*

(1) *A_{ik} is K-parallel to an annulus on H_j in (V_{j1}, t_{j1}) for $k = 1$ or $k = 2$.*

(2) *There is a cancelling disc C_1 of t_{j1} in (V_{j1}, t_{j1}) with $(\partial C_1 \cap H_j) \cap (H_1 \cap H_2) = \emptyset$. ($\mathrm{int}\, C_1$ may intersect H_i.)*

Moreover, if the essential loops of $H_1 \cap H_2$ are of a longitudinal slope of V_{j1}, then the conclusion (1) holds.

Proof. By using Lemma 2.10 repeatedly, performing K-compressing and K-∂-compressing operations on $A_{i1} \cup A_{i2}$, we can obtain discs.

Suppose that there are peripheral discs. Let Q_1 be the outermost one cutting off a ball B_1 from V_{j1} such that B_1 is disjoint from the other discs. If B_1 is disjoint from K, then Q_1 is yielded by a K-∂-compressing operation since $H_1 \cap H_2$ does not contain a K-inessential loop. Thus we obtain the conclusion (1). If B_1 contains t_{j1}, then the conclusion (2) holds. Note that the interior of the disc $B_1 \cap H_j$ is disjoint from $H_1 \cap H_2$ because Q_1 is outermost.

Suppose that there are no peripheral discs. Then the argument as in the proof of Lemma A.2 allows us to assume that the operations are all K-compressing ones, and that we obtain four meridian discs which are bounded by $H_1 \cap H_2$, and together divide V_{j1} into four balls. One of the balls contain t_{j1}, and the conclusion (2) holds.

If the essential loops of $\partial(A_{i1} \cup A_{i2})$ are of a longitudinal slope of V_{j1}, then Lemma A.1 implies that the conclusion (1) holds. $\square$

Lemma 11.2. *Suppose that, for* $(i, j) = (1, 2)$ *or* $(2, 1)$, $H_i \cap V_{j2}$ *contains a peripheral disc* Q_i *which intersects* K *in two points. Then one of the three conditions below holds.*
(1) A_{ik} *is* K*-parallel to an annulus on* H_j *in* (V_{j1}, t_{j1}) *for* $k = 1$ *or* 2.
(2) Q_i *is* K*-parallel to a disc in* H_j.
(3) H_j *is weakly* K*-reducible.*
In cases (1) *and* (2), Lemma 4.2 *allows us to isotope* H_1 *and* H_2 *in* (M, K) *so that* H_1 *and* H_2 *intersect each other in smaller number of non-empty collection of loops which are* K*-essential on both* H_1 *and* H_2.

Proof. The conclusion (1) of Lemma 11.1 is the conclusion (1) of this lemma. Hence we can assume that there is a cancelling disc C_1 of t_{j1} with $(\partial C_1 \cap H_j) \cap (H_1 \cap H_2) = \emptyset$ as in (2) of Lemma 11.1.

By Lemma 2.10, Q_i is K-compressible or K-∂-compressible in (V_{j1}, t_{j1}). Suppose first that Q_i is K-compressible. By K-compressing Q_i, we obtain a K-compressing disc D_i of H_j with $D_i \cap K = \emptyset$. Then D_i and C_1 are disjoint, and show that H_j is K-reducible.

Suppose that Q_i has a K-∂-compressing disc, say D, in (V_{j2}, t_{j2}). Since Q_i is peripheral, ∂Q_i bounds a disc Q_j on H_j. If the arc $\partial D \cap H_j$ is contained in Q_j, then we obtain the conclusion (2) by Lemma B.4. Hence we can assume that the arc $\partial D \cap H_j$ is disjoint from $\operatorname{int} Q_j$. The arc $\partial D \cap Q_i$ divides Q_i into two discs, and let R_i be one of them. R_i intersects K in a single point, and $R_i' = R_i \cup D$ forms a meridian disc of V_{j2} by the unusual definition of K-∂-compressibility. After an adequate small isotopy, R_i' is disjoint from Q_i. Since the two points $K \cap H_j$ is contained in Q_j, $\partial C_1 \cap H_j \subset \operatorname{int} Q_j$ and $\partial R_i' \cap \partial C_1 = \emptyset$. Hence H_j is weakly K-reducible. $\square$

Lemma 11.3. *Suppose that, for* $(i, j) = (1, 2)$ *or* $(2, 1)$, *a component of* $H_i \cap V_{j2}$ *forms a meridian disc* Q_i *of* V_{j2} *such that* Q_i *intersects* K *in two points. Then one of the two conditions below holds.*
(1) *We can isotope* H_1 *and* H_2 *in* (M, K) *so that* H_1 *and* H_2 *intersect each other in*

smaller number of non-empty collection of loops which are K-essential on both H_1 and H_2.
(2) *H_j is weakly K-reducible.*

Proof. The conclusion (1) of Lemma 11.1 implies the conclusion (1) of this lemma by Lemma 4.2. Hence we can assume that there is a cancelling disc C_1 of t_{j1} with $(\partial C_1 \cap H_j) \cap (H_1 \cap H_2) = \emptyset$ as in (2) of Lemma 11.1.

H_i intersects V_{j2} in a disjoint union of Q_i and a 2-sphere with three holes, say P. By Lemma 2.10, $Q_i \cup P$ is K-compressible or K-∂-compressible in (V_{j2}, t_{j2}). Suppose first that $Q_i \cup P$ has a K-compressing disc D. Then K-compressing $Q_i \cup P$, we obtain a disc component D' disjoint from K and bounded by a component of $H_1 \cap H_2$. Since $\partial D' \cap C_1 = \emptyset$, H_j is K-reducible. Note that similar argument shows H_j is weakly K-reducible when P is meridionally K-compressible in (V_{j2}, t_{j2}).

Suppose that $Q_i \cup P$ has a K-∂-compressing disc D. If D is incident to Q_i, then performing a K-∂-compressing operation on a copy of Q_i, we obtain a meridian disc D_1 and a peripheral disc D_2 such that each of them intersects K in a single point. Let D_2' be the disc bounded by ∂D_2 on H_j. Then we can see that D_2' also intersects K in a single point, considering the 2-sphere $D_2 \cup D_2'$. If $H_1 \cap H_2$ contains an inessential loop on H_j, then such a loop bounds a disc intersecting K in two points, and hence must intersect the arc $\partial D \cap H_j$, which is a contradiction. Hence no loop of $H_1 \cap H_2$ is inessenital on H_j, and ∂D_1 and a loop, say l, of $H_1 \cap H_2$ cobound an annulus, say R, disjoint from K on H_j. (Recall that $H_j - K$ does not contain three parallel loops of $H_1 \cap H_2$.) After an adequate small isotopy, the disc $R' = R \cup D_1$ gives a meridian disc of V_{j2} with $|R' \cap K| = 1$ and $\partial R' = l$. Since $\partial R' \cap \partial C_1 = \emptyset$, H_j is weakly K-reducible.

Hence we can assume that D is incident to P. Suppose that a K-∂-compressing operation on a copy of P along D yields a K-inessential boundary loop. Then this loop bounds a disc intersecting K in at most one point on H_j, and we isotope this disc near its boundary along the copies of D, to obtain a K-compressing disc or meridionally compressing disc of P in (V_{j2}, t_{j2}). Hence we obtain the conclusion (2) by a similar argument in the second paragraph.

Hence we can assume that we can isotope H_i in (M, K) along D slightly beyond the arc $\partial D \cap H_j$, so that H_1 and H_2 intersect each other in K-essential loops after the isotopy. When the arc $\partial D \cap H_j$ connects distinct components of ∂P, this isotopy decreases the number of intersection loops, and we obtain the conclusion (1). When the arc $\partial D \cap H_j$ has both endpoints in the same component of ∂P, this isotopy increases the number of intersection loops by one. Note that $H_1 \cap H_2$ has three parallel loops on H_i. We apply Proposition 4.1. The conclusion (1) of Proposition 4.1 implies that we can further isotope H_1 and H_2 so that H_1 and H_2 intersects each other in non-empty collection of three or less number of loops which are K-essential both on H_1 and H_2. The conclusion (2) of Proposition 4.1 is the conclusion (2) of this lemma. The conclusion (3) of Proposition 4.1 is impossible, since $H_1 \cap H_2$ contains ∂Q_i of the meridional slope of V_{j2}. □

Lemma 11.4. *Suppose that, for* $(i, j) = (1, 2)$ *and* $(2, 1)$, $H_i \cap V_{j2}$ *consists of two annuli, say* R_{i1} *and* R_{i2}, *such that each of them intersects* K *in a single point. Then one of the four conditions below holds.*
(1) A_{ik} *is* K*-parallel to an annulus on* H_j *in* (V_{j1}, t_{j1}) *for* $k = 1$ *or* 2.
(2) *One of* R_{i1} *and* R_{i2} *is* K*-parallel to an annulus on* H_j *in* (V_{j2}, t_{j2}).
(3) *One of* H_1 *and* H_2 *is* K*-reducible.*
(4) *One of* H_1 *and* H_2 *has a satellite diagram of non-meridional and non-longitudinal slope given by a loop of* $H_1 \cap H_2$.
In cases (1) *and* (2), Lemma 4.2 *allows us to isotope* H_1 *and* H_2 *in* (M, K) *so that* H_1 *and* H_2 *intersect each other in smaller number of non-empty collection of loops which are* K*-essential on both* H_1 *and* H_2.

Proof. Note that the loops $H_1 \cap H_2$ are essential both on H_1 and H_2.

The conclusion (1) of Lemma 11.1 is the conclusion (1) of this lemma. Hence we can assume that there is a cancelling disc C_1 of t_{j1} with $(\partial C_1 \cap H_j) \cap (H_1 \cap H_2) = \emptyset$ as in (2) of Lemma 11.1.

By Lemma 2.10, $R_{i1} \cup R_{i2}$ is K-compressible or K-∂-compressible. Suppose first that it is K-compressible. The K-compression yields a meridian disc disjoint from K and C_1. Then H_j is K-reducible.

Suppose that $R_{i1} \cup R_{i2}$ is K-incompressible, and has a K-∂-compressing disc, say D, in (V_{j2}, t_{j2}). We can assume, without loss of generality, that D is incident to R_{i1}. If the arc $\partial D \cap R_{i1}$ is essential on R_{i1}, then we obtain the conclusion (2) by Lemma C.2. Hence we can assume that the arc $\partial D \cap R_{i1}$ is inessential on R_{i1}. Then by Lemma C.3 either the conclusion (2) holds, or there is a cancelling disc C_2 of t_{j2} in (V_{j2}, t_{j2}) such that $\partial C_2 \cap H_j$ is disjoint from a component of ∂R_{i1}. Hence C_1 and C_2 together show that H_j has a satellite diagram. If the slope of the satellite diagram is meridional, then K is the trivial knot. If the slope of the satellite diagram is longitudinal, then one of the conclusions (1) and (2) of this lemma holds by Lemmas A.1 and C.3. □

12. Semi-satellite diagrams

In this section, we will show that the conclusion of Lemma 6.3 implies that either H_i admits a satellite diagram or K is a torus knot.

Lemma 12.1. *Let* M, K, H_i, V_{ik}, t_{ik}, A_{jk} *as in* Section 6. *Suppose that the loops* $H_1 \cap H_2$ *are non-meridional and non-longitudinal with respect to* V_{ik} *for* i, $k \in \{1, 2\}$. *For* $(i, j) = (1, 2)$ *and* $(2, 1)$, *if* A_{j1} *and* A_{j2} *are of the form as in the conclusion* Lemma C.3 (2) *in* Appendix C, *then either*
(1) H_i *admits a satellite diagram of non-meridional and non-longitudinal slope, or*

(2) *K has a* 1*-bridge diagram with no crossings on H_i such that the diagram intersects each component of $H_1 \cap H_2$ in a single point.*

Proof. Since A_{j1} and A_{j2} are of the form of Lemma C.3 (2), H_i has a semi-satellite diagram of non-longitudinal and non-meridional slope. More precisely, there is a cancelling disc C_{ik} of t_{ik} in (V_{ik}, t_{ik}) such that $\partial C_{ik} \cap H_i$ is disjoint from a component, say l_{ik}, of $H_1 \cap H_2$ for $k = 1$ and 2. If l_{i1} and l_{i2} are the same component of $H_1 \cap H_2$, then the semi-satellite diagram is a satellite diagram, which is the conclusion (1). Hence we can assume $l_{i1} \neq l_{i2}$.

We retake the cancelling disc C_{ik} so that it intersects l_{ih} in minimum number of points for $(k, h) = (1, 2)$ and $(2, 1)$. Let n_k be the number of intersection points of $\partial C_{ik} \cap l_{ih}$. We will show that $n_k = 1$ for $k = 1$ and 2.

For $(i, j) = (1, 2)$ and $(2, 1)$ and $k \in \{1, 2\}$, A_{jk} is isotopic in (V_{ik}, t_{ik}) to the annulus which is the union of the two annuli R_{k1} and R_{k2} as below. R_{k1} is obtained by cutting a copy of H_i along l_{ik} and isotoping along C_{ik}. R_{k2} is obtained from a copy of one of the two annuli on H_i between the loops $H_1 \cap H_2 = l_{i1} \cup l_{i2}$ by slightly isotoping into $\operatorname{int} V_{ik}$. R_{k1} is disjoint from K, and R_{k2} intersects K in a single point.

Sublemma 12.2. $n_k = 1$ *for* $k = 1$ *and* 2.

Proof. If R_{12} and R_{22} are copies of the same annulus A_{i1} or A_{i2}, say A_{i1}, then H_j is isotopic to a union of two parallel copies of A_{i2} in M (ignoring K), and hence the loops ∂A_{i2} are of longitudinal slope of one of V_{i1} and V_{i2} because H_j is a Heegaard splitting torus. This is a contradiction. See Fig. 12.1.

Hence we can assume that R_{k2} is a copy of A_{ik} for $k \in \{1, 2\}$. See Fig. 12.2. We can isotope H_j so that R_{k2} is contained in A_{ik}, that one component of ∂R_{k2} coincides with l_{ih} and the other is contained in $\operatorname{int} A_{ik}$ and is very close to l_{ik} for $(k, h) = (1, 2)$ and $(2, 1)$.

Set $A'_{ik} = \operatorname{cl}(A_{ik} - R_{k2})$ after the isotopy. Then $t_{ih} = t_{jk}$ for $(i, j) = (1, 2)$ and $(2, 1)$ and for $(k, h) = (1, 2)$ and $(2, 1)$. Note that $(\partial C_{ih} \cap H_i) \subset (R_{12} \cup A'_{ik} \cup R_{22})$ for $(k, h) = (1, 2)$ and $(2, 1)$.

A'_{ik} is K-incompressible and K-∂-incompressible in $(V_{jk} \cap V_{ik}, K \cap (V_{jk} \cap V_{ik}))$ since $H_1 \cap H_2$ is of non-meridional and non-longitudinal slope of V_{ik}. It is also K-incompressible in $(V_{jk} \cap V_{ih}, K \cap (V_{jk} \cap V_{ih}))$ since $H_1 \cap H_2$ is of non-meridional slope of V_{ih}. Since A'_{ik} is K-compressible or K-∂-compressible in (V_{jk}, t_{jk}) by Lemma 2.10, A'_{ik} has a K-∂-compressing disc P in $V_{jk} \cap V_{ih}$. We leave the proof of the next claim to readers. See Fig. 12.3, where A'_{ik} is contracted to l_{ik}.

CLAIM 12.3. Let D be a disc properly embedded in the solid torus $V_{jk} \cap V_{ih}$. Suppose that D is a K-compressing disc of the toral boundary of the solid torus, and is disjoint from C_{ih}. Then ∂D intersects A'_{ik} in n_h or larger number of arcs. If ∂D in-

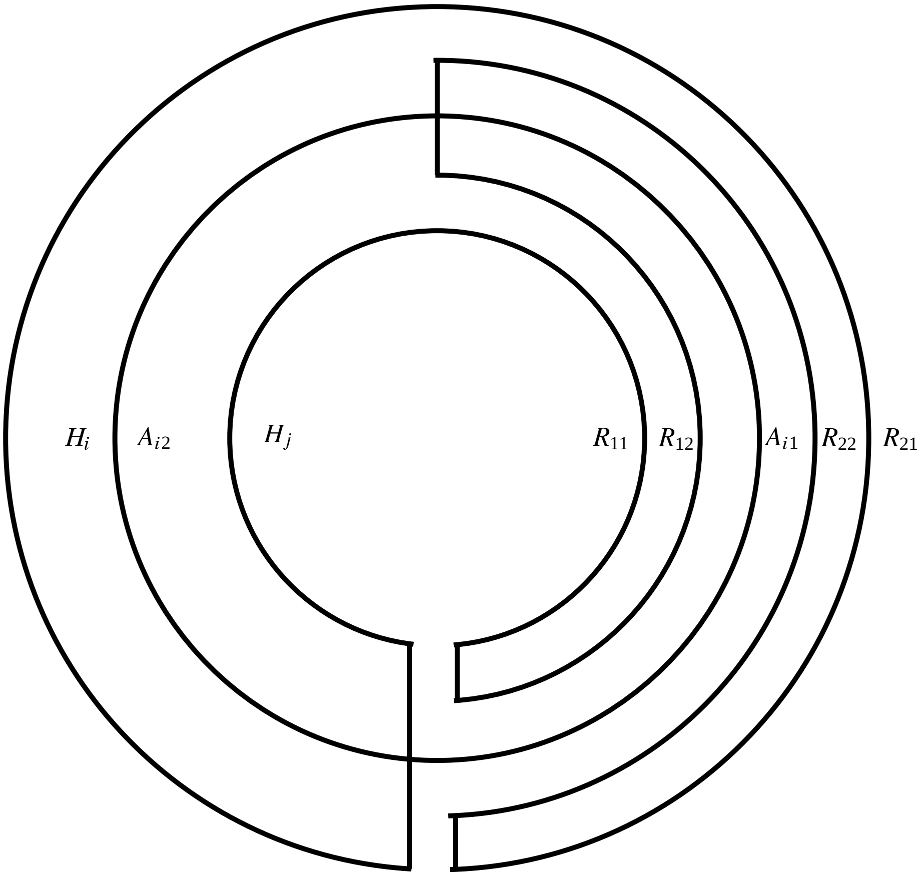

Fig. 12.1.

tersects A'_{ik} in just n_h arcs, then D is a meridian disc of $V_{jk} \cap V_{ih}$.

Recall that P is a K-∂-compressing disc of A'_{ik} in $V_{jk} \cap V_{ih}$. ∂P intersects A'_{ik} in a single essential arc. We retake P among all such K-∂-compressing discs so that ∂P intersects $\partial C_{ih} \cap H_i$ in minimum number of points. A standard innermost loop argument allows us to isotope P so that each component of $P \cap C_{ih}$ is an arc rather than a loop.

If P is disjoint from C_{ih}, then Claim shows that ∂P intersect A'_{ik} in n_h or larger number of arcs. In fact, $\partial P \cap A'_{ik}$ consists of a single arc, and hence we obtain $n_h = 1$ as desired. We consider the case $P \cap C_{ih} \neq \emptyset$. We can isotope P near ∂P so that $\partial P \cap \partial C_{ih} \cap A'_{ij} = \emptyset$. Let ρ be an arc of $P \cap C_{ih}$ such that ρ is outermost away from A'_{ik} on P. That is, ρ cuts off a disc P' from P such that $\partial P'$ is disjoint from A'_{ik}. The arc ρ divides C_{ih} into two discs C' and C'' where $\partial C'$ is disjoint from t_{ih} and

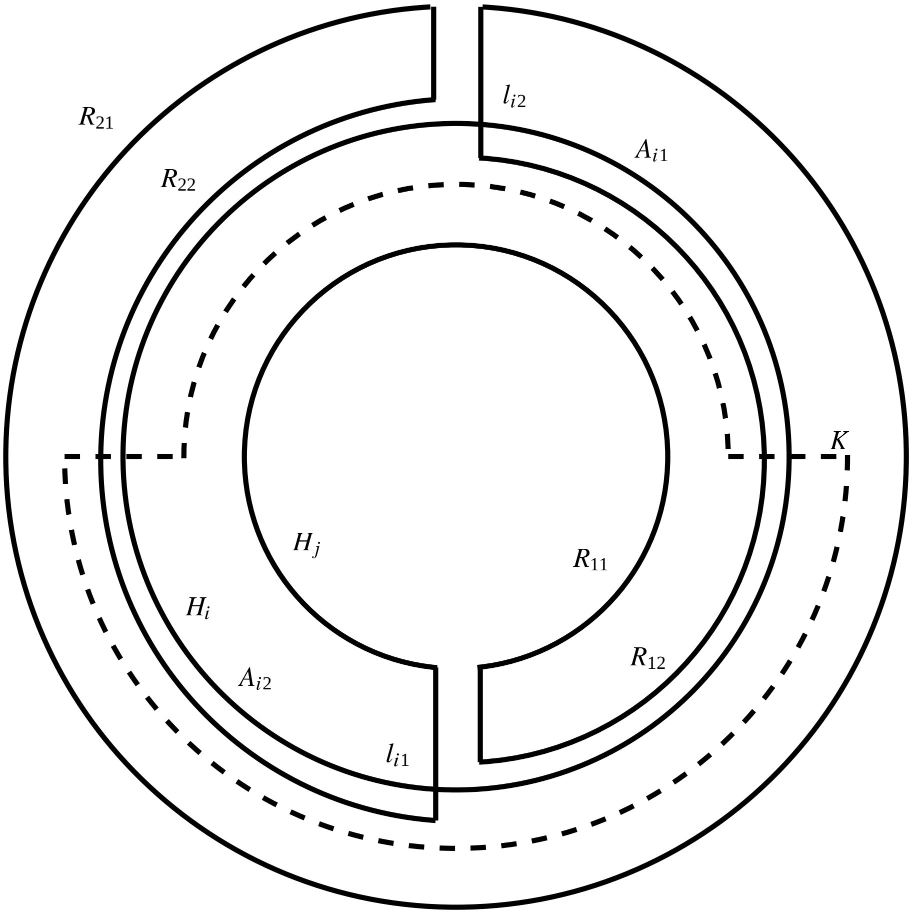

Fig. 12.2.

$\partial C''$ contains t_{ih} entirely. Let D be a disc obtained from the disc $C' \cup P'$ by isotoping slightly to be disjoint from C_{ih}.

We first suppose that D is not a K-compressing disc of the torus $\partial V = \partial(V_{jk} \cap V_{ih})$. Then ∂D bounds a disc D' on ∂V such that D' is disjoint from K. Note that $\partial P'$ may intersect R_{h1}, while $\partial C'$ does not intersect R_{h1}, and that $\partial C'$ may intersect A'_{ik}, while $\partial P'$ does not intersect A'_{ik}. Hence we can isotope P' near the arc $\partial P' \cap \partial V$ so that P' does not intersect $R_{h1} \cup A'_{ik}$. See Fig. 12.4. If $\partial C' \cap A'_{ik} \neq \emptyset$, then the disc $C'' \cup P'$ gives a cancelling disc of t_{ih} which intersects A'_{ik} in smaller number of arcs. This is a contradiction.

If $\partial C' \cap A'_{ik} = \emptyset$, then let μ be an outermost arc of $P \cap C_{ih}$ on C', and C''' the outermost disc. μ cuts P into two discs, one of which, say P'', intersects A'_{ik} in a single arc. Then $P'' \cup C'''$ gives a K-∂-compressing disc of A'_{ik}, intersecting ∂C_{ih} in

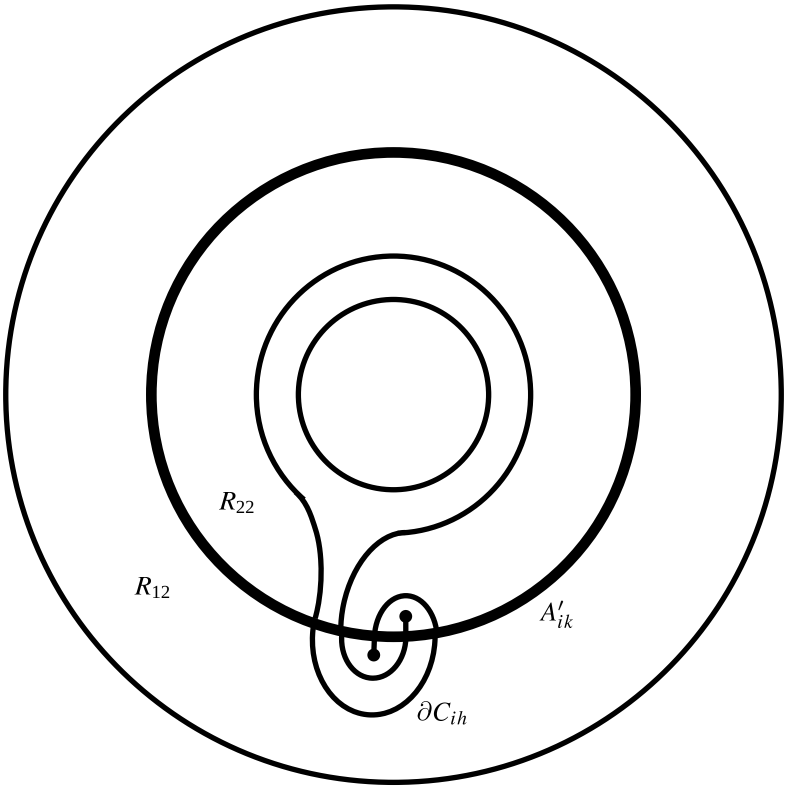

Fig. 12.3.

smaller number of points than P. This is again a contradiction.

Hence we can assume that D is a K-compressing disc of ∂V. By Claim 12.3, ∂D intersects A'_{ik} in n_h or more arcs. Since $n_h = |\partial C_{ih} \cap l_{jk}|$, and since $\partial P' \cap A'_{ik} = \emptyset$, the arc $\partial C' \cap H_i$ intersects A'_{ik} in n_h arcs, and also ∂D does in only n_h arcs. Hence ∂D is a meridian disc of V, and the arc $\partial P' \cap \partial V$ must intersect R_{h1}. Moreover, the two arcs $(\partial C_{ih} - \partial C') \cap H_i$ do not intersect A'_{ik}. One of them and a subarc of $\partial P' \cap \partial V$ give an arc in R_{12} such that it connects a point of $K \cap H_i$ and a component of ∂R_{h1}, and is disjoint from the arc $\partial C_{ih} \cap H_i$. Hence there is no arc component of $\partial C_{ih} \cap R_{12}$ with both endpoints in the same component of ∂R_{12}. Then we obtain $n_h = 1$. □

Thus we have shown that ∂C_{ih} intersects the loop l_{ik} in a single point for $(k, h) = (1, 2)$ and $(2, 1)$. Then we can take the discs C_{i1} and C_{i2} so that $\partial C_{i1} \cap \partial C_{i2} = K \cap H_i$. (In general, there is only a single isotopy class of arcs connecting a fixed point in an annulus and a fixed boundary component of the annulus.) This implies that K has a 1-bridge diagram on H_i with no crossing points. □

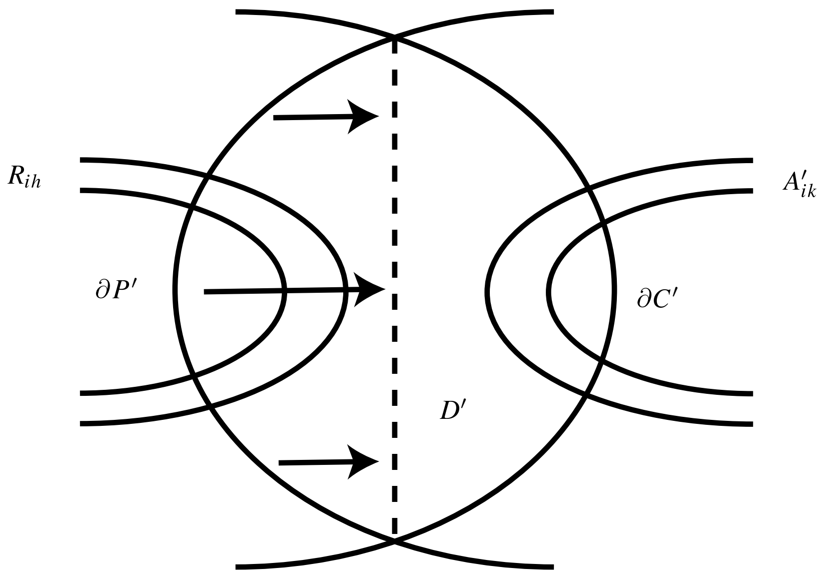

Fig. 12.4.

In the case of Lemma 12.1 (2), K is a torus knot. If K is the trivial knot or a core knot, then H_1 and H_2 are weakly K-reducible as noted in Section 3. Hence we obtain the conclusion (2) of Theorem 1.2. If K is neither the trivial knot nor a core knot, then H_i is "cancellable" by the result of Theorem 3 in [27]. That is, for $i = 1$ and 2, there is a cancelling disc C_{ik} of t_{ik} for $k = 1$ and 2 such that $\partial C_{i1} \cap \partial C_{i2} = K \cap H_i$. The loop $K' = (\partial C_{i1} \cup \partial C_{i2}) \cap H_i$ is isotopic to K and is of non-meridional and non-longitudinal slope. Its exterior $E(K') = M - \operatorname{int} N(K')$ has a Seifert fibering structure over a disc with two exceptional fibres such that $H_i \cap E(K')$ is a vertical essential annulus with respect to the fibering. Note that such an annulus is unique up to isotopy in $E(K')$. Hence H_1 and H_2 are isotopic in (M, K), and we obtain the conclusion (1) of Theorem 1.2.

13. An example of case (b)

We give an example of a pair of a once punctured lens space X and two disjoint arcs s_1 and s_2 properly embedded in X as described in the conclusion (b) of Theorem 1.3. That is, the exterior $E_i = \mathrm{cl}(X - N(s_i))$ of the string s_i is a solid torus, and the other arc s_j is trivial in E_i for $(i, j) = (1, 2)$ and $(2, 1)$. We will give an example where s_1 and s_2 are not "parallel".

The exterior $E = \mathrm{cl}(X - N(S))$ of the two strings is homeomorphic to a handlebody of genus two since E is the exterior of s_2 in E_1. In Fig. 13.1 we can find the boundary of the ball B obtained from E by cutting along two discs D_1 and D_2. We

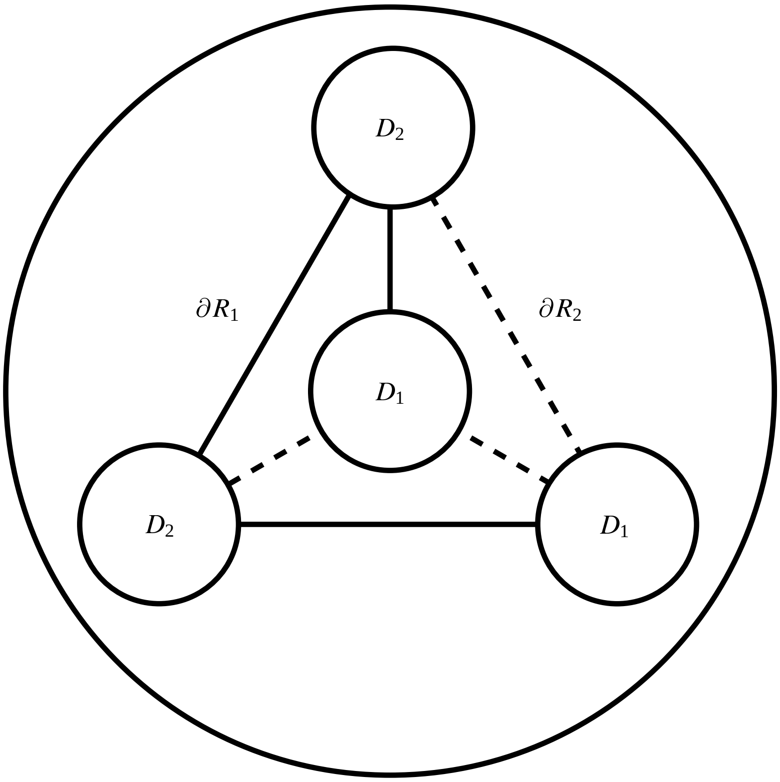

Fig. 13.1.

can find two copies of D_i in the figure for $i = 1$ and 2. The neighbourhood $N(s_i)$ of s_i contains a meridian disc R_i of s_i. That is, by some homeomorphism $N(s_i) \cong D^2 \times s_i$ with $D^2 \times \partial s_i = N(s_i) \cap \partial X$, the disc R_i is mapped to $D^2 \times p$, where p is a point of $\operatorname{int} s_i$. In the figure, ∂R_1 is described by three solid lines, and ∂R_2 is described by three broken lines. The four copies of D_1 and D_2 and these six lines together form vertices and edges of a 1-skelton Δ of a tetrahedron. Note that neither the union of the three solid lines nor the union of the three broken lines forms a triangle. Each copy of D_i intersects ∂R_i in a single point, and intersects ∂R_j in two points for $(i, j) = (1, 2)$ and $(2, 1)$. We can recover E_j by attaching the 2-handle $N(s_i)$ on E along ∂R_i. D_i shows that s_i is trivial in E_j.

We will show that s_1 and s_2 are not "parallel" in (X, S). The arcs s_1 and s_2 are *parallel* if there is a disc P properly embedded in E such that $P \cap \partial R_i$ is a single point for $i = 1$ and 2. We assume, for a contradiction, that there is such a disc P. We take P so that it intersects $\partial D_1 \cup \partial D_2$ in minimum number of points over all the discs of parallelism. A standard innermost argument allows us to isotope P so that P intersects D_1 and D_2 in arcs only. Then there is an outermost arc α of $(D_1 \cup D_2) \cap P$

on P, that is, α cuts off a disc, say Q, from P such that $Q \cap (D_1 \cup D_2) = \alpha$. We can take α so that $\partial Q \cap (\partial R_1 \cup \partial R_2)$ is empty or a single point. The arc $\beta = \partial Q \cap \partial P$ is contained in the 2-sphere ∂B, and has both endpoints in the same copy D of D_1 or D_2. If β does not intersect $\partial R_1 \cup \partial R_2$, then it is entirely contained in a single face of the tetrahedron, and hence we can isotope P near β in E so that ∂P intersects $\partial D_1 \cup \partial D_2$ in smaller number of points. This is a contradiction. If β intersects $\partial R_1 \cup \partial R_2$ in a single point, then it is contained in a union of two adjacent faces of the tetrahedron. Hence β and a subarc, say γ, of ∂D cobound a disc O intersecting $\partial R_1 \cup \partial R_2$ in a single arc connecting β and γ. We can isotope P along O so that ∂P intersects $\partial D_1 \cup \partial D_2$ in smaller number of points, which is a contradiction. Thus P is disjoint from D_1 and D_2. ∂P is contained in ∂B so that it is disjoint from the four vertices of the tetrahedron. Since ∂P intersects the 1-skelton Δ of the tetrahedron in two points, it is contained in a union of two adjacent faces and bounds a disc intersecting Δ in a subarc of an edge. Then ∂P intersects ∂R_i in two points for $i = 1$ or 2, which is a contradiction.

Similar argument shows that s_i is not trivial in (X, S) for $i = 1$ and 2, that is, there is a disc P_i properly embedded in E so that ∂P_i intersects ∂R_i in a single point and is disjoint from ∂R_j neither for $(i, j) = (1, 2)$ nor $(2, 1)$.

Similar situations are studied in Lemma 2.3.2 in [2], [10], [12].

A. Annuli disjoint from t

Let V be a solid torus, and t a trivial arc in V. Let A be an annulus properly embedded in V with $A \cap t = \emptyset$.

Lemma A.1. *Suppose that the loops of ∂A are essential on ∂V. Let A_1 and A_2 be the annuli obtained by cutting ∂V along ∂A. Suppose $\partial t \subset A_1$. Let R_i be the region bounded by the torus $A_i \cup A$ in V. Then one of the three conditions below holds.*
(1) *A is t-compressible, the loops ∂A bound meridian discs of V disjoint from a cancelling disc of t. Moreover, if R_1 contains a t-compressing disc of A, then A_2 is t-incompressible in R_2.*
(2) *A has a t-∂-compressing disc in R_2, and is t-parallel to A_2.*
(3) *A has a t-∂-compressing disc D in R_1, and there is a cancelling disc C of t such that $\partial C \cap \partial V \subset \operatorname{int} A_1$ and $C \cap (A \cup D) = \emptyset$.*
Moreover, if the slope of ∂A is longitudinal on ∂V, then (2) *holds.*

Proof. Lemma 2.10 implies that A is t-compressible or t-∂-compressible in (V, t). Suppose that A has a t-compressing disc Q. Since $A \cap t = \emptyset$, ∂Q divides A into two annuli. We perform a t-compressing operation on A along Q, that is, take a tubular neighbourhood $N \cong Q \times [0, 1]$ of Q so that $N \cap A = \partial Q \times [0, 1]$, and we deform A into the surface $(A - (\partial Q \times [0, 1])) \cup (Q \times \{0\} \cup Q \times \{1\})$. Then we obtain meridian discs D_1, D_2 disjoint from t and bounded by ∂A. Then a standard innermost loop and outermost arc argument allows us to take a cancelling disc of t to be disjoint from D_1, D_2.

Suppose $Q \subset R_1$. We take a straight arc α in $N \subset R_1$ connecting $Q \times \{0\}$ and $Q \times \{1\}$. After the t-compressing operation, α connects D_1 and D_2, and is contained in the ball bounded by $D_1 \cup D_2 \cup A_2$ in V. If A_2 has a t-compressing disc P, then ∂P separates ∂D_1 and ∂D_2 on A_2 because $A_2 \cap t = \emptyset$. Hence P must intersect $D_1 \cup D_2$ or α, and cannot be contained in R_2 entirely.

Suppose that A is t-∂-compressible. Since A is disjoint from t, any t-∂-compressing disc intersects A in an essential arc. Performing a t-∂-compressing operation on A, we obtain a disc, say D, in V. Since the loops ∂A are essential on ∂V, ∂D bounds a disc, say D', on ∂V, and the 2-sphere $D \cup D'$ bounds a ball, say B, in V. If B does not contain t, then D is t-parallel to D', and A is also t-parallel to an annulus in ∂V. If $t \subset B$, then a standard innermost loop and outermost arc argument allows us to take a cancelling disc C of t with $C \cap D = \emptyset$ in B.

If the slope of ∂A is longitudinal, then it is well-known that in both regions R_1 and R_2 A has ∂-compressing discs, which in R_2 is disjoint from t. □

Lemma A.2. *Suppose that a component, say l_1, of ∂A is essential on ∂V and the other component, say l_2, is not. Let Q be the disc bounded by l_2 on ∂V. Suppose $\partial t \subset Q$. Then* (1) *l_1 bounds a meridian disc, say D_1, with $D_1 \cap (t \cup (A - l_1)) = \emptyset$,* (2) *$l_2$ bounds a peripheral disc, say D_2, with $D_2 \cap (t \cup (A - l_2)) = \emptyset$, and* (3) *there is a cancelling disc C of t with $\partial C \cap \partial V \subset Q$ and $C \cap D_2 = \emptyset$.*

Proof. By Lemma 2.10, A is t-compressible or t-∂-compressible in (V, t).

Suppose that A is t-compressible. Since $A \cap t = \emptyset$, performing a t-compressing operation on a copy of A, we obtain a meridian disc D_1 bounded by l_1, and a peripheral disc D_2 bounded by l_2 disjoint from t. An adequate small isotopy makes their interior be disjoint from A. A standard innermost loop and outermost arc argument allows us to take a cancelling disc C of t with $C \cap D_2 = \emptyset$. Since $\partial t \subset Q$, the arc $\partial C \cap H_1$ is contained in Q.

Suppose that A is t-∂-compressible. Since $A \cap t = \emptyset$, performing a t-∂-compressing operation on A, we obtain a meridian disc, say D, of V with $D \cap A = \emptyset$ after an adequate small isotopy. Then ∂D and l_1 together divide ∂V into two annuli, one of which, say R, does not contain Q. Pushing the disc $D \cup R$ slightly into $\operatorname{int} V$, we obtain a t-compressing disc of A. Then the previous paragraph shows the lemma. □

Lemma A.3. *Suppose that both components of A, say l_1 and l_2, are inessential and t-essential on ∂V. Assume that l_1 bounds a disc Q disjoint from l_2 on ∂V. Let A' be the annulus on ∂V such that $\partial A' = \partial A$. Then one of two conditions below holds.*
(1) *A is t-parallel to A' in (V, t).*

(2) *A is t-compressible in (V, t), l_1 and l_2 bound peripheral discs Q_1, Q_2 disjoint from t, and there is a cancelling disc C of t with $\partial C \cap \partial V \subset Q$ and $C \cap (Q_1 \cup Q_2) = \emptyset$.*

Proof. By Lemma 2.10, A is t-compressible or t-∂-compressible in (V, t).

Suppose that A is t-compressible. Performing a t-compressing operation on A, we obtain peripheral discs Q_1, Q_2 disjoint from t and bounded by l_1 and l_2 respectively. Since l_1 is t-essential, $\partial t \subset Q$. Then a standard innermost loop and outermost arc argument allows us to take a cancelling disc C of t with $C \cap Q_1 = \emptyset$ and $\partial C \cap \partial V \subset Q$.

Suppose that A has a t-∂-compressing disc D. The arc $D \cap \partial V$ connects l_1 and l_2, and hence is contained in the annulus A'. Performing a t-∂-compressing operation on A, we obtain a peripheral disc, which is t-parallel to a disc in A' since $A' \cap \partial t = \emptyset$. Hence A is t-parallel to A'. $\square$

B. Discs with two punctures

Let V be a solid torus, and t a trivial arc in V. Let Q be a peripheral disc properly embedded in V such that t intersects Q transversely in two points. Let B be the ball cut off from V by Q, and set $Q' = \partial B \cap \partial V$ the disc, $V' = \mathrm{cl}(V - B)$ and $t' = t \cap V'$.

Lemma B.1. *Assume $\partial t \subset Q'$. Then t' is an arc. Suppose that Q has a t-compressing disc D in (V, t). Let D' be the disc bounded by ∂D on Q, and set $R = (Q - D') \cup D$. Then* (1) *t intersects D' in two points, $R \cap t = \emptyset$ and D is contained in V',* (2) *we can take a cancelling disc C of t in (V, t) with $C \cap R = \emptyset$ and $\partial C \cap \partial V \subset Q'$ and* (3) *we can take a cancelling disc C' of the arc t' in (V', t') with $C' \cap D = \emptyset$ and $\partial C' \cap \partial V' \subset D'$.*

Proof. ∂D is essential on $Q - t$. If t intersects D' at a single point, then t intersects the 2-sphere $D \cup D'$ in a single point, which contradicts that V is irreducible. Hence t intersects D' in two points, and ∂D is parallel to ∂Q on $Q - t$. Suppose, for a contradiction, that D is contained in B. Then D divides B into two balls, both of which intersect t. This contradicts $D \cap t = \emptyset$. Thus we have shown that D is contained in V', and (1) follows.

A standard innermost loop and outermost arc argument allows us to take a cancelling disc C of t in V with $C \cap R = \emptyset$. Then (2) follows from $\partial t \subset Q'$. Similar argument shows (3). $\square$

Lemma B.2. *Suppose that Q is t-incompressible in (V, t). Then*
(1) *$\partial t \subset Q'$, and hence t' is an arc in V',*
(2) *any cancelling disc C of t can be isotoped in (V, t) so that $C \cap Q$ consists of arcs only and that the two points $t \cap Q$ are contained in distinct arc components of $C \cap Q$,*
(3) *$(B, t \cap B)$ is a rational tangle, that is, a trivial 2-string tangle and*

(4) *the punctured torus* $\partial V \cap \partial V' = \mathrm{cl}(\partial V - Q')$ *is* t*-incompressible in* (V', t').

Proof. If Q' contains a single endpoint of ∂t, then t intersects the 2-sphere $Q \cup Q'$ in three points, which is a contradiction. If $Q' \cap \partial t = \emptyset$, then Q' gives a t-compressing disc of Q, which contradicts our assumption. Hence $\partial t \subset Q'$, which is the conclusion (1).

Let C be a cancelling disc of t in V. Since Q is t-incompressible, a standard innermost loop argument allows us to isotope C in (V, t) so that $C \cap Q$ contains no loops. If there is an arc α of $C \cap Q$ connecting the two points $t \cap Q$, then it cuts off from C a disc D with $\partial D = \alpha \cup t'$, $D \cap Q = \alpha$ and $D \subset V'$. We take a small regular neighbourhood N of D in V'. Then the disc $\mathrm{cl}(\partial N - Q)$ is a t-compressing disc of Q, which is a contradiction. Hence the two points $t \cap Q$ are contained in distinct arc components of $C \cap Q$. This is the conclusion (2). Let t_1 and t_2 be the two components of $t \cap B$. The arcs $C \cap Q$ divide C into subdiscs. There are two subdiscs C_1, C_2 such that C_i contains a copy of t_i and that $C_i \cap t_j = \emptyset$ for $(i, j) = (1, 2)$ and $(2, 1)$. These two subdiscs show that $(B, t \cap B)$ is a trivial 2-string tangle. Thus we obtain the conclusion (3).

Suppose for a contradiction that the punctured torus $T = \partial V \cap \partial V'$ is t-compressible in (V', t'). Then we compress T and obtain a disc whose boundary coincides with ∂Q. This disc is disjoint from the arc t', which contradicts that Q is t-incompressible. □

Lemma B.3. *Assume* $\partial t \subset Q'$. *Then* t' *is an arc. Suppose that* Q *has a* t*-*∂*-compressing disc* D *in* (V', t'). *Then*
(1) *the arc* $D \cap Q$ *separates the two points* $t \cap Q$ *on* Q *and* D *is a meridian disc of* V',
(2) V *has a meridian disc* R *intersecting* t *in a sigle point and disjoint from* Q,
(3) Q *is* t*-incompressible in* (V, t) *and*
(4) *there is a cancelling disc* P *of* t' *in* (V', t') *such that*
  (a) P *is disjoint from* D,
  (b) $P \cap Q$ *consists of two arcs each of which contains a point of* $t \cap Q$ *and*
  (c) $P \cap R$ *is a single arc.*

Proof. The arc $D \cap Q$ divides Q into two discs Q_1, Q_2, each of which intersects t in a sigle point. However, the two points $t \cap Q$ are connected by the arc of t' in V'. Hence D is not separating in V', and is a meridian disc of V'. This is the conclusion (1). Then we can obtain a disc R as desired by isotoping the disc $D \cap Q_1$ off of Q slightly. Note that $|R \cap t| = |Q_1 \cap t| = 1$. Thus we obtain the conclusion (2).

We assume, for a contradiction, that Q is t-compressible in (V, t). Then, by Lemma B.1, there is a cancelling disc C' of t' in (V', t') with $\partial C' \cap \partial V' \subset Q$. We can isotope C' near the arc $\partial C' \cap Q$ so that $\partial C' \cap \partial D \cap Q$ is a single point p. We can isotope C' slightly fixing $\partial C'$ so that it is transverse to D. Since D is disjoint from t, $(\partial C') \cap D$ consists of the only one point p. This contradicts that C' and D intersect properly embedded 1-manifold in the disc C'. Hence Q is t-incompressible in (V, t). Thus we obtain the conclusion (3).

Let C be a cancelling disc of t in (V, t) as in (2) of Lemma B.2. That is, $C \cap Q$ consists of arcs only and the two points $t \cap Q$ are contained in distinct arc components of $C \cap Q$. Moreover, we can take C so that the number of the arc components of $C \cap Q$ is minimal. Then a standard outermost arc argument shows that an arc of $C \cap Q$ separates the two points $t \cap Q$ on Q if it is disjoint from $t \cap Q$. Let C' be one of subdiscs obtained by cutting C along the arcs $C \cap Q$ such that C' contains t'. Note that C' is a cancelling disc of t' in (V', t'). We can isotope C' so that $\partial C'$ is disjoint from the arc $D \cap Q$ since the arc $D \cap Q$ separates the two points $t \cap Q$ on Q. Then every arc of $C' \cap Q$ is parallel to the arc $D \cap Q$ in $Q - t$ if it is disjoint from $t \cap Q$. We can retake C' so that it is disjoint from D by a standard innermost loop and outermost arc argument on D. Then we add a copy of D along every arc component of $C' \cap Q$ if it is disjoint from $t \cap Q$. A standard innermost loop and outermost arc argument allows us to retake C' so that it intersects R in a single arc. Thus we have obtained a cancelling disc of t' as desired, and we obtain the conclusion (4). □

Lemma B.4. *Suppose that Q is t-∂-compressible in $(B, t \cap B)$. Then $\partial t \subset Q'$, and Q and Q' are t-parallel in (V, t).*

Proof. Let D be a t-∂-compressing disc of Q in $(B, t \cap B)$. The arc $D \cap Q$ is essential in $Q - T$, and hence it divides Q into two discs, say Q_1 and Q_2, each of which intersects t in a single point. The arc $D \cap Q'$ divides Q' into two discs, say Q'_1 and Q'_2, such that $\partial Q_i \cap \partial Q = \partial Q'_i \cap \partial Q$ for $i = 1$ and 2. For $i = 1$ and 2, the 2-sphere $Q_i \cup Q'_i \cup D$ must intersect t in even number of points. Because Q' contains at most two endpoints ∂t, Q'_i contains a single point of ∂t for $i = 1$ and 2. Thus $\partial t \subset Q'$. Let B_i be the ball bounded by the 2-sphere $Q_i \cup Q'_i \cup D$. Then $t \cap B_i$ is a trivial arc in B_i for $i = 1$ and 2 by Lemma 2.1. Thus $(B, t \cap B)$ gives a parallelism between Q and Q' in (V, t). □

C. Annuli with a puncture

Throughout Appendix C, we consider the situation as below. Let V be a solid torus, and t a trivial arc in V. Let A be an annulus properly embedded in V intersecting t transversely in precisely one point. Suppose that ∂A are essential in ∂V (ignoring the points ∂t). Then they divide ∂V into two annuli A_1 and A_2. Let R_i be the region bounded by $A_i \cup A$, and set $t_i = t \cap R_i$ for $i = 1$ and 2.

When A has a t-∂-compressing disc D in (V, t), we say D is *essential* if $\partial D \cap A$ is an essential arc on A (ignoring the point $t \cap A$), and say D is *inessential* if $\partial D \cap A$ is inessential.

Lemma C.1. *Suppose that A has a t-compressing disc D in R_1. Then ∂D is essential in A (ignoring the point $t \cap A$), A_1 is also t-compressible in (R_1, t_1), and A_2 is t-incompressible in (R_2, t_2). Moreover, R_1 is a solid torus, and the arc $t \cap R_1$ has a cancelling disc C' in R_1 such that each of $\partial C' \cap A$ and $\partial C' \cap \partial V$ is an arc.*

Proof. Suppose, for a contradiction, that ∂D bounds a disc, say D', in A. Since ∂D is essential in $A - t$, D' contains $t \cap A$. Then the 2-sphere $D \cup D'$ intersects t in a single point in V, which is a contradiction. Hence ∂D is essential.

A t-compressing operation on A along D yields a disc D_1 disjoint from t and a disc D_2 intersecting t transversely in a single point. Then D_1 gives a t-compressing disc of A_1.

Similar argument as in the latter half of the first paragraph in the proof of Lemma A.1 shows that A_2 is t-incompressible in (R_2, t_2).

R_1 is a solid torus because it is obtained from the ball between D_1 and D_2 by gluing the two copies of the t-compressing disc D. t has a cancelling disc C in V. As in the proof of Lemma 2.7, a standard innermost loop and outermost arc argument allows us to take C so that it intersects $D_1 \cup D_2$ in a single arc connecting the point $t \cap D_2$ and ∂D_2. Moreover, we can take C to be disjoint from the copies of D. Then the disc $C' = C \cap R_1$ gives a cancelling disc of the arc $t \cap R_1$ as desired. □

Lemma C.2. *Suppose that A has an essential t-∂-compressing disc D in R_1. Then A is t-parallel to A_1 in (V, t).*

Proof. Performing a t-∂-compressing operation on A along D, we obtain a peripheral disc Q. Q cuts off a ball B from R_1 with $t \cap B = t_1$. t_1 is trivial in B by Lemma 2.1. Hence A and A_1 are t-parallel. □

Lemma C.3. *Suppose that A is t-incompressible and has an inessential t-∂-compressing disc D in R_1. Let A' be the annulus obtained by performing a t-∂-compressing operation on A along D. Then either*

(1) *A is t-parallel to A_1, or*

(2) *there is a cancelling disc C of t in (V, t) with $C \cap A' = \emptyset$.*

If ∂A is of a longitudinal slope of V, then the condition (1) *holds. In case* (2), *A is isotopic in (V, t) to the annulus which is the union of the two annuli Z_1 and Z_2 as below. See* Fig. C.1. *Let l be the component of ∂A disjoint from D.*

(a) *Z_1 is obtained by cutting a copy of ∂V along l and isotoping along C and*

(b) *Z_2 is obtained from a copy of A_1 by slightly isotoping into* $\operatorname{int} V$.

Proof. The arc $\partial D \cap A$ cuts off a disc, say D_A, from A. with $t \cap A \subset D_A$. The arc $\partial D \cap A_1$ also cuts off a disc, say D_1, from A_1 with $\partial t \cap A_1 \subset D_1$. Then the 2-sphere $D \cup D_A \cup D_1$ bounds a ball, say B, in R_1, with $t \cap B = t_1$ trivial in B by

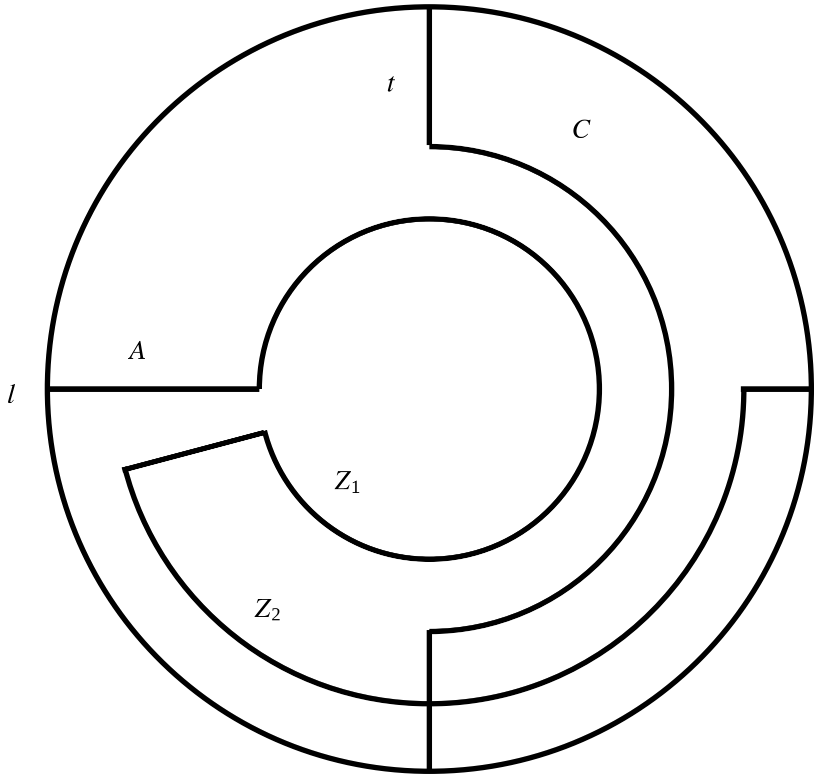

Fig. C.1.

Lemma 2.1. We can recover A by pushing the interior of $A' \cup D_1$ into int V. We can recover A also by taking the union of a copy of A', two copies of $\mathrm{cl}(A_1 - D_1)$ and a copy of D_1.

A' is t-compressible or t-∂-compressible by Lemma 2.10. First, we suppose that A' is t-compressible. Let P be a t-compressing disc of A'. We isotope P near ∂P so that $P \cap D = \emptyset$. Then a standard innermost loop argument allows us to retake P to be disjoint from D and D_A. Then P forms a t-compressing disc of A, contradicting our assumption.

Hence A' has a t-∂-compressing disc D'. When D' is contained in $R_2 \cup B$, we perform a t-∂-compressing operation on a copy of A' along D', and obtain a peripheral disc Q which cuts off a ball B' from $R_2 \cup B$ with $t \subset B'$. Hence we can take a cancelling disc C of t with $C \cap Q = \emptyset$ and $C \subset B'$. Then A' is obtained from $A_2 \cup D_1$ by isotoping along C. This implies the conclusion (2) by setting $Z_1 = A' \cup (A_1 - D_1)$ and $Z_2 = A_1$.

We consider the case where D' is contained in $\mathrm{cl}(R_1 - B)$. Note that such a t-∂-compressing disc always exists if ∂A is of a longitudinal slope of V. We can take D'

so that D' is disjoint from the copy of D. Hence D' is also a t-∂-compressing disc of A. Note that the arc $D' \cap A$ is essential in A (ignoring the point $t \cap A$). Hence A and A_1 are t-parallel in (V, t) by Lemma C.2. □

D. Annuli with two punctures

In Appendix D, we consider the situation as below. Let V be a solid torus V, and t a trivial arc in V. Let A be an annulus properly embedded in V intersecting t transversely in two points. Suppose that the loops ∂A are essential in ∂V (ignoring the points ∂t). Then they divide ∂V into two annuli A_1 and A_2, one of which, say A_1 contains the two points ∂t. The annulus A separates V into two regions R_1 and R_2 with $\partial R_i = A_i \cup A$ for $i = 1$ and 2. When A has a K-compressing disc D in (V, t), we say that D is *essential* if $\partial D \cap A$ is essential on A, otherwise it is *inessential*.

Lemma D.1. *Suppose that A has an inessential t-compressing disc D in (V, t) and is t-∂-incompressible in $(R_2, t \cap R_2)$. Then there is a cancelling disc C of t with $\partial C \cap \partial V \subset A_1$.*

Proof. Note that D is contained in R_2. By performing a compressing operation on A along D, we obtain a 2-sphere and an annulus, say A'. A' is disjoint from t, and separates V into two regions R'_1 and R'_2, one of which, say R'_1 contains t. By Lemma 2.10, A' is t-compressible or t-∂-compressible in (V, t).

If A' is t-compressible, then, compressing A', we obtain two meridian discs of V. A standard innermost loop and outermost arc argument allows us to take a cancelling disc of t disjoint from these discs. This implies the conclusion.

If A' has a t-∂-compressing disc Q in R'_2, then we can isotope Q near the arc $\partial Q \cap A'$ in R'_2 so that ∂Q is disjoint from the copy of the t-compressing disc D. This implies that A is t-∂-compressible in $(R_2, t \cap R_2)$, which contradicts our assumption.

If A' is t-∂-compressible in R'_1, then, performing a t-∂-compressing operation on A', we obtain a peripheral disc which cuts off a ball containing t from R'_1. We can take a cancelling disc of t entirely contained in the ball. This implies the conclusion. □

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