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ON SMOOTH $SO_0(p, q)$-ACTIONS ON S^{p+q-1}

Dedicated to Professor Shôrô Araki on his 60th birthday

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0. Introduction

Consider the standard $\boldsymbol{SO}(p)\times\boldsymbol{SO}(q)$-action on S^{p+q-1}. This action has codimension one principal orbits with $\boldsymbol{SO}(p-1)\times\boldsymbol{SO}(q-1)$ as principal isotropy group. Furthermore, the fixed point set of restricted $\boldsymbol{SO}(p-1)\times\boldsymbol{SO}(q-1)$-action is diffeomorphic to S^1.

In this paper, we shall study smooth $\boldsymbol{SO}_0(p, q)$-actions on S^{p+q-1}, each of which is an extension of the above action, and we shall show that such an action is characterized by a pair (ϕ, f) satisfying certain conditions, where ϕ is a smooth one-parameter group on S^1 and $f: S^1\to P_1(\boldsymbol{R})$ is a smooth function.

In his paper [1], T. Asoh has classified smooth $\boldsymbol{SL}(2, \boldsymbol{C})$-actions on S^3 topologically. In particular, he has introduced such a pair to study the case that the restricted $\boldsymbol{SU}(2)$-action has codimension one orbits. We shall show that Asoh's method is useful to our problem.

1. Subgroups of $SO(p, q)$

Let $\boldsymbol{SO}(p, q)$ denote the group of matrices in $\boldsymbol{SL}(p+q, \boldsymbol{R})$ which leave invariant the quadratic form

$$-x_1^2-\cdots-x_p^2+x_{p+1}^2+\cdots+x_{p+q}^2 .$$

In particular, $\boldsymbol{SO}(p, q)$ contains $\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))$ as a maximal compact subgroup.

Put

$$I_{p,q}=\begin{bmatrix}-I_p & 0\\ 0 & I_q\end{bmatrix},$$

where I_n denotes the unit matrix of order n. It is clear that for a real matrix g of order $p+q$, $g\in\boldsymbol{SO}(p, q)$ if and only if ${}^tgI_{p,q}g=I_{p,q}$ and $\det g=1$.

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Let $\mathfrak{so}(p, q)$ denote the Lie algebra of $\boldsymbol{SO}(p, q)$. Then, for a real matrix X of order $p+q$, $X\in\mathfrak{so}(p, q)$ if and only if

$$ {}^{t}XI_{p,q}+I_{p,q}X=0\,. \tag{1.1}$$

Writing X in the form

$$X=\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

where X_1 is of order p and X_4 is of order q, we see that the condition (1.1) is equivalent to $X_3={}^{t}X_2$ and X_1, X_4 are skew-symmetric.

Here we consider the standard representations of $\boldsymbol{SO}(p, q)$ and $\mathfrak{so}(p, q)$ on $\boldsymbol{R}^{p+q}$. Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{p+q}$ denote the standard basis of $\boldsymbol{R}^{p+q}$. Let $H(a\colon b)$ (resp. $\mathfrak{h}(a\colon b)$) denote the isotropy group (resp. the isotropy algebra) at $a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}$ for $(a, b)\neq(0, 0)$. It is clear that $\mathfrak{h}(a\colon b)$ is the subalgebra of $\mathfrak{so}(p, q)$ consisting of matrices in the form

$$\left(\begin{array}{c|c|c|c} 0 & -b\,{}^{t}U & 0 & b\,{}^{t}V \\ \hline bU & * & -aU & * \\ \hline 0 & -a\,{}^{t}U & 0 & a\,{}^{t}V \\ \hline bV & * & -aV & * \end{array}\right)\;;\quad U\in\boldsymbol{R}^{p-1},\; V\in\boldsymbol{R}^{q-1}\,. \tag{1.2}$$

Moreover, we see $H(1\colon 0)=\boldsymbol{SO}(p-1, q)$ and $H(0\colon 1)=\boldsymbol{SO}(p, q-1)$. Put

$$m(\theta)=\left(\begin{array}{c|c|c|c} \cosh\theta & & \sinh\theta & \\ \hline & I_{p-1} & & \\ \hline \sinh\theta & & \cosh\theta & \\ \hline & & & I_{q-1} \end{array}\right)\;;\quad \theta\in\boldsymbol{R}\,. \tag{1.3}$$

It is clear that $m(\theta)\in\boldsymbol{SO}(p, q)$ and

$$m(\theta)\,(a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1})=a'\boldsymbol{e}_1+b'\boldsymbol{e}_{p+1}\,, \tag{1.4}$$

where $a'=a\cosh\theta+b\sinh\theta$, $b'=a\sinh\theta+b\cosh\theta$. Let $M(p, q)$ denote the subgroup of $\boldsymbol{SO}(p, q)$ consisting of matrices $m(\theta)$, $\theta\in\boldsymbol{R}$.

Lemma 1.5. $\boldsymbol{SO}(p, q)=\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))M(p, q)\boldsymbol{SO}(p-1, q)$
$=\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))M(p, q)\boldsymbol{SO}(p, q-1)$.
The coset space $\boldsymbol{SO}(p, q)/\boldsymbol{SO}(p-1, q)$ *(resp.* $\boldsymbol{SO}(p, q)/\boldsymbol{SO}(p, q-1)$*) is homeomorphic*

to $S^{p-1}\times \boldsymbol{R}^q$(*resp.* $\boldsymbol{R}^p\times S^{q-1}$).

Proof. Let $g\in \boldsymbol{SO}(p, q)$ and $g\boldsymbol{e}_1=\boldsymbol{u}\oplus\boldsymbol{v}\in\boldsymbol{R}^p\oplus\boldsymbol{R}^q$. There exist $k\in\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))$ and $\varepsilon=\pm 1$ such that

$$k^{-1}g\boldsymbol{e}_1=||\boldsymbol{u}||\boldsymbol{e}_1+\varepsilon||\boldsymbol{v}||\boldsymbol{e}_{p+1}\,.$$

Since $||\boldsymbol{u}||^2-||\boldsymbol{v}||^2=1$, there exists $\theta\in\boldsymbol{R}$ such that

$$||\boldsymbol{u}||=\cosh\theta,\quad \varepsilon||\boldsymbol{v}||=\sinh\theta\,.$$

Then we see that $m(-\theta)k^{-1}g\in\boldsymbol{SO}(p-1, q)$, and hence we obtain the first equation. The correspondence $g\boldsymbol{SO}(p-1, q)\to(||\boldsymbol{u}||^{-1}\boldsymbol{u}, \boldsymbol{v})$ gives a homeomorphism from $\boldsymbol{SO}(p, q)/\boldsymbol{SO}(p-1, q)$ onto $S^{p-1}\times\boldsymbol{R}^q$. The second half can be proved similarly by considering the orbit of $\boldsymbol{e}_{p+1}$. q.e.d.

Let $\boldsymbol{SO}_0(p, q)$ denote the identity component of $\boldsymbol{SO}(p, q)$. By the above lemma, we see that $\boldsymbol{SO}(p, q)$ has two connected components for $p, q\geqq 1$. Writing $g\in\boldsymbol{SO}(p, q)$ in the form

$$g=\begin{bmatrix} A & B \\ C & D\end{bmatrix},$$

where A is of order p and D is of order q, we see that $g\in\boldsymbol{SO}_0(p, q)$ if and only if $\det A>0$.

Considering the orbit of $a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}$, we obtain

$$\boldsymbol{SO}(p, q)=\boldsymbol{S}(\boldsymbol{O}(p)\times\boldsymbol{O}(q))M(p, q)H(a\colon b) \tag{1.6}$$

for each $(a, b)\neq(0, 0)$. It is clear that

$$\bigcap_{(a,b)} H(a\colon b)=\boldsymbol{SO}(p-1, q-1)\,,$$

where the intersection is taken over all $(a, b)\neq(0, 0)$.

Lemma 1.7. *Suppose* $p, q\geqq 3$. *Let* $\mathfrak{g}$ *be a proper subalgebra of* $\mathfrak{so}(p, q)$ *which contains* $\mathfrak{so}(p-1)\oplus\mathfrak{so}(q-1)$. If

$$\dim\mathfrak{so}(p, q)-\dim\mathfrak{g}\leqq p+q-1\,. \tag{*}$$

then $\mathfrak{g}=\mathfrak{h}(a\colon b)$ *for some* $(a, b)\neq(0, 0)$ *or* $\mathfrak{g}=\mathfrak{h}(1\colon\varepsilon)\oplus\theta^1$ *for* $\varepsilon=\pm 1$, *where the one-dimensional space* θ^1 *is generated by a matrix* $E_{1,p+1}+E_{p+1,1}$.

Proof. By considering the adjoint representation of $\boldsymbol{SO}(p-1)\times\boldsymbol{SO}(q-1)$ on $\mathfrak{so}(p, q)$, we see first that $\mathfrak{g}$ contains $\mathfrak{so}(p-1, q-1)$ under the condition (*). Next, we obtain the desired result by considering the bracket operations on $\boldsymbol{SO}(p-1)\times\boldsymbol{SO}(q-1)$-invariant subspaces. We omit the detail (cf. [4], §2). q.e.d.

2. Smooth $SO_0(p, q)$ actions on S^{p+q-1}

Let Φ_0: $SO_0(p, q)\times S^{p+q-1}\to S^{p+q-1}$ denote the standard action defined by

$$\Phi_0(g, u) = ||gu||^{-1}gu \,. \tag{2.1}$$

Its restricted $SO(p)\times SO(q)$-action ψ is of orthogonal transformations and has codimension one principal orbits with $SO(p-1)\times SO(q-1)$ as principal isotropy group. Moreover, the fixed point set of its restricted $SO(p-1)\times SO(q-1)$-action is one-dimensional. Put

$$\begin{aligned} &G = SO_0(p, q),\ K = SO(p)\times SO(q),\ H = SO(p-1)\times SO(q-1)\,, \\ &\psi = \Phi_0\,|\,K\times S^{p+q-1},\ F(H) = \{xe_1+ye_{p+1}\,|\,x^2+y^2 = 1\}\ , \end{aligned} \tag{2.2}$$

where $F(H)$ is the fixed point set of the restricted H-action. In the following, we shall identify $F(H)$ with the circle S^1 by the natural diffeomorphism h: $S^1\to F(H)$ defined by $h(x, y)=xe_1+ye_{p+1}$.

Let $\Phi: G\times S^{p+q-1}\to S^{p+q-1}$ be a smooth G-action on $S^{p+q-1}(p, q\geqq 3)$ such that its restricted K-action coincides with the action ψ, i.e. $\Phi\,|\,K\times S^{p+q-1}=\psi$.

First, we shall show that there exists a smooth function f: $F(H)\to P_1(\boldsymbol{R})$ uniquely determined by the condition

$$\mathfrak{h}(f(z))\subset\mathfrak{g}_z; \quad z\in F(H)\,, \tag{2.3}$$

where $P_1(\boldsymbol{R})$ is the real projective line, $\mathfrak{g}_z$ is the isotropy algebra at z with respect to the given G-action Φ, and $\mathfrak{h}(f(z))$ is a subalgebra of $\mathfrak{so}(p, q)$ defined by (1.2).

Because $\mathfrak{g}_z$ is a proper subalgebra of $\mathfrak{so}(p, q)$ which contains Lie $H=\mathfrak{so}(p-1)\oplus\mathfrak{so}(q-1)$, there exists uniquely $(a: b)\in P_1(\boldsymbol{R})$ such that

$$\mathfrak{h}(a: b)\subset\mathfrak{g}_z \tag{2.4}$$

by Lemma 1.7. It remains only to show the smoothness of f. By (1.2), (2.4), we obtain

$$\begin{aligned} &b(E_{i1}-E_{1i})-a(E_{i,p+1}+E_{p+1,i})\in\mathfrak{g}_z \qquad (2\leqq i\leqq p)\,, \\ &b(E_{1,p+j}+E_{p+j,1})+a(E_{p+1,p+j}-E_{p+j,p+1})\in\mathfrak{g}_z \qquad (2\leqq j\leqq q)\,, \end{aligned}$$

and hence

$$\begin{aligned} &b||E_{i1}-E_{1i}||_z^2-a\langle E_{i,p+1}+E_{p+1,i},\ E_{i1}-E_{1i}\rangle_z = 0\,, \\ &b\langle E_{1,p+j}+E_{p+j,1},\ E_{p+1,p+j}-E_{p+j,p+1}\rangle_z+a||E_{p+1,p+j}-E_{p+j,p+1}||_z^2 = 0\,, \end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the standard Riemannian metric on S^{p+q-1} and each element of $\mathfrak{so}(p, q)$ can be considered naturally as a smooth vector field on S^{p+q-1} (cf. [3], ch. II, Th. II). These equations assure the smoothness of f.

Comparing $\mathfrak{h}(a\colon b)$ with isotropy algebras of the restricted K-action, we obtain

$$
(2.5)\qquad \begin{aligned} f(z) &= (1\colon 0) \Leftrightarrow z = \pm e_1\,, \\ f(z) &= (0\colon 1) \Leftrightarrow z = \pm e_{p+1}\,. \end{aligned}
$$

Let $m(\theta)$ be the matrix defined by (1.3). Then, the set $F(H)$ is invariant under the transformation $\Phi(m(\theta), -)$, because $m(\theta)$ commutes with each element of H. Let $\phi\colon \boldsymbol{R}\times F(H)\to F(H)$ denote the smooth $\boldsymbol{R}$-action on $F(H)$ defined by $\phi(\theta, z)=\Phi(m(\theta), z)$. Then, we obtain

$$
(2.6)\qquad f(z) = (a\colon b) \Rightarrow f(\phi(\theta, z)) = (a'\colon b')\,,
$$

where $a'=a\cosh\theta+b\sinh\theta$, $b'=a\sinh\theta+b\cosh\theta$. This follows from (1.4), (2.3) and the definition of $\mathfrak{h}(a\colon b)$.

Let $J_i\colon F(H)\to F(H)$ $(i=1, 2)$ denote involutions defined by $J_1(x, y)=(-x, y)$ and $J_2(x, y)=(x, -y)$. Then, we obtain

$$
(2.7)\qquad f(z) = (a\colon b) \Rightarrow f(J_1(z)) = f(J_2(z)) = (a\colon -b)\,.
$$

This follows from the fact $J_i(z)=\psi(j_i, z)$ $(i=1, 2)$, where

$$
(2.8)\qquad j_1 = \begin{bmatrix} -I_2 & \\ & I_{p+q-2} \end{bmatrix},\quad j_2 = \begin{bmatrix} I_p & & \\ & -I_2 & \\ & & I_{q-2} \end{bmatrix}.
$$

There is a following relation of the involution J_i with the transformation $\phi(\theta, -)=\Phi(m(\theta), -)$:

$$
(2.9)\qquad J_i(\phi(\theta, z)) = \phi(-\theta, J_i(z)) \qquad (i=1, 2)\,.
$$

This follows from the fact: $j_i m(\theta)=m(-\theta)j_i$.

Let $\sigma\colon \boldsymbol{SO}_0(p, q)\to \boldsymbol{SO}_0(p, q)$ denote an automorphism defined by $\sigma(g)={}^t g^{-1}=I_{p,q}gI_{p,q}$. We may give a new G-action Φ^σ defined by $\Phi^\sigma(g, u)=\Phi(\sigma(g), u)$. It is clear that

$$
\Phi^\sigma | K\times S^{p+q-1} = \Phi | K\times S^{p+q-1}\,.
$$

Let f^σ, ϕ^σ denote the smooth function $f\colon F(H)\to P_1(\boldsymbol{R})$ and the smooth $\boldsymbol{R}$-action $\phi\colon \boldsymbol{R}\times F(H)\to F(H)$, respectively, with respect to the G-action Φ^σ. Then we see that

$$
(2.10)\qquad \begin{aligned} &\phi^\sigma(\theta, z) = \phi(-\theta, z)\,, \\ &f(z) = (a\colon b) \Rightarrow f^\sigma(z) = (a\colon -b)\,. \end{aligned}
$$

3. Properties of (ϕ, f)

Let P be a symmetric matrix of order $p+q$, and let $U(P)$ denote a closed subgroup of $G=\boldsymbol{SO}_0(p, q)$ defined by

$$U(P) = \{g \in G \mid gP{}^tg = P\} .$$

Let $f: F(H) \to P_1(\boldsymbol{R})$ be a smooth function. Let $P(z)$ denote a symmetric matrix defined by

$$(3.1) \qquad P(z) = (a^2+b^2)^{-1}(a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1}){}^t(a\boldsymbol{e}_1+b\boldsymbol{e}_{p+1})$$

for $f(z)=(a: b)$, and let $U(z)$ denote the identity component of $U(P(z))$. Then, it is clear that (see §1)

$$(3.2) \qquad U(z) = \textit{the identity component of } H(a: b) .$$

Let (ϕ, f) be a pair of a smooth $\boldsymbol{R}$-action ϕ on $F(H)$ and a smooth function $f: F(H) \to P_1(\boldsymbol{R})$ satisfying the following conditions:

(i) $J_i(\phi(\theta, z)) = \phi(-\theta, J_i(z))$,

(ii) $f(z) = (a: b) \Rightarrow f(J_i(z)) = (a: -b)$,

where J_1, J_2 are involutions on $F(H)$ defined in §2,

(iii) $f(z) = (a: b) \Rightarrow f(\phi(\theta, z)) = (a': b')$,

where $a'=a\cosh\theta+b\sinh\theta$, $b'=a\sinh\theta+b\cosh\theta$,

(iv) $f(z) = (1: 0) \Leftrightarrow z = \pm\boldsymbol{e}_1$; $f(z) = (0: 1) \Leftrightarrow z = \pm\boldsymbol{e}_{q+1}$.

By (1.4), (3.1) and the condition (iii), we obtain

$$(3.3) \qquad m(\theta)P(z)m(\theta) = \lambda(\theta, z)P(\phi(\theta, z)) ,$$

where $\lambda(\theta, z)$ is a positive real number defined by

$$\lambda(\theta, z) = (a^2+b^2)^{-1}\{(a \cosh\theta+b \sinh\theta)^2+(a \sinh\theta+b \cosh\theta)^2\} ,$$

for $f(z)=(a: b)$. By the conditinon (iv), we obtain

$$(3.4) \qquad K \cap U(z) = K_z ,$$

where K_z denotes the isotropy group at $z \in F(H)$ with respect to the K-action ψ.

Lemma 3.5. *Suppose* $kP(z){}^tk=P(w)$ *for some* $k \in K$ *and* $z, w \in F(H)$. *Put* $f(z)=(a: b)$.

(1) *If* $ab \neq 0$, *then* $f(z)=f(w)$ *and* $k \in H \cup j_1j_2H$, *or* $f(z)=f(J_i(w))$ *and* $k \in j_1H \cup j_2H$.

(2) *If* $ab=0$, *then* $f(z)=f(w)$ *and* $k \in U(z) \cup j_1j_2U(z)$.

Proof. The result follows by a routine work from the fact that $X^tX=Y^tY$ implies $X=\pm Y$ for column vectors X, Y. So we omit the detail. q.e.d.

Lemma 3.6. *Put* $f(z)=(a:b)$. *If* $f(\phi(\theta, z))=f(J_i(z))$, *then* $|a| \neq |b|$,

$\phi(\theta, z)=J_1(z)$ *for* $|a|<|b|$, *and* $\phi(\theta, z)=J_2(z)$ *for* $|a|>|b|$.

Proof. $f(J_i(z)) = (a: -b)$ by the condition (ii). On the other hand, if $|a|=|b|$, then $f(\phi(\theta, z))=f(z)=(a: b)\neq(a: -b)$ by the condition (iii). Hence we obtain $|a|\neq|b|$. Suppose $|a|<|b|$. Then $z=\phi(\tau, \varepsilon e_{p+1})$ for some $\tau\in \boldsymbol{R}$ and $\varepsilon=\pm 1$ by the conditions (iii), (iv). Hence we obtain

$$\begin{aligned} &J_1(z) = \phi(-\tau, \varepsilon e_{p+1}), \quad J_2(z) = \phi(-\tau, -\varepsilon e_{p+1}), \\ &f(J_i(z)) = (-\tanh \tau: 1), \quad \phi(\theta, z) = \phi(\theta+\tau, \varepsilon e_{p+1}), \\ &f(\phi(\theta, z)) = (\tanh(\theta+\tau): 1). \end{aligned}$$

Therefore, $\tau=-\theta/2$ and $\phi(\theta, z)=J_1(z)$. The remaining case is similarly proved. q.e.d.

Lemma 3.7. *Put* $f(z)=(a:b)$. *If* $j_i m(\theta)\in U(z)$, *then* $|a|\neq|b|$, $i=1$ *for* $|a|<|b|$, *and* $i=2$ *for* $|a|>|b|$.

Proof. By (3.2) and our assumption, we obtain

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (-1)^i \begin{bmatrix} a \\ -b \end{bmatrix}.$$

This implies $(-1)^i(a^2+b^2)=(a^2-b^2)\cosh\theta$. Hence we obtain the desired result. q.e.d.

4. Construction of $SO_0(p, q)$-actions

4.1. Let (ϕ, f) be a pair of a smooth $\boldsymbol{R}$-action ϕ on $F(H)$ and a smooth function $f: F(H)\to P_1(\boldsymbol{R})$ satisfying the four conditions in §3. We shall show how to construct a smooth $G=\boldsymbol{SO}_0(p, q)$-action on S^{p+q-1} from the pair (ϕ, f). We use the notations (2.2), (2.8).

By (1.6), (3.2), we obtain

$$G = KM(p, q)U(z) \tag{4.1}$$

for each $z\in F(H)$. Take $(g, p)\in G\times S^{p+q-1}$. Let us choose

$$\begin{aligned} &k\in K,\ z\in F(H):\ p = \psi(k, z), \\ &k'\in K,\ \theta\in \boldsymbol{R},\ u\in U(z):\ gk = k'm(\theta)u, \end{aligned} \tag{4.2}$$

and put

$$\Phi(g, p) = \psi(k', \phi(\theta, z))\in S^{p+q-1}. \tag{4.3}$$

We shall show that Φ is a smooth G-action on S^{p+q-1}. To show this, we prepare the followings.

Lemma 4.4. *Suppose* $km(\theta)u=k'm(\theta')u'$ *for* $k, k'\in K$ *and* $u, u'\in U(z)$.

Then, $\psi(k, \phi(\theta, z))=\psi(k', \phi(\theta', z))$.

Proof. We obtain

$$km(\theta)P(z)m(\theta)^t k = k'm(\theta')P(z)m(\theta')^t k' .$$

Then, by (3.3)

$$\lambda(\theta, z)kP(\phi(\theta, z))^t k = \lambda(\theta', z)k'P(\phi(\theta', z))^t k' .$$

Comparing traces of both sides, we obtain

$$\lambda(\theta, z) = \lambda(\theta', z) ,$$
$$kP(\phi(\theta, z))^t k = k'P(\phi(\theta', z))^t k' .$$

By the second equation, Lemma 3.5 and the conditions (i), (iii), we obtain the following possibilities:

(a) $$f(\phi(\theta-\theta', z)) = f(z) , \qquad \text{or}$$

(b) $$f(\phi(\theta+\theta', z)) = f(J_i(z)) .$$

Put $f(z)=(a: b)$. We see that if $|a|=|b|$ (resp. $|a|\neq|b|$), then the equation $\lambda(\theta, z)=\lambda(\theta', z)$ (resp. $f(\phi(\theta-\theta', z))=f(z)$) implies $\theta=\theta'$. Suppose $\theta=\theta'$. Then

$$k^{-1}k' = m(\theta)uu'^{-1}m(\theta)^{-1}\in m(\theta)U(z)m(\theta)^{-1} = U(\phi(\theta, z))$$

by (3.3), and hence $\psi(k^{-1}k', \phi(\theta, z))=\phi(\theta, z)$ by (3.4). Therefore, if $\theta=\theta'$ then $\psi(k, \phi(\theta, z))=\psi(k', \phi(\theta', z))$.

Finally, we consider the case (b). Then $k^{-1}k'\in j_1H\cup j_2H$ by Lemma 3.5, and hence $k'=kj_ih$ for some i and $h\in H$. Then

$$m(\theta)u = j_ihm(\theta')u' = j_im(\theta')hu' = m(-\theta')j_ihu' ,$$

and hence $j_im(\theta+\theta')=hu'u^{-1}\in U(z)$. Therefore, we obtain $|a|\neq|b|$, $i=1$ for $|a|<|b|$, and $i=2$ for $|a|>|b|$ by Lemma 3.7. On the other hand, the equation (b) implies $\phi(\theta+\theta', z)=J_1(z)$ for $|a|<|b|$ and $\phi(\theta+\theta', z)=J_2(z)$ for $|a|>|b|$ by Lemma 3.6. Therefore, we obtain $k'=kj_ih$ and $\phi(\theta+\theta', z)=J_i(z)$ for some i and $h\in H$. Then

$$\begin{aligned}\psi(k', \phi(\theta', z)) &= \psi(kj_ih, \phi(\theta', z))\\ &= \psi(k, J_i\,\phi(\theta', z)) = \psi(k, \phi(-\theta', J_i(z)))\\ &= \psi(k, \phi(-\theta', \phi(\theta+\theta', z))) = \psi(k, \phi(\theta, z)) .\end{aligned}$$ q.e.d.

Proposition 4.5. Φ *of* (4.3) *defines an abstract G-action on* S^{p+q-1} *such that* $\Phi|K\times S^{p+q-1}=\psi$.

Proof. For $(g, p)\in G\times S^{p+q-1}$, let us choose as in (4.2);

$$p = \psi(k_1, z_1) = \psi(k_2, z_2)\,,$$
$$gk_i = k_i' m(\theta_i) u_i,\; u_i \in U(z_i)\,.$$

By the first equation, we obtain $z_1 = J_1^s J_2^t(z_2)$ for some integers s, t. Then, $k_2^{-1} k_1 j_1^s j_2^t \in K_{z_2} \subset U(z_2)$ by (3.4). Therefore, $k_2 = k_1 j_1^s j_2^t u_2'$ for some $u_2' \in U(z_2)$. Then, we obtain

$$\begin{aligned} k_2' m(\theta_2) u_2 = gk_2 = gk_1 j_1^s j_2^t u_2' &= k_1' m(\theta_1) u_1 j_1^s j_2^t u_2' \\ &= (k_1' j_1^s j_2^t) m((-1)^{s+t}\theta_1)\,(j_1^s j_2^t u_1 j_1^s j_2^t) u_2'\,. \end{aligned}$$

It is clear $k_2'' = k_1' j_1^s j_2^t \in K$, and we see

$$(j_1^s j_2^t u_1 j_1^s j_2^t) u_2' \in U(z_2)$$

by the equation $P(J_i(z)) = j_i P(z) j_i$. Then,

$$\psi(k_2', \phi(\theta_2, z_2)) = \psi(k_2'', \phi((-1)^{s+t}\theta_1, z_2))$$

by Lemma 4.4. On the other hand,

$$\begin{aligned} \psi(k_2'', \phi((-1)^{s+t}\theta_1, z_2)) &= \psi(k_1', J_1^s J_2^t \phi((-1)^{s+t}\theta_1, z_2)) \\ &= \psi(k_1', \phi(\theta_1, J_1^s J_2^t(z_2))) = \psi(k_1', \phi(\theta_1, z_1))\,. \end{aligned}$$

This shows that Φ of (4.3) is a well-defined mapping.

Take $g, g' \in G$ and $p \in S^{p+q-1}$. Let us choose as in (4.2);

$$p = \psi(k, z), \quad gk = k' m(\theta) u, \quad g'k' = k'' m(\theta') u'\,,$$

where $u \in U(z)$ and $u' \in U(\phi(\theta, z))$. Then,

$$\begin{aligned} \Phi(g', \Phi(g, p)) &= \Phi(g', \psi(k', \phi(\theta, z))) \\ &= \psi(k'', \phi(\theta', \phi(\theta, z))) \\ &= \psi(k'', \phi(\theta+\theta', z)) = \Phi(g'g, p)\,. \end{aligned}$$

Because

$$\begin{aligned} g'gk = g'k' m(\theta) u &= k'' m(\theta') u' m(\theta) u \\ &= k'' m(\theta+\theta')\,(m(-\theta) u' m(\theta)) u\,, \end{aligned}$$

and $m(-\theta) u' m(\theta) \in U(z)$ by (3.3). This shows that Φ of (4.3) is an abstract G-action.

Finally, take $(k, p) \in K \times S^{p+q-1}$ and put $p = \psi(k', z)$ as in (4.2). Then,

$$\Phi(k, p) = \psi(kk', z) = \psi(k, \psi(k', z)) = \psi(k, p)\,. \qquad \text{q.e.d.}$$

Notice that the continuity of Φ is unknown in this stage. In the remaining of this section, we shall show the smoothness of the G-action Φ.

4.2. Put $f(z)=(a: b)$ and $z=(x, y)$. It is clear that $ab\neq 0$ if and only if $xy\neq 0$ by the condition (iv). To simplify the following discussion, we add a condition on the pair (ϕ, f)

(v) $xy>0 \Rightarrow ab>0$.

Notice that the condition (v) is inessential, by (2.10).

Define

$$S_+ = \{z = (x, y)\in F(H) \mid x>0, y>0\} .$$

By the condition (v), there is a smooth positive valued function β on S_+ such that $f(z)=(1: \beta(z))$.

Lemma 4.6. *For $(\theta, z)\in \boldsymbol{R}\times S_+$, $\phi(\theta, z)\in S_+$ if and only if*

$$(1+\beta(z)\tanh\theta)(\beta(z)+\tanh\theta)>0 . \tag{4.6}$$

Proof. $f(\phi(\theta, z))=(1+\beta(z)\tanh\theta: \beta(z)+\tanh\theta)$ by the condition (iii). Then, only if part is clear. Suppose (4.6). Then,

$$\phi(\theta, z)\in S_+\cup J_1J_2(S_+)$$

and we see that $\phi(\theta, z)\notin J_1J_2(S_+)$ by considering orbits of the $\boldsymbol{R}$-action ϕ.
q.e.d.

Define

$$D_+ = \{(\theta, z)\in \boldsymbol{R}\times S_+ \mid \phi(\theta, z)\in S_+\} ,$$
$$W_+ = \{(g, z)\in G\times S_+ \mid \pm\mathrm{trace}\,(gP(z)^tg)\neq(1-\beta(z)^2)(1+\beta(z)^2)^{-1}\} .$$

Lemma 4.7. *For any $(g, z)\in W_+$, there exist uniquely $kH\in K/H$ and $\theta\in\boldsymbol{R}$ such that*

$$g = km(\theta)u\,;\ u\in U(z),\ (\theta, z)\in D_+ . \tag{4.7}$$

Furthermore, the correspondence Δ: $W_+\to K/H\times D_+$ defined by $\Delta(g, z)=(kH, (\theta, z))$ is smooth.

Proof. First, we show the uniqueness of the decomposition (4.7). Suppose

$$g = km(\theta)u = k'm(\theta')u'$$

for $k, k'\in K$, $u, u'\in U(z)$ and $(\theta, z), (\theta', z)\in D_+$. Then, $\psi(k, \phi(\theta, z))=\psi(k', \phi(\theta', z))$ by Lemma 4.4. Since $\phi(\theta, z)$ and $\phi(\theta', z)$ are contained in S_+, we see $\phi(\theta, z)=\phi(\theta', z)$. Then, $k^{-1}k'\in K_{\phi(\theta,z)}=H$, and hence $kH=k'H$. Furthermore, we obtain $\theta=\theta'$ by the same argument as in the proof of Lemma 4.4.

Next, we show the existence of the decomposition (4.7). Choose $k\in K$,

$\theta\in\boldsymbol{R}$ and $u\in U(z)$ such that $g=km(\theta)u$. Then,

$$(*)\quad \operatorname{trace}(gP(z)^tg) = \cosh 2\theta+2\beta(z)(1+\beta(z)^2)^{-1}\sinh 2\theta\,.$$

Suppose $(\theta, z)\notin D_+$. If $\beta(z)=1$, then $\phi(\theta, z)\in S_+$ for any $\theta\in\boldsymbol{R}$. Hence we see $\beta(z)\neq 1$. (i) Suppose $0<\beta(z)<1$. We can find $\tau\in\boldsymbol{R}$ satisfying $z=\phi(\tau, \boldsymbol{e}_1)$ and $\beta(z)=\tanh\tau$. The assumption $\phi(\theta, z)\notin S_+$ implies $\tanh(\theta+\tau)\leqq 0$ by Lemma 4.6, and hence $\theta+\tau\leqq 0$. If $\theta+\tau=0$, then we obtain

$$\operatorname{trace}(gP(z)^tg) = (1+\beta(z)^2)^{-1}(1-\beta(z)^2)\,.$$

This is a contradiction to $(g, z)\in W_+$, and hence $\theta+\tau<0$. Then,

$$\phi(-\theta-2\tau, z) = \phi(-\theta-\tau, \boldsymbol{e}_1) = J_2\phi(\theta+\tau, \boldsymbol{e}_1)$$

and $\phi(-\theta-2\tau, z)\in S_+$ by Lemma 4.6. Furthermore,

$$j_2m(-2\tau) = m(\tau)j_2m(-\tau)\in U(z)\,,$$

by $j_2\in U(\boldsymbol{e}_1)$. Then

$$g = km(\theta)u = (kj_2)m(-\theta-2\tau)\,(j_2m(-2\tau)u)\,,$$

where $kj_2\in K$, $j_2m(-2\tau)u\in U(z)$ and $(-\theta-2\tau, z)\in D_+$. (ii) Suppose $\beta(z)>1$. We can find $\tau\in\boldsymbol{R}$ satisfying $z=\phi(\tau, \boldsymbol{e}_{p+1})$ and $\beta(z)^{-1}=\tanh\tau$. Then we obtain similarly

$$g = km(\theta)u = (kj_1)m(-\theta-2\tau)\,(j_1m(-2\tau)u)\,,$$

where $kj_1\in K$, $j_1m(-2\tau)u\in U(z)$ and $(-\theta-2\tau, z)\in D_+$.

Finally, we shall show the smoothness of Δ. Put $\theta=\theta(g, z)$ and $kH=\delta(g, z)$ for $\Delta(g, z)=(kH, (\theta, z))$, and we shall show the smoothness of $\theta(g, z)$ and $\delta(g, z)$.

Consider the smooth function γ on $W_+\times\boldsymbol{R}$ defined by

$$\gamma(g, z, \theta) = \cosh 2\theta+2\beta(z)(1+\beta(z)^2)^{-1}\sinh 2\theta-\operatorname{trace}(gP(z)^tg)\,.$$

Then, $\gamma(g, z, \theta(g, z))=0$ by (4.7) and (*). Furthermore, if $\gamma(g, z, \theta)=0$, then

$$\frac{\partial\gamma}{\partial\theta}(g, z, \theta) = 2\cosh 2\theta\,(\tanh 2\theta+2\beta(z)(1+\beta(z)^2)^{-1})\neq 0$$

by the definition of W_+. Then, we see that the function $\theta(g, z)$ is smooth by Lemma 4.6 and the implicit function theorem.

Consider the smooth function $\delta_1\colon W_+\to\boldsymbol{R}^{p+q}$ defined by

$$\delta_1(g, z) = (1+\beta(z)^2)^{-1/2}g(\boldsymbol{e}_1+\beta(z)\boldsymbol{e}_{p+1})\,.$$

Put $\Delta(g, z)=(kH, (\theta, z))$, and define

$$x = (1+\beta(z)^2)^{-1/2}(\cosh\theta+\beta(z)\sinh\theta)\,,$$
$$y = (1+\beta(z)^2)^{-1/2}(\sinh\theta+\beta(z)\cosh\theta)\,.$$

Then, we see that $\delta_1(g, z)=k(x\boldsymbol{e}_1+y\boldsymbol{e}_{p+1})$ and $x>0$, $y>0$ by Lemma 4.6. Since the correpsondence of kH to $k(\boldsymbol{e}_1+\boldsymbol{e}_{p+1})$ defines an embedding of K/H into $\boldsymbol{R}^{p+q}$, we see that the function $\delta(g, z)$ is smooth, by considering a correspondence of $\boldsymbol{u}\oplus\boldsymbol{v}$ $(\boldsymbol{u}\neq 0, \boldsymbol{v}\neq 0)$ to $||\boldsymbol{u}||^{-1}\boldsymbol{u}\oplus||\boldsymbol{v}||^{-1}\boldsymbol{v}$. q.e.d.

4.3. Define

$$S_0(\Phi) = \{\Phi(g, \boldsymbol{e}_1)\,|\,g\in G\}, \quad S_0(\Phi_0) = \{\Phi_0(g, \boldsymbol{e}_1)\,|\,g\in G\}$$

for the G-action Φ of (4.3) and the standard G-action Φ_0 of (2.1), respectively. By (4.3) and the conditions of (ϕ, f), there exists a positive real number $r<1$ such that

$$S_0(\Phi) = \{\boldsymbol{u}\oplus\boldsymbol{v}\in S(\boldsymbol{R}^p\oplus\boldsymbol{R}^q)\,|\,||\boldsymbol{v}||<r\}\,.$$

On the other hand, it is clear that

$$S_0(\Phi_0) = \{\boldsymbol{u}\oplus\boldsymbol{v}\in S(\boldsymbol{R}^p\oplus\boldsymbol{R}^q)\,|\,||\boldsymbol{u}||>||\boldsymbol{v}||\}\,.$$

Lemma 4.8. *The restriction of Φ to $G\times S_0(\Phi)$ is smooth.*

Proof. Put $D^q(\delta)=\{\boldsymbol{v}\in\boldsymbol{R}^q\,|\,||\boldsymbol{v}||<\delta\}$, and define a diffeomorphism $\alpha\colon S^{p-1}\times D^q(1)\to S_0(\Phi_0)$ by

$$\alpha(\boldsymbol{u}, \boldsymbol{v}) = (1+||\boldsymbol{v}||^2)^{-1/2}(\boldsymbol{u}\oplus\boldsymbol{v})\,.$$

Let us define a diffeomorphism $F_0\colon S_0(\Phi)\to S_0(\Phi_0)$ by $F_0(\boldsymbol{u}\oplus\boldsymbol{v})=\alpha(||\boldsymbol{u}||^{-1}\boldsymbol{u}, F(\boldsymbol{v}))$, where $F\colon D^q(r)\to D^q(1)$ is a diffeomorphism not yet introduced.

There is a smooth real valued function h on $(-r, r)$ such that $f((1-y^2)^{1/2}, y)=(1\colon h(y))$. It is clear that $h(y)>0$ for $0<y<r$ by the condition (v). Furthermore, h is a diffeomorphism from $(-r, r)$ onto $(-1, 1)$ by the conditions (iii), (iv). Since

$$(1\colon h(-y)) = f((1-y^2)^{1/2}, -y) = f(J_2((1-y^2)^{1/2}, y)) = (1\colon -h(y))\,,$$

we obtain $h(-y)=-h(y)$, and hence $y\to y^{-1}h(y)$ is a smooth even function. Therefore, $\boldsymbol{v}\to||\boldsymbol{v}||^{-1}h(||\boldsymbol{v}||)$ is a smooth function on $D^q(r)$ (cf. [2], ch. VIII, §14, Problem 6-c). Then we can define $F(\boldsymbol{v})=||\boldsymbol{v}||^{-1}(h||\boldsymbol{v}||)\boldsymbol{v}$.

Now we shall show that the diffeomorphism $F_0\colon S_0(\Phi)\to S_0(\Phi_0)$ is G-equivariant. It is clear that F_0 is K-equivariant. By definition of h and the conditions (iii), (iv), we obtain

$$F_0(\phi(\theta, \boldsymbol{e}_1)) = \Phi_0(m(\theta), \boldsymbol{e}_1);\ \theta\in\boldsymbol{R}\,.$$

Take $g\in G$ and put $g=km(\theta)u$ for $k\in K$, $u\in U(\boldsymbol{e}_1)=\boldsymbol{SO}_0(p-1, q)$. Then,

$$F_0(\Phi(g, e_1)) = F_0(\psi(k, \phi(\theta, e_1))) = \Phi_0(k, F_0(\phi(\theta, e_1)))$$
$$= \Phi_0(k, \Phi_0(m(\theta), e_1)) = \Phi_0(km(\theta), e_1) = \Phi_0(g, e_1)\,.$$

Therefore, the diffeomorphism F_0 is G-equivariant, and hence the restriction $\Phi | G\times S_0(\Phi)$ is smooth. q.e.d.

Now we can prove the smoothness of Φ. By Lemma 4.8 and a similar argument, we see that the restrictions of Φ to

$$G\times \{\Phi(g, e_1) | g\in G\} \quad \text{and} \quad G\times \{\Phi(g, e_{p+1}) | g\in G\}$$

are smooth. Define $W(\Phi) = \{(g, \psi(k, z)) | (gk, z)\in W_+\}$. Then, we see that $W(\Phi)$ is an open set of $G\times S^{p+q-1}$, since W_+ is an open set of $G\times S_+$. Furthermore, we see that $\Phi | W(\Phi)$ is smooth, since Δ is smooth by Lemma 4.7. Consequently, we obtain the smoothness of Φ on $G\times S^{p+q-1}$, because three open sets $G\times \{\Phi(g, e_1) | g\in G\}$, $G\times \{\Phi(g, e_{p+1}) | g\in G\}$ and $W(\Phi)$ cover $G\times S^{p+q-1}$.

5. Conclusion

Theorem. *Suppose $p\geqq 3$, $q\geqq 3$. Then, there is a one-to-one correspondence between the set of smooth $\boldsymbol{SO}_0(p, q)$-actions Φ on S^{p+q-1} whose restricted $\boldsymbol{SO}(p)\times \boldsymbol{SO}(q)$-action is the standard orthogonal action and the set of pairs (ϕ, f) satisfying the conditions (i) to (iv) in §3, where ϕ is a smooth one-parameter group on S^1 and $f: S^1 \to P_1(\boldsymbol{R})$ is a smooth function.*

Proof. The correspondence of Φ to (ϕ, f) is given in §2, and its reversed correspondence of (ϕ, f) to Φ is given in §4. q.e.d.

By Asoh's consideration (cf. [1], §9–§11), we can show that there are infinitely many topologically distinct smooth $\boldsymbol{SO}_0(p, q)$-actions on S^{p+q-1} whose restricted $\boldsymbol{SO}(p)\times \boldsymbol{SO}(q)$-action is the standard orthogonal action. We omit the proof.

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