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## ON GENERALIZATION OF ASANO'S MAXIMAL ORDERS IN A RING

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As an extension of maximal orders in a central simple algebra  $\Sigma$  over  $K$  of finite dimension, the author has studied structure of hereditary orders in  $\Sigma$  in [4], [5]. On the other hand in [1], [1'] the theory of maximal orders in  $\Sigma$  was extended to the theory in any ring by Asano. Following the method given by Asano in [1], [1'] we shall generalize the notion of hereditary order in  $\Sigma$ .

Let  $S$  be a ring with unit element 1 and  $\Lambda$  a subring in  $S$  containing 1 such that  $S$  is the right and left quotient ring of  $\Lambda$  with respect to  $\Lambda \cap S^*$ , where  $S^*$  consists of all non zero-divisors in  $S$ . We call  $\Lambda$  an *order* in  $S$ . Asano showed in [1], [1'] that  $\Lambda$  is a maximal order which satisfies two conditions  $(A_2')$ ,  $(A_3)$  (see below) if and only if the set of two-sided ideals is a group with respect to multiplication. In this case

(H) *every two-sided ideal<sup>1)</sup> is finitely generated  $\Lambda$ -projective as a right and left  $\Lambda$ -module.*

Thus, we shall generalize the notion of the Asano's maximal order to orders which satisfies (H).

In this note we shall show that many results of hereditary orders in a central simple algebra in [4], [5] and [6] are valid in the above generalized orders.

We shall call briefly an order  $\Lambda$  in  $S$  which satisfies (H) an  $H$ -order. Furthermore, we call elements in  $S^*$  *regular*.

### 1. Definitions and lemmas

DEFINITION 1. Let  $\Lambda$  be an order in  $S$ . A subset  $\mathfrak{A}$  of  $S$  is called *left (right) ideal* of  $\Lambda$  if  $\mathfrak{A}$  satisfies the following conditions:

- 1)  $\mathfrak{A}$  is a left (right)  $\Lambda$ -module,
- 2)  $\mathfrak{A}$  contains a regular element,

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1) See Definition 1,

3) there exists a regular element  $\lambda(\in \Lambda)$  such that  $\mathfrak{A}\lambda \subseteq \Lambda$  ( $\lambda\mathfrak{A} \subseteq \Lambda$ ). If  $\mathfrak{A}$  is a right and left ideal of  $\Lambda$ , we call  $\mathfrak{A}$  a *two-sided ideal* of  $\Lambda$ .

**DEFINITION 2.** Let  $\Lambda$  and  $\Gamma$  be orders in  $S$ . If there exist regular elements  $\alpha, \alpha', \beta$  and  $\beta'$  such that  $\alpha\Lambda\alpha' \subseteq \Gamma$  and  $\beta\Gamma\beta' \subseteq \Lambda$  then we call  $\Lambda$  and  $\Gamma$  are *similar* and we denote by  $\Lambda \sim \Gamma$ .

**Lemma 1.** ([4], Lemma 1.2). *Let  $\Lambda$  be an order in  $S$  and  $\mathfrak{U}, \mathfrak{U}'$  left ideals of  $\Lambda$ . Then  $\text{Hom}'_{\Lambda}(\mathfrak{U}, \mathfrak{U}') = \{x \mid x \in S, \mathfrak{U}x \subseteq \mathfrak{U}'\}$ .*

*Proof.* It is clear that  $\{x \mid x \in S, \mathfrak{U}x \subseteq \mathfrak{U}'\} \subseteq \text{Hom}'_{\Lambda}(\mathfrak{U}, \mathfrak{U}')$ . Since  $S\mathfrak{U} = S$ , we have, for any element  $x$  in  $S$ , that  $x = \sum s_i l_i$ ,  $s_i \in S$ ,  $l_i \in \mathfrak{U}$ . We define  $\bar{f}(x) = \sum s_i f(l_i)$  for  $f \in \text{Hom}'_{\Lambda}(\mathfrak{U}, \mathfrak{U}')$ . Let  $x = \sum s'_i l'_i$  be another expression, then there exists a regular element  $\gamma$  in  $\Lambda$  such that  $\gamma s_i, \gamma s'_i \in \Lambda$  for all  $i$  by [1']. Hence,  $\gamma \sum s_i f(l_i) = \sum f(\gamma s_i l_i) = \sum \bar{f}(\gamma s'_i l'_i) = \gamma \sum s'_i f(l'_i)$ . Therefore,  $\bar{f}$  is well defined and  $\bar{f} \in \text{Hom}_S(S\mathfrak{U}, S\mathfrak{U}') = S$ . Hence,  $\bar{f}(x) = xy$  for some  $y \in S$ . It is clear that  $\bar{f}\mathfrak{U} = f$ .

If  $\mathfrak{U} = \mathfrak{U}'$ , then  $\text{Hom}'_{\Lambda}(\mathfrak{U}, \mathfrak{U})$  is a similar order to  $\Lambda$  by [1'], Theorem 4.4 and we call it *the right order* of  $\mathfrak{U}$  and denote it by  $\Lambda^r(\mathfrak{U})$ . Similarly, we can define *the left order* of  $\mathfrak{U}$  and denote it by  $\Lambda^l(\mathfrak{U})$ .

**DEFINITION 3.** Let  $\Lambda$  be an order in  $S$ . If there exist regular elements  $\alpha, \beta$  in  $\Lambda$  for  $x$  in  $S$  such that  $x\Lambda\alpha \subseteq \Lambda$ ,  $\beta\Lambda x \subseteq \Lambda$ , then  $\Lambda$  is called *regular*.

**Lemma 2.** *Let  $\Lambda$  be a regular order in  $S$  and  $\Gamma$  a similar order to  $\Lambda$ . Then there exist regular elements  $\alpha, \beta$  in  $\Lambda$  such that  $\alpha\Gamma \subseteq \Lambda$  and  $\Gamma\beta \subseteq \Lambda$ , and  $\Gamma$  is also regular.*

It is clear (cf. [1'] pp. 163-165).

**Corollary.** *Let  $\Lambda$  be a regular order and  $\Gamma$  an order containing  $\Lambda$ . If  $\Gamma$  is a finitely generated left or right  $\Lambda$ -module, then  $\Gamma \sim \Lambda$ . Hence,  $\Gamma$  is a two-sided ideal of  $\Lambda$  and is regular.*

It is clear by [1'] and [2].

**Lemma 3.** *Let  $\Lambda$  be an order and  $\mathfrak{U}$  a left ideal. Then  $\mathfrak{U}\mathfrak{U}^{-1} = \Lambda^l(\mathfrak{U})$  if and only if  $\mathfrak{U}$  is a finitely generated projective  $\Lambda^r(\mathfrak{U})$ -module, where  $\mathfrak{U}^{-1} = \{x \mid x \in S, \mathfrak{U}x \subseteq \mathfrak{U}\}$ .*

*Proof.* We put  $\Gamma = \Lambda^r(\mathfrak{U})$ . We define  $\varphi: \mathfrak{U} \otimes_{\Gamma} \text{Hom}(\mathfrak{U}, \Gamma) = \mathfrak{U} \otimes_{\Gamma} \mathfrak{U}^{-1} \rightarrow \text{Hom}_{\Gamma}(\mathfrak{U}, \mathfrak{U}) = \Lambda^l(\mathfrak{U})$  by setting  $\varphi(l \otimes f)(l') = f(l')l$ , where  $l, l' \in \mathfrak{U}$  and  $f \in \mathfrak{U}^{-1}$ . Then the lemma is clear by [3], Proposition A.1.

**Lemma 4.** *Let  $\Lambda$  be an order and  $\mathfrak{A}$  a two-sided ideal of  $\Lambda$  which is*

*finitely generated as left, right module. If  $\mathfrak{A}$  is a projective left  $\Lambda$ -module, then  $\Lambda'(\mathfrak{A})$  is a right finitely generated  $\Lambda$ -projective module.*

**Proof.** The operation of  $\Lambda$  on  $\Lambda'(\mathfrak{A})$  from the right side coincides with the operation of  $\Lambda$  on  $\text{Hom}_{\Lambda}^l(\mathfrak{A}, \mathfrak{A})$  with respect to the second  $\mathfrak{A}$ . Since  $\mathfrak{A}$  is left  $\Lambda$ -projective,  $M = \Sigma \oplus \Lambda u_i \rightarrow \mathfrak{A} \rightarrow 0$  splits. Hence,  $\text{Hom}_{\Lambda}(M, \mathfrak{A}) = \Sigma \oplus \mathfrak{A} \leftarrow \text{Hom}_{\Lambda}^l(\mathfrak{A}, \mathfrak{A}) \leftarrow 0$  splits as a usual right  $\Lambda$ -module. Since  $\mathfrak{A}$  is a finitely generated right  $\Lambda$ -module, so is  $\Lambda'(\mathfrak{A})$ .

## 2. *H*-orders

We shall quote here Asano's axioms. Let  $\Lambda$  be an order in  $S$ .

( $A_2'$ )  $\Lambda$  satisfies a minimal condition for two-sided ideals in  $\Lambda$  which contains a fixed two-sided ideal.

( $A_3$ ) Every prime ideal in  $\Lambda$  is a maximal two-sided ideal.

**Proposition 1.** *If a regular order  $\Lambda$  satisfies ( $A_2'$ ) and ( $A_3$ ) and  $\Lambda$  is maximal among similar orders to it, then  $\Lambda$  is an *H*-order.*

**Proof.** By [1], [1'] we know that the set of two-sided ideals is a group with respect to multiplication. Hence,  $\Lambda$  satisfies (H) by Lemma 3.

From Lemma 2 and the proof of [4], Lemma 1.2, we have

**Theorem 1.** *Let  $\Lambda$  be a regular *H*-order in  $S$ . If  $\Gamma$  is an order containing  $\Lambda$  which is similar to  $\Lambda$ , then  $\Gamma$  is an *H*-order.*

**Proposition 2.** *Let  $\Lambda$  be an *H*-order and  $\mathfrak{A}$  a two-sided ideal of  $\Lambda$ . Then  $\mathfrak{A}\mathfrak{A}^{-1} = \Lambda'(\mathfrak{A})$  and  $\mathfrak{A}^{-1}\mathfrak{A} = \Lambda^r(\mathfrak{A})$ .*

**Proof.** It is clear from the fact  $\Lambda'(\mathfrak{A}) = \tau_{\Lambda'(\mathfrak{A})}(\mathfrak{A}) = \mathfrak{A}\mathfrak{A}^{-1}$  and  $\Lambda^r(\mathfrak{A}) = \tau_{\Lambda^r(\mathfrak{A})}(\mathfrak{A}) = \mathfrak{A}^{-1}\mathfrak{A}$  which is obtained by [3], Proposition A. 3.

**Corollary.** *Let  $\Lambda$  be a regular *H*-order and maximal order among similar orders to  $\Lambda$ . Then  $\Lambda$  is a maximal order satisfying ( $A_2'$ ) and ( $A_3$ ).*

**Proof.** Let  $\mathfrak{A}$  be a two-sided ideal in  $\Lambda$ . Since  $\Lambda'(\mathfrak{A}) \sim \Lambda$ ,  $\Lambda'(\mathfrak{A}) = \Lambda$ . Hence  $\mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{A}^{-1}\mathfrak{A} = \Lambda$ . Let  $\mathfrak{B}$  be any two-sided ideal of  $\Lambda$ . Since  $\mathfrak{B}\lambda \subseteq \Lambda$  for some regular element  $\lambda$  in  $\Lambda$ ,  $\mathfrak{B}\lambda\lambda\Lambda \subseteq \Lambda$ . By [1'], Theorem 4.12  $\Lambda\lambda\Lambda$  is a two-sided ideal in  $\Lambda$ . Hence the set of two-sided ideals of  $\Lambda$  is a group.

We note that in the proof of [4], Proposition 1.6 we have only used the facts that  $\mathfrak{A}$  is a finitely generated projective module and  $\Lambda'(\mathfrak{A})$ ,  $\Lambda^r(\mathfrak{A}) \subseteq S$  for a two-sided ideal  $\mathfrak{A}$  of  $\Lambda$  and that if an order  $\Gamma \supseteq \Lambda$  is a finitely generated left  $\Lambda$ -module,  $C(\Gamma) = \{x \mid x \in S, \Gamma x \subseteq \Lambda\}$  is a two-sided

ideal in  $\Lambda$ . Hence, we have by Lemma 4

**Theorem 2.** ([4], Theorem 1.7). *Let  $\Lambda$  be an  $H$ -order in  $S$  and  $\Gamma$  an order containing  $\Lambda$ . If  $\Gamma$  is finitely generated left  $\Lambda$ -projective, then  $C(\Gamma)$  is an idempotent two-sided ideal in  $\Lambda$  and  $\Gamma = \Lambda^l(C(\Gamma))$ . Conversely, if  $\mathfrak{A}$  is an idempotent two-sided ideal in  $\Lambda$  then  $\Lambda^l(\mathfrak{A})$  is finitely generated left  $\Lambda$ -projective and  $\mathfrak{A} = C(\Lambda^l(\mathfrak{A}))$ . This correspondence is anti-lattice isomorphic.*

**Corollary 1.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$ . Then there exists the above one-to-one correspondence between two-sided idempotent ideals in  $\Lambda$  and orders  $\Gamma$  containing  $\Lambda$  which is finitely generated as a right or left  $\Lambda$ -module.*

Proof. It is clear from Theorem 2 and Lemma 2.

**Corollary 2.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$ . If there exists only a finite number of maximal orders containing  $\Lambda$  which are similar to  $\Lambda$ , then  $\Lambda$  is equal to the intersection of them.*

Proof. Let  $\{\Omega_i\}_{i=1}^n$  be the set of maximal orders in the corollary. Then  $\{C(\Omega_i)\}_{i=1}^n$  is the set of minimal ones among idempotent two-sided ideals by Theorem 2. We put  $\mathfrak{D} = \sum C(\Omega_i)$ .  $\mathfrak{D}$  is also idempotent. Let  $\Gamma = \cap \Omega_i = \Lambda^l(\mathfrak{D})$ , and  $\Gamma' = \Lambda^r(\mathfrak{D})$ . Since  $C(\Omega_i)$  is minimal idempotent,  $\Lambda^r(C(\Omega_i))$  is also a maximal order. Hence,  $\{\Omega_i\}_{i=1}^n = \{\Lambda^r(C(\Omega_i))\}_{i=1}^n$  and  $\Gamma = \cap \Omega_i = \Gamma'$  by Theorem 2. Therefore,  $\Lambda = \Gamma$  by [4], Corollary 1.9.

**Proposition 3.** *Let  $\Lambda$  be an  $H$ -order in  $S$  and  $\mathfrak{U}$  a left ideal of  $\Lambda$  such that  $\Lambda = \Lambda^l(\mathfrak{U})$ . Then the following statements are equivalent.*

- 1)  $\Lambda^r(\mathfrak{U}^{-1}) = \Lambda$ .
- 2)  $\mathfrak{U}\mathfrak{U}^{-1} = \Lambda$ .
- 3)  $\mathfrak{U}$  is a finitely generated  $\Lambda^r(\mathfrak{U})$ -module.

Proof. 2) is equivalent to 3) by Lemma 3. It is clear that 2) implies 1).

1)  $\rightarrow$  3). We set  $\Gamma = \Lambda^r(\mathfrak{U})$ .  $\text{Hom}_{\Lambda}^l(\mathfrak{U} \otimes_{\Gamma} \mathfrak{U}^{-1}, \Lambda) = \text{Hom}_{\Gamma}^l(\mathfrak{U}^{-1}, \text{Hom}_{\Lambda}^l(\mathfrak{U}, \Lambda)) = \text{Hom}_{\Gamma}^l(\mathfrak{U}^{-1}, \mathfrak{U}^{-1}) = \Lambda$ . From an exact sequence  $\mathfrak{U} \otimes_{\Gamma} \mathfrak{U}^{-1} \rightarrow \mathfrak{U}\mathfrak{U}^{-1} \rightarrow 0$  we obtain an exact sequence  $0 \rightarrow \text{Hom}_{\Lambda}^l(\mathfrak{U}\mathfrak{U}^{-1}, \Lambda) \rightarrow \text{Hom}_{\Lambda}^l(\mathfrak{U} \otimes_{\Gamma} \mathfrak{U}^{-1}, \Lambda) = \Lambda$ . Since  $\mathfrak{U}\mathfrak{U}^{-1} \subseteq \Lambda$ ,  $\text{Hom}_{\Lambda}^l(\mathfrak{U}\mathfrak{U}^{-1}, \Lambda) \supseteq \Lambda$ . Therefore,  $\text{Hom}_{\Lambda}^l(\mathfrak{U}\mathfrak{U}^{-1}, \Lambda) = \Lambda$ , which implies  $\Lambda^r(\mathfrak{U}\mathfrak{U}^{-1}) = \Lambda$  and  $\tau_{\Lambda}^l(\mathfrak{U}\mathfrak{U}^{-1}) = \mathfrak{U}\mathfrak{U}^{-1}\Lambda = \mathfrak{U}\mathfrak{U}^{-1}$ . Hence  $\mathfrak{U}\mathfrak{U}^{-1}$  is idempotent by [4], Lemma 1.5. Therefore, we obtain by Theorem 2 that  $\Lambda = D(\Lambda)^{(2)} = D(\Lambda^r(\mathfrak{U}\mathfrak{U}^{-1})) = \mathfrak{U}\mathfrak{U}^{-1}$ .

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2)  $D(\Gamma) = \{x \mid x \in S, x\Gamma \subseteq \Lambda\}$ .

**Corollary.** *Let  $\Lambda$  be an order in  $S$ . We assume the set of two-sided ideals of  $\Lambda$  is a group. Then every left ideal  $\mathfrak{L}$  of  $\Lambda$  is finitely generated  $\Lambda$ -projective and  $\mathfrak{L}\mathfrak{L}^{-1}=\Lambda$ .*

*Proof.*  $\Lambda$  is a maximal order among similar to  $\Lambda$  by [1'], Theorem 4.22. It is clear that  $\Lambda \sim \Lambda'(\mathfrak{L}^{-1})$ . Hence,  $\Lambda'(\mathfrak{L}^{-1})=\Lambda$ .

**Proposition 4.** ([6], Theorem 1.1). *Let  $\Lambda$  be an  $H$ -order in  $S$  and  $\mathfrak{L}$  a left ideal of  $\Lambda$ . If  $\Lambda'(\mathfrak{L})=\Lambda$  and  $\mathfrak{L}$  is finitely generated  $\Lambda$ -projective, then  $\Lambda'(\mathfrak{L})$  is an  $H$ -order and  $\mathfrak{L}$  is finitely generated  $\Lambda'(\mathfrak{L})$ -projective.*

*Proof.* Let  $\Gamma=\Lambda'(\mathfrak{L})$ . Since  $\mathfrak{L}$  is finitely generated  $\Lambda$ -projective, we have  $\Gamma=\mathfrak{L}^{-1}\mathfrak{L}$  by Lemma 3, and  $\tau_{\Lambda}^{\iota}(\mathfrak{L})\mathfrak{L}=\mathfrak{L}$  by [3], Proposition A.5. Hence,  $\tau_{\Lambda}^{\iota}(\mathfrak{L})=\mathfrak{L}\mathfrak{L}^{-1}=\tau_{\Lambda}^{\iota}(\mathfrak{L})\mathfrak{L}\mathfrak{L}^{-1}=\tau(\mathfrak{L})^2$ . We obtain  $\Lambda'(\tau_{\Lambda}^{\iota}(\mathfrak{L}))=\Lambda$  from the facts  $\Lambda'(\mathfrak{L})=\Lambda$  and  $\tau_{\Lambda}^{\iota}(\mathfrak{L})\mathfrak{L}=\mathfrak{L}$ . Therefore,  $\mathfrak{L}\mathfrak{L}^{-1}=\tau_{\Lambda}(\mathfrak{L})=\Lambda$  by Theorem 2. We can easily check that a correspondence  $\mathfrak{A} \leftrightarrow \mathfrak{L}^{-1}\mathfrak{A}\mathfrak{L}$  of two-sided ideals  $\mathfrak{A}$  of  $\Lambda$  and those of  $\Gamma$  is one-to-one and preserves projectivity and finiteness, because  $\mathfrak{L}^{-1}\mathfrak{A}\mathfrak{L} \approx \mathfrak{L}^{-1} \otimes \mathfrak{A} \otimes \mathfrak{L}$ .

From the similar argument to [4], Lemma 2.1 we have

**Proposition 5.** ([4], Proposition 2.2). *Let  $\Lambda$  be an  $H$ -order in  $S$ . If  $S=S_1 \oplus \cdots \oplus S_n$ , then  $\Lambda=\Lambda_1 \oplus \cdots \oplus \Lambda_n$  and  $\Lambda_i$  is an  $H$ -order in  $S_i$ , where the  $S_i$ 's are subring of  $S$ .*

### 3. Inversible ideals in an $H$ -order

Finally we shall consider two-sided ideals  $\mathfrak{A}$  in  $\Lambda$  such that  $\mathfrak{A}\mathfrak{A}^{-1}=\mathfrak{A}^{-1}\mathfrak{A}=\Lambda$ . We call those ideals *inversible ideals* of  $\Lambda$ .

**Proposition 6.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$  satisfying  $(A_2')$ . If every maximal two-sided ideal in  $\Lambda$  is invertible, then  $\Lambda$  is a maximal order.*

*Proof.* We assume that there exists an ideal in  $\Lambda$  which is not invertible. Let  $\mathfrak{C}$  be a maximal one among ideals in  $\Lambda$  which are not invertible, and  $\mathfrak{N}$  be a maximal two-sided ideal containing  $\mathfrak{C}$ . Since  $\Lambda$  satisfies  $(A_2')$ ,  $\mathfrak{C} \not\subseteq \mathfrak{N}^n$  for some  $n$ . Then  $\mathfrak{C} \not\subseteq \mathfrak{N}^{-1}\mathfrak{C} \subseteq \Lambda$ , because if  $\mathfrak{N}\mathfrak{C}=\mathfrak{C}$ ,  $\mathfrak{C}=\mathfrak{N}^n\mathfrak{C} \subseteq \mathfrak{N}^n$ . Thus,  $\mathfrak{N}^{-1}\mathfrak{C}$  must be invertible, which is a contradiction.

**Lemma 5.** *Let  $\Lambda$  be an  $H$ -order in  $S$  and  $\mathfrak{M}$  a maximal two-sided ideal. Then  $\mathfrak{M}$  is either invertible in  $\Lambda$  or idempotent.*

*Proof.* Since  $\mathfrak{C}(\Lambda'(\mathfrak{M})) \supseteq \mathfrak{M}$ ,  $C(\Lambda'(\mathfrak{M}))=\Lambda$  or  $=\mathfrak{M}$ . If  $C(\Lambda'(\mathfrak{M}))=\Lambda$ , then  $\Lambda=\Lambda'(\mathfrak{M})$ . Hence,  $\mathfrak{M}$  is not idempotent by Theorem 2. Therefore,  $D(\Lambda'(\mathfrak{M}))=\Lambda$ , which implies  $\Lambda'(\mathfrak{M})=\Lambda$ . Hence,  $\mathfrak{M}$  is invertible by Pro-

position 2. If  $C(\Lambda'(\mathfrak{M})) = \mathfrak{M}$  then  $\mathfrak{M}$  is idempotent by Theorem 2.

**REMARK 1.** Let  $\Omega$  be a maximal order among similar orders to  $\Lambda$  and satisfy  $(A_2')$  and  $(A_3)$ . If  $\Lambda$  is an  $H$ -order in  $\Omega$  which is similar to  $\Lambda$ , then the above idempotent and maximal two-sided ideals divide a unique maximal one among two-sided ideals of  $\Omega$  in  $\Lambda$  by [2], Lemma 1.

**Lemma 6.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$  and  $\mathfrak{M}$  a maximal and idempotent two-sided ideal in  $\Lambda$ . Then  $C(\Lambda'(\mathfrak{M}))$  is also a maximal and idempotent two-sided ideal in  $\Lambda$ .*

**Proof.** We set  $\Gamma_1 = \Lambda'(\mathfrak{M})$  and  $\Gamma_2 = \Lambda^r(\mathfrak{M})$ . From Corollary 1 to Theorem 2 we know that  $\mathfrak{C} = C(\Gamma_2)$  is idempotent and that there are no orders between  $\Gamma_2$  and  $\Lambda$  which is a finitely generated  $\Lambda$ -module. Hence,  $\mathfrak{C}$  is a maximal one among idempotent two-sided ideal. We assume that  $\mathfrak{C}$  is not a maximal ideal. Let  $\mathfrak{N} \supsetneq \mathfrak{C}$  be a maximal two-sided ideal in  $\Lambda$ . Then  $\mathfrak{N}$  is invertible by the above observation and Lemma 5. If  $\mathfrak{N}^{-1}\mathfrak{C} = \mathfrak{C}$ , then  $\mathfrak{N}^{-1} \subseteq \Gamma_2$  by Theorem 2. Hence  $\mathfrak{M} \subseteq \mathfrak{M}\mathfrak{N}^{-1} \subseteq \mathfrak{M}\Gamma_2 = \mathfrak{M}$ . Therefore,  $\mathfrak{M} = \mathfrak{M}\mathfrak{N} \subseteq \mathfrak{N}$ , which implies  $\mathfrak{M} = \mathfrak{N}$ . It is a contradiction. Thus, we know  $\mathfrak{C} \subsetneq \mathfrak{N}^{-1}\mathfrak{C} \subsetneq \Lambda$ , and  $\Lambda'(\mathfrak{N}^{-1}\mathfrak{C}) = \Lambda^r(\mathfrak{N}^{-1}\mathfrak{C}) = \Lambda$ . Therefore,  $\mathfrak{N}^{-1}\mathfrak{C}$  is invertible in  $\Lambda$  and hence, so is  $\mathfrak{C}$  which is a contradiction.

By using the same argument as [4], Theorem 5.3 we shall prove

**Proposition 7.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$  and  $\{\mathfrak{M}_i\}_{i=1}$  a set of maximal and idempotent two-sided ideals in  $\Lambda$  such that  $\Lambda^r(\mathfrak{M}_i) = \Lambda'(\mathfrak{M}_{i+1})$  for all  $i$ . If all the  $\mathfrak{M}_i$ 's are distinct then  $\Lambda'(\mathfrak{N}_i) = \Lambda'(\mathfrak{M}_1)$ ,  $\Lambda^r(\mathfrak{N}_i) = \Lambda^r(\mathfrak{M}_i)$  for  $\mathfrak{N}_i = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \cdots \cap \mathfrak{M}_i$ . If  $\mathfrak{M}_1 = \mathfrak{M}_n$  for some  $n > 1$ , then  $\mathfrak{N}_{n-1} = \mathfrak{M}_1 \cap \cdots \cap \mathfrak{M}_{n-1}$  is an invertible two-sided ideal. Furthermore, if  $\mathfrak{A}$  is an invertible two-sided ideal in  $\Lambda$ , which is contained in  $\mathfrak{M}_i$  for some  $i$ , then  $\mathfrak{A} \subseteq \mathfrak{M}_i \cap \cdots \cap \mathfrak{M}_r$  for any  $r > i$ .*

**Proof.** We denote  $\Lambda'(\mathfrak{M}_i)$  and  $\Lambda^r(\mathfrak{M}_i)$  by  $\Gamma_i$  and  $\Gamma_{i+1}$ . Let  $\mathfrak{N} = \mathfrak{M}_1 \cap \cdots \cap \mathfrak{M}_i$ . We know from argument of [4], Corollary 1.9 that  $\mathfrak{N}\Gamma_i = \Gamma_i$ ,  $\Gamma_{i+1}\mathfrak{N} = \Gamma_{i+1}$ . Since  $\mathfrak{M}_j$  is maximal,  $\mathfrak{N} = \sum \mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\cdots\mathfrak{M}_{p_i}$  where  $\Sigma$  runs through all elements of symmetric group  $S_i$ .

$$(*) \quad \Lambda \supseteq \mathfrak{M}_{j-1}\Gamma_j \supseteq \mathfrak{N}\Gamma_j \supseteq \mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\cdots\mathfrak{M}_{p_{j-1}}\mathfrak{M}_j\Gamma_j = \mathfrak{M}_{p_1}\cdots\mathfrak{M}_{p_{j-2}}\mathfrak{M}_{j-1} \quad \text{if } j \neq 1.$$

Hence  $\Gamma_j \subseteq \text{Hom}_{\Lambda}^l(\mathfrak{N}, \Lambda)$  and  $\tau_{\Lambda}^l(\mathfrak{N}) \supseteq \mathfrak{N}\Gamma_j \supseteq \mathfrak{M}_{p_1}\cdots\mathfrak{M}_{p_{j-2}}\mathfrak{M}_{j-1}^{3)} + \mathfrak{N}$  for  $j \neq 1$ . Therefore, if  $\mathfrak{M}_n = \mathfrak{M}_1$ , then  $\Lambda/\mathfrak{N} = \Lambda/\mathfrak{M}_1 \oplus \cdots \oplus \Lambda/\mathfrak{M}_{n-1} \supseteq \tau(\mathfrak{N})/\mathfrak{N} = \Lambda/\mathfrak{M}_1 \oplus \cdots \oplus \Lambda/\mathfrak{M}_{n-1} = \Lambda/\mathfrak{N}$ , and hence  $\tau_{\Lambda}^l(\mathfrak{N}) = \Lambda$ . Similarly, we obtain  $\tau_{\Lambda}^r(\mathfrak{N}) = \Lambda$ . Therefore,  $\mathfrak{N}$  is invertible. Let  $\mathfrak{A}$  be the ideal in the

3)  $\hat{j}$  means that the  $j$ th factor is omitted.

proposition. We may assume  $\mathfrak{A} \subseteq \mathfrak{M}_1$ .  $\mathfrak{A}\Gamma_2\mathfrak{A} \subseteq \mathfrak{M}_1\Gamma_2\mathfrak{A} \subseteq \mathfrak{M}_1\mathfrak{A} \subseteq \mathfrak{A}$ . Hence  $\Gamma_2\mathfrak{A} \subseteq \Lambda'(\mathfrak{A}) = \Lambda$ , which implies  $\mathfrak{A} \subseteq C(\Gamma_2) = \mathfrak{M}_2$ . Therefore,  $\mathfrak{A} \subseteq \bigcap_{i=1}^r \mathfrak{M}_i$ . Finally we assume that all the  $\mathfrak{M}_i$ 's are distinct. From the fact (\*) we obtain  $\tau_\Lambda^i(\mathfrak{N}) \supseteq \mathfrak{M}_1$ . If  $\tau_\Lambda^i(\mathfrak{N}) \neq \mathfrak{M}_1$ , then  $\tau_\Lambda^i(\mathfrak{N}) = \Lambda$ . By replacing  $\mathfrak{A}$  by  $\mathfrak{N}$  in the above argument, we obtain  $\bigcap \mathfrak{M}_i = \mathfrak{N} \subseteq D(\Gamma_1)$ . On the other hand,  $D(\Gamma_1)$  is a maximal two-sided ideal by Lemma 6. Hence,  $\mathfrak{M}_i = D(\Gamma_1)$  for some  $i$ . Then  $\Lambda'(\mathfrak{M}_i) = \Gamma_1 = \Gamma_{i+1}$  and hence,  $\mathfrak{M}_i = \mathfrak{M}_{i+1}$ , which is a contradiction. Therefore,  $\tau_\Lambda^i(\mathfrak{N}) = \mathfrak{M}_1$ . Thus, we obtain  $\Lambda'(\mathfrak{N}) = \Lambda'(\mathfrak{M}_1)$  by the similar argument to [4], Proposition 1.6, 2). Similarly, we have  $\Lambda'(\mathfrak{N}) = \Lambda'(\mathfrak{M}_i)$ .

**Lemma 7.** *Let  $\Lambda$  be a regular  $H$ -order in  $S$  which satisfies  $(A_2')$ . Then every invertible two-sided ideal in  $\Lambda$  is contained in one of the following ideals: 1) maximal non-idempotent two-sided ideals, 2)  $\mathfrak{N} = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_n$ , where  $\mathfrak{M}_i$ 's are as in Proposition 7 and  $\Lambda'(\mathfrak{M}_n) = \Lambda'(\mathfrak{M}_1)$ .*

*Proof.* Let  $\mathfrak{A}$  be an invertible ideal in  $\Lambda$  and  $\mathfrak{M}$  a maximal ideal containing  $\mathfrak{A}$ . If  $\mathfrak{M}$  is not idempotent, then  $\mathfrak{M}_2 = C(\Lambda'(\mathfrak{M}))$ ,  $\mathfrak{M}_3 = C(\Lambda'(\mathfrak{M}_2))$ , ... are maximal. By Proposition 5 we know  $\mathfrak{A} \subseteq \mathfrak{M} \cap \mathfrak{M}_2 \cap \dots \cap \mathfrak{M}_r$ . Since  $\Lambda$  satisfies  $(A_2')$ , we can find  $n$  such that  $\mathfrak{M}_n = \mathfrak{M}_{n'}$  for some  $n \leq n'$ .

By  $\mathfrak{Q}$  we shall denote either the maximal and non-idempotent ideals or  $\mathfrak{N}$  as in the Lemma 7, 2).

**Theorem 3.** ([4], Theorem 7.5). *Let  $\Lambda$  be a regular  $H$ -order in  $S$  which satisfies  $(A_2')$ . Then the set of invertible two-sided ideals in  $\Lambda$  is uniquely written as a product of maximal ones among invertible ideals in  $\Lambda$ , which are commutative.*

*Proof.* First we shall show that  $\mathfrak{Q}_1\mathfrak{Q}_2 = \mathfrak{Q}_2\mathfrak{Q}_1$ . We may assume  $\mathfrak{Q}_1 \not\subseteq \mathfrak{Q}_2$ .  $\mathfrak{Q}_1\mathfrak{Q}_2 = \mathfrak{Q}_2\mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$ . It is clear that  $\mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$  is an invertible ideal in  $\Lambda$ . If  $\mathfrak{Q}_1$  is maximal, then  $\mathfrak{Q}_1 \not\subseteq \mathfrak{Q}_2$  since if  $\mathfrak{Q}_2 = \mathfrak{M}_1 \cap \dots \cap \mathfrak{M}_i \subseteq \mathfrak{Q}_1$ , then  $\mathfrak{M}_j = \mathfrak{Q}_1$ . However  $\mathfrak{M}_j$  is not invertible, which is a contradiction. Since  $\mathfrak{Q}_1$  is prime,  $\mathfrak{Q}_1 \supseteq \mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$ . Therefore,  $\mathfrak{Q}_2\mathfrak{Q}_1 \supseteq \mathfrak{Q}_1\mathfrak{Q}_2$ . If  $\mathfrak{Q}_1 = \mathfrak{M}_1 \cap \dots \cap \mathfrak{M}_i$ , then  $\mathfrak{Q}_1 \not\subseteq \mathfrak{Q}_2$ . Because if  $\mathfrak{Q}_1 \supseteq \mathfrak{Q}_2$ , then  $\mathfrak{Q}_1 = \mathfrak{Q}_2$  by Proposition 5. Hence, we have as above that  $\mathfrak{Q}_2\mathfrak{Q}_1 \supseteq \mathfrak{Q}_1\mathfrak{Q}_2$ . Similarly we obtain  $\mathfrak{Q}_1\mathfrak{Q}_2 \supseteq \mathfrak{Q}_2\mathfrak{Q}_1$ . Since the set of  $\mathfrak{Q}_i$ 's consists of maximal ones among invertible ideals in  $\Lambda$  and  $\Lambda$  satisfies  $(A_2')$ , we can easily show that  $\mathfrak{A} = \prod \mathfrak{Q}_i^{e_i}$  for an invertible ideal  $\mathfrak{A}$ . The uniqueness of this expression is easily proved by making use of the same argument as above.

**REMARK 2.** We may replace  $(A_2')$  in Theorem 3 by a condition that  $\Lambda$  satisfies a minimal condition with respect to two-sided ideals in  $\Lambda$  con-



taining an inversible ideal.

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