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ON GENERALIZATION OF ASANO'S MAXIMAL ORDERS IN A RING

MANABU HARADA

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As an extension of maximal orders in a central simple algebra $\Sigma$ over $K$ of finite dimension, the author has studied structure of hereditary orders in $\Sigma$ in [4], [5]. On the other hand in [1], [1'] the theory of maximal orders in $\Sigma$ was extended to the theory in any ring by Asano. Following the method given by Asano in [1], [1'] we shall generalize the notion of hereditary order in $\Sigma$.

Let $S$ be a ring with unit element 1 and $\Lambda$ a subring in $S$ containing 1 such that $S$ is the right and left quotient ring of $\Lambda$ with respect to $\Lambda \cap S^*$, where $S^*$ consists of all non zero-divisors in $S$. We call $\Lambda$ an order in $S$. Asano showed in [1], [1'] that $\Lambda$ is a maximal order which satisfies two conditions $(A_1')$, $(A_2)$ (see below) if and only if the set of two-sided ideals is a group with respect to multiplication. In this case

$$(H) \text{ every two-sided ideal}^{1)} \text{ is finitely generated } \Lambda \text{-projective as a right and left } \Lambda \text{-module.}$$

Thus, we shall generalize the notion of the Asano's maximal order to orders which satisfies (H).

In this note we shall show that many results of hereditary orders in a central simple algebra in [4], [5] and [6] are valid in the above generalized orders.

We shall call briefly an order $\Lambda$ in $S$ which satisfies (H) an $H$-order. Furthermore, we call elements in $S^*$ regular.

1. Definitions and lemmas

**Definition 1.** Let $\Lambda$ be an order in $S$. A subset $\mathcal{I}$ of $S$ is called left (right) ideal of $\Lambda$ if $\mathcal{I}$ satisfies the following conditions:

1) $\mathcal{I}$ is a left (right) $\Lambda$-module,
2) $\mathcal{I}$ contains a regular element,

$^{1)}$ See Definition 1.
3) there exists a regular element $\lambda(\in \Lambda)$ such that $\mathfrak{A}\lambda\subseteq\Lambda$ ($\lambda\mathfrak{A}\subseteq\Lambda$).

If $\mathfrak{A}$ is a right and left ideal of $\Lambda$, we call $\mathfrak{A}$ a two-sided ideal of $\Lambda$.

**Definition 2.** Let $\Lambda$ and $\Gamma$ be orders in $S$. If there exist regular elements $\alpha$, $\alpha'$, $\beta$ and $\beta'$ such that $\alpha\Lambda\alpha'\subseteq\Gamma$ and $\beta\Gamma\beta'\subseteq\Lambda$ then we call $\Lambda$ and $\Gamma$ are similar and we denote by $\Lambda\sim\Gamma$.

**Lemma 1.** ([4], Lemma 1.2). Let $\Lambda$ be an order in $S$ and $\mathcal{Q}$, $\mathcal{Q}'$ left ideals of $\Lambda$. Then $\text{Hom}_\Lambda^\Lambda(\mathcal{Q}, \mathcal{Q}') = \{x \mid x \in S, \mathcal{Q}x \subseteq \mathcal{Q}'\}$.

**Proof.** It is clear that $\{x \mid x \in S, \mathcal{Q}x \subseteq \mathcal{Q}'\} \subseteq \text{Hom}_\Lambda^\Lambda(\mathcal{Q}, \mathcal{Q}')$. Since $\mathcal{S}=S$, we have, for any element $x$ in $S$, that $x=\Sigma s_i l_i$, $s_i \in \mathcal{Q}$, $l_i \in \mathcal{Q}$. We define $f(x)=\Sigma s_i f(l_i)$ for $f \in \text{Hom}_\Lambda^\Lambda(\mathcal{Q}, \mathcal{Q}')$. Let $x=\Sigma s'_i l'_i$ be another expression, then there exists a regular element $\gamma$ in $\Lambda$ such that $\gamma s_i$, $\gamma s'_i \in \Lambda$ for all $i$ by [1']. Hence, $\gamma \Sigma s_i f(l_i) = \Sigma f(\gamma s_i l_i) = \Sigma f(\gamma s'_i l'_i) = \gamma \Sigma s'_i f(l'_i)$. Therefore, $f$ is well defined and $f \in \text{Hom}_S(\mathcal{S}, \mathcal{S})$. Hence, $f(x)=xy$ for some $y \in S$. It is clear that $f^\mathcal{Q}=f$.

If $\mathcal{Q}=\mathcal{Q}'$, then $\text{Hom}_\Lambda^\Lambda(\mathcal{Q}, \mathcal{Q})$ is a similar order to $\Lambda$ by [1'], Theorem 4.4 and we call it the right order of $\mathcal{Q}$ and denote it by $\Lambda^\mathcal{Q}$. Similarly, we can define the left order of $\mathcal{Q}$ and denote it by $\Lambda_\mathcal{Q}$.

**Definition 3.** Let $\Lambda$ be an order in $S$. If there exist regular elements $\alpha$, $\beta$ in $\Lambda$ for $x$ in $S$ such that $x\Lambda\alpha\subseteq\Lambda$, $\beta\Lambda x\subseteq\Lambda$, then $\Lambda$ is called regular.

**Lemma 2.** Let $\Lambda$ be a regular order in $S$ and $\Gamma$ a similar order to $\Lambda$. Then there exist regular elements $\alpha$, $\beta$ in $\Lambda$ such that $\alpha\Gamma\subseteq\Lambda$ and $\Gamma\beta\subseteq\Lambda$, and $\Gamma$ is also regular.

It is clear (cf. [1'] pp. 163-165).

**Corollary.** Let $\Lambda$ be a regular order and $\Gamma$ an order containing $\Lambda$. If $\Lambda$ is a finitely generated left or right $\Lambda$-module, then $\Gamma\sim\Lambda$. Hence, $\Gamma$ is a two-sided ideal of $\Lambda$ and is regular.

It is clear by [1'] and [2].

**Lemma 3.** Let $\Lambda$ be an order and $\mathcal{Q}$ a left ideal. Then $\mathcal{Q}\mathcal{Q}^{-1} = \Lambda^\mathcal{Q}(\mathcal{Q})$ if and only if $\mathcal{Q}$ is a finitely generated projective $\Lambda^\mathcal{Q}(\mathcal{Q})$-module, where $\mathcal{Q}^{-1} = \{x \mid x \in S, \mathcal{Q}x \subseteq \mathcal{Q}\}$.

**Proof.** We put $\Gamma = \Lambda^\mathcal{Q}(\mathcal{Q})$. We define $\varphi : \mathcal{Q} \otimes S, \text{Hom}_S(\mathcal{Q}, \Gamma) = \mathcal{Q} \otimes S, \mathcal{Q}^{-1} \rightarrow \text{Hom}_S(\mathcal{Q}, \mathcal{Q}) = \Lambda^\mathcal{Q}(\mathcal{Q})$ by setting $\varphi((l \otimes f)(l') = f(l')l$, where $l, l' \in \mathcal{Q}$ and $f \in \mathcal{Q}^{-1}$. Then the lemma is clear by [3], Proposition A.1.

**Lemma 4.** Let $\Lambda$ be an order and $\mathfrak{A}$ a two-sided ideal of $\Lambda$ which is
finitely generated as left, right module. If \( \mathfrak{A} \) is a projective left \( \Lambda \)-module, then \( \Lambda'(\mathfrak{A}) \) is a right finitely generated \( \Lambda \)-projective module.

Proof. The operation of \( \Lambda \) on \( \Lambda'(\mathfrak{A}) \) from the right side coincides with the operation of \( \Lambda \) on \( \text{Hom}_\Lambda(\mathfrak{A}, \mathfrak{A}) \) with respect to the second \( \mathfrak{A} \). Since \( \mathfrak{A} \) is left \( \Lambda \)-projective, \( M = \Sigma \oplus \Delta u_i \rightarrow \mathfrak{A} \rightarrow 0 \) splits. Hence, \( \text{Hom}_\Lambda(M, \mathfrak{A}) = \Sigma \oplus \mathfrak{A} \leftarrow \text{Hom}_\Lambda(\mathfrak{A}, \mathfrak{A}) \leftarrow 0 \) splits as a usual right \( \Lambda \)-module. Since \( \mathfrak{A} \) is a finitely generated right \( \Lambda \)-module, so is \( \Lambda'(\mathfrak{A}) \).

2. \( H \)-orders

We shall quote here Asano’s axioms. Let \( \Lambda \) be an order in \( S \).

\( (A_1') \) \( \Lambda \) satisfies a minimal condition for two-sided ideals in \( \Lambda \) which contains a fixed two-sided ideal.

\( (A_2) \) Every prime ideal in \( \Lambda \) is a maximal two-sided ideal.

Proposition 1. If a regular order \( \Lambda \) satisfies \( (A_1') \) and \( (A_2) \) and \( \Lambda \) is maximal among similar orders to it, then \( \Lambda \) is an \( H \)-order.

Proof. By \([1],[1']\) we know that the set of two-sided ideals is a group with respect to multiplication. Hence, \( \Lambda \) satisfies (H) by Lemma 3.

From Lemma 2 and the proof of \([4]\), Lemma 1.2, we have

Theorem 1. Let \( \Lambda \) be a regular \( H \)-order in \( S \). If \( \Gamma \) is an order containing \( \Lambda \) which is similar to \( \Lambda \), then \( \Gamma \) is an \( H \)-order.

Proposition 2. Let \( \Lambda \) be an \( H \)-order and \( \mathfrak{A} \) a two-sided ideal of \( \Lambda \). Then \( \mathfrak{A}\Lambda^{-1} = \Lambda'(\mathfrak{A}) \) and \( \Lambda^{-1}\mathfrak{A} = \Lambda'(\mathfrak{A}) \).

Proof. It is clear from the fact \( \Lambda'(\mathfrak{A}) = \tau_{\Lambda'\mathfrak{A}}(\mathfrak{A}) = \mathfrak{A}\Lambda^{-1} \) and \( \Lambda'(\mathfrak{A}) = \tau_{\Lambda'\mathfrak{A}}(\mathfrak{A}) = \Lambda^{-1}\mathfrak{A} \) which is obtained by \([3]\), Proposition A.3.

Corollary. Let \( \Lambda \) be a regular \( H \)-order and maximal order among similar orders to \( \Lambda \). Then \( \Lambda \) is a maximal order satisfying \( (A_1') \) and \( (A_2) \).

Proof. Let \( \mathfrak{A} \) be a two-sided ideal in \( \Lambda \). Since \( \Lambda'(\mathfrak{A}) \sim \Lambda \), \( \Lambda'(\mathfrak{A}) = \Lambda \). Hence \( \mathfrak{A}\Lambda^{-1} = \Lambda^{-1}\mathfrak{A} = \Lambda \). Let \( \mathfrak{B} \) be any two-sided ideal of \( \Lambda \). Since \( \mathfrak{B}\lambda \mathfrak{A} \subseteq \Lambda \) for some regular element \( \lambda \) in \( \Lambda \), \( \mathfrak{B}\Lambda\Lambda \mathfrak{A} \subseteq \Lambda \). By \([1']\), Theorem 4.12 \( \Lambda\Lambda\Lambda \mathfrak{A} \) is a two-sided ideal in \( \Lambda \). Hence the set of two-sided ideals of \( \Lambda \) is a group.

We note that in the proof of \([4]\), Proposition 1.6 we have only used the facts that \( \mathfrak{A} \) is a finitely generated projective module and \( \Lambda'(\mathfrak{A}) \), \( \Lambda'(\mathfrak{A}) \subseteq S \) for a two-sided ideal \( \mathfrak{A} \) of \( \Lambda \) and that if an order \( \Gamma \supseteq \Lambda \) is a finitely generated left \( \Lambda \)-module, \( C(\Gamma) = \{ x | \in S, \Gamma x \subseteq \Lambda \} \) is a two-sided
ideal in \( \Lambda \). Hence, we have by Lemma 4.

**Theorem 2.** ([4], Theorem 1.7). Let \( \Lambda \) be an \( H \)-order in \( S \) and \( \Gamma \) an order containing \( \Lambda \). If \( \Gamma \) is finitely generated left \( \Lambda \)-projective, then \( C(\Gamma) \) is an idempotent two-sided ideal in \( \Lambda \) and \( \Gamma = \Lambda'C(\Gamma) \). Conversely, if \( \mathfrak{A} \) is an idempotent two-sided ideal in \( \Lambda \) then \( \Lambda'(\mathfrak{A}) \) is finitely generated left \( \Lambda \)-projective and \( \mathfrak{A} = C(\Lambda'(\mathfrak{A})) \). This correspondence is anti-lattice isomorphic.

**Corollary 1.** Let \( \Lambda \) be a regular \( H \)-order in \( S \). Then there exists the above one-to-one correspondence between two-sided idempotent ideals in \( \Lambda \) and orders \( \Gamma \) containing \( \Lambda \) which is finitely generated as a right or left \( \Lambda \)-module.

Proof. It is clear from Theorem 2 and Lemma 2.

**Corollary 2.** Let \( \Lambda \) be a regular \( H \)-order in \( S \). If there exists only a finite number of maximal orders containing \( \Lambda \) which are similar to \( \Lambda \), then \( \Lambda \) is equal to the intersection of them.

Proof. Let \( \{\Omega_i\}_{i=1}^n \) be the set of maximal orders in the corollary. Then \( \{C(\Omega_i)\}_{i=1}^n \) is the set of minimal ones among idempotent two-sided ideals by Theorem 2. We put \( \Xi = \Sigma C(\Omega_i) \). \( \Xi \) is also idempotent. Let \( \Gamma = \bigcap \Omega_i = \Lambda'(\Xi) \), and \( \Gamma' = \Lambda'(\Xi) \). Since \( C(\Omega_i) \) is minimal idempotent, \( \Lambda'(C(\Omega_i)) \) is also a maximal order. Hence, \( \{\Omega_i\}_{i=1}^n = \{\Lambda'(C(\Omega_i))\}_{i=1}^n \) and \( \Gamma = \bigcap \Omega_i = \Gamma' \) by Theorem 2. Therefore, \( \Lambda = \Gamma \) by [4], Corollary 1.9.

**Proposition 3.** Let \( \Lambda \) be an \( H \)-order in \( S \) and \( \mathfrak{S} \) a left ideal of \( \Lambda \) such that \( \Lambda = \Lambda'(\mathfrak{S}) \). Then the following statements are equivalent.

1) \( \Lambda'(\mathfrak{S}^{-1}) = \Lambda \).
2) \( \mathfrak{S}^{-1} = \Lambda \).
3) \( \mathfrak{S} \) is a finitely generated \( \Lambda'(\mathfrak{S}) \)-module.

Proof. 2) is equivalent to 3) by Lemma 3. It is clear that 2) implies 1).

1) \( \rightarrow \) 3). We set \( \Gamma = \Lambda'(\mathfrak{S}) \). \( \text{Hom}_\Lambda(\mathfrak{S} \otimes_\mathfrak{S} \mathfrak{S}^{-1}, \Lambda) = \text{Hom}_\mathfrak{S}(\mathfrak{S}^{-1}, \text{Hom}_\Lambda(\mathfrak{S}, \Lambda)) = \text{Hom}_\mathfrak{S}(\mathfrak{S}^{-1}, \mathfrak{S}^{-1}) = \Lambda \). From an exact sequence \( \mathfrak{S} \otimes \mathfrak{S}^{-1} \to \mathfrak{S}^{-1} \to 0 \) we obtain an exact sequence \( 0 \to \text{Hom}_\Lambda(\mathfrak{S}, \mathfrak{S}^{-1}, \Lambda) \to \text{Hom}_\Lambda(\mathfrak{S} \otimes_\mathfrak{S} \mathfrak{S}^{-1}, \Lambda) = \Lambda \). Since \( \mathfrak{S} \otimes \mathfrak{S}^{-1} \subseteq \Lambda \), \( \text{Hom}_\Lambda(\mathfrak{S}, \Lambda) \supseteq \Lambda \). Therefore, \( \text{Hom}_\Lambda(\mathfrak{S}, \Lambda) = \Lambda \), which implies \( \Lambda'(\mathfrak{S}^{-1}) = \Lambda \) and \( \tau_\Lambda(\mathfrak{S}^{-1}) = \mathfrak{S}^{-1} = \mathfrak{S}^{-1} \). Hence \( \mathfrak{S}^{-1} \) is idempotent by [4], Lemma 1.5. Therefore, we obtain by Theorem 2 that \( \Lambda = D(\Lambda)^{\otimes} = D(\Lambda')(\mathfrak{S}^{-1}) = \mathfrak{S}^{-1} \).

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2) \( D(\Gamma) = \{x \in S \mid x \Gamma \subseteq \Lambda\} \).
Corollary. Let $\Lambda$ be an order in $S$. We assume the set of two-sided ideals of $\Lambda$ is a group. Then every left ideal $\mathfrak{G}$ of $\Lambda$ is finitely generated $\Lambda$-projective and $\mathfrak{G}^{-1}=\Lambda$.

Proof. $\Lambda$ is a maximal order among similar to $\Lambda$ by [1'], Theorem 4.22. It is clear that $\Lambda \sim \Lambda'\mathfrak{G}^{-1}$. Hence, $\Lambda'\mathfrak{G}^{-1}=\Lambda$.

Proposition 4. ([6], Theorem 1.1). Let $\Lambda$ be an $H$-order in $S$ and $\mathfrak{G}$ a left ideal of $\Lambda$. If $\Lambda'\mathfrak{G}=\Lambda$ and $\mathfrak{G}$ is finitely generated $\Lambda$-projective, then $\Lambda'\mathfrak{G}$ is an $H$-order and $\mathfrak{G}$ is finitely generated $\Lambda'\mathfrak{G}$-projective.

Proof. Let $\Gamma=\Lambda'\mathfrak{G}$. Since $\mathfrak{G}$ is finitely generated $\Lambda$-projective, we have $\Gamma=\mathfrak{G}^{-1}\mathfrak{G}$ by Lemma 3, and $\tau_\Lambda^\mathfrak{G}(\mathfrak{G})=\mathfrak{G}$ by [3], Proposition A.5. Hence, $\tau_\Lambda^\mathfrak{G}(\mathfrak{G})=\mathfrak{G}^{-1}=\tau_\Lambda^\mathfrak{G}(\mathfrak{G})\mathfrak{G}^{-1}=\tau_\Lambda(\mathfrak{G})'$. We obtain $\Lambda'(\tau_\Lambda^\mathfrak{G}(\mathfrak{G}))=\Lambda$ from the facts $\Lambda'\mathfrak{G}=\Lambda$ and $\tau_\Lambda^\mathfrak{G}(\mathfrak{G})=\mathfrak{G}$. Therefore, $\mathfrak{G}^{-1}=\tau_\Lambda(\mathfrak{G})=\Lambda$ by Theorem 2. We can easily check that a correspondence $\mathfrak{A} \leftrightarrow \mathfrak{G}^{-1}\mathfrak{A}$ of two-sided ideals $\mathfrak{A}$ of $\Lambda$ and those of $\Gamma$ is one-to-one and preserves projectivity and finiteness, because $\mathfrak{G}^{-1}\mathfrak{A}=\mathfrak{A}\mathfrak{G}^{-1}\mathfrak{A}$.

From the similar argument to [4], Lemma 2.1 we have

Proposition 5. ([4], Proposition 2.2). Let $\Lambda$ be an $H$-order in $S$. If $S=S_1 \oplus \cdots \oplus S_n$, then $\Lambda=\Lambda_1 \oplus \cdots \oplus \Lambda_n$ and $\Lambda_i$ is an $H$-order in $S_i$, where the $S_i$'s are subring of $S$.

3. Inversible ideals in an $H$-order

Finally we shall consider two-sided ideals $\mathfrak{A}$ in $\Lambda$ such that $\mathfrak{A}^{-1}=\mathfrak{A}^{-1}\mathfrak{A}=\Lambda$. We call those ideals inversible ideals of $\Lambda$.

Proposition 6. Let $\Lambda$ be a regular $H$-order in $S$ satisfying $(A_\mathfrak{A})$. If every maximal two-sided ideal in $\Lambda$ is inversible, then $\Lambda$ is a maximal order.

Proof. We assume that there exists an ideal in $\Lambda$ which is not inversible. Let $C$ be a maximal one among ideals in $\Lambda$ which are not inversible, and $N$ be a maximal two-sided ideal containing $C$. Since $\Lambda$ satisfies $(A_\mathfrak{A})$, $C \subseteq N^n$ for some $n$. Then $C \subseteq N^{-1}C \subseteq \Lambda$, because if $NC=C$, $C=N^{-1}C \subseteq N^n$. Thus, $N^{-1}C$ must be inversible, which is a contradiction.

Lemma 5. Let $\Lambda$ be an $H$-order in $S$ and $\mathfrak{M}$ a maximal two-sided ideal. Then $\mathfrak{M}$ is either inversible in $\Lambda$ or idempotent.

Proof. Since $C(\Lambda'\mathfrak{M}) \subseteq \mathfrak{M}$, $C(\Lambda'\mathfrak{M})=\Lambda$ or $=\mathfrak{M}$. If $C(\Lambda'\mathfrak{M})=\Lambda$, then $\Lambda=\Lambda'\mathfrak{M}$. Hence, $\mathfrak{M}$ is not idempotent by Theorem 2. Therefore, $D(\Lambda'\mathfrak{M})=\Lambda$, which implies $\Lambda'\mathfrak{M}=\Lambda$. Hence, $\mathfrak{M}$ is inversible by Pro-
position 2. If $C(\Lambda'(M)) = M$ then $M$ is idempotent by Theorem 2.

REMARK 1. Let $\Omega$ be a maximal order among similar orders to $\Lambda$ and satisfy $(A')$ and $(A_3)$. If $\Lambda$ is an $H$-order in $\Omega$ which is similar to $\Lambda$, then the above idempotent and maximal two-sided ideals divide a unique maximal one among two-sided ideals of $\Omega$ in $\Lambda$ by [2], Lemma 1.

**Lemma 6.** Let $\Lambda$ be a regular $H$-order in $S$ and $\mathcal{M}$ a maximal and idempotent two-sided ideal in $\Lambda$. Then $C(\Lambda'(\mathcal{M}))$ is also a maximal and idempotent two-sided ideal in $\Lambda$.

**Proof.** We set $\Gamma_1 = \Lambda'(\mathcal{M})$ and $\Gamma_2 = \Lambda'(M)$. From Corollary 1 to Theorem 2 we know that $C(\Gamma_2) = C(\Gamma_1)$ is idempotent and that there are no orders between $\Gamma_1$ and $\Lambda$ which is a finitely generated $\Lambda$-module. Hence, $C$ is a maximal one among idempotent two-sided ideal. We assume that $C$ is not a maximal ideal. Let $\mathcal{N} \supseteq C$ be a maximal two-sided ideal in $\Lambda$. Then $\mathcal{N}$ is invertible by the above observation and Lemma 5. If $\mathcal{N}^{-1} \subseteq C$, then $\mathcal{N}^{-1} \subseteq \Gamma_1$ by Theorem 2. Hence $\mathcal{N} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_{\Gamma_2} = \mathcal{M}$. Therefore, $\mathcal{M} = \mathcal{M} \mathcal{N} \mathcal{M} \subseteq \mathcal{N}$, which implies $\mathcal{M} = \mathcal{N}$. It is a contradiction. Thus, we know $\mathcal{N}^{-1} \subseteq C \subseteq \Lambda$, and $\Lambda'(\mathcal{N}^{-1}C) = \Lambda'(\mathcal{N}^{-1}C) = \Lambda$. Therefore, $\mathcal{N}^{-1}C$ is invertible in $\Lambda$ and hence, so is $C$ which is a contradiction.

By using the same argument as [4], Theorem 5.3 we shall prove

**Proposition 7.** Let $\Lambda$ be a regular $H$-order in $S$ and $\{\mathcal{M}_i\}_{i=1}^n$ a set of maximal and idempotent two-sided ideals in $\Lambda$ such that $\Lambda'/(\mathcal{M}_i) = \Lambda'/(\mathcal{M}_{i+1})$ for all $i$. If all the $\mathcal{M}_i$'s are distinct then $\Lambda'/(\mathcal{M}_i) = \Lambda'(\mathcal{M}_i)$, $\Lambda'/(\mathcal{M}_i) = \Lambda'(\mathcal{M}_i)$ for $\mathcal{M}_i = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots \wedge \mathcal{M}_i$. If $\mathcal{M}_i = \mathcal{M}_n$ for some $n > 1$, then $\mathcal{N}_n = \mathcal{M}_1 \wedge \cdots \wedge \mathcal{M}_{n-1}$ is an invertible two-sided ideal. Furthermore, if $\mathcal{N}$ is an invertible two-sided ideal in $\Lambda$, which is contained in $\mathcal{M}_i$ for some $i$, then $\mathcal{N} \subseteq \mathcal{M}_1 \wedge \cdots \wedge \mathcal{M}_r$ for any $r > i$.

**Proof.** We denote $\Lambda'(\mathcal{M}_i)$ and $\Lambda'(\mathcal{M}_i)$ by $\Gamma_i$ and $\Gamma_{i+1}$. Let $\mathcal{N} = \mathcal{M}_1 \wedge \cdots \wedge \mathcal{M}_i$. We know from argument of [4], Corollary 1.9 that $\mathcal{M}_i \subseteq \Gamma_i$, $\Gamma_{i+1} \subseteq \Gamma_i$. Since $\mathcal{M}_i$ is maximal, $\mathcal{M}_1 = \Sigma \mathcal{M}_{\rho_1} \mathcal{M}_{\rho_2} \cdots \mathcal{M}_{\rho_1}$ where $\Sigma$ runs through all elements of symmetric group $S_i$.

\[(*) \quad \Lambda \supseteq \mathcal{M}_{\rho_{i-1}} \Gamma_j \supseteq \mathcal{M}_{\rho_i} \mathcal{M}_{\rho_{i+1}} \cdots \mathcal{M}_{\rho_{i-j+1}} \mathcal{M}_{\rho_i} = \mathcal{M}_{\rho_{i-j+1}} \mathcal{M}_{\rho_i} \quad \text{if } j + 1.
\]

Hence $\Gamma_j \subseteq \text{Hom}(\mathcal{N}, \Lambda)$ and $\tau_j(\mathcal{N}) \supseteq \mathcal{M}_{\rho_{i-1}} \supseteq \mathcal{M}_{\rho_i} \cdots \mathcal{M}_{\rho_{i-j+1}} \mathcal{M}_{\rho_i} \supseteq \mathcal{M}_{\rho_{i-j+1}}$ for $j + 1$. Therefore, if $\mathcal{M}_n = \mathcal{M}_1$, then $\Lambda/\mathcal{N} = \Lambda/\mathcal{M}_i \cdots \Lambda/\mathcal{M}_n = \tau(\mathcal{N})/\mathcal{N} = \Lambda/\mathcal{M}_i \cdots \Lambda/\mathcal{M}_{n-1} = \Lambda/\mathcal{N}$, and hence $\tau(\mathcal{N}) = \Lambda$. Similarly, we obtain $\tau(\mathcal{N}) = \Lambda$. Therefore, $\mathcal{N}$ is invertible. Let $\mathcal{I}$ be the ideal in the

3) $j$ means that the $j$th factor is omitted.
proposition. We may assume \( A \subseteq M_1 \). Hence \( \Gamma_2 M_1 \subseteq \Lambda^*(A) = \Lambda \), which implies \( A \subseteq C(\Gamma_2) = M_2 \). Therefore, \( A \subseteq \bigcap_{i=1}^{n} M_i \).

Finally we assume that all the \( M_i \)'s are distinct. From the fact (*) we obtain \( \tau_1(A) = M_i \). If \( \tau_1(A) = \Lambda \), then \( \tau_1(A) = \Lambda \). By replacing \( A \) by \( R \) in the above argument, we obtain \( \bigcap_{i=1}^{n} M_i = R \subseteq D(\Gamma_1) \). On the other hand, \( D(\Gamma_1) \) is a maximal two-sided ideal by Lemma 6. Hence, \( M_i = D(\Gamma_1) \) for some \( i \). Then \( \Lambda^*(M_i) = \Gamma_i = \Gamma_{i+1} \), and hence, \( M_i = M_{i+1} \), which is a contradiction. Therefore, \( \tau_1(A) = M_i \). Thus, we obtain \( \Lambda'(A) = \Lambda'(M_i) \) by the similar argument to \([4], \text{Proposition 1.6, 2}\). Similarly, we have \( \Lambda'(A) = \Lambda'(M_i) \).

**Lemma 7.** Let \( A \) be a regular \( H \)-order in \( S \) which satisfies \( (A_*) \). Then every inversible two-sided ideal in \( A \) is contained in one of the following ideals: 1) maximal non-idempotent two-sided ideals, 2) \( R = M_1 \cap M_2 \cap M_n \), where \( M_i \)'s are as in Proposition 7 and \( \Lambda^*(M_i) = \Lambda'(M_i) \).

**Proof.** Let \( A \) be an inversible ideal in \( A \) and \( R \) a maximal ideal containing \( A \). If \( M_n \) is not idempotent, then \( M_n = C(\Lambda'(M_i)) \), \( M_2 = C(\Lambda'(M_i)) \), ... are maximal. By Proposition 5 we know \( A \subseteq M_i \cap M_2 \cap M_n \). Since \( A \) satisfies \( (A_*) \), we can find \( n \) such that \( M_n = M_{n'} \) for some \( n \leq n' \).

By \( \Omega \) we shall denote either the maximal and non-idempotent ideals or \( R \) as in the Lemma 7, 2).

**Theorem 3.** ([4], Theorem 7.5). Let \( A \) be a regular \( H \)-order in \( S \) which satisfies \( (A_*) \). Then the set of inversible two-sided ideals in \( A \) is uniquely written as a product of maximal ones among inversible ideals in \( A \), which are commutative.

**Proof.** First we shall show that \( \Omega_i \cap \Omega_j = \Omega_i \cap \Omega_j \). We may assume \( \Omega_i \subseteq \Omega_j \). \( \Omega_i \cap \Omega_j = \Omega_i \cap \Omega_j \cap \Omega_i \cap \Omega_j \). It is clear that \( \Omega_i : \Omega_j \Omega_j \) is an inversible ideal in \( A \). If \( \Omega_i \) is maximal, then \( \Omega_i \cap \Omega_j \cap \Omega_i \cap \Omega_j \subseteq \Omega_1 \), then \( M_j = \Omega_1 \). However \( M_j \) is not inversible, which is a contradiction. Since \( \Omega_i \) is prime, \( \Omega_i \subseteq \Omega_i \cap \Omega_i \cap \Omega_i \cap \Omega_j \). Therefore, \( \Omega_i \cap \Omega_i \subseteq \Omega_i \cap \Omega_i \). If \( \Omega_i = M_i \cap \cdots \cap M_i \), then \( \Omega_i \subseteq \Omega_1 \). Because if \( \Omega_i \subseteq \Omega_1 \), then \( \Omega_i = \Omega_1 \) by Proposition 5. Hence, we have as above that \( \Omega_i : \Omega_i \cap \Omega_i \). Similarly we obtain \( \Omega_i : \Omega_i \subseteq \Omega_i : \Omega_i \). Since the set of \( \Omega_i \)'s consists of maximal ones among inversible ideals in \( A \), which satisfies \( (A_*) \), we can easily show that \( A = \pi \Omega_i \) for an inversible ideal \( A \). The uniqueness of this expression is easily proved by making use of the same argument as above.

**Remark 2.** We may replace \( (A_*) \) in Theorem 3 by a condition that \( A \) satisfies a minimal condition with respect to two-sided ideals in \( A \). con-
taining an inversible ideal.

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