Harada, M.
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ON GENERALIZATION OF ASANO'S MAXIMAL ORDERS IN A RING

MANABU HARADA

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As an extension of maximal orders in a central simple algebra Σ over K of finite dimension, the arthor has studied structure of hereditary orders in Σ in [4], [5]. On the other hand in [1], [1′] the theory of maximal orders in Σ was extended to the theory in any ring by Asano. Following the method given by Asano in [1], [1′] we shall generalize the notion of hereditary order in Σ.

Let S be a ring with unit element 1 and Λ a subring in S containing 1 such that S is the right and left quotient ring of Λ with respect to $\Lambda \cap S^*$, where S^* consists of all non zero-divisors in S. We call Λ *an order* in S. Asano showed in [1], [1′] that Λ is a maximal order which satisfies two conditions (A_2'), (A_3) (see below) if and only if the set of two-sided ideals is a group with respect to multiplication. In this case

(H) *every two-sided ideal*[1] *is finitely generated Λ-projective as a right and left Λ-module.*

Thus, we shall generalize the notion of the Asano's maximal order to orders which satisfies (H).

In this note we shall show that many results of hereditary orders in a central simple algebra in [4], [5] and [6] are valid in the above generalized orders.

We shall call briefly an order Λ in S which satisfies (H) an *H*-order. Furthermore, we call elements in S^* *regular*.

1. Definitions and lemmas

DEFINITION 1. Let Λ be an order in S. A subset $\mathfrak{A}$ of S is called *left* (*right*) *ideal* of Λ if $\mathfrak{A}$ satisfies the following conditions:

1) $\mathfrak{A}$ is a left (right) Λ-module,
2) $\mathfrak{A}$ contains a regular element,

1) See Definition 1,

3) there exists a regular element $\lambda(\in\Lambda)$ such that $\mathfrak{A}\lambda\subseteq\Lambda$ ($\lambda\mathfrak{A}\subseteq\Lambda$).
If $\mathfrak{A}$ is a right and left ideal of Λ, we call $\mathfrak{A}$ a *two-sided ideal* of Λ.

DEFINITION 2. Let Λ and Γ be orders in S. If there exist regular elements α, α', β and β' such that $\alpha\Lambda\alpha'\subseteq\Gamma$ and $\beta\Gamma\beta'\subseteq\Lambda$ then we call Λ and Γ are *similar* and we denote by $\Lambda\sim\Gamma$.

Lemma 1. ([4], Lemma 1.2). *Let Λ be an order in S and $\mathfrak{L}$, $\mathfrak{L}'$ left ideals of Λ. Then* $\mathrm{Hom}_\Lambda^l(\mathfrak{L}, \mathfrak{L}')=\{x\mid\in S, \mathfrak{L}x\subseteq\mathfrak{L}'\}$.

Proof. It is clear that $\{x\mid\in S, \mathfrak{L}x\subseteq\mathfrak{L}'\}\subseteq\mathrm{Hom}_\Lambda^l(\mathfrak{L}, \mathfrak{L}')$. Since $S\mathfrak{L}=S$, we have, for any element x in S, that $x=\Sigma s_il_i$, $s_i\in S$, $l_i\in\mathfrak{L}$. We define $\bar{f}(x)=\Sigma s_if(l_i)$ for $f\in\mathrm{Hom}_\Lambda^l(\mathfrak{L}, \mathfrak{L}')$. Let $x=\Sigma s_i'l_j'$ be another expression, then there exists a regular element γ in Λ such that γs_i, $\gamma s_i'\in\Lambda$ for all i by [1′]. Hence, $\gamma\Sigma s_if(l_i)=\Sigma f(\gamma s_il_i)=\Sigma\bar{f}(\gamma s_i'l_i')=\gamma\Sigma s_i'f(l_i')$. Therefore, $\bar{f}$ is well defined and $\bar{f}\in\mathrm{Hom}_S(S\mathfrak{L}, S\mathfrak{L})=S$. Hence, $\bar{f}(x)=xy$ for some $y\in S$. It is clear that $\bar{f}|\mathfrak{L}=f$.

If $\mathfrak{L}=\mathfrak{L}'$, then $\mathrm{Hom}_\Lambda^l(\mathfrak{L}, \mathfrak{L})$ is a similar order to Λ by [1′], Theorem 4.4 and we call it *the right order* of $\mathfrak{L}$ and denote it by $\Lambda^r(\mathfrak{L})$. Similarly, we can define *the left order* of $\mathfrak{L}$ and denote it by $\Lambda^l(\mathfrak{L})$.

DEFINITION 3. Let Λ be an order in S. If there exist regular elements α, β in Λ for x in S such that $x\Lambda\alpha\subseteq\Lambda$, $\beta\Lambda x\subseteq\Lambda$, then Λ is called *regular*.

Lemma 2. *Let Λ be a regular order in S and Γ a similar order to Λ. Then there exist regular elements α, β in Λ such that $\alpha\Gamma\subseteq\Lambda$ and $\Gamma\beta\subseteq\Lambda$, and Γ is also regular.*

It is clear (cf. [1′] pp. 163–165).

Corollary. *Let Λ be a regular order and Γ an order containing Λ. If Λ is a finitely generated left or right Λ-module, then $\Gamma\sim\Lambda$. Hence, Γ is a two-sided ideal of Λ and is regular.*

It is clear by [1′] and [2].

Lemma 3. *Let Λ be an order and $\mathfrak{L}$ a left ideal. Then $\mathfrak{L}\mathfrak{L}^{-1}=\Lambda^l(\mathfrak{L})$ if and only if $\mathfrak{L}$ is a finitely generated projective $\Lambda^r(\mathfrak{L})$-module, where* $\mathfrak{L}^{-1}=\{x\mid\in S, \mathfrak{L}x\mathfrak{L}\subseteq\mathfrak{L}\}$.

Proof. We put $\Gamma=\Lambda^r(\mathfrak{L})$. We define $\varphi:\mathfrak{L}\otimes_\Gamma\mathrm{Hom}(\mathfrak{L}, \Gamma)=\mathfrak{L}\otimes_\Gamma\mathfrak{L}^{-1}\to\mathrm{Hom}_\Gamma(\mathfrak{L}, \mathfrak{L})=\Lambda^l(\mathfrak{L})$ by setting $\varphi(l\otimes f)(l')=f(l')l$, where $l, l'\in\mathfrak{L}$ and $f\in\mathfrak{L}^{-1}$. Then the lemma is clear by [3], Proposition A.1.

Lemma 4. *Let Λ be an order and $\mathfrak{A}$ a two-sided ideal of Λ which is* *finitely generated as left, right module. If $\mathfrak{A}$ is a projective left Λ-module, then $\Lambda^r(\mathfrak{A})$ is a right finitely generated Λ-projective module.*

Proof. The operation of Λ on $\Lambda^r(\mathfrak{A})$ from the right side coincides with the operation of Λ on $\mathrm{Hom}^l_\Lambda(\mathfrak{A}, \mathfrak{A})$ with respect to the second $\mathfrak{A}$. Since $\mathfrak{A}$ is left Λ-projective, $M=\Sigma\oplus\Lambda u_i \to \mathfrak{A} \to 0$ splits. Hence, $\mathrm{Hom}_\Lambda(M, \mathfrak{A})=\Sigma\oplus\mathfrak{A} \leftarrow \mathrm{Hom}^l_\Lambda(\mathfrak{A}, \mathfrak{A}) \leftarrow 0$ splits as a usual right Λ-module. Since $\mathfrak{A}$ is a finitely generated right Λ-module, so is $\Lambda^r(\mathfrak{A})$.

2. *H*-orders

We shall quote here Asano's axioms. Let Λ be an order in S.

(A_2') *Λ satisfies a minimal condition for two-sided ideals in Λ which contains a fixed two-sided ideal.*

(A_3) *Every prime ideal in Λ is a maximal two-sided ideal.*

Proposition 1. *If a regular order Λ satisfies* (A_2') *and* (A_3) *and Λ is maximal among similar orders to it, then Λ is an H-order.*

Proof. By [1], [1′] we know that the set of two-sided ideals is a group with respect to multiplication. Hence, Λ satisfies (H) by Lemma 3.

From Lemma 2 and the proof of [4], Lemma 1.2, we have

Theorem 1. *Let Λ be a regular H-order in S. If Γ is an order containing Λ which is similar to Λ, then Γ is an H-order.*

Proposition 2. *Let Λ be an H-order and $\mathfrak{A}$ a two-sided ideal of Λ. Then $\mathfrak{A}\mathfrak{A}^{-1}=\Lambda^l(\mathfrak{A})$ and $\mathfrak{A}^{-1}\mathfrak{A}=\mathfrak{A}^r(\mathfrak{A})$.*

Proof. It is clear from the fact $\Lambda^l(\mathfrak{A})=\tau_{\Lambda^l(\mathfrak{A})}(\mathfrak{A})=\mathfrak{A}\mathfrak{A}^{-1}$ and $\Lambda^r(\mathfrak{A})=\tau_{\Lambda^r(\mathfrak{A})}(\mathfrak{A})=\mathfrak{A}^{-1}\mathfrak{A}$ which is obtained by [3], Proposition A.3.

Corollary. *Let Λ be a regular H-order and maximal order among similar orders to Λ. Then Λ is a maximal order satisfying* (A_2') *and* (A_3).

Proof. Let $\mathfrak{A}$ be a two-sided ideal in Λ. Since $\Lambda^l(\mathfrak{A})\sim\Lambda$, $\Lambda^l(\mathfrak{A})=\Lambda$. Hence $\mathfrak{A}\mathfrak{A}^{-1}=\mathfrak{A}^{-1}\mathfrak{A}=\Lambda$. Let $\mathfrak{B}$ be any two-sided ideal of Λ. Since $\mathfrak{B}\lambda\subseteq\Lambda$ for some regular element λ in Λ, $\mathfrak{B}\Lambda\lambda\Lambda\subseteq\Lambda$. By [1′], Theorem 4.12 $\Lambda\lambda\Lambda$ is a two-sided ideal in Λ. Hence the set of two-sided ideals of Λ is a group.

We note that in the proof of [4], Proposition 1.6 we have only used the facts that $\mathfrak{A}$ is a finitely generated projective module and $\Lambda^l(\mathfrak{A})$, $\Lambda^r(\mathfrak{A})\subseteq S$ for a two-sided ideal $\mathfrak{A}$ of Λ and that if an order $\Gamma\supseteq\Lambda$ is a finitely generated left Λ-module, $C(\Gamma)=\{x \mid \in S, \Gamma x\subseteq\Lambda\}$ is a two-sided ideal in Λ. Hence, we have by Lemma 4

Theorem 2. ([4], Theorem 1.7). *Let Λ be an H-order in S and Γ an order containing Λ. If Γ is finitely generated left Λ-projective, then $C(\Gamma)$ is an idempotent two-sided ideal in Λ and $\Gamma=\Lambda^l(C(\Gamma))$. Conversely, if $\mathfrak{A}$ is an idempotent two-sided ideal in Λ then $\Lambda^l(\mathfrak{A})$ is finitely generated left Λ-projective and $\mathfrak{A}=C(\Lambda^l(\mathfrak{A}))$. This correspondence is anti-lattice isomorphic.*

Corollary 1. *Let Λ be a regular H-order in S. Then there exists the above one-to-one correspondence between two-sided idempotent ideals in Λ and orders Γ containing Λ which is finitely generated as a right or left Λ-module.*

Proof. It is clear from Theorem 2 and Lemma 2.

Corollary 2. *Let Λ be a regular H-order in S. If there exists only a finite number of maximal orders containing Λ which are similar to Λ, then Λ is equal to the intersection of them.*

Proof. Let $\{\Omega_i\}_{i=1}^n$ be the set of maximal orders in the corollary. Then $\{C(\Omega_i)\}_{i=1}^n$ is the set of minimal ones among idempotent two-sided ideals by Theorem 2. We put $\mathfrak{D}=\Sigma\, C(\Omega_i)$. $\mathfrak{D}$ is also idempotent. Let $\Gamma=\cap\Omega_i=\Lambda^l(\mathfrak{D})$, and $\Gamma'=\Lambda^r(\mathfrak{D})$. Since $C(\Omega_i)$ is minimal idempotent, $\Lambda^r(C(\Omega_i))$ is also a maximal order. Hence, $\{\Omega_i\}_{i=1}^n=\{\Lambda^r(C(\Omega_i))\}_{i=1}^n$ and $\Gamma=\cap\Omega_i=\Gamma'$ by Theorem 2. Therefore, $\Lambda=\Gamma$ by [4], Corollary 1.9.

Proposition 3. *Let Λ be an H-order in S and $\mathfrak{L}$ a left ideal of Λ such that $\Lambda=\Lambda^l(\mathfrak{L})$. Then the following statements are equivalent.*

1) $\Lambda^r(\mathfrak{L}^{-1})=\Lambda$.
2) $\mathfrak{L}\mathfrak{L}^{-1}=\Lambda$.
3) *$\mathfrak{L}$ is a finitely generated $\Lambda^r(\mathfrak{L})$-module.*

Proof. 2) is equivalent to 3) by Lemma 3. It is clear that 2) implies 1).

1)→3). We set $\Gamma=\Lambda^r(\mathfrak{L})$. $\mathrm{Hom}^l_\Lambda(\mathfrak{L}\otimes_\Gamma\mathfrak{L}^{-1},\Lambda)=\mathrm{Hom}^l_\Gamma(\mathfrak{L}^{-1},\mathrm{Hom}^l_\Lambda(\mathfrak{L},\Lambda))$ $=\mathrm{Hom}^l_\Gamma(\mathfrak{L}^{-1},\mathfrak{L}^{-1})=\Lambda$. From an exact sequence $\mathfrak{L}\otimes\mathfrak{L}^{-1}\to\mathfrak{L}\mathfrak{L}^{-1}\to 0$ we obtain an exact sequence $0\to\mathrm{Hom}^l_\Lambda(\mathfrak{L}\mathfrak{L}^{-1},\Lambda)\to\mathrm{Hom}^l_\Lambda(\mathfrak{L}\otimes_\Gamma\mathfrak{L}^{-1},\Lambda)=\Lambda$. Since $\mathfrak{L}\mathfrak{L}^{-1}\subseteq\Lambda$, $\mathrm{Hom}^l_\Lambda(\mathfrak{L}\mathfrak{L}^{-1},\Lambda)\supseteq\Lambda$. Therefore, $\mathrm{Hom}^l_\Lambda(\mathfrak{L}\mathfrak{L}^{-1},\Lambda)=\Lambda$, which implies $\Lambda^r(\mathfrak{L}\mathfrak{L}^{-1})=\Lambda$ and $\tau^l_\Lambda(\mathfrak{L}\mathfrak{L}^{-1})=\mathfrak{L}\mathfrak{L}^{-1}\Lambda=\mathfrak{L}\mathfrak{L}^{-1}$. Hence $\mathfrak{L}\mathfrak{L}^{-1}$ is idempotent by [4], Lemma 1.5. Therefore, we obtain by Theorem 2 that $\Lambda=D(\Lambda)$[2] $=D(\Lambda^r(\mathfrak{L}\mathfrak{L}^{-1}))=\mathfrak{L}\mathfrak{L}^{-1}$.

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2) $D(\Gamma)=\{x|\in S,\ x\Gamma\subseteq\Lambda\}$.

Corollary. *Let Λ be an order in S. We assume the set of two-sided ideals of Λ is a group. Then every left ideal $\mathfrak{L}$ of Λ is finitely generated Λ-projective and $\mathfrak{L}\mathfrak{L}^{-1}=\Lambda$.*

Proof. Λ is a maximal order among similar to Λ by [1′], Theorem 4.22. It is clear that $\Lambda\sim\Lambda^l(\mathfrak{L}^{-1})$. Hence, $\Lambda^r(\mathfrak{L}^{-1})=\Lambda$.

Proposition 4. ([6], Theorem 1.1). *Let Λ be an H-order in S and $\mathfrak{L}$ a left ideal of Λ. If $\Lambda^l(\mathfrak{L})=\Lambda$ and $\mathfrak{L}$ is finitely generated Λ-projective, then $\Lambda^r(\mathfrak{L})$ is an H-order and $\mathfrak{L}$ is finitely generated $\Lambda^r(\mathfrak{L})$-projective.*

Proof. Let $\Gamma=\Lambda^r(\mathfrak{L})$. Since $\mathfrak{L}$ is finitely generated Λ-projective, we have $\Gamma=\mathfrak{L}^{-1}\mathfrak{L}$ by Lemma 3, and $\tau_\Lambda^l(\mathfrak{L})\mathfrak{L}=\mathfrak{L}$ by [3], Proposition A.5. Hence, $\tau_\Lambda^l(\mathfrak{L})=\mathfrak{L}\mathfrak{L}^{-1}=\tau_\Lambda^l(\mathfrak{L})\mathfrak{L}\mathfrak{L}^{-1}=\tau(\mathfrak{L})^2$. We obtain $\Lambda^l(\tau_\Lambda^l(\mathfrak{L}))=\Lambda$ from the facts $\Lambda^l(\mathfrak{L})=\Lambda$ and $\tau_\Lambda^l(\mathfrak{L})\mathfrak{L}=\mathfrak{L}$. Therefore, $\mathfrak{L}\mathfrak{L}^{-1}=\tau_\Lambda(\mathfrak{L})=\Lambda$ by Theorem 2. We can easily check that a correspondence $\mathfrak{A}\leftrightarrow\mathfrak{L}^{-1}\mathfrak{A}\mathfrak{L}$ of two-sided ideals $\mathfrak{A}$ of Λ and those of Γ is one-to-one and preserves projectivity and finiteness, because $\mathfrak{L}^{-1}\mathfrak{A}\mathfrak{L}\approx\mathfrak{L}^{-1}\otimes\mathfrak{A}\otimes\mathfrak{L}$.

From the similar regument to [4], Lemma 2.1 we have

Proposition 5. ([4], Proposition 2.2). *Let Λ be an H-order in S. If $S=S_1\oplus\cdots\oplus S_n$, then $\Lambda=\Lambda_1\oplus\cdots\oplus\Lambda_n$ and Λ_i is an H-order in S_i, where the S_i's are subring of S.*

3. Inversible ideals in an *H*-order

Finally we shall consider two-sided ideals $\mathfrak{A}$ in Λ such that $\mathfrak{A}\mathfrak{A}^{-1}=\mathfrak{A}^{-1}\mathfrak{A}=\Lambda$. We call those ideals *inversible ideals* of Λ.

Proposition 6. *Let Λ be a regular H-order in S satisfying* (A_2'). *If every maximal two-sided ideal in Λ is inversible, then Λ is a maximal order.*

Proof. We assume that there exists an ideal in Λ which is not inversible. Let $\mathfrak{C}$ be a maximal one among ideals in Λ which are not inversible, and $\mathfrak{N}$ be a maximal two-sided ideal containing $\mathfrak{C}$. Since Λ satisfies (A_2'), $\mathfrak{C}\not\supseteq\mathfrak{N}^n$ for some n. Then $\mathfrak{C}\subsetneqq\mathfrak{N}^{-1}\mathfrak{C}\subseteq\Lambda$, because if $\mathfrak{N}\mathfrak{C}=\mathfrak{C}$, $\mathfrak{C}=\mathfrak{N}^n\mathfrak{C}\subseteq\mathfrak{N}^n$. Thus, $\mathfrak{N}^{-1}\mathfrak{C}$ must be inversible, which is a contradiction.

Lemma 5. *Let Λ be an H-order in S and $\mathfrak{M}$ a maximal two-sided ideal. Then $\mathfrak{M}$ is either inversible in Λ or idempotent.*

Proof. Since $\mathfrak{C}(\Lambda^l(\mathfrak{M}))\supseteq\mathfrak{M}$, $C(\Lambda^l(\mathfrak{M}))=\Lambda$ or $=\mathfrak{M}$. If $C(\Lambda^l(\mathfrak{M}))=\Lambda$, then $\Lambda=\Lambda^l(\mathfrak{M})$. Hence, $\mathfrak{M}$ is not idempotent by Theorem 2. Therefore, $D(\Lambda^r(\mathfrak{M}))=\Lambda$, which implies $\Lambda^r(\mathfrak{M})=\Lambda$. Hence, $\mathfrak{M}$ is inversible by Proposition 2. If $C(\Lambda^l(\mathfrak{M}))=\mathfrak{M}$ then $\mathfrak{M}$ is idempotent by Theorem 2.

REMARK 1. Let Ω be a maximal order among similar orders to Λ and satisfy (A_2') and (A_3). If Λ is an H-order in Ω which is similar to Λ, then the above idempotent and maximal two-sided ideals divide a unique maximal one among two-sided ideals of Ω in Λ by [2], Lemma 1.

Lemma 6. *Let Λ be a regular H-order in S and $\mathfrak{M}$ a maximal and idempotent two-sided ideal in Λ. Then $C(\Lambda^r(\mathfrak{M}))$ is also a maximal and idempotent two-sided ideal in Λ.*

Proof. We set $\Gamma_1=\Lambda^l(\mathfrak{M})$ and $\Gamma_2=\Lambda^r(\mathfrak{M})$. From Corollary 1 to Theorem 2 we know that $\mathfrak{C}=C(\Gamma_2)$ is idempotent and that there are no orders between Γ_2 and Λ which is a finitely generated Λ-module. Hence, $\mathfrak{C}$ is a maximal one among idempotent two-sided ideal. We assume that $\mathfrak{C}$ is not a maximal ideal. Let $\mathfrak{N}\supsetneqq\mathfrak{C}$ be a maximal two-sided ideal in Λ. Then $\mathfrak{N}$ is inversible by the above observation and Lemma 5. If $\mathfrak{N}^{-1}\mathfrak{C}=\mathfrak{C}$, then $\mathfrak{N}^{-1}\subseteq\Gamma_2$ by Theorem 2. Hence $\mathfrak{M}\subseteq\mathfrak{M}\mathfrak{N}^{-1}\subseteq\mathfrak{M}\Gamma_2=\mathfrak{M}$. Therefore, $\mathfrak{M}=\mathfrak{M}\mathfrak{N}\subseteq\mathfrak{N}$, which implies $\mathfrak{M}=\mathfrak{N}$. It is a contradiction. Thus, we know $\mathfrak{C}\subsetneqq\mathfrak{N}^{-1}\mathfrak{C}\subsetneqq\Lambda$, and $\Lambda^l(\mathfrak{N}^{-1}\mathfrak{C})=\Lambda^r(\mathfrak{N}^{-1}\mathfrak{C})=\Lambda$. Therefore, $\mathfrak{N}^{-1}\mathfrak{C}$ is inversible in Λ and hence, so is $\mathfrak{C}$ which is a contradiction.

By using the same argument as [4], Theorem 5.3 we shall prove

Proposition 7. *Let Λ be a regular H-order in S and $\{\mathfrak{M}_i\}_{i=1}$ a set of maximal and idempotent two-sided ideals in Λ such that $\Lambda^r(\mathfrak{M}_i)=\Lambda^l(\mathfrak{M}_{i+1})$ for all i. If all the $\mathfrak{M}_i$'s are distinct then $\Lambda^l(\mathfrak{N}_i)=\Lambda^l(\mathfrak{M}_1)$, $\Lambda^r(\mathfrak{N}_i)=\Lambda^r(\mathfrak{M}_i)$ for $\mathfrak{N}_i=\mathfrak{M}_1\cap\mathfrak{M}_2\cap\cdots\cap\mathfrak{M}_i$. If $\mathfrak{M}_1=\mathfrak{M}_n$ for some $n>1$, then $\mathfrak{N}_{n-1}=\mathfrak{M}_1\cap\cdots\cap\mathfrak{M}_{n-1}$ is an inversible two-sided ideal. Furthermore, if $\mathfrak{A}$ is an inversible two-sided ideal in Λ, which is contained in $\mathfrak{M}_i$ for some i, then $\mathfrak{A}\subseteq\mathfrak{M}_i\cap\cdots\cap\mathfrak{M}_r$ for any $r>i$.*

Proof. We denote $\Lambda^l(\mathfrak{M}_i)$ and $\Lambda^r(\mathfrak{M}_i)$ by Γ_i and Γ_{i+1}. Let $\mathfrak{N}=\mathfrak{M}_1\cap\cdots\cap\mathfrak{M}_i$. We know from argument of [4], Corollary 1.9 that $\mathfrak{M}_i\Gamma_i=\Gamma_i$, $\Gamma_{i+1}\mathfrak{M}_i=\Gamma_{i+1}$. Since $\mathfrak{M}_j$ is maximal, $\mathfrak{N}=\Sigma\,\mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\cdots\mathfrak{M}_{p_i}$ where Σ runs through all elements of symmetric group S_i.

$$(*)\quad \Lambda\supseteq\mathfrak{M}_{j-1}\Gamma_j\supseteq\mathfrak{N}\Gamma_j\supseteq\mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\cdots\mathfrak{M}_{j-1}\mathfrak{M}_j\Gamma_j=\mathfrak{M}_{p_1}\cdots\mathfrak{M}_{p_{i-2}}\mathfrak{M}_{j-1}\ \text{ if } j\neq 1.$$

Hence $\Gamma_j\subseteq\mathrm{Hom}^l_\Lambda(\mathfrak{N},\Lambda)$ and $\tau^l_\Lambda(\mathfrak{N})\supseteq\mathfrak{N}\Gamma_j\supseteq\mathfrak{M}_{p_1}\cdots\mathfrak{M}_{p_{i-2}}\overset{j}{\vee}\mathfrak{M}_{j-1}$[3]$+\mathfrak{N}$ for $j\neq 1$. Therefore, if $\mathfrak{M}_n=\mathfrak{M}_1$, then $\Lambda/\mathfrak{N}=\Lambda/\mathfrak{M}_1\oplus\cdots\oplus\Lambda/\mathfrak{M}_{n-1}\supseteq\tau(\mathfrak{N})/\mathfrak{N}=\Lambda/\mathfrak{M}_1\oplus\cdots\oplus\Lambda/\mathfrak{M}_{n-1}=\Lambda/\mathfrak{N}$, and hence $\tau^l_\Lambda(\mathfrak{N})=\Lambda$. Similarly, we obtain $\tau^r_\Lambda(\mathfrak{N})=\Lambda$. Therefore, $\mathfrak{N}$ is inversible. Let $\mathfrak{A}$ be the ideal in the

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3) $\overset{j}{\vee}$ means that the jth factor is omitted.

proposition. We may assume $\mathfrak{A}\subseteq\mathfrak{M}_1$. $\mathfrak{A}\Gamma_2\mathfrak{A}\subseteq\mathfrak{M}_1\Gamma_2\mathfrak{A}\subseteq\mathfrak{M}_1\mathfrak{A}\subseteq\mathfrak{A}$. Hence $\Gamma_2\mathfrak{A}\subseteq\Lambda^r(\mathfrak{A})=\Lambda$, which implies $\mathfrak{A}\subseteq C(\Gamma_2)=\mathfrak{M}_2$. Therefore, $\mathfrak{A}\subseteq\bigcap_{i=1}^{r}\mathfrak{M}_i$. Finally we assume that all the $\mathfrak{M}_i$'s are distinct. From the fact (*) we obtain $\tau_\Lambda^l(\mathfrak{N})\supseteq\mathfrak{M}_1$. If $\tau_\Lambda^l(\mathfrak{N})\neq\mathfrak{M}_1$, then $\tau_\Lambda^l(\mathfrak{N})=\Lambda$. By replacing $\mathfrak{A}$ by $\mathfrak{N}$ in the above argument, we obtain $\bigcap\mathfrak{M}_i=\mathfrak{N}\subseteq D(\Gamma_1)$. On the other hand, $D(\Gamma_1)$ is a maximal two-sided ideal by Lemma 6. Hence, $\mathfrak{M}_i=D(\Gamma_1)$ for some i. Then $\Lambda^r(\mathfrak{M}_i)=\Gamma_1=\Gamma_{i+1}$ and hence, $\mathfrak{M}_1=\mathfrak{M}_{i+1}$, which is a contradication. Therefore, $\tau_\Lambda^l(\mathfrak{N})=\mathfrak{M}_1$. Thus, we obtain $\Lambda^l(\mathfrak{N})=\Lambda^l(\mathfrak{M}_1)$ by the similar argument to [4], Proposition 1.6, 2). Similarly, we have $\Lambda^r(\mathfrak{N})=\Lambda^r(\mathfrak{M}_i)$.

Lemma 7. *Let Λ be a regular H-order in S which satisfies* (A_2'). *Then every inversible two-sided ideal in Λ is contained in one of the following ideals*: 1) *maximal non-idempotent two-sided ideals,* 2) $\mathfrak{N}=\mathfrak{M}_1\cap\mathfrak{M}_2\cap\mathfrak{M}_n$, *where $\mathfrak{M}_i$'s are as in Proposition* 7 *and* $\Lambda^r(\mathfrak{M}_n)=\Lambda^l(\mathfrak{M}_1)$.

Proof. Let $\mathfrak{A}$ be an inversible ideal in Λ and $\mathfrak{M}$ a maximal ideal containing $\mathfrak{A}$. If $\mathfrak{M}$ is not idempotent, then $\mathfrak{M}_2=C(\Lambda^r(\mathfrak{M}))$, $\mathfrak{M}_3=C(\Lambda^r(\mathfrak{M}_2))$, $\cdots$ are maximal. By Proposition 5 we know $\mathfrak{A}\subseteq\mathfrak{M}\cap\mathfrak{M}_2\cap\cdots\cap\mathfrak{M}_r$. Since Λ satisfies (A_2'), we can find n such that $\mathfrak{M}_n=\mathfrak{M}_{n'}$ for some $n\lneqq n'$.

By $\mathfrak{Q}$ we shall denote either the maximal and non-idempotent ideals or $\mathfrak{N}$ as in the Lemma 7, 2).

Theorem 3. ([4], Theorem 7.5). *Let Λ be a regular H-order in S which satisfies* (A_2'). *Then the set of inversible two-sided ideals in Λ is uniquely written as a product of maximal ones among inversible ideals in Λ, which are commutative.*

Proof. First we shall show that $\mathfrak{Q}_1\mathfrak{Q}_2=\mathfrak{Q}_2\mathfrak{Q}_1$. We may assume $\mathfrak{Q}_1\neq\mathfrak{Q}_2$. $\mathfrak{Q}_1\mathfrak{Q}_2=\mathfrak{Q}_2\mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$. It is clear that $\mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$ is an inversible ideal in Λ. If $\mathfrak{Q}_1$ is maximal, then $\mathfrak{Q}_1\not\supseteq\mathfrak{Q}_2$ since if $\mathfrak{Q}_2=\mathfrak{M}_1\cap\cdots\cap\mathfrak{M}_i\subseteq\mathfrak{Q}_1$, then $\mathfrak{M}_j=\mathfrak{Q}_1$. However $\mathfrak{M}_j$ is not inversible, which is a contradiction. Since $\mathfrak{Q}_1$ is prime, $\mathfrak{Q}_1\supseteq\mathfrak{Q}_2^{-1}\mathfrak{Q}_1\mathfrak{Q}_2$. Therefore, $\mathfrak{Q}_2\mathfrak{Q}_1\supseteq\mathfrak{Q}_1\mathfrak{Q}_2$. If $\mathfrak{Q}_1=\mathfrak{M}_1\cap\cdots\cap\mathfrak{M}_i$, then $\mathfrak{Q}_1\not\supseteq\mathfrak{Q}_2$. Because if $\mathfrak{Q}_1\supseteq\mathfrak{Q}_2$, then $\mathfrak{Q}_1=\mathfrak{Q}_2$ by Proposition 5. Hence, we have as above that $\mathfrak{Q}_2\mathfrak{Q}_1\supseteq\mathfrak{Q}_1\mathfrak{Q}_2$. Similarly we obtain $\mathfrak{Q}_1\mathfrak{Q}_2\supseteq\mathfrak{Q}_2\mathfrak{Q}_1$. Since the set of $\mathfrak{Q}_i$'s consists of maximal ones among inversible ideals in Λ and Λ satisfies (A_2'), we can easily show that $\mathfrak{A}=\Pi\,\mathfrak{Q}_i^{e_i}$ for an inversible ideal $\mathfrak{A}$. The uniqueness of this expression is easily proved by making use of the same argument as above.

REMARK 2. We may replace (A_2') in Theorem 3 by a condition that Λ satisfies a minimal condition with respect to two-sided ideals in Λ containing an inversible ideal.

OSAKA CITY UNIVERSITY

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