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THE STRUCTURE OF PRIMITIVE GAMMA RINGS

JIANG LUH

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1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [7]. The class of Γ-rings contains not only all rings but also all Hestenes ternary rings. In [7], Nobusawa generalized the Wedderburn-Artin Theorem for simple Γ-rings and for semi-simple Γ-rings. Barnes [1] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings. The author [5] gave a characterization of primitive Γ-rings with minimal one-sided ideals by means of certain Γ-rings of continuous semilinear transformations. He [6] also established several structure theorems for simple Γ-rings having minimal one-sided ideals. Recently, Coppage and the author [2] introduced the notions of Jacobson radical, Levitzki radical, nil radical for Γ-rings and obtained some basic radical properties and inclusion relations for these radicals together with the prime radical defined by Barnes [1].

The object of this paper is to study the structure of primitive Γ-rings. One of its main results is a generalization of the Jacobson - Chevalley density theorem. This generalizes further a result given by Smiley and Stephenson for Hestenes ternary rings [8].

We refer to [4] for all notions relevant to ring theory.

2. Preliminaries

Let $M$ and $\Gamma$ be two additive abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$ the conditions

(1) $x\alpha y \in M$

(2) $(x+y)\alpha z = x\alpha z + y\alpha z,$
   $x(\alpha + \beta)z = x\alpha z + x\beta z,$
   $x\alpha(y+z) = x\alpha y + x\alpha z,$

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied then we call $M$ a Γ-ring.

If these conditions are strengthened to
Let $M$ be a $\Gamma$-ring. If $S, T \subseteq M$, we write $ST \subseteq M$ for the set of finite sums $\sum s_i \alpha_i t_i$, where $s_i \in S$, $t_i \in T$, $\alpha_i \in \Gamma$. A subgroup $I$ of $M$ is a left (right) ideal of $M$ if $M \Gamma I \subseteq I (I \Gamma M \subseteq I)$. If $I$ is both a left and a right ideal of $M$, then $I$ is an ideal of $M$. A one-sided ideal $I$ is strongly nilpotent if $I^n = I \Gamma I \cdots \Gamma I = 0$ for some positive integer $n$. A non-zero right (left) ideal is minimal if the only right (left) ideals of $M$ contained in $I$ are 0 and $I$ itself. It has been shown that every minimal right ideal which is not strongly nilpotent can be expressed as the form $e \gamma M$, where $\gamma \in \Gamma$, $e \in M$ and $e \gamma e = e$ (see [5] Theorem 3.2).

Let $F$ be the free abelian group generated by the set of all ordered pairs $(\alpha, x)$ where $\alpha \in \Gamma$, $x \in M$. Let $K$ be the subgroup of elements $\sum m_i (\alpha_i, x_i) \in F$, where $m_i$ are integers such that $\sum m_i (x \alpha, x_i) = 0$ for all $x \in M$. Denote by $R$ the factor group $F/K$ and by $[\alpha, x]$ the coset $K + (\alpha, x)$. Clearly every element in $R$ can be expressed as a finite sum $\sum_\alpha \xi \alpha_\xi$. We define multiplication in $R$ by

$$\sum_\alpha \xi \alpha_\xi \cdot \sum_\beta \eta \beta_\eta = \sum_\alpha \xi_\alpha \xi_\beta \eta_\beta \beta_\eta .$$

Then $R$ forms a ring. Furthermore, $M$ is a right $R$-module with the definition

$$sx \sum_\alpha \xi \alpha_\xi = \sum_\alpha s \alpha_\alpha \alpha_\xi, \quad \text{for } x \in M.$$

We call the ring $R$ the right operator ring of $M$. Similarly, we can define the left operator ring $L$. Every element in $L$ can be expressed as a finite sum $\sum_\beta \beta_\beta \beta_j \beta_j \beta_j$ where $x_j \in M$, $\beta_j \in \Gamma$. These two operator rings play important roles in studying the structure of $\Gamma$-rings. We recall that a $\Gamma$-ring $M$ is right primitive if (i) $M \Gamma x = 0$ implies $x = 0$ and (ii) the right operator ring $R$ of $M$ is a right primitive ring.

**Theorem 1.** If $M$ is a right primitive $\Gamma$-ring, then the left operator ring of $M$ is a right primitive ring.

**Proof.** Let $R$ and $L$ be respectively the right and left operator rings of $M$.

Let $G$ be a faithful irreducible right $R$-module. Let $A$ be the free abelian group generated by the set of ordered pairs $(g, \gamma)$, where $g \in G$, $\gamma \in \Gamma$, and let $B$ be the subgroup of elements $\sum m_i (g_i, \gamma_i) \in A$ where $m_i$ are integers such that $\sum m_i (g, \gamma, x) = 0$ for all $x \in M$. Denote by $H$ the factor group $A/B$ and, without causing any ambiguity, by $[g, \gamma]$ the coset $B + (g, \gamma)$. Every element in $H$ therefore can be expressed as a finite sum $\sum_\gamma \xi \gamma \gamma_\gamma$. $H$ forms a right $L$-module with the definition
\[ \Sigma_i[g_i, \gamma_i] \cdot \Sigma_j[x_j, \beta_j] = \Sigma_i, j[g_i[\gamma_i, x_j], \beta_j] \]

for \( \Sigma_i[g_i, \gamma_i] \in H \) and \( \Sigma_j[x_j, \beta_j] \in L \). We claim that \( H \) is a faithful irreducible right \( L \)-module. Assume \( H \Sigma_j[x_j, \beta_j] = 0 \). Then for all \( \gamma \in \Gamma \), \( g \in G \), we have \( \Sigma_j[g[\gamma, x_j], \beta_j] = [g, \gamma] \Sigma_j[x_j, \beta_j] = 0 \), i.e. \( g \Sigma_j[\gamma, x_j] [\beta_j, x] = 0 \) for all \( x \in M \). By the faithfulness of the \( R \)-module of \( G \), \( [\gamma, \Sigma_j x_j \beta_j x] = \Sigma_j[\gamma, x_j] [\beta_j, x] = 0 \), so \( MT \Sigma_j x_j \beta_j x = 0 \). By the condition (i), \( \Sigma_j x_j \beta_j x = 0 \) for all \( x \in M \). This means that \( \Sigma_j[x_j, \beta_j] = 0 \) and \( H \) is faithful. To see that \( H \) is irreducible, let \( \Sigma_i[g_i, \gamma_i] \) be an arbitrary non-zero element in \( H \). Then the set \( G' = \{ \Sigma_i g_i[\gamma_i, x] : x \in M \} \) is a non-zero \( R \)-submodule of \( G \). Since \( G \) is irreducible, \( G' = G \). For any \( \Sigma_j[g_j', \gamma_j'] \in H \), we may write \( g_j' = \Sigma_i g_i[\gamma_i, x_j] \) where \( x_j \in M \). Thus \( \Sigma_j[g_j', \gamma_j'] = \Sigma_i \Sigma_j g_i[\gamma_i, x_j, \gamma_j'] = \Sigma_i[g_i, \gamma_i] \Sigma_j[x_j, \gamma_j'] \in \Sigma_i[g_i, \gamma_i] L \). Hence \( H \) is irreducible and \( L \) is a right primitive ring.

3. Irreducible \( \Gamma \)-rings of homomorphisms on groups

Let \( G \) and \( H \) be non-zero additive abelian groups. If \( M \) and \( \Gamma \) are respectively subgroups of Hom \( (H, G) \) and Hom \( (G, H) \) such that \( g \Gamma = H \) and \( h M = G \) whenever \( 0 \neq g \in G \) and \( 0 \neq h \in H \), and moreover if \( x a y \in M \) and \( a \beta \gamma \in \Gamma \) for all \( x, y \in M \), then \( M \) forms a \( \Gamma \)-ring in the sense of Nobusawa under the composition of mappings. We shall call such a \( \Gamma \)-ring an irreducible \( \Gamma \)-ring of homomorphisms on groups.

A \( \Gamma \)-ring \( M \) and a \( \Gamma' \)-ring \( M' \) are said to be isomorphic if there exist a group isomorphism \( \theta \) of \( M \) onto \( M' \) and a group isomorphism \( \phi \) of \( \Gamma \) onto \( \Gamma' \) such that \( (xa \gamma) \theta = (x \theta) (a \phi) (\gamma \theta) \) for all \( x, \gamma \in M, a \in \Gamma \). It is clear that \( M \) is right primitive if and only if \( M' \) is right primitive.

**Theorem 2.** A \( \Gamma \)-ring \( M \) is a right primitive \( \Gamma \)-ring in the sense of Nobusawa if and only if it is isomorphic to an irreducible \( \Gamma \)-ring of homomorphisms on groups.

Proof. Necessity. Let \( M \) be a right primitive \( \Gamma \)-ring in the sense of Nobusawa with right operator ring \( R \) and left operator ring \( L \) and let \( G \) be a faithful irreducible right \( R \)-module, from the proof of Theorem 1, we can construct the faithful irreducible right \( L \)-module \( H \). Now, for each \( \gamma \in \Gamma \) let \( g \phi \in \text{Hom}(G, H) \) defined by \( g(\gamma \phi) = [g, \gamma] \). Clearly \( \phi \) is a group homomorphism of \( \Gamma \) into \( \text{Hom}(G, H) \). Moreover, if \( g \phi = x \phi \), then \( [g, \gamma_1 - \gamma_2] = 0 \) i.e. \( g[\gamma_1 - \gamma_2, x] = 0 \) for all \( g \in G, x \in M \). By the faithfulness of \( G \) as an \( R \)-module, \( [\gamma_1 - \gamma_2, x] = 0 \) for all \( x \in M \). Consequently \( M(\gamma_1 - \gamma_2) = 0 \) and, by the condition (4') in the definition of \( \Gamma \)-ring in the sense of Nobusawa, \( \gamma_1 = \gamma_2 \). Thus \( \phi \) is a group isomorphism of \( \Gamma \) onto \( \Gamma' = \Gamma \phi \).

Likewise, for each \( x \in M \), let \( x \theta \) be the mapping of \( H \) into \( G \) defined by \( \Sigma_i[g_i, \gamma_i](x \theta) = \Sigma_i g_i[\gamma_i, x] \). It can be shown easily that \( x \theta \in \text{Hom}(H, G) \) and
that $\theta$ is a group homomorphism of $M$ into $\text{Hom}(H, G)$. We claim that $\theta$ is one-to-one. Indeed, if $x\theta = y\theta$, where $x, y \in M$, then $g[\gamma, x - y] = g[\gamma, x] - g[\gamma, y] = 0$ for all $g \in G$, $\gamma \in \Gamma$. Again by the faithfulness of $G$, $[\gamma, x - y] = 0$ for all $\gamma \in \Gamma$, or equivalently that $M\Gamma(x - y) = 0$. Hence $x = y$ and $\theta$ is a group isomorphism of $M$ onto $M' = M\theta$. It is easy to see that the $\Gamma$-ring $M$ is isomorphic to the $\Gamma'$-ring $M'$.

It remains to show that $M'$ is an irreducible $\Gamma'$-ring of homomorphisms on groups. Let $0 \neq g \in G$. Since $gR = G$, every element in $H$ can be expressed as $\sum [\gamma_i, x_i, \beta_j] = g(\gamma\phi)$ where $\gamma_i, \beta_j \in \Gamma$, $x_i \in M$ and $\gamma = \sum_i \gamma_i x_i \beta_j$. Hence $H = g\Gamma'$. Now, let $h$ be an arbitrary non-zero element in $H$. Then $h = g(\gamma\phi) = [g, \gamma]$ for some $\gamma \in \Gamma$. It follows that $h(x\theta) = [g, \gamma](x\theta) = g[\gamma, x]$ for all $x \in M$. Thus $hM'$ is a non-zero $R$-submodule of $G$ and hence $hM' = G$.

Sufficiency. We may assume that $M$ is an irreducible $\Gamma$-ring of homomorphisms on groups, and that $0 \neq \Gamma \subset \text{Hom}(G, H), 0 \neq M \subset \text{Hom}(H, G)$ where $H$ and $G$ are abelian groups with the property that for any $0 \neq g \in G$ and $0 \neq h \in H$, $g\Gamma = H$ and $hM = G$. Clearly, $MTx = 0$ for $x \in M$ implies $x = 0$. For $g \in G$ and $\Sigma[\gamma_i, x_i] \in R$, the right operator ring of $M$, we define composition

$$g\Sigma[\gamma_i, x_i] = \Sigma_i(g\gamma_i)x_i.$$  

This composition is well defined. For if $\Sigma_j[\gamma_i, x_i] = \Sigma_j[\beta_j, y_j]$ in $R$, then $\Sigma_i \gamma_i x_i = \Sigma_j \beta_j y_j = 0$ for all $x \in M$. By noting that $g \in g\Gamma M$, we obtain $\Sigma_i (g\gamma_i)x_i - \Sigma_j (g\beta_j)y_j = g(\Sigma_i \gamma_i x_i - \beta_j y_j) = 0$, so $g\Sigma_i[\gamma_i, x_i] = g\Sigma_j[\beta_j, y_j]$. Clearly $G$ forms an irreducible right $R$-module. Moreover, if $\Sigma_i[\gamma_i, x_i] \in R$ and if $G\Sigma_i[\gamma_i, x_i] = 0$, then $HM\Sigma_i[\gamma_i, x_i] = 0$. Consequently, $\Sigma_i[\gamma_i, x_i] = 0$ and $G$ is a faithful $R$-module. Thus, $R$ is a right primitive ring and $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa.

Observe the definition of irreducible $\Gamma'$-rings of homomorphisms on groups. We can easily see that $M$ is irreducible $\Gamma$-rings of homomorphisms on groups if and only if $\Gamma$ is a irreducible $\Gamma'$-ring of homomorphisms on groups, where $\Gamma' = \Gamma$. Thus from Theorem 2, we immediately have the following

**Corollary.** Let $M$ be a $\Gamma$-ring. Then $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa if and only if $\Gamma$ is a right primitive $\Gamma'$-ring in the sense of Nobusawa, where $\Gamma' = \Gamma$.

4. **Chevalley-Jacobson density theorem**

Let $G$ and $H$ be non-zero right vector spaces over division rings $\Delta$ and $\Delta'$ respectively, and let $\sigma$ be an isomorphism of $\Delta$ onto $\Delta'$. A group $N$ of semilinear transformations (associated with $\sigma$) of $G$ into $H$ is said to be dense if, for every positive integer $n$ and every $n$ linearly independent elements $g_1, g_2,$
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•••>£" in $G$ and every $n$ elements $h_1, h_2, \ldots, h_n$ in $H$, there exists $x \in N$ such that $g_i x = h_i, i = 1, 2, \ldots, n$.

Now, if $\Gamma$ is a dense group of semilinear transformations (associated with $\sigma$) of $G$ into $H$ and $M$ is a dense group of semilinear transformations (associated with $\sigma^{-1}$) of $H$ into $G$, and if the compositions of mappings $x \alpha y \in M$ and $\alpha \beta \in \Gamma$ for all $x, y \in M$, $\alpha, \beta \in \Gamma$, then $M$ forms a $\Gamma$-ring in the sense of Nobusawa under the composition of mappings. We shall call such a $\Gamma$-ring a dense $\Gamma$-ring of semilinear transformations.

Following is a generalization of the well known Chevalley-Jacobson density theorem.

**Theorem 3.** Let $M$ be a $\Gamma$-ring. Then $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa if and only if it is isomorphic to a dense $\Gamma$-ring of semilinear transformations.

Proof. Sufficiency. It is an immediate consequence of Theorem 2, since a dense $\Gamma$-ring of semilinear transformations evidently is an irreducible $\Gamma$-ring of homomorphisms on groups.

Necessity. We assume that $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa. According to the proof of Theorem 1 we can construct a faithful irreducible right $R$-module $G$ and a faithful irreducible right $L$-module $H$, where $R$ and $L$ are respectively the right operator ring and the left operator ring of $M$. Set $\Delta = \text{Hom}_R(G, G)$ and $\Delta' = \text{Hom}_L(H, H)$. By Schur's Lemma, $\Delta$ and $\Delta'$ are division rings.

First, we shall show that $\Delta$ and $\Delta'$ are isomorphic. For $\delta \in \Delta$, we define the mapping $\delta^*: H \rightarrow H$ by

$$(\Sigma_i [g_i, \gamma_i]) \delta^* = \Sigma_i [g_i \delta, \gamma_i]$$

for $\Sigma_i [g_i, \gamma_i] \in H$. Here $\delta^*$ is well defined. For, if $\Sigma_i [g_i, \gamma_i] = \Sigma_j [g_j', \gamma_j']$ then for all $x \in M$, $\Sigma_i [g_i, \gamma_i, x] = \Sigma_j [g_j', \gamma_j', x]$, and hence $\Sigma_i (g_i \delta) [\gamma_i, x] = (\Sigma_i g_i [\gamma_i, x]) \delta = (\Sigma_j g_j' [\gamma_j', x]) \delta = \Sigma_j (g_j' \delta) [\gamma_j', x]$. Thus $\Sigma_i [g_i \delta, \gamma_i] = \Sigma_j [g_j' \delta, \gamma_j']$ as we desired. Clearly, $\delta^*$ preserves addition. Moreover, for $\Sigma_i [g_i, \gamma_i] \in H$ and $\Sigma_j [x_j, \beta_j] \in L$, we have $(\Sigma_i [g_i, \gamma_i] \Sigma_j [x_j, \beta_j]) \delta^* = (\Sigma_i [g_i, \gamma_i, x_j]) \delta^* = (\Sigma_i, j [g_i, \gamma_i, x_j]) \delta^* = (\Sigma_i, j [g_i, \gamma_i, x_j, \beta_j]) \delta^* = \Sigma_i, j [g_i \delta, \gamma_i, x_j, \beta_j] = \Sigma_i, j [g_i \delta, \gamma_i] [x_j, \beta_j] = (\Sigma_i, j [g_i, \gamma_i]) \delta^* \Sigma_i [x_j, \beta_j]$.

Hence $\delta^* \in \Delta'$. It can be easily verified that $\delta: \delta \rightarrow \delta^*$ is a monomorphism of $\Delta$ into $\Delta'$. To show that $\sigma$ is an onto mapping, we note that since $H$ is a faithful irreducible right $L$-module and $G$ is a faithful irreducible right $R$-module there exist $g_0 \in G$ and $\gamma_0 \in \Gamma$ such that $\{(g_0, \gamma_0): \gamma \in \Gamma\} = H$ and $\{g_0[\gamma_0, x]: x \in M\} = G$.

Let $\delta'$ be an arbitrary element in $\Delta'$ and $[g_0, \gamma_0] \delta' = [g_0, \gamma_0]$ where $\gamma \in \Gamma$. Let $\delta: G \rightarrow G$ be defined by $(g_0[\gamma_0, x]) \delta = g_0[\gamma_0, x]$ for $x \in M$. This is well defined. In fact, if $g_0[\gamma_0, x] = g_0[\gamma_0, y]$, then, for any $\gamma \in \Gamma$, $[g_0, \gamma_0] [x, \gamma] = [g_0, \gamma_0] [y, \gamma] \delta' = [g_0, \gamma_0] [y, \gamma] \delta' = [g_0, \gamma_0] [y, \gamma]$.
and hence, by the construction of $H$, $g_0[\gamma,x\gamma,x]=g_0[\gamma,y\gamma,x]$ for all $\gamma \in \Gamma$, $x \in M$. It follows that $(g_0[\gamma,x]-g_0[\gamma,y])R=0$. Since $G$ is a faithful irreducible right $R$-module, $g_0[\gamma,x]=g_0[\gamma,y]$. Clearly $\delta \in \Delta$ and $\delta^*=\delta'$. Therefore $\Delta \cong \Delta'$.

In the proof of Theorem 2 we have known already that the $\Gamma$-ring $M$ is isomorphic to a $\Gamma'$-ring $M'$, where $\Gamma'$ is a subgroup of Hom $(G,H)$ and $M'$ is a subgroup of Horn $(H,G)$. More precisely, two group isomorphisms $\theta: M \rightarrow M'$ and $\phi: \Gamma \rightarrow \Gamma'$ exist such that $\Sigma_i[g_i, \gamma_i](x\theta)=\Sigma_i[g_i, \gamma_i, x]$ and $g(\gamma\phi)=\sigma^\delta \sigma$ for all $g_i, g \in G$, $\gamma_i, \gamma \in \Gamma, x \in M$.

Now we consider $G$ and $H$ as right $\Delta$-vector space and right $\Delta'$-vector space respectively. For any $g \in G$, $\delta \in \Delta$ and $\gamma \in \Gamma$, we have $(g\delta) (\gamma\phi)=\sigma^\delta \sigma$ and $((g, \gamma)\delta')(x\theta)=\sigma^\delta \sigma$. Thus $g\phi$ and $x\theta$ are semilinear transformations (associated with $\sigma$ and $\sigma^{-1}$ respectively).

It remains to show the density property for $\Gamma'$. The density property for $M'$ can be obtained similarly. We shall show that for any $n$ $\Delta$-independent elements $g_1, g_2, \ldots, g_n \in G$ and any $n$ elements $h_1, h_2, \ldots, h_n \in H$ there exists $\gamma \in \Gamma$ such that $g_i(\gamma\phi)=h_i, i=1, 2, \ldots, n$. We proceed by induction on $n$.

From Theorem 2, the assertion is obviously true for $n=1$. Now we assume that the assertion is true for $n-1$. We want first to show the existence of $\gamma \in \Gamma$ such that $g_i(\gamma\phi)=0$ for $i<n$ and $g_n(\gamma\phi) \neq 0$. Suppose such a $\gamma \in \Gamma$ does not exist. Then, for any $\gamma \in \Gamma$, $g_i(\gamma\phi)=0, 1 \leq i \leq n-1$, implies $g_n(\gamma\phi)=0$. Thus for any $h \in H$, $\delta \in \Delta$ and $\gamma \in \Gamma$, we have $(g\delta)(\gamma\phi)=\sigma^\delta \sigma$ and $((g, \gamma)\delta')(x\theta)=\sigma^\delta \sigma$. Hence the mapping $\psi: H \rightarrow H$ defined by $h\psi=g_n(\gamma\phi)$ whenever $g_n(\gamma\phi)=h$ and $g_n(\gamma\phi)=0$ for $1<i<n$, is well defined. It is easy to see that $\psi$ preserves addition. Let us recall that $g_0$ is an element in $G$ with $[\{g_0, \gamma\}: \gamma \in \Gamma]=H$. Let $[g_0, \gamma] \in H$ and $\Sigma_i[x_i, \gamma_i] \in L$. Then $[g_0, \gamma] \psi=g_n(\gamma\phi)$ for some $\gamma \in \Gamma$, where $g_i(\gamma\phi)=\sigma^\delta \sigma$. Thus, $[(g_0, \gamma)] \Sigma_i[x_i, \gamma_i] \psi=(g_n(\gamma\phi)) \Sigma_i[x_i, \gamma_i] \psi=g_n(\gamma\phi)$. Consequently, $[(g_0, \gamma)] \Sigma_i[x_i, \gamma_i] \psi=(g_n(\gamma\phi)) \Sigma_i[x_i, \gamma_i] \psi$ and hence $\psi \in \Delta'$. Let $\psi=\delta'$ where $\delta \in \Delta$. Since $g\delta-g_n, g_2, \ldots, g_n$ are $\Delta$-linearly independent, by the induction hypothesis, there exists $\gamma \in \Gamma$ such that $(g, \delta-g_n) (\gamma\phi) \neq 0$ and $g_i(\gamma'\phi)=0$ for $1<n$. But by the definition of $\psi$, $(g, \delta-g_n) (\gamma'\phi)=(g, \delta) (\gamma'\phi)-g_n(\gamma'\phi)=(g(\gamma'\phi)) \psi-g_n(\gamma'\phi)=0$, a contradiction. This proves the existence of $\gamma \in \Gamma$ such that $g_n(\gamma\phi) \neq 0$ and $g_i(\gamma\phi)=0$ for $1<i<n$. Since $g_n(\gamma\phi)L=H$, there exists $\gamma_n \in \Gamma$ such that $g_n(\gamma_n\phi)=h_n$, and $g_i(\gamma_n\phi)=0$ for $1<i<n$. 
Likewise, there exist $\gamma_i \in \Gamma, 1 \leq i \leq n$, such that $g_i(\gamma_i \phi) = h_i$ and $g_j(\gamma_i \phi) = 0$ for $i \neq j$. Now let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. Then $g_i(\gamma \phi) = h_i, 1 \leq i \leq n$ as we desired. This completes the proof of the theorem.

We recall the definition of Hestenes ternary rings. Let $G$ and $H$ be additive abelian groups. $M$ and $\Gamma$ be subgroups of $\text{Hom}(H, G)$ and $\text{Hom}(G, H)$ respectively. If there is a mapping $\ast$ of $M$ onto $\Gamma$ such that $a_i \ast c^M$ whenever $a, b, c \in M$ then $M$ is called a Hestenes ternary ring. The set of all finite sums $\sum a_i \ast b_i$ with $a_i, b_i \in M$ form a ring $R$ and the set of all finite sums $\sum c_i \ast d_i$ with $c_i, d_i \in M$ form a ring $L$. Clearly $M$ is a right $R$-module and is a left $L$-module. If $M$ is irreducible as a $R$-module and as an $L$-module then $M$ is called an irreducible Hestenes ternary ring. Evidently, if $M$ is an irreducible Hestenes ternary ring then $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa and the rings $R$ and $L$ are respectively the right operator ring and the left operator ring of $M$. Therefore Theorem 3 generalizes further the extension of the Chevalley-Jacobson density theorem given by Smiley and Stephenson (see [8, 9]).

5. Primitive $\Gamma$-rings with non-zero socles

In [6], we have introduced the notion of socles for $\Gamma$-rings. The right (left) socle $S_r(S_l)$ of a $\Gamma$-ring $M$ is the sum of all minimal right (left) ideals of $M$. In the case $M$ has no minimal right (left) ideals, the right (left) socle of $M$ is defined to be 0. It has been shown that if $M$ is an one-sided primitive $\Gamma$-ring having minimal one-sided ideals then $M$ is a two sided primitive and its right socle and left socle coincide (see [5, Theorem 4.2] and [6, Theorem 4.3]).

In this section we shall present a characterization for primitive $\Gamma$-ring with non-zero socle which is different from the one given in [5].

**Theorem 4.** A $\Gamma$-ring $M$ in the sense of Nobusawa is primitive with non-zero socle if and only if it is isomorphic to a dense $\Gamma'$-ring $M'$ of semi-linear transformations containing non-zero semilinear transformations of finite rank. Moreover, the socle of $M'$ is the set of semilinear transformations of finite rank contained in $M'$.

**Proof.** Necessity. Assume that $M$ is a primitive $\Gamma$-ring in the sense of Nobusawa with non-zero socle. According to Theorem 3, $M$ can be regarded as a dense $\Gamma$-ring of semilinear transformations. Let $G$ and $H$ be vector spaces over division rings $\Delta$ and $\Delta'$, $\sigma: \Delta \rightarrow \Delta'$ be an isomorphism, $M$ be a dense group of semilinear transformations of $H$ into $G$ (associated with $\sigma^{-1}$) and $\Gamma$ be a dense group of semilinear transformations of $G$ into $H$ (associated with $\sigma$). Let $e \gamma M$ be a minimal right ideal of $M$, where $e \in M, \gamma \in \Gamma$ and $e \gamma e = e$. We claim that $e$ is a rank 1, for otherwise, there would exist $h, h \in H$ such that $h, e$ and $h, e$ are $\Delta$-linearly independent. By the density property of $\Gamma$ and $M$, there would exists $\gamma_0 \in \Gamma$ such that $h, e \gamma_0 = 0$ and $h, e \gamma_0 \neq 0$ and $h, e \gamma_0, M = G$. 


Since \( e\gamma M \) is minimal and \( h_i e\gamma(e\gamma_0 M) = h_i e\gamma M = 0 \), the right ideal \( \{ x \in e\gamma M : h_i x = 0 \} = e\gamma M \), i.e. \( h_i e\gamma M = 0 \). Particularly, \( h_i e = h_i e\gamma e = 0 \), a contradiction. Thus \( M \) contains non-zero semilinear transformations of finite rank. In addition, since the socle \( S \) of \( M \) is the sum of minimal right ideals, every element in \( S \) is of finite rank.

Sufficiency. Assume that \( M \) is a dense \( \Gamma \)-ring of semilinear transformations on vector spaces \( G \) and \( H \) described above, and assume that \( M \) contains semilinear transformations of finite rank. By density property, \( M \) contains semilinear transformations of rank 1. Let \( a \in M \) be of rank 1, and let \( Ha = \langle g_i \rangle \), the subspace of \( G \) generated by \( g_i \). Consider \( I = \{ x \in M : Hx \subseteq \langle g_i \rangle \} \), a left ideal of \( M \). We claim that \( I \) is minimal. Let \( 0 \neq x_i \in I \). Then \( Hx_i = \langle g_i \rangle \) and \( h_i x_i = g_i \), for some \( h_i \in H \). By the density property of \( \Gamma \), there exists \( \gamma_i \in \Gamma \) such that \( g_i \gamma_i = h_i \). Thus \( g_i = g_i \gamma_i x_i \). Now let \( x \) be an arbitrary element in \( I \). For any \( h \in H \), there exists \( \delta \in \Delta \) such that \( hx = gh \delta = (g_i \gamma_i x_i) \delta = (g, \delta) \gamma_i x_i = h\gamma_i x_i \). Hence \( x = x\gamma_i x_i \in M\Gamma x_i \), so \( I = M\Gamma x_i \) for every \( 0 \neq x_i \in I \). Therefore \( I \) is a minimal left ideal containing \( a \), \( a \) is in the socle of \( M \), and \( M \) has a non-zero socle \( S \).

The argument just used shows that every element in \( M \) of rank 1 is in \( S \). But the density property of \( M \) and \( \Gamma \) insures that every element in \( M \) of finite rank is a sum of finitely many elements in \( M \) of rank 1. Therefore \( S \) contains all elements in \( M \) of finite rank. This completes the proof.

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References