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THE STRUCTURE OF PRIMITIVE GAMMA RINGS

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1. Introduction

The notion of a Γ -ring was first introduced by Nobusawa [7]. The class of Γ -rings contains not only all rings but also all Hestenes ternary rings. In [7], Nobusawa generalized the Wedderburn-Artin Theorem for simple Γ -rings and for semi-simple Γ -rings. Barnes [1] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ -rings. The author [5] gave a characterization of primitive Γ -rings with minimal one-sided ideals by means of certain Γ -rings of continuous semilinear transformations. He [6] also established several structure theorems for simple Γ -rings having minimal one-sided ideals. Recently, Coppage and the author [2] introduced the notions of Jacobson radical, Levitzki radical, nil radical for Γ -rings and obtained some basic radical properties and inclusion relations for these radicals together with the prime radical defined by Barnes [1].

The object of this paper is to study the structure of primitive Γ -rings. One of its main results is a generalization of the Jacobson - Chevalley density theorem. This generalizes further a result given by Smiley and Stephenson for Hestenes ternary rings [8].

We refer to [4] for all notions relevant to ring theory.

2. Preliminaries

Let M and Γ be two additive abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$ the conditions

- (1) $x\alpha y \in M$
- (2) $(x+y)\alpha z = x\alpha z + y\alpha z,$
 $x(\alpha + \beta)z = x\alpha z + x\beta z,$
 $x\alpha(y+z) = x\alpha y + x\alpha z,$
- (3) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied then we call M a Γ -ring.

If these conditions are strengthened to

- (1') $x\alpha y \in M, \alpha x \beta \in \Gamma,$
- (2') the same as (2),
- (3') $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$
- (4') $x\alpha y = 0$ for all $x, y \in M$ implies $\alpha = 0$, then M is called a Γ -ring in the sense of Nobusawa.

Let M be a Γ -ring. If $S, T \subseteq M$, we write $S\Gamma T$ for the set of finite sums $\sum_i s_i \alpha_i t_i$ where $s_i \in S, t_i \in T, \alpha_i \in \Gamma$. A subgroup I of M is a left (right) ideal of M if $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). If I is both a left and a right ideal of M , then I is an ideal of M . A one-sided ideal I is strongly nilpotent if $I^n = I\Gamma I \dots \Gamma I = 0$ for some positive integer n . A non-zero right (left) ideal is minimal if the only right (left) ideals of M contained in I are 0 and I itself. It has been shown that every minimal right ideal which is not strongly nilpotent can be expressed as the form $e\gamma M$, where $\gamma \in \Gamma, e \in M$ and $e\gamma e = e$ (see [5] Theorem 3.2).

Let F be the free abelian group generated by the set of all ordered pairs (α, x) where $\alpha \in \Gamma, x \in M$. Let K be the subgroup of elements $\sum_i m_i (\alpha_i, x_i) \in F$, where m_i are integers such that $\sum_i m_i (x\alpha_i x_i) = 0$ for all $x \in M$. Denote by R the factor group F/K and by $[\alpha, x]$ the coset $K + (\alpha, x)$. Clearly every element in R can be expressed as a finite sum $\sum_i [\alpha_i, x_i]$. We define multiplication in R by

$$\sum_i [\alpha_i, x_i] \cdot \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].$$

Then R forms a ring. Furthermore, M is a right R -module with the definition

$$x \sum_i [\alpha_i, x_i] = \sum_i x \alpha_i x_i, \quad \text{for } x \in M.$$

We call the ring R the right operator ring of M . Similarly, we can define the left operator ring L . Every element in L can be expressed as a finite sum $\sum_j [x_j, \beta_j]$ where $x_j \in M, \beta_j \in \Gamma$. These two operator rings play important roles in studying the structure of Γ -rings. We recall that a Γ -ring M is right primitive if (i) $M\Gamma x = 0$ implies $x = 0$ and (ii) the right operator ring R of M is a right primitive ring.

Theorem 1. *If M is a right primitive Γ -ring, then the left operator ring of M is a right primitive ring.*

Proof. Let R and L be respectively the right and left operator rings of M .

Let G be a faithful irreducible right R -module. Let A be the free abelian group generated by the set of ordered pairs (g, γ) , where $g \in G, \gamma \in \Gamma$, and let B be the subgroup of elements $\sum_i m_i (g_i, \gamma_i) \in A$ where m_i are integers such that $\sum_i m_i g_i [\gamma_i, x] = 0$ for all $x \in M$. Denote by H the factor group A/B and, without causing any ambiguity, by $[g, \gamma]$ the coset $B + (g, \gamma)$. Every element in H therefore can be expressed as a finite sum $\sum_i [g_i, \gamma_i]$. H forms a right L -module with the definition

$$\Sigma_i[g_i, \gamma_i] \cdot \Sigma_j[x_j, \beta_j] = \Sigma_{i,j}[g_i[\gamma_i, x_j], \beta_j]$$

for $\Sigma_i[g_i, \gamma_i] \in H$ and $\Sigma_j[x_j, \beta_j] \in L$. We claim that H is a faithful irreducible right L -module. Assume $H \Sigma_j[x_j, \beta_j] = 0$. Then for all $\gamma \in \Gamma, g \in G$, we have $\Sigma_j[g[\gamma, x_j], \beta_j] = [g, \gamma] \Sigma_j[x_j, \beta_j] = 0$, i.e. $g \Sigma_j[\gamma, x_j] [\beta_j, x] = 0$ for all $x \in M$. By the faithfulness of the R -module of $G, [\gamma, \Sigma_j x_j \beta_j x] = \Sigma_j[\gamma, x_j] [\beta_j, x] = 0$, so $M \Gamma \Sigma_j x_j \beta_j x = 0$. By the condition (i), $\Sigma_j x_i \beta_j x = 0$ for all $x \in M$. This means that $\Sigma_j[x_j, \beta_j] = 0$ and H is faithful. To see that H is irreducible, let $\Sigma_i[g_i, \gamma_i]$ be an arbitrary non-zero element in H . Then the set $G' = \{\Sigma_i g_i[\gamma_i, x] : x \in M\}$ is a non-zero R -submodule of G . Since G is irreducible, $G' = G$. For any $\Sigma_j[g_j', \gamma_j'] \in H$, we may write $g_j' = \Sigma_i g_i[\gamma_i, x_j]$ where $x_j \in M$. Thus $\Sigma_j[g_j', \gamma_j'] = \Sigma_j[\Sigma_i g_i[\gamma_i, x_j], \gamma_j'] = \Sigma_i[g_i, \gamma_i] \Sigma_j[x_j, \gamma_j] \in \Sigma_i[g_i, \gamma_i]L$. Hence H is irreducible and L is a right primitive ring.

3. Irreducible Γ -rings of homomorphisms on groups

Let G and H be non-zero additive abelian groups. If M and Γ are respectively subgroups of $\text{Hom}(H, G)$ and $\text{Hom}(G, H)$ such that $g\Gamma = H$ and $hM = G$ whenever $0 \neq g \in G$ and $0 \neq h \in H$, and moreover if $x\alpha y \in M$ and $\alpha x \beta \in \Gamma$ for all $x, y \in M$, then M forms a Γ -ring in the sense of Nobusawa under the composition of mappings. We shall call such a Γ -ring an irreducible Γ -ring of homomorphisms on groups.

A Γ -ring M and a Γ' -ring M' are said to be isomorphic if there exist a group isomorphism θ of M onto M' and a group isomorphism ϕ of Γ onto Γ' such that $(x\alpha y)\theta = (x\theta)(\alpha\phi)(y\theta)$ for all $x, y \in M, \alpha \in \Gamma$. It is clear that M is right primitive if and only if M' is right primitive.

Theorem 2. *A Γ -ring M is a right primitive Γ -ring in the sense of Nobusawa if and only if it is isomorphic to an irreducible Γ -ring of homomorphisms on groups.*

Proof. Necessity. Let M be a right primitive Γ -ring in the sense of Nobusawa with right operator ring R and left operator ring L and let G be a faithful irreducible right R -module, from the proof of Theorem 1, we can construct the faithful irreducible right L -module H . Now, for each $\gamma \in \Gamma$ let $\gamma\phi \in \text{Hom}(G, H)$ defined by $g(\gamma\phi) = [g, \gamma]$. Clearly ϕ is a group homomorphism of Γ into $\text{Hom}(G, H)$. Moreover, if $\gamma_1\phi = \gamma_2\phi$, then $[g, \gamma_1 - \gamma_2] = 0$ i.e. $g[\gamma_1 - \gamma_2, x] = 0$ for all $g \in G, x \in M$. By the faithfulness of G as an R -module, $[\gamma_1 - \gamma_2, x] = 0$ for all $x \in M$. Consequently $M(\gamma_1 - \gamma_2)M = 0$ and, by the condition (4') in the definition of Γ -ring in the sense of Nobusawa, $\gamma_1 = \gamma_2$. Thus ϕ is a group isomorphism of Γ onto $\Gamma' = \Gamma\phi$.

Likewise, for each $x \in M$, let $x\theta$ be the mapping of H into G defined by $\Sigma_i[g_i, \gamma_i](x\theta) = \Sigma_i g_i[\gamma_i, x]$. It can be shown easily that $x\theta \in \text{Hom}(H, G)$ and

that θ is a group homomorphism of M into $\text{Hom}(H, G)$. We claim that θ is one-to-one. Indeed, if $x\theta=y\theta$, where $x, y \in M$, then $g[\gamma, x-y]=g[\gamma, x]-g[\gamma, y]=0$ for all $g \in G, \gamma \in \Gamma$. Again by the faithfulness of $G, [\gamma, x-y]=0$ for all $\gamma \in \Gamma$, or equivalently that $M\Gamma(x-y)=0$. Hence $x=y$ and θ is a group isomorphism of M onto $M'=M\theta$. It is easy to see that the Γ -ring M is isomorphic to the Γ' -ring M' .

It remains to show that M' is an irreducible Γ' -ring of homomorphisms on groups. Let $0 \neq g \in G$. Since $gR=G$, every element in H can be expressed as $\sum_j [g \sum_i [\gamma_{ij}, x_{ij}], \beta_j]=g(\gamma\phi)$ where $\gamma_{ij}, \beta_j \in \Gamma, x_{ij} \in M$ and $\gamma = \sum_{i,j} \gamma_{ij} x_{ij} \beta_j$. Hence $H=g\Gamma'$. Now, let h be an arbitrary non-zero element in H . Then $h=g(\gamma\phi)=[g, \gamma]$ for some $\gamma \in \Gamma$. It follows that $h(x\theta)=[g, \gamma](x\theta)=g[\gamma, x]$ for all $x \in M$. Thus hM' is a non-zero R -submodule of G and hence $hM'=G$.

Sufficiency. We may assume that M is an irreducible Γ -ring of homomorphisms on groups, and that $0 \neq \Gamma \subseteq \text{Hom}(G, H), 0 \neq M \subseteq \text{Hom}(H, G)$ where H and G are abelian groups with the property that for any $0 \neq g \in G$ and $0 \neq h \in H, g\Gamma=H$ and $hM=G$. Clearly, $M\Gamma x=0$ for $x \in M$ implies $x=0$. For $g \in G$ and $\sum_i [\gamma_i, x_i] \in R$, the right operator ring of M , we define composition

$$g \sum_i [\gamma_i, x_i] = \sum_i (g\gamma_i)x_i.$$

This composition is well defined. For if $\sum_j [\gamma_j, x_j] = \sum_j [\beta_j, y_j]$ in R , then $\sum_i x\gamma_i x_i - \sum_j x\beta_j y_j = 0$ for all $x \in M$. By noting that $g \in g\Gamma M$, we obtain $\sum_i (g\gamma_i)x_i - \sum_j (g\beta_j)y_j = g(\sum_i \gamma_i x_i - \sum_j \beta_j y_j) \in g\Gamma M(\sum_i \gamma_i x_i - \sum_j \beta_j y_j) = 0$, so $g \sum_i [\gamma_i, x_i] = g \sum_j [\beta_j, y_j]$. Clearly G forms an irreducible right R -module. Moreover, if $\sum_i [\gamma_i, x_i] \in R$ and if $G \sum_i [\gamma_i, x_i] = 0$, then $HM \sum_i [\gamma_i, x_i] = G\Gamma M \sum_i [\gamma_i, x_i] = G \sum_i [\gamma_i, x_i] = 0$, and hence $M \sum_i [\gamma_i, x_i] = 0$. Consequently, $\sum_i [\gamma_i, x_i] = 0$ and G is a faithful R -module. Thus, R is a right primitive ring and M is a right primitive Γ -ring in the sense of Nobusawa.

Observe the definition of irreducible Γ -rings of homomorphisms on groups. We can easily see that M is irreducible Γ -rings of homomorphisms on groups if and only if Γ is an irreducible Γ' -ring of homomorphisms on groups, where $\Gamma'=M$. Thus from Theorem 2, we immediately have the following

Corollary. *Let M be a Γ -ring. Then M is a right primitive Γ -ring in the sense of Nobusawa if and only if Γ is a right primitive Γ' -ring in the sense of Nobusawa, where $\Gamma'=M$.*

4. Chevalley-Jacobson density theorem

Let G and H be non-zero right vector spaces over division rings Δ and Δ' respectively, and let σ be an isomorphism of Δ onto Δ' . A group N of semilinear transformations (associated with σ) of G into H is said to be dense if, for every positive integer n and every n linearly independent elements $g_1, g_2,$

\dots, g_n in G and every n elements h_1, h_2, \dots, h_n in H , there exists $x \in N$ such that $g_i x = h_i, i=1, 2, \dots, n$.

Now, if Γ is a dense group of semilinear transformations (associated with σ) of G into H and M is a dense group of semilinear transformations (associated with σ^{-1}) of H into G , and if the compositions of mappings $x\alpha y \in M$ and $\alpha x \beta \in \Gamma$ for all $x, y \in M, \alpha, \beta \in \Gamma$, then M forms a Γ -ring in the sense of Nobusawa under the composition of mappings. We shall call such a Γ -ring a dense Γ -ring of semilinear transformations.

Following is a generalization of the well known Chevalley-Jacobson density theorem.

Theorem 3. *Let M be a Γ -ring. Then M is a right primitive Γ -ring in the sense of Nobusawa if and only if it is isomorphic to a dense Γ -ring of semilinear transformations.*

Proof. Sufficiency. It is an immediate consequence of Theorem 2, since a dense Γ -ring of semilinear transformations evidently is an irreducible Γ -ring of homomorphisms on groups.

Necessity. We assume that M is a right primitive Γ -ring in the sense of Nobusawa. According to the proof of Theorem 1 we can construct a faithful irreducible right R -module G and a faithful irreducible right L -module H , where R and L are respectively the right operator ring and the left operator ring of M . Set $\Delta = \text{Hom}_R(G, G)$ and $\Delta' = \text{Hom}_L(H, H)$. By Schur's Lemma, Δ and Δ' are division rings.

First, we shall show that Δ and Δ' are isomorphic. For $\delta \in \Delta$, we define the mapping $\delta^\sigma: H \rightarrow H$ by

$$(\sum_i [g_i, \gamma_i])\delta^\sigma = \sum_i [g_i \delta, \gamma_i]$$

for $\sum_i [g_i, \gamma_i] \in H$. Here δ^σ is well defined. For, if $\sum_i [g_i, \gamma_i] = \sum_j [g_j', \gamma_j']$ then for all $x \in M, \sum_i g_i [\gamma_i, x] = \sum_j g_j' [\gamma_j', x]$, and hence $\sum_i (g_i \delta) [\gamma_i, x] = (\sum_i g_i [\gamma_i, x])\delta = (\sum_j g_j' [\gamma_j', x])\delta = \sum_j (g_j' \delta) [\gamma_j', x]$. Thus $\sum_i [g_i \delta, \gamma_i] = \sum_j [g_j' \delta, \gamma_j']$ as we desired. Clearly, δ^σ preserves addition. Moreover, for $\sum_i [g_i, \gamma_i] \in H$ and $\sum_j [x_j, \beta_j] \in L$, we have $(\sum_i [g_i, \gamma_i] \sum_j [x_j, \beta_j])\delta^\sigma = (\sum_{i,j} [g_i [\gamma_i, x_j], \beta_j])\delta^\sigma = (\sum_{i,j} [g_i, \gamma_i x_j \beta_j])\delta^\sigma = \sum_{i,j} [g_i \delta, \gamma_i x_j \beta_j] = \sum_{i,j} [g_i \delta, \gamma_i] [x_j, \beta_j] = (\sum_i [g_i, \gamma_i] \delta^\sigma) \sum_j [x_j, \beta_j]$. Hence $\delta^\sigma \in \Delta'$. It can be easily verified that $\sigma: \delta \rightarrow \delta^\sigma$ is a monomorphism of Δ into Δ' . To show that σ is an onto mapping, we note that since H is a faithful irreducible right L -module and G is a faithful irreducible right R -module there exist $g_0 \in G$ and $\gamma_0 \in \Gamma$ such that $\{[g_0, \gamma]: \gamma \in \Gamma\} = H$ and $\{g_0 [\gamma_0, x]: x \in M\} = G$. Let δ' be an arbitrary element in Δ' and $[g_0, \gamma_0] \delta' = [g_0, \gamma_1]$, where $\gamma_1 \in \Gamma$. Let $\delta: G \rightarrow G$ be defined by $(g_0 [\gamma_0, x])\delta = g_0 [\gamma_1, x]$ for $x \in M$. This is well defined. In fact, if $g_0 [\gamma_0, x] = g_0 [\gamma_0, y]$, then, for any $\gamma \in \Gamma, [g_0, \gamma_1] [x, \gamma] = ([g_0, \gamma_0] \delta') [x, \gamma] = ([g_0, \gamma_0] [x, \gamma])\delta' = [g_0, \gamma_0] [y, \gamma] \delta' = ([g_0, \gamma_1] \delta') [y, \gamma] = [g_0, \gamma_0] [y, \gamma]$

and hence, by the construction of H , $g_0[\gamma_1 x \gamma, z] = g_0[\gamma_1 y \gamma, z]$ for all $\gamma \in \Gamma, z \in M$. It follows that $(g_0[\gamma_1, x] - g_0[\gamma_1, y])R = 0$. Since G is a faithful irreducible right R -module, $g_0[\gamma_1, x] = g_0[\gamma_1, y]$. Clearly $\delta \in \Delta$ and $\delta^\sigma = \delta'$. Therefore $\Delta \simeq \Delta'$.

In the proof of Theorem 2 we have known already that the Γ -ring M is isomorphic to a Γ' -ring M' , where Γ' is a subgroup of $\text{Hom}(G, H)$ and M' is a subgroup of $\text{Hom}(H, G)$. More precisely, two group isomorphisms $\theta: M \rightarrow M'$ and $\phi: \Gamma \rightarrow \Gamma'$ exist such that $\sum_i [g_i, \gamma_i](x\theta) = \sum_i g_i[\gamma_i, x]$ and $g(\gamma\phi) = [g, \gamma]$ for all $g, g \in G, \gamma_i, \gamma \in \Gamma, x \in M$.

Now we consider G and H as right Δ -vector space and right Δ' -vector space respectively. For any $g \in G, \delta \in \Delta$ and $\gamma \in \Gamma$, we have $(g\delta)(\gamma\phi) = [g\delta, \gamma] = [g, \gamma]\delta^\sigma = (g(\gamma\phi))\delta^\sigma$ and $([g, \gamma]\delta^\sigma)(x\theta) = [g\delta, \gamma](x\theta) = g\delta[\gamma, x] = (g[\gamma, x])\delta = ([g, \gamma](x\theta))\delta$. Thus $\gamma\phi$ and $x\theta$ are semilinear transformations (associated with σ and σ^{-1} respectively).

It remains to show the density property for Γ' . The density property for M' can be obtained similarly. We shall show that for any n Δ -independent elements $g_1, g_2, \dots, g_n \in G$ and any n elements $h_1, h_2, \dots, h_n \in H$ there exists $\gamma \in \Gamma$ such that $g_i(\gamma\phi) = h_i, i = 1, 2, \dots, n$. We proceed by induction on n .

From Theorem 2, the assertion is obviously true for $n = 1$. Now we assume that the assertion is true for $n - 1$. We want first to show the existence of $\gamma \in \Gamma$ such that $g_i(\gamma\phi) = 0$ for $i < n$ and $g_n(\gamma\phi) \neq 0$. Suppose such a $\gamma \in \Gamma$ does not exist. Then, for any $\gamma \in \Gamma, g_i(\gamma\phi) = 0, 1 \leq i \leq n - 1$, implies $g_n(\gamma\phi) = 0$. Thus for any $h \in H$, by the induction hypothesis, there exists $\gamma_0 \in \Gamma$ such that $g_i(\gamma_0\phi) = h$ and $g_i(\gamma_0\phi) = 0, 1 < i \leq n - 1$. If also $g_1(\gamma_1\phi) = h$ and $g_i(\gamma_1\phi) = 0, 1 < i \leq n - 1$, for some $\gamma_1 \in \Gamma$, then since $g_i((\gamma_0 - \gamma_1)\phi) = 0, 1 \leq i \leq n - 1$ it follows that $g_n((\gamma_0 - \gamma_1)\phi) = 0$, i.e. $g_n(\gamma_0\phi) = g_n(\gamma_1\phi)$. Hence the mapping $\psi: H \rightarrow H$ defined by $h\psi = g_n(\gamma_0\phi)$ whenever $g_1(\gamma_0\phi) = h$ and $g_i(\gamma_0\phi) = 0$ for $1 < i < n$, is well defined. It is easy to see that ψ preserves addition. Let us recall that g_0 is an element in G with $\{[g_0, \gamma]: \gamma \in \Gamma\} = H$. Let $[g_0, \gamma] \in H$ and $\sum_i [x_i, \gamma_i] \in L$. Then $[g_0, \gamma]\psi = g_n(\gamma_0\phi)$ for some $\gamma_0 \in \Gamma$, where $g_1(\gamma_0\phi) = [g_0, \gamma]$ and $g_i(\gamma_0\phi) = 0, 2 \leq i \leq n - 1$. Thus, $([g_0, \gamma]\psi) \sum_i [x_i, \gamma_i] = (g_n(\gamma_0\phi)) \sum_i [x_i, \gamma_i] = [g_n, \gamma_0] \sum_i [x_i, \gamma_i] = g_n(\gamma_1\phi)$, where $\gamma_1 = \sum_i \gamma_0 x_i \gamma_i$. On the other hand, since $g_1(\gamma_1\phi) = [g_0, \gamma] \sum_i [x_i, \gamma_i]$ and $g_i(\gamma_1\phi) = 0, 2 \leq i \leq n - 1$, by the definition of ψ , $([g_0, \gamma] \sum_i [x_i, \gamma_i])\psi = g_n(\gamma_1\phi)$. Consequently, $([g_0, \gamma] \sum_i [x_i, \gamma_i])\psi = ([g_0, \gamma]\psi) \sum_i [x_i, \gamma_i]$ and hence $\psi \in \Delta'$. Let $\psi = \delta^\sigma$ where $\delta \in \Delta$. Since $g_1\delta - g_n, g_2, \dots, g_{n-1}$ are Δ -linearly independent, by the induction hypothesis, there exists $\gamma' \in \gamma$ such that $(g_1\delta - g_n)(\gamma'\phi) \neq 0$ and $g_i(\gamma'\phi) = 0$ for $1 < i < n$. But by the definition of ψ , $(g_1\delta - g_n)(\gamma'\phi) = (g_1\delta)(\gamma'\phi) - g_n(\gamma'\phi) = (g_1(\gamma'\phi))\psi - g_n(\gamma'\phi) = 0$, a contradiction. This proves the existence of $\gamma \in \Gamma$ such that $g_n(\gamma\phi) \neq 0$ and $g_i(\gamma\phi) = 0$ for $1 \leq i < n$. Since $g_n(\gamma\phi)L = H$, there exists $\gamma_n \in \Gamma$ such that $g_n(\gamma_n\phi) = h_n$, and $g_i(\gamma_n\phi) = 0$ for $1 \leq i < n$.

Likewise, there exist $\gamma_i \in \Gamma, 1 \leq i < n$, such that $g_i(\gamma_i \phi) = h_i$ and $g_j(\gamma_i \phi) = 0$ for $i \neq j$. Now let $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$. Then $g_i(\gamma \phi) = h_i, 1 \leq i \leq n$ as we desired. This completes the proof of the theorem.

We recall the definition of Hestense ternary rings. Let G and H be additive abelian groups. M and Γ be subgroups of $\text{Hom}(H, G)$ and $\text{Hom}(G, H)$ respectively. If there is a mapping $*$ of M onto Γ such that a $b^* c \in M$ whenever $a, b, c \in M$ then M is called a Hestenes ternary ring. The set of all finite sums $\sum_i a_i^* b_i$ with $a_i, b_i \in M$ form a ring R and the set of all finite sums $\sum_i c_i d_i^*$ with $c_i, d_i \in M$ form a ring L . Clearly M is a right R -module and is a left L -module. If M is irreducible as a R -module and as an L -module then M is called an irreducible Hestenes ternary ring. Evidently, if M is an irreducible Hestenes ternary ring then M is a right primitive Γ -ring in the sense of Nobusawa and the rings R and L are respectively the right operator ring and the left operator ring of M . Therefore Theorem 3 generalizes further the extension of the Chevalley-Jacobson density theorem given by Smiley and Stephenson (see [8, 9]).

5. Primitive Γ -rings with non-zero socles

In [6], we have introduced the notion of socles for Γ -rings. The right (left) socle $S_r(S_l)$ of a Γ -ring M is the sum of all minimal right (left) ideals of M . In the case M has no minimal right (left) ideals, the right (left) socle of M is defined to be 0. It has been shown that if M is an one-sided primitive Γ -ring having minimal one-sided ideals then M is a two sided primitive and its right socle and left socle coincide (see [5, Theorem 4.2] and [6, Theorem 4.3]).

In this section we shall present a characterization for primitive Γ -ring with non-zero socle which is different from the one given in [5].

Theorem 4. *A Γ -ring M in the sense of Nobusawa is primitive with non-zero socle if and only if it is isomorphic to a dense Γ' -ring M' of semi-linear transformations containing non-zero semilinear transformations of finite rank. Moreover, the socle of M' is the set of semilinear transformations of finite rank contained in M' .*

Proof. Necessity. Assume that M is a primitive Γ -ring in the sense of Nobusawa with non-zero socle. According to Theorem 3, M can be regarded as a dense Γ -ring of semilinear transformations. Let G and H be vector spaces over division rings Δ and Δ' , $\sigma: \Delta \rightarrow \Delta'$ be an isomorphism, M be a dense group of semilinear transformations of H into G (associated with σ^{-1}) and Γ be a dense group of semilinear transformations of G into H (associated with σ). Let $e\gamma M$ be a minimal right ideal of M , where $e \in M$, $\gamma \in \Gamma$ and $e\gamma e = e$. We claim that e is a rank 1, for otherwise, there would exist $h_1, h_2 \in H$ such that $h_1 e$ and $h_2 e$ are Δ -linearly independent. By the density property of Γ and M , there would exist $\gamma_0 \in \Gamma$ such that $h_1 e \gamma_0 = 0$ and $h_2 e \gamma_0 \neq 0$ and $h_2 e \gamma_0 M = G$.

Since $e\gamma M$ is minimal and $h_1 e\gamma(e\gamma_0 M) = h_1 e\gamma_0 M = 0$, the right ideal $\{x \in e\gamma M : h_1 x = 0\} = e\gamma M$, i.e. $h_1 e\gamma M = 0$. Particularly, $h_1 e = h_1 e\gamma e = 0$, a contradiction. Thus M contains non-zero semilinear transformations of finite rank. In addition, since the socle S of M is the sum of minimal right ideals, every element in S is of finite rank.

Sufficiency. Assume that M is a dense Γ -ring of semilinear transformations on vector spaces G and H described above, and assume that M contains semilinear transformations of finite rank. By density property, M contains semilinear transformations of rank 1. Let $a \in M$ be of rank 1, and let $Ha = \langle g_1 \rangle$, the subspace of G generated by g_1 . Consider $I = \{x \in M : Hx \subseteq \langle g_1 \rangle\}$, a left ideal of M . We claim that I is minimal. Let $0 \neq x_1 \in I$. Then $Hx_1 = \langle g_1 \rangle$ and $h_1 x_1 = g_1$ for some $h_1 \in H$. By the density property of Γ , there exists $\gamma_1 \in \Gamma$ such that $g_1 \gamma_1 = h_1$. Thus $g_1 = g_1 \gamma_1 x_1$. Now let x be an arbitrary element in I . For any $h \in H$, there exists $\delta \in \Delta$ such that $hx = g_1 \delta = (g_1 \gamma_1 x_1) \delta = (g_1 \delta) \gamma_1 x_1 = hx \gamma_1 x_1$. Hence $x = x \gamma_1 x_1 \in M \Gamma x_1$, so $I = M \Gamma x_1$ for every $0 \neq x_1 \in I$. Therefore I is a minimal left ideal containing a , a is in the socle of M , and M has a non-zero socle S .

The argument just used shows that every element in M of rank 1 is in S . But the density property of M and Γ insures that every element in M of finite rank is a sum of finitely many elements in M of rank 1. Therefore S contains all elements in M of finite rank. This completes the proof.

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