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THE STRUCTURE OF PRIMITIVE GAMMA RINGS

Jiang Luh

(Received November 4, 1969)

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [7]. The class of Γ-rings contains not only all rings but also all Hestenes ternary rings. In [7], Nobusawa generalized the Wedderburn-Artin Theorem for simple Γ-rings and for semi-simple Γ-rings. Barnes [1] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings. The author [5] gave a characterization of primitive Γ-rings with minimal one-sided ideals by means of certain Γ-rings of continuous semilinear transformations. He [6] also established several structure theorems for simple Γ-rings having minimal one-sided ideals. Recently, Coppage and the author [2] introduced the notions of Jacobson radical, Levitzki radical, nil radical for Γ-rings and obtained some basic radical properties and inclusion relations for these radicals together with the prime radical defined by Barnes [1].

The object of this paper is to study the structure of primitive Γ-rings. One of its main results is a generalization of the Jacobson - Chevalley density theorem. This generalizes further a result given by Smiley and Stephenson for Hestenes ternary rings [8].

We refer to [4] for all notions relevent to ring theory.

2. Preliminaries

Let M and Γ be two additive abelian groups. If for all x, y, z ∈ M and all α, β ∈ Γ the conditions

1. \( xαy ∈ M \)
2. \( (x+y)αz = xαz + yαz, \)
   \( x(α+β)z = xαz + xβz, \)
   \( xα(y+z) = xαy + xαz, \)
3. \( (xαy)βz = xα(yβz) \)

are satisfied then we call M a Γ-ring.

If these conditions are strengthened to
(1') \( x\alpha y \in M, \alpha x\beta \in \Gamma, \)
(2') the same as (2),
(3') \( (x\alpha y)\beta z = x(\alpha y)\beta z = x\alpha (y\beta z) \)
(4') \( x\alpha y = 0 \) for all \( x, y \in M \) implies \( \alpha = 0 \), then \( M \) is called a \( \Gamma \)-ring in the sense of Nobusawa.

Let \( M \) be a \( \Gamma \)-ring. If \( S, T \subseteq M \), we write \( ST \subseteq M \) for the set of finite sums \( \sum_{i} s_{i} \alpha_{i} t_{i} \) where \( s_{i} \in S, \alpha_{i} \in \Gamma, \) and \( t_{i} \in T \). A subgroup \( I \) of \( M \) is a left (right) ideal of \( M \) if \( MI \subseteq I \) (\( IM \subseteq I \)). If \( I \) is both a left and a right ideal of \( M \), then \( I \) is an ideal of \( M \). A one-sided ideal \( I \) is strongly nilpotent if \( I^{n+1} = I \Gamma I \cdots \Gamma I = 0 \) for some positive integer \( n \). A non-zero right (left) ideal is minimal if the only right (left) ideals of \( M \) contained in \( I \) are 0 and \( I \) itself. It has been shown that every minimal right ideal which is not strongly nilpotent can be expressed as the form \( e \gamma M \), where \( \gamma \in \Gamma, e \in M \) and \( e e = e \) (see [5] Theorem 3.2).

Let \( F \) be the free abelian group generated by the set of all ordered pairs \( (\alpha, x) \) where \( \alpha \in \Gamma, x \in M \). Let \( K \) be the subgroup of elements \( \sum m_{i} (\alpha_{i}, x_{i}) \in F \), where \( m_{i} \) are integers such that \( \sum m_{i} (x_{i}, x_{i}) = 0 \) for all \( x \in M \). Denote by \( R \) the factor group \( F/K \) and by \( [\alpha, x] \) the coset \( K+(\alpha, x) \). Clearly every element in \( R \) can be expressed as a finite sum \( \sum_{i} [\alpha_{i}, x_{i}] \). We define multiplication in \( R \) by

\[
\sum_{i} [\alpha_{i}, x_{i}] \cdot \sum_{j} [\beta_{j}, y_{j}] = \sum_{i,j} [\alpha_{i}, x_{i}\beta_{j}, y_{j}].
\]

Then \( R \) forms a ring. Furthermore, \( M \) is a right \( R \)-module with the definition

\[
x \sum_{i} [\alpha_{i}, x_{i}] = \sum_{i} x \alpha_{i} x_{i}, \quad \text{for } x \in M.
\]

We call the ring \( R \) the right operator ring of \( M \). Similarly, we can define the left operator ring \( L \). Every element in \( L \) can be expressed as a finite sum \( \sum [x_{j}, \beta_{j}] \) where \( x_{j} \in M, \beta_{j} \in \Gamma \). These two operator rings play important roles in studying the structure of \( \Gamma \)-rings. We recall that a \( \Gamma \)-ring \( M \) is right primitive if (i) \( MTx = 0 \) implies \( x = 0 \) and (ii) the right operator ring \( R \) of \( M \) is a right primitive ring.

**Theorem 1.** If \( M \) is a right primitive \( \Gamma \)-ring, then the left operator ring of \( M \) is a right primitive ring.

**Proof.** Let \( R \) and \( L \) be respectively the right and left operator rings of \( M \).

Let \( G \) be a faithful irreducible right \( R \)-module. Let \( A \) be the free abelian group generated by the set of ordered pairs \( (g, \gamma) \), where \( g \in G, \gamma \in \Gamma \), and let \( B \) be the subgroup of elements \( \sum m_{i} (g_{i}, \gamma_{i}) \in A \) where \( m_{i} \) are integers such that \( \sum m_{i} (\gamma_{i} x_{i}) = 0 \) for all \( x \in M \). Denote by \( H \) the factor group \( A/B \) and, without causing any ambiguity, by \([g, \gamma]\) the coset \( B+(g, \gamma) \). Every element in \( H \) therefore can be expressed as a finite sum \( \sum_{i} [g_{i}, \gamma_{i}] \). \( H \) forms a right \( L \)-module with the definition
\[\Sigma_i[g, \gamma_i] \cdot \Sigma_j[x, \beta_j] = \Sigma_i, [g, [\gamma_i, x], \beta_j]\]
for \(\Sigma_i[g, \gamma_i] \in H\) and \(\Sigma_j[x, \beta_j] \in L\). We claim that \(H\) is a faithful irreducible right \(L\)-module. Assume \(H \cdot \Sigma_j[x, \beta_j] = 0\). Then for all \(\gamma \in \Gamma\), \(g \in G\), we have \(\Sigma_j[g, \gamma, x, \beta_j] = [g, \gamma] \cdot \Sigma_j[x, \beta_j] = 0\), i.e. \(g \cdot \Sigma_j[\gamma, x, \beta_j, x] = 0\) for all \(x \in M\).

By the faithfulness of the \(R\)-module of \(G\), \([\gamma, \Sigma_j[x, \beta_j, x]] = \Sigma_j[\gamma, x, \beta_j, x] = 0\), so \(M \cdot \Sigma_j[x, \beta_j, x] = 0\). By the condition (i), \(\Sigma_j[x, \beta_j, x] = 0\) for all \(x \in M\). This means that \(\Sigma_j[x, \beta_j] = 0\) and \(H\) is faithful. To see that \(H\) is irreducible, let \(\Sigma_i[g_i, \gamma_i]\) be an arbitrary non-zero element in \(H\). Then the set \(G' = \{\Sigma_i[g_i, \gamma_i, x] : x \in M\}\) is a non-zero \(R\)-submodule of \(G\). Since \(G\) is irreducible, \(G' = G\). For any \(\Sigma_i[g_i', \gamma_i'] \in H\), we may write \(g_i' = \Sigma_i[g_i, \gamma_i, x_j]\) where \(x_j \in M\). Thus \(\Sigma_i[g_i', \gamma_i'] = \Sigma_i[\Sigma_i[g_i, \gamma_i, x_j], \gamma_i'] = \Sigma_i[g_i, \gamma_i] \cdot \Sigma_i[x_j, \gamma_i] \in \Sigma_i[g_i, \gamma_i]L\). Hence \(H\) is irreducible and \(L\) is a right primitive ring.

3. Irreducible \(\Gamma\)-rings of homomorphisms on groups

Let \(G\) and \(H\) be non-zero additive abelian groups. If \(M\) and \(\Gamma\) are respectively subgroups of \(\text{Hom}(H, G)\) and \(\text{Hom}(G, H)\) such that \(g \Gamma = H\) and \(h M = G\) whenever \(0 \neq g \in G\) and \(0 \neq h \in H\), and moreover if \(xy \in M\) and \(\alpha \beta \in \Gamma\) for all \(x, y \in M\), then \(M\) forms a \(\Gamma\)-ring in the sense of Nobusawa under the composition of mappings. We shall call such a \(\Gamma\)-ring an irreducible \(\Gamma\)-ring of homomorphisms on groups.

A \(\Gamma\)-ring \(M\) and a \(\Gamma'\)-ring \(M'\) are said to be isomorphic if there exist a group isomorphism \(\theta\) of \(M\) onto \(M'\) and a group isomorphism \(\phi\) of \(\Gamma\) onto \(\Gamma'\) such that \((xy) \theta = (x \phi) \phi (y \theta)\) for all \(x, y \in M, \alpha \in \Gamma\) and \(H\). It is clear that \(M\) is right primitive if and only if \(M'\) is right primitive.

**Theorem 2.** A \(\Gamma\)-ring \(M\) is a right primitive \(\Gamma\)-ring in the sense of Nobusawa if and only if it is isomorphic to an irreducible \(\Gamma\)-ring of homomorphisms on groups.

Proof. Necessity. Let \(M\) be a right primitive \(\Gamma\)-ring in the sense of Nobusawa with right operator ring \(R\) and left operator ring \(L\) and let \(G\) be a faithful irreducible right \(R\)-module, from the proof of Theorem 1, we can construct the faithful irreducible right \(L\)-module \(H\). Now, for each \(\gamma \in \Gamma\) let \(g \phi \in \text{Hom}(G, H)\) defined by \(g(\gamma \phi) = [g, \gamma]\). Clearly \(\phi\) is a group homomorphism of \(\Gamma\) into \(\text{Hom}(G, H)\). Moreover, if \(g, \gamma, \phi \phi\) then \([g, \gamma, - \gamma] = 0\) i.e. \(g[\gamma, - \gamma, x] = 0\) for all \(g \in G, x \in M\). By the faithfulness of \(G\) as an \(R\)-module, \([\gamma, - \gamma, x] = 0\) for all \(x \in M\). Consequently \(M(\gamma, - \gamma) \cdot M = 0\) and, by the condition (4') in the definition of \(\Gamma\)-ring in the sense of Nobusawa, \(\gamma = \gamma\). Thus \(\phi\) is a group isomorphism of \(\Gamma\) onto \(\Gamma' = \Gamma\).

Likewise, for each \(x \in M\), let \(x \theta\) be the mapping of \(H\) into \(G\) defined by \(\Sigma_i[g_i, \gamma_i] \cdot (x \theta) = \Sigma_i[g_i, \gamma_i, x]\). It can be shown easily that \(x \theta \in \text{Hom}(H, G)\) and
that \( \theta \) is a group homomorphism of \( M \) into \( \text{Hom}(H, G) \). We claim that \( \theta \) is one-
to-one. Indeed, if \( x\theta=y\theta \), where \( x, y \in M \), then \( g[\gamma, x-y]=g[\gamma, x]-g[\gamma, y] \)=0 for all \( g \in G, \gamma \in \Gamma \). Again by the faithfulness of \( G \), \( [\gamma, x-y]=0 \) for all \( \gamma \in \Gamma \), or equivalently that \( M\Gamma(x-y)=0 \). Hence \( x=y \) and \( \theta \) is a group iso-
morphism of \( M \) onto \( M'\theta \). It is easy to see that the \( \Gamma \)-ring \( M \) is isomorphic to the \( \Gamma' \)-ring \( M' \).

It remains to show that \( M' \) is an irreducible \( \Gamma' \)-ring of homomorphisms
on groups. Let \( 0 \neq g \in G \). Since \( gR=G \), every element in \( H \) can be expressed
as \( \Sigma_j[g\Sigma_i[\gamma_j, x_i], \beta_j]=g(\gamma\phi) \) where \( \gamma_i, \beta_j \in \Gamma \), \( x_i \in M \) and \( \gamma=\Sigma_i, \gamma_i x_i \beta_j \). Hence \( H=g\Gamma' \). Now, let \( h \) be an arbitrary non-zero element in \( H \). Then \( h=g(\gamma\phi)=[g, \gamma] \) for some \( \gamma \in \Gamma \). It follows that \( h(x\theta)=[g, \gamma](x\theta)=g[\gamma, x] \) for all \( x \in M \). Thus \( hM' \) is a non-zero \( R \)-submodule of \( G \) and hence \( hM'=G \).

 Sufficiency. We may assume that \( M \) is an irreducible \( \Gamma \)-ring of homomorphisms
on groups, and that \( 0 \neq \Gamma \subseteq \text{Hom}(G, H), 0 \neq M \subseteq \text{Hom}(H, G) \) where \( H \) and \( G \) are abelian groups with the property that for any \( 0 \neq g \in G \) and \( 0 \neq h \in H \),
\( g\Gamma=H \) and \( hM=G \). Clearly, \( MTx=0 \) for \( x \in M \) implies \( x=0 \). For \( g \in G \) and
\( \Sigma_i[\gamma_i, x_i] \in R \), the right operator ring of \( M \), we define composition
\[
g\Sigma_i[\gamma_i, x_i] = \Sigma_i[g(\gamma_i)x_i] .
\]
This composition is well defined. For if \( \Sigma_j[\gamma_j, x_j] = \Sigma_j[\beta_j, y_j] \) in \( R \), then \( \Sigma_i\gamma_i x_i - \Sigma_j \beta_j y_j = 0 \) for all \( x \in M \). By noting that \( g \in g\Gamma M \), we obtain \( \Sigma_i(g\gamma_i)x_i - \Sigma_j(g\beta_j)y_j = g(\Sigma_i\gamma_i x_i - \Sigma_j \beta_j y_j) = 0 \), so \( g\Sigma_i[\gamma_i, x_i] = g\Sigma_j[\beta_j, y_j] \). Clearly \( G \) forms an irreducible right \( R \)-module. Moreover, if \( \Sigma_i[\gamma_i, x_i] \in R \) and if \( G\Sigma_i[\gamma_i, x_i]=0 \), then \( HM\Sigma_i[\gamma_i, x_i]=G\Gamma M \Sigma_i[\gamma_i, x_i]=G \Sigma_i[\gamma_i, x_i]=0 \), and hence \( M\Sigma_i[\gamma_i, x_i]=0 \). Consequently, \( \Sigma_i[\gamma_i, x_i] = 0 \) and \( G \) is a faithful \( R \)-module. Thus, \( R \) is a right primitive ring and \( M \) is a
right primitive \( \Gamma \)-ring in the sense of Nobusawa.

Observe the definition of irreducible \( \Gamma \)-rings of homomorphisms on groups.
We can easily see that \( M \) is irreducible \( \Gamma \)-rings of homomorphisms on groups
if and only if \( \Gamma \) is a irreducible \( \Gamma' \)-ring of homomorphisms on groups, where \( \Gamma'=M \). Thus from Theorem 2, we immediately have the following

**Corollary.** Let \( M \) be a \( \Gamma \)-ring. Then \( M \) is a right primitive \( \Gamma \)-ring in the sense of Nobusawa if and only if \( \Gamma \) is a right primitive \( \Gamma' \)-ring in the sense of Nobusawa, where \( \Gamma'=M \).

4. Chevalley-Jacobson density theorem

Let \( G \) and \( H \) be non-zero right vector spaces over division rings \( \Delta \) and \( \Delta' \) respectively, and let \( \sigma \) be an isomorphism of \( \Delta \) onto \( \Delta' \). A group \( N \) of semilinear transformations (associated with \( \sigma \)) of \( G \) into \( H \) is said to be dense if, for every positive integer \( n \) and every \( n \) linearly independent elements \( g_1, g_2, \)
\[ \cdots, g_n \text{ in } G \text{ and every } n \text{ elements } h_1, h_2, \cdots, h_n \text{ in } H, \text{ there exists } x \in N \text{ such that } g_i x = h_i, i = 1, 2, \cdots, n. \]

Now, if \( \Gamma \) is a dense group of semilinear transformations (associated with \( \sigma \)) of \( G \) into \( H \) and \( M \) is a dense group of semilinear transformations (associated with \( \sigma^{-1} \)) of \( H \) into \( G \), and if the compositions of mappings \( x \alpha y \in M \) and \( \alpha x \beta \in \Gamma \) for all \( x, y \in M, \alpha, \beta \in \Gamma \), then \( M \) forms a \( \Gamma \)-ring in the sense of Nobusawa under the composition of mappings. We shall call such a \( \Gamma \)-ring a dense \( \Gamma \)-ring of semilinear transformations.

Following is a generalization of the well known Chevalley-Jacobson density theorem.

**Theorem 3.** Let \( M \) be a \( \Gamma \)-ring. Then \( M \) is a right primitive \( \Gamma \)-ring in the sense of Nobusawa if and only if it is isomorphic to a dense \( \Gamma \)-ring of semilinear transformations.

Proof. Sufficiency. It is an immediate consequence of Theorem 2, since a dense \( \Gamma \)-ring of semilinear transformations evidently is an irreducible \( \Gamma \)-ring of homomorphisms on groups.

Necessity. We assume that \( M \) is a right primitive \( \Gamma \)-ring in the sense of Nobusawa. According to the proof of Theorem 1 we can construct a faithful irreducible right \( R \)-module \( G \) and a faithful irreducible right \( L \)-module \( H \), where \( R \) and \( L \) are respectively the right operator ring and the left operator ring of \( M \). Set \( \Delta = \text{Hom}_R(G, G) \) and \( \Delta' = \text{Hom}_L(H, H) \). By Schur’s Lemma, \( \Delta \) and \( \Delta' \) are division rings.

First, we shall show that \( \Delta \) and \( \Delta' \) are isomorphic. For \( \delta \in \Delta \), we define the mapping \( \delta^* : H \to H \) by

\[
(\Sigma_i [g_i, \gamma_i])\delta^* = \Sigma_i [g_i, \gamma_i]
\]

for \( \Sigma_i [g_i, \gamma_i] \in H \). Here \( \delta^* \) is well defined. For, if \( \Sigma_i [g_i, \gamma_i] = \Sigma_j [g_j', \gamma_j'] \) then for all \( x \in M, \Sigma_i [g_i, \gamma_i, x] = \Sigma_j [g_j', \gamma_j', x] \), and hence \( \Sigma_i (g_i \delta) [\gamma_i, x] = (\Sigma_i g_i, [\gamma_i, x]) \delta = (\Sigma_j g_j, [\gamma_j', x]) \delta = \Sigma_j (g_j' \delta) [\gamma_j', x] \). Thus \( \Sigma_i (g_i \delta, \gamma_i) = \Sigma_j (g_j', \delta, \gamma_j') \) as we desired. Clearly, \( \delta^* \) preserves addition. Moreover, for \( \Sigma_i (g_i, \gamma_i) \in H \) and \( \Sigma_j (x_j, \beta_j) \in L \), we have \( (\Sigma_i (g_i, \gamma_i)) \Sigma_j (x_j, \beta_j) \delta^* = (\Sigma_i (g_i, \gamma_i)) (\Sigma_j (x_j, \beta_j)) \delta^* = (\Sigma_i [g_i, \gamma_i] \Sigma_j [x_j, \beta_j]) \delta^* = \Sigma_i [g_i, \Delta, \gamma_i, x_j, \beta_j] = \Sigma_i [g_i, \delta, \gamma_i] [x_j, \beta_j] = (\Sigma_i [g_i, \gamma_i]) \delta^* \Sigma_j [x_j, \beta_j] \). Hence \( \delta^* \in \Delta' \). It can be easily verified that \( \sigma : \delta \to \delta^* \) is a monomorphism of \( \Delta \) into \( \Delta' \). To show that \( \sigma \) is an onto mapping, we note that since \( H \) is a faithful irreducible right \( \Delta' \)-module and \( G \) is a faithful irreducible right \( \Delta \)-module there exist \( g_o \in G \) and \( \gamma_o \in \Gamma \) such that \( \{[g_o, \gamma] : \gamma \in \Gamma\} = H \) and \( \{g_o [\gamma, x] : x \in M\} = G \).

Let \( \delta : G \to G \) be defined by \( (g_o, \gamma) \delta = g_o [\gamma, x] \) for \( x \in M \). This is well defined. In fact, if \( g_o [\gamma, x] = g_o [\gamma, y] \), then, for any \( \gamma \in \Gamma, \{g_o, \gamma{[x, \gamma]} = (g_o, \gamma) \delta^* [x, \gamma] = ([g_o, \gamma_o] [x, \gamma]) \delta^* = ([g_o, \gamma_o] [x, \gamma]) = ([g_o, \gamma_o] [x, \gamma]) \delta^* = ([g_o, \gamma_o] [x, \gamma]) \delta^* = ([g_o, \gamma_o] [x, \gamma]) \delta^* \).
and hence, by the construction of $H$, $g_0[\gamma, \chi, z] = g_0[\gamma, y, \gamma, z]$ for all $\gamma \in \Gamma$, $z \in M$. It follows that $(g_4[\gamma, z] - g_2[\gamma, y])[R] = 0$. Since $G$ is a faithful irreducible right $R$-module, $g_0[\gamma, x] = g_0[\gamma, y]$. Clearly $\delta \in \Delta$ and $\delta^\ast = \delta^\prime$. Therefore $\Delta = \Delta^\prime$.

In the proof of Theorem 2 we have known already that the $\Gamma$-ring $M$ is isomorphic to a $\Gamma'$-ring $M'$, where $\Gamma'$ is a subgroup of $\text{Hom}(G, H)$ and $M'$ is a subgroup of $\text{Horn}(H, G)$. More precisely, two group isomorphisms $\theta: M \to M'$ and $\phi: \Gamma \to \Gamma'$ exist such that $\Sigma_4[g_i, \gamma_i] = \Sigma_4 g_i[\gamma_i, x]$ and $g(\gamma \phi) = [g, \gamma]$ for all $g_i, \gamma_i \in G, \gamma_i \in \Gamma, x \in M$.

Now we consider $G$ and $H$ as right $\Delta$-vector space and right $\Delta'$-vector space respectively. For any $g \in G, \delta \in \Delta$ and $\gamma \in \Gamma$, we have $(g \delta)(\gamma \phi) = [g \delta, \gamma]$ and $(g \gamma \delta^\ast)(x \theta) = [g \delta, \gamma] = g \delta[\gamma, x] = (g \gamma, x) \delta = ([g, \gamma](x \theta)) \delta$. Thus $\gamma \phi$ and $x \theta$ are semilinear transformations (associated with $\sigma$ and $\sigma^{-1}$ respectively).

It remains to show the density property for $\Gamma'$. The density property for $M'$ can be obtained similarly. We shall show that for any $n$ $\Delta$-independent elements $g_1, g_2, \ldots, g_n \in G$ and any $n$ elements $h_1, h_2, \ldots, h_n \in H$ there exists $\gamma \in \Gamma$ such that $g_i(\gamma \phi) = h_i, i = 1, 2, \ldots, n$. We proceed by induction on $n$.

From Theorem 2, the assertion is obviously true for $n = 1$. Now we assume that the assertion is true for $n - 1$. We want first to show the existence of $\gamma \in \Gamma$ such that $g_i(\gamma \phi) = 0$ for $i < n$ and $g_n(\gamma \phi) \neq 0$. Suppose such a $\gamma \in \Gamma$ does not exist. Then, for any $\gamma \in \Gamma, g_i(\gamma \phi) = 0, 1 \leq i \leq n - 1$, implies $g_n(\gamma \phi) = 0$. Thus for any $h \in H$, by the induction hypothesis, there exists $\gamma_o \in \Gamma$ such that $g_o(\gamma \phi) = h$ and $g_i(\gamma \phi) = 0, 1 < i < n - 1$. If also $g_i(\gamma \phi) = h$ and $g_i(\gamma \phi) = 0, 1 < i < n - 1$, for some $\gamma_i \in \Gamma$, then since $g_i((\gamma_o - \gamma_i) \phi) = 0$, for $1 \leq i < n - 1$ it follows that $g_i((\gamma_o - \gamma_i) \phi) = 0$, i.e. $g_o(\gamma \phi) = g_o(\gamma \phi)$. Hence the mapping $\psi: H \to H$ defined by $h \psi = g_i(\gamma \phi)$ whenever $g_i(\gamma \phi) = h$ and $g_i(\gamma \phi) = 0$ for $1 < i < n$, is well defined. It is easy to see that $\psi$ preserves addition. Let us recall that $g_i$ is an element in $G$ with $\{[g_0, \gamma]: \gamma \in \Gamma\} = H$. Let $[g_0, \gamma] \in H$ and $\Sigma_4[x_i, \gamma_i] \in L$. Then $[g_0, \gamma] \psi = g_n(\gamma \phi)$ for some $\gamma_o \in \Gamma$, where $g_o(\gamma \phi) = [g_0, \gamma]$ and $g_i(\gamma \phi) = 0, 2 \leq i \leq n - 1$. Thus, $([g_0, \gamma] \psi \Sigma_4[x_i, \gamma_i] = [g_o(\gamma \phi), \Sigma_4[x_i, \gamma_i]] = [g_n, \gamma_3] \Sigma_4[x_i, \gamma_i] = g_n(\gamma \phi)$. Consequently, $([g_0, \gamma] \psi \Sigma_4[x_i, \gamma_i]) \psi = ([g_0, \gamma] \psi \Sigma_4[x_i, \gamma_i]) \psi = \Sigma_4[x_i, \gamma_i]$ and hence $\psi \in \Delta'$. Let $\psi = \delta^\ast \phi$ where $\delta \in \Delta$. Since $g_i \delta = g_o, g_2, \ldots, g_n$ are $\Delta$-linearly independent, by the induction hypothesis, there exists $\gamma \in \Gamma$ such that $g_i \delta = g_n(\gamma \phi) \neq 0$ and $g_i(\gamma \phi) = 0$ for $1 < i < n$. But by the definition of $\psi$, $(g_i \delta - g_n(\gamma \phi)) \psi = \Sigma_4[x_i, \gamma_i]$ and hence $\psi \in \Delta'$. Therefore $\psi = \delta^\ast \phi$. Since $g_i \delta - g_n(\gamma \phi) = 0$ for $1 < i < n$. Thus $g_n(\gamma \phi) = H$, there exists $\gamma_n \in \Gamma$ such that $g_n(\gamma_n \phi) = h_n$, and $g_i(\gamma_n \phi) = 0$ for $1 < i < n$. 
Likewise, there exist $\gamma_i \in \Gamma, 1 \leq i \leq n$, such that $g_i(\gamma_i \phi) = h_i$ and $g_j(\gamma_i \phi) = 0$ for $i \neq j$. Now let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. Then $g_i(\gamma \phi) = h_i, 1 \leq i \leq n$ as we desired. This completes the proof of the theorem.

We recall the definition of Hestenes ternary rings. Let $G$ and $H$ be additive abelian groups. $M$ and $\Gamma$ be subgroups of $\text{Hom}(H, G)$ and $\text{Hom}(G, H)$ respectively. If there is a mapping $* \in M$ onto $\Gamma$ such that $a \ast \gamma \in M$ whenever $a, \gamma \in M$ then $M$ is called a Hestenes ternary ring. The set of all finite sums $\sum_a \sum_b \sum c_i d_i e_i$ with $a_i, b_i, c_i, d_i \in M$ form a ring $R$ and the set of all finite sums $\sum c_i d_i e_i$ with $c_i, d_i \in M$ form a ring $L$. Clearly $M$ is a right $R$-module and is a left $L$-module. If $M$ is irreducible as a $R$-module and as an $L$-module then $M$ is called an irreducible Hestenes ternary ring. Evidently, if $M$ is an irreducible Hestenes ternary ring then $M$ is a right primitive $\Gamma$-ring in the sense of Nobusawa and the rings $R$ and $L$ are respectively the right operator ring and the left operator ring of $M$. Therefore Theorem 3 generalizes further the extension of the Chevalley-Jacobson density theorem given by Smiley and Stephenson (see [8, 9]).

5. Primitive $\Gamma$-rings with non-zero socles

In [6], we have introduced the notion of socles for $\Gamma$-rings. The right (left) socle $S_r(S_l)$ of a $\Gamma$-ring $M$ is the sum of all minimal right (left) ideals of $M$. In the case $M$ has no minimal right (left) ideals, the right (left) socle of $M$ is defined to be 0. It has been shown that if $M$ is an one-sided primitive $\Gamma$-ring having minimal one-sided ideals then $M$ is a two sided primitive and its right socle and left socle coincide (see [5, Theorem 4.2] and [6, Theorem 4.3]).

In this section we shall present a characterization for primitive $\Gamma$-ring with non-zero socle which is different from the one given in [5].

**Theorem 4.** A $\Gamma$-ring $M$ in the sense of Nobusawa is primitive with non-zero socle if and only if it is isomorphic to a dense $\Gamma'$-ring $M'$ of semi-linear transformations containing non-zero semilinear transformations of finite rank. Moreover, the socle of $M'$ is the set of semilinear transformations of finite rank contained in $M'$.

**Proof.** Necessity. Assume that $M$ is a primitive $\Gamma$-ring in the sense of Nobusawa with non-zero socle. According to Theorem 3, $M$ can be regarded as a dense $\Gamma$-ring of semilinear transformations. Let $G$ and $H$ be vector spaces over division rings $\Delta$ and $\Delta'$, $\sigma: \Delta \to \Delta'$ be an isomorphism, $M$ be a dense group of semilinear transformations of $H$ into $G$ (associated with $\sigma^{-1}$) and $\Gamma$ be a dense group of semilinear transformations of $G$ into $H$ (associated with $\sigma$). Let $e \gamma M$ be a minimal right ideal of $M$, where $e \in M$, $\gamma \in \Gamma$ and $e \gamma e = e$. We claim that $e$ is a rank 1, for otherwise, there would exist $h_1, h_2 \in H$ such that $h_1 e$ and $h_2 e$ are $\Delta$-linearly independent. By the density property of $\Gamma$ and $M$, there would exists $\gamma_0 \in \Gamma$ such that $h_1 e \gamma_0 = 0$ and $h_2 e \gamma_0 \neq 0$ and $h e \gamma_0 M = G$. 


Since $e\gamma M$ is minimal and $h_1 e\gamma(e\gamma_0 M) = h_1 e\gamma M = 0$, the right ideal $\{x \in e\gamma M: h_1 x = 0\} = e\gamma M$, i.e., $h_1 e\gamma M = 0$. Particularly, $h_1 e = h_1 e\gamma e = 0$, a contradiction. Thus $M$ contains non-zero semilinear transformations of finite rank. In addition, since the socle $S$ of $M$ is the sum of minimal right ideals, every element in $S$ is of finite rank.

Sufficiency. Assume that $M$ is a dense $\Gamma$-ring of semilinear transformations on vector spaces $G$ and $H$ described above, and assume that $M$ contains semilinear transformations of finite rank. By density property, $M$ contains semilinear transformations of rank 1. Let $a \in M$ be of rank 1, and let $H a = \langle g_i \rangle$, the subspace of $G$ generated by $g_i$. Consider $I = \{x \in M: H x \subseteq \langle g_i \rangle\}$, a left ideal of $M$. We claim that $I$ is minimal. Let $0 + x_i \in I$. Then $H x_i = \langle g_i \rangle$ and $h_i x_i = g_i$ for some $h_i \in H$. By the density property of $\Gamma$, there exists $\gamma_i \in \Gamma$ such that $g_i \gamma_i = h_i$. Thus $g_i = g_i \gamma_i x_i$. Now let $x$ be an arbitrary element in $I$. For any $h \in H$, there exists $\delta \in \Delta$ such that $h x = g, \delta = (g, \gamma_i x_i) \delta = (g, \delta) \gamma_i x_i = h x \gamma_i x_i$. Hence $x = x \gamma_i x_i \in M \Gamma x_i$, so $I = M \Gamma x_i$ for every $0 + x_i \in I$. Therefore $I$ is a minimal left ideal containing $a$, $a$ is in the socle of $M$, and $M$ has a non-zero socle $S$.

The argument just used shows that every element in $M$ of rank 1 is in $S$. But the density property of $M$ and $\Gamma$ insures that every element in $M$ of finite rank is a sum of finitely many elements in $M$ of rank 1. Therefore $S$ contains all elements in $M$ of finite rank. This completes the proof.

References