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LARGE TIME BEHAVIOR OF SOLUTIONS TO THE GENERALIZED BURGERS EQUATIONS

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Abstract
We study large time behavior of the solutions to the initial value problem for the generalized Burgers equation. It is known that the solution tends to a self-similar solution to the Burgers equation at the rate $t^{-1}\log t$ in L^∞ as $t \to \infty$. The aim of this paper is to show that the rate is optimal under suitable assumptions and to obtain the second asymptotic profile of large time behavior of the solutions.

1. Introduction

This paper is concerned with large time behavior of the global solutions to the generalized Burgers equations:

$$u_t + (f(u))_x = u_{xx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $u_0 \in L^1(\mathbb{R})$ and $f(u) = (b/2)u^2 + (c/3)u^3$ with $b \neq 0$, $c \in \mathbb{R}$. The subscripts t and x stand for the partial derivatives with respect to t and x, respectively. It is well-known that the solution of (1.1) and (1.2) tends to a nonlinear diffusion wave defined by

$$\chi(x, t) \equiv \frac{1}{\sqrt{1+t}}\chi_*\left(\frac{x}{\sqrt{1+t}}\right), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b}\frac{(e^{b\delta/2} - 1)e^{-(x^2/4)}}{\sqrt{\pi} + (e^{b\delta/2} - 1)\int_{x/2}^{\infty} e^{-y^2}\,dy}, \tag{1.4}$$

$$\delta \equiv \int_{\mathbb{R}} u_0(x)\,dx. \tag{1.5}$$

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By the Hopf-Cole transformation in Hopf [4] and Cole [1], we see that it is a solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2}\chi^2\right)_x = \chi_{xx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.6}$$

satisfying

$$\int_{\mathbb{R}} \chi(x, 0)\, dx = \delta. \tag{1.7}$$

Concerning the convergence rate of the nonliner diffusion wave $\chi(x,t)$ to the original solution $u(x, t)$, we can infer the following result from the argument given in Kawashima [7] and Nishida [10], which deal with a class of system, in the case where $u(x, t)$ is a scalar unknow function without any essential difficulty: If $u_0 \in L^1_\beta(\mathbb{R}) \cap H^1(\mathbb{R})$ for some $\beta \in (0, 1)$ and $\|u_0\|_{H^1} + \|u_0\|_{L^1}$ is small, then we have

$$\|u(\,\cdot\,, t) - \chi(\,\cdot\,, t)\|_{L^\infty} \le C(1+t)^{-1+\alpha}(\|u_0\|_{H^1} + \|u_0\|_{L^1_\beta}), \quad t \ge 0, \tag{1.8}$$

where $\alpha = (1-\beta)/2$. Here, for an integer $k \ge 0$, $H^k(\mathbb{R})$ denotes the space of functions $u = u(x)$ such that ∂_x^l are L^2-functions on $\mathbb{R}$ for $0 \le l \le k$, endowed, with the norm $\|\cdot\|_{H^k}$, while $L^1_\beta(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u\|_{L^1_\beta} \equiv \int_{\mathbb{R}} |u|(1 + |x|)^\beta\, dx < \infty$.

This observation lead to a natural question whether it is possible to take $\alpha = 0$ in (1.8) for the extreme case $\beta = 1$ or not. An attempt to answer the question can be found in Matsumura and Nishihara [9]. To be more precise, we put $w_0(x) = \exp(-(b/2)\int_{-\infty}^x u_0(y)\, dy) - \exp(-(b/2)\int_{-\infty}^x \chi(y, 0)\, dy)$ and we suppose that $w_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\|w_0\|_{H^2} + \|w_0\|_{L^1} + \|u_0\|_{L^1}$ is small. Then the estimate

$$\|u(\,\cdot\,, t) - \chi(\,\cdot\,, t)\|_{L^\infty} \le C(1+t)^{-1}\log(2+t)(\|w_0\|_{H^2} + \|w_0\|_{L^1} + |\delta|^{3/2}), \quad t \ge 0 \tag{1.9}$$

holds instead of (1.8). The aim of this paper is to show that the decay rate of estimate (1.9) is actually optimal, unless $\delta = 0$ or $c = 0$. Indeed, the second asymptotic profile of large time behavior of the solutions is given by

$$V(x, t) \equiv -\frac{cd}{12\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(2+t), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.10}$$

where

$$V_*(x) \equiv (b\chi_*(x) - x)e^{-x^2/4}\eta_*(x), \tag{1.11}$$

$$\eta_*(x) \equiv \exp\left(\frac{b}{2}\int_{-\infty}^x \chi_*(y)\, dy\right), \tag{1.12}$$

$$d \equiv \int_{\mathbb{R}} \eta_*^{-1}(y)\chi_*^3(y)\,dy. \tag{1.13}$$

Bisides, we can take the initial data from $L_1^1(\mathbb{R}) \cap H^1(\mathbb{R})$, analogously to the works of [7], [10]. And we set, for $k \geq 0$, $E_{k,\beta} \equiv \|u_0\|_{H^k} + \|u_0\|_{L_\beta^1}$. Then we have the following result.

Theorem 1.1. *Assume that* $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ *and* $E_{1,0}$ *is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution* $u(x,t)$ *satisfying* $u \in C^0([0,\infty); H^1)$ *and* $\partial_x u \in L^2(0,\infty; H^1)$. *Moreover, if* $u_0 \in L_1^1(\mathbb{R}) \cap H^1(\mathbb{R})$ *and* $E_{1,1}$ *is small, then the solution satisfies the estimate*

$$\|u(\,\cdot\,,t) - \chi(\,\cdot\,,t) - V(\,\cdot\,,t)\|_{L^\infty} \leq CE_{1,1}(1+t)^{-1}, \quad t \geq 1. \tag{1.14}$$

Here $\chi(x,t)$ *is defined by* (1.3), *while* $V(x,t)$ *is defined by* (1.10).

REMARK 1.2. In Liu [8], the initial value problem for the Burgers equations (1.1) and (1.2) is studied, provided $c = 0$ implicity at p.42. After the proof of Theorem 2.2.1, it is mentioned, without proof, that if we assume $(1+|x|)^2|u_0(x)| \leq \tilde{\delta}$ and $\tilde{\delta}$ is small, then the estimate

$$\|u(\,\cdot\,,t) - \chi(\,\cdot\,,t)\|_{L^\infty} \leq C\tilde{\delta}(1+t)^{-1}, \quad t \geq 1$$

holds. However, from our result, the above estimate fails true for the case $c\delta \neq 0$.

We remark that the estimate similar to (1.14) was obtained for other types of Burgers equation such as KdV-Burgers in Hayashi and Naumkin [3] and Kaikina and Ruiz-Paredes [5], and Benjamin-Bona-Mahony-Burgers in Hayashi, Kaikina and Naumkin [2].

2. Preliminaries

In order to prove the basic estimates given by Lemma 3.2, Lemma 3.3 and Lemma 3.5, we prepare the following two lemmas. The first one is concerned with the decay estimates for semigroup $e^{t\Delta}$ associated with the heat equation. For the proof, see Kawashima [6].

Lemma 2.1. *Let* k *be a positive integer. Suppose* $q_0 \in H^k(\mathbb{R}) \cap L^1(\mathbb{R})$. *Then the estimate*

$$\|\partial_x^l e^{t\Delta} q_0\|_{L^2} \leq C(1+t)^{-(1/4+l/2)}\|q_0\|_{L^1} + Ce^{-C't}\|\partial_x^l q_0\|_{L^2}, \quad t \geq 0 \tag{2.1}$$

holds for any $l = 0, 1, \ldots, k$.

The second one is related to the diffusion wave $\chi(x,t)$ and the heat kernel $G(x,t)$. The explicit formula of $\chi(x, t)$ and $G(x, t)$ are given by (1.3) and

$$G(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-(x^2/4t)}, \quad t > 0, \ x \in \mathbb{R}, \tag{2.2}$$

respectively. It is easy to see that

$$|\chi(x, t)| \leq C|\delta|(1 + t)^{-(1/2)} e^{-x^2/(4(1+t))}, \quad t \geq 0, \ x \in \mathbb{R}. \tag{2.3}$$

Moreover, we get the following (see e.g. [9]).

Lemma 2.2. *Let α and β be positive integers. Then, for $p \in [1, \infty]$, the estimates*

$$\|\partial_x^\alpha \partial_t^\beta \chi(\,\cdot\,, t)\|_{L^p} \leq C|\delta|(1 + t)^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t \geq 0, \tag{2.4}$$

$$\|\partial_x^\alpha \partial_t^\beta G(\,\cdot\,, t)\|_{L^p} \leq C t^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t > 0 \tag{2.5}$$

hold.

For the latter sake, we introduce η_1, η_2 defined by

$$\eta_1(x, t) \equiv \eta_*\left(\frac{x}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi(y, t)\, dy\right), \tag{2.6}$$

$$\eta_2(x, t) \equiv \eta_1^{-1}(x, t). \tag{2.7}$$

We easily have

$$\min\{1, e^{b\delta/2}\} \leq \eta_1(x, t) \leq \max\{1, e^{b\delta/2}\}, \tag{2.8}$$

$$\min\{1, e^{-b\delta/2}\} \leq \eta_2(x, t) \leq \max\{1, e^{-b\delta/2}\}. \tag{2.9}$$

Moreover, we get

Corollary 2.3. *Let l be a positive integer. For $i = 1, 2$, if $|\delta| \leq 1$, then we have*

$$\|\partial_x^l \eta_i(\,\cdot\,, t)\|_{L^1} \leq C|\delta|(1 + t)^{-(l-1)/2}, \tag{2.10}$$

$$\|\partial_x^l \eta_i(\,\cdot\,, t)\|_{L^\infty} \leq C|\delta|(1 + t)^{-1/2}, \tag{2.11}$$

$$\|\partial_x^l \eta_i(\,\cdot\,, t)\|_{L^2} \leq C|\delta|(1 + t)^{-(l/2-1/4)}. \tag{2.12}$$

Proof. We shall prove only (2.10), (2.11) and (2.12) for $i = 1$, since we can prove (2.10), (2.11) and (2.12) for $i = 2$ in a similar way. We put $q_1(x) = e^{(b/2)x}$,

$q_2(x,t) = \int_{-\infty}^x \chi(y,t)\,dy$. Then we have, for $l \geq 1$,

$$\begin{aligned}\partial_x^l \eta_1(x,t) &= \sum C_{l,m_1,\ldots,m_l} \frac{d^p q_1}{dx^p}(q_2(x,t))(\partial_x q_2(x,t))^{m_1} \cdots (\partial_x^l q_2(x,t))^{m_l} \\ &= \sum C_{l,m_1,\ldots,m_l} \left(\frac{b}{2}\right)^p \eta_1(x,t)(\chi(x,t))^{m_1} \cdots (\partial_x^{l-1}\chi(x,t))^{m_l},\end{aligned} \tag{2.13}$$

where the sum is taken for all $(m_1,\ldots,m_l) \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \cdots + lm_l = l$, and $p = m_1 + \cdots + m_l$, $1 \leq p \leq l$.

First we derive (2.10). Since $m_1 + 2m_2 + \cdots + lm_l = l$, there exists $j \in \{1,2,\ldots,l\}$ such that $m_j \geq 1$. Therefore, by Lemma 2.2 and (2.13), we have

$$\begin{aligned}&\|\partial_x^l \eta_1(\cdot,t)\|_{L^1} \\ &\leq C \sum \|\chi(\cdot,t)\|_{L^\infty}^{m_1} \|\partial_x \chi(\cdot,t)\|_{L^\infty}^{m_2} \cdots \|\partial_x^{j-1}\chi(\cdot,t)\|_{L^{m_j}}^{m_j} \cdots \|\partial_x^{l-1}\chi(\cdot,t)\|_{L^\infty}^{m_l} \\ &\leq C|\delta| \sum (1+t)^{-m_1/2}(1+t)^{-2m_2/2} \cdots (1+t)^{-(jm_j-1)/2} \cdots (1+t)^{-lm_l/2} \\ &\leq C|\delta|(1+t)^{-(l-1)/2}.\end{aligned}$$

Hence we get (2.10).

Next we derive (2.11). By Lemma 2.2 and (2.13), we have

$$\begin{aligned}\|\partial_x^l \eta_1(\cdot,t)\|_{L^\infty} &\leq C \sum \|\chi(\cdot,t)\|_{L^\infty}^{m_1} \|\partial_x \chi(\cdot,t)\|_{L^\infty}^{m_2} \cdots \|\partial_x^{l-1}\chi(\cdot,t)\|_{L^\infty}^{m_l} \\ &\leq C|\delta| \sum (1+t)^{-m_1/2}(1+t)^{-2m_2/2} \cdots (1+t)^{-lm_l/2} \\ &\leq C|\delta|(1+t)^{-l/2}.\end{aligned}$$

Hence we get (2.11).

Finally, by the interpolation inequality $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{1-1/p} \|f\|_{L^1}^{1/p}$ $(1 \leq p \leq \infty)$, we have from (2.10) and (2.11)

$$\begin{aligned}\|\partial_x^l \eta_1(\cdot,t)\|_{L^2} &\leq \|\partial_x^l \eta_1(\cdot,t)\|_{L^\infty}^{1/2} \|\partial_x^l \eta_1(\cdot,t)\|_{L^1}^{1/2} \\ &\leq C|\delta|(1+t)^{-(l/2-1/4)}.\end{aligned}$$

This completes the proof. □

3. Basic estimates

We deal with the following linearized equations which coressponds to (4.10), (4.11) below:

$$z_t = z_{xx} - (b\chi z)_x, \quad t > 0,\ x \in \mathbb{R}, \tag{3.1}$$

$$z(x,0) = z_0(x). \tag{3.2}$$

The explicit representation formula (3.4) below plays a crucial role in our analysis.

Lemma 3.1. *If we set*

$$U[w](x,t,\tau)=\int_{\mathbb{R}}\partial_x(G(x-y,t-\tau)\eta_1(x,t))\eta_2(y,\tau)\int_{-\infty}^{y}w(\xi)\,d\xi\,dy, \tag{3.3}$$
$$0\le\tau<t,\ x\in\mathbb{R},$$

then the solutions for (3.1) *and* (3.2) *is given by*

$$z(x,t)=U[z_0](x,t,0),\quad t>0,\ x\in\mathbb{R}. \tag{3.4}$$

Proof. If we put

$$r(x,t)=\int_{-\infty}^{x}z(y,t)\,dy, \tag{3.5}$$

then we see from (3.1), (3.2) that $r(x,t)$ satisfies

$$r_t=r_{xx}-b\chi r_x,\quad t>0,\ x\in\mathbb{R}, \tag{3.6}$$

$$r(x,0)=\int_{-\infty}^{x}z_0(y)\,dy. \tag{3.7}$$

Then a direct computation yields

$$\left(\frac{r(x,t)}{\eta_1(x,t)}\right)_t=\left(\frac{r(x,t)}{\eta_1(x,t)}\right)_{xx}, \tag{3.8}$$

where η_1 is defined by (2.6). Therefore, we have

$$r(x,t)=\eta_1(x,t)\int_{\mathbb{R}}G(x-y,t)\eta_1(y,0)^{-1}r(y,0)\,dy. \tag{3.9}$$

Hence (3.9), (3.5), (3.7) and (2.7) yield (3.4). □

Next we derive the decay estimates (3.10) and (3.22) below for the homogenous equation (3.1). Here, to prove the estimate, we follow the argument which is used to show the estimate for the semigroup $e^{t\Delta}$ in [7].

Lemma 3.2. *Let* $\beta\in[0,1]$, k *be a positive interger and* $p\in[1,\infty]$. *Assume that* $|\delta|\le 1$, $z_0\in L^1_\beta(\mathbb{R})$ *and* $\int_{\mathbb{R}}z_0(x)\,dx=0$. *Then, the estimate*

$$\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^p}\le Ct^{-(1-1/p+\beta+l)/2}\|z_0\|_{L^1_\beta},\quad t>0 \tag{3.10}$$

holds for any $l=0,1,\dots,k$.

Proof. Since $\int_{\mathbb{R}} z_0(x)\,dx = 0$, by the integration by parts with respect to y, we have from (3.3)

$$U[z_0](x,t,0) = -\int_{\mathbb{R}} z_0(y) \int_{-\infty}^{y} \partial_x (G(x-\xi,t)\eta_1(x,t))\eta_2(\xi,0)\,d\xi\,dy. \tag{3.11}$$

Using this expression, we shall show that

$$\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^\infty} \le Ct^{-(1+\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t>0, \tag{3.12}$$

$$\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^1} \le Ct^{-(\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t>0. \tag{3.13}$$

The desired estimate follows from the interpolation inequality. Actually, we have

$$\begin{aligned}\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^p} &\le \|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^\infty}^{1-1/p}\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^1}^{1/p}\\ &\le Ct^{-(1-1/p+\beta+l)/2}\|z_0\|_{L^1_\beta}.\end{aligned}$$

First we shall prove (3.12). From (3.11), we have

$$\begin{aligned}|\partial_x^l U[z_0](x,t,0)| &\le \int_{\mathbb{R}} |z_0(y)| \left| \int_0^y \partial_x^{l+1}(G(x-\xi,t)\eta_1(x,t))\eta_2(\xi,0)\,d\xi \right| dy\\ &\equiv \int_{\mathbb{R}} |z_0(y)| I_1(x,y,t)\,dy.\end{aligned} \tag{3.14}$$

By using (2.11), (2.9), (2.5) and the Hölder inequality, we have

$$\begin{aligned} I_1 &\le C\sum_{m=0}^{l+1} \|\partial_x^{l+1-m}\eta_1(\,\cdot\,,t)\|_{L^\infty}\left|\int_0^y \partial_x^m G(x-\xi,t)\eta_2(\xi,0)\,d\xi\right|\\ &\le C\sum_{m=0}^{l+1}(1+t)^{-(l+1-m)/2}\|\partial_x^m G(x-\,\cdot\,,t)\|_{L^{1/(1-\beta)}}|y|^\beta\\ &\le Ct^{-(1+\beta+l)/2}|y|^\beta.\end{aligned} \tag{3.15}$$

Hence (3.14) and (3.15) yield (3.12).

Next we shall prove (3.13). From (3.11), we have

$$\begin{aligned}\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^1} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |z_0(y)| \left|\int_0^y \partial_x^{l+1}(G(x-\xi,t)\eta_1(x,t))\eta_2(\xi,0)\,d\xi\right| dy\,dx\\ &\le \int_{\mathbb{R}} |z_0(y)| I_2(y,t)\,dy,\end{aligned}$$

where we put

$$I_2(y,t) = \int_{\mathbb{R}} \left|\int_0^y \partial_x^{l+1}(G(x-\xi,t)\eta_1(x,t))\eta_2(\xi,0)\,d\xi\right| dx. \tag{3.16}$$

In order to get (3.13), it suffices to show

$$I_2 \le Ct^{-l/2}, \tag{3.17}$$

$$I_2 \le C|y|t^{-(1+l)/2}. \tag{3.18}$$

In fact, if we would have these estimates, then

$$\begin{aligned}\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^1} &\le \int_{\mathbb{R}} |z_0(y)| I_2^{1-\beta} I_2^{\beta}\,dy\\ &\le C\int_{\mathbb{R}} |z_0(y)| t^{-l(1-\beta)/2} t^{-(1+l)\beta/2} |y|^{\beta}\,dy \le Ct^{-(\beta+l)/2}\|z_0\|_{L^1_\beta}.\end{aligned}$$

First we shall prove (3.17). From (3.16) we have

$$\begin{aligned}I_2 &\le C\sum_{m=0}^{l+1}\int_{\mathbb{R}}\left|\partial_x^{l+1-m}\eta_1(x,t)\int_0^y \partial_x^m G(x-\xi,t)\eta_2(\xi,0)\,d\xi\right|dx\\ &= C\int_{\mathbb{R}}\left|\eta_1(x,t)\int_0^y \partial_x^{l+1} G(x-\xi,t)\eta_2(\xi,0)\,d\xi\right|dx\\ &\quad + C\sum_{m=0}^{l}\int_{\mathbb{R}}\left|\partial_x^{l+1-m}\eta_1(x,t)\int_0^y \partial_x^m G(x-\xi,t)\eta_2(\xi,0)\,d\xi\right|dx\\ &\equiv I_{2,1}+I_{2,2}.\end{aligned} \tag{3.19}$$

By using the integration by parts with respect to ξ, we have

$$\begin{aligned}I_{2,1} &\le C\int_{\mathbb{R}} |\partial_x^l G(x-y,t)\eta_2(y,0) - \partial_x^l G(x,t)\eta_2(0,0)|\,dx\\ &\quad + C\int_{\mathbb{R}}\left|\int_0^y \partial_x^l G(x-\xi,t)\partial_\xi\eta_2(\xi,0)\,d\xi\right|dx\\ &\le C\int_{\mathbb{R}} |\partial_x^l G(x,t)|\,dx + C\int_{\mathbb{R}} |\partial_\xi\eta_2(\xi,0)|\int_{\mathbb{R}} |\partial_x^l G(x-\xi,t)|\,dx\,d\xi\\ &\le Ct^{-l/2},\end{aligned} \tag{3.20}$$

where we have used (2.8), (2.9), (2.10) and (2.5). On the other hand, from (2.9), (2.10) and (2.5), we have

$$\begin{aligned}I_{2,2} &\le C\sum_{m=0}^{l}\|\partial_x^{l+1-m}\eta_1(\,\cdot\,,t)\|_{L^1}\|\partial_x^m G(\,\cdot\,,t)\|_{L^1}\\ &\le C\sum_{m=0}^{l}(1+t)^{-(l-m)/2}t^{-m/2} \le Ct^{-l/2}.\end{aligned} \tag{3.21}$$

Hence we obtain (3.17).

Next we shall prove (3.18). From (3.16), (2.9), (2.11) and (2.5) we have

$$\begin{aligned} I_2 &\le C \sum_{m=0}^{l+1} \|\partial_x^{l+1-m}\eta_1(\,\cdot\,,t)\|_{L^\infty}\|\partial_x^m G(\,\cdot\,,t)\|_{L^1}|y| \\ &\le C \sum_{m=0}^{l+1}(1+t)^{-(l+1-m)/2}t^{-m/2}|y| \le Ct^{-(1+l)/2}|y|. \end{aligned}$$

Hence we get (3.18). This completes the proof. □

Lemma 3.3. *Let k be a positive interger. Assume that $|\delta| \le 1$, $z_0 \in H^k(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)\,dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^2} \le C(1+t)^{-(1/4+l/2)}\|z_0\|_{L^1} + Ce^{-t}\|z_0\|_{H^l}, \quad t > 0 \tag{3.22}$$

holds for any $l = 0, 1, \ldots, k$.

Proof. We have from (3.3)

$$\begin{aligned} \partial_x^l U[z_0](x,t,0) &= \int_{\mathbb{R}} \partial_x^{l+1}(G(x-y,t)\eta_1(x,t))\eta_2(y,0)\int_{-\infty}^{y} z_0(\xi)\,d\xi\,dy \\ &= \sum_{m=0}^{l+1}\binom{l+1}{m}\partial_x^{l+1-m}\eta_1(x,t)\int_{\mathbb{R}}\partial_x^m G(x-y,t)\eta_2(y,0)\int_{-\infty}^{y} z_0(\xi)\,d\xi\,dy \\ &= \partial_x^{l+1}\eta_1(x,t)\int_{\mathbb{R}} G(x-y,t)\eta_2(y,0)\int_{-\infty}^{y} z_0(\xi)\,d\xi\,dy \\ &\quad + \sum_{m=1}^{l+1}\binom{l+1}{m}\partial_x^{l+1-m}\eta_1(x,t)\int_{\mathbb{R}}\partial_x^{m-1}G(x-y,t)\partial_y J(y)\,dy, \end{aligned} \tag{3.23}$$

where we put

$$J(y) = \eta_2(y,0)\int_{-\infty}^{y} z_0(\xi)\,d\xi. \tag{3.24}$$

Therefore we have from (2.9), (2.5) and Corollary 2.3

$$\begin{aligned} \|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^2} &\le C\|\partial_x^{l+1}\eta_1(\,\cdot\,,t)\|_{L^2}\|G(\,\cdot\,,t)\|_{L^1}\|z_0\|_{L1} \\ &\quad + C\sum_{m=1}^{l+1}\|\partial_x^{l+1-m}\eta_1(\,\cdot\,,t)\|_{L^\infty}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2} \\ &\le C(1+t)^{-(1/4+l/2)}\|z_0\|_{L1} \\ &\quad + C\sum_{m=1}^{l+1}(1+t)^{-(l+1-m)/2}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2}. \end{aligned} \tag{3.25}$$

From (3.24), (2.9), (2.4) and Corollary 2.3, we have

$$\begin{aligned}\|\partial_x J\|_{L^1} &\le C\left\|\chi(x,t)\int_{-\infty}^{x} z_0(\xi)\,d\xi\right\|_{L^1(\mathbb{R}_x)} + C\|z_0\|_{L^1}\\ &\le C\|z_0\|_{L^1},\end{aligned}\tag{3.26}$$

and for $m \ge 1$

$$\begin{aligned}\|\partial_x^m J\|_{L^2} &\le C\sum_{n=0}^{m}\left\|\partial_x^{m-n}\eta_2(x,0)\partial_x^n\int_{-\infty}^{x} z_0(\xi)\,d\xi\right\|_{L^2(\mathbb{R}_x)}\\ &\le C\|\partial_x^m\eta_2(\,\cdot\,,0)\|_{L^2}\|z_0\|_{L^1} + C\sum_{n=1}^{m}\|\partial_x^{m-n}\eta_2(\,\cdot\,,0)\|_{L^\infty}\|\partial_x^{n-1}z_0\|_{L^2}\\ &\le C(\|z_0\|_{L^1} + \|z_0\|_{H^{m-1}}).\end{aligned}\tag{3.27}$$

Hence, for $1 \le m \le l+1$, from (3.26), (3.27) and Lemma 2.1, we have

$$\begin{aligned}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2} &\le C(1+t)^{-(1/4+(m-1)/2)}\|\partial_x J\|_{L^1} + Ce^{-t}\|\partial_x^m J\|_{L^2}\\ &\le C(1+t)^{-(1/4+(m-1)/2)}\|z_0\|_{L^1} + Ce^{-t}(\|z_0\|_{L^1} + \|z_0\|_{H^{m-1}}).\end{aligned}\tag{3.28}$$

Therefore by (3.25) and (3.28), we obtain (3.22). This completes the proof. □

From Lemma 3.2 and Lemma 3.3, we get the following uniform estimate.

Corollary 3.4. *Let k be a positive integer. Assume that $|\delta| \le 1$, $z_0 \in L^1_1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)\,dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\,\cdot\,,t,0)\|_{L^2} \le CE_{l,1}(1+t)^{-(3/4+l/2)}, \quad t > 0$$

holds for any $l = 0, 1, \ldots, k$. Here $E_{l,1} = \|z_0\|_{H^l} + \|z_0\|_{L^1_1}$.

Next we derive the decay estimate (3.29) below in the same way as Lemma 3.6 of [6]. The estimate will be used to get the decay rate of the solution $w(x,t)$ for the problem (4.10) and (4.11). Bisides, we denote the Fourier transform of $f(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by $\hat{f}(\xi) \equiv \mathcal{F}[f](\xi) = (1/\sqrt{2\pi})\int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$, and set

$$H^k_1(\mathbb{R}) \equiv \left\{ f : L^1_{\mathrm{loc}}(\mathbb{R}) \;\middle|\; \|f\|_{H^k_1} \equiv \sum_{m=0}^{k}\|\partial_x^m f\|_{L^1} < \infty \right\}.$$

Lemma 3.5. *Let k be a positive integer. Suppose $|\delta| \le 1$ and $w \in C^0(0,\infty;H^k)\cap C^0(0,\infty;H_1^k)$. Then the estimate*

$$\begin{aligned}
&\left\| \partial_x^l \int_0^t U[\partial_x w(\tau)](x,t,\tau)\,d\tau \right\|_{L^2(\mathbb{R}_x)} \\
&\le C\int_0^{t/2} (1+t-\tau)^{-(3/4+l/2)} \|w(\cdot,\tau)\|_{L^1}\,d\tau \\
&\quad + C\sum_{m=0}^{l} \int_{t/2}^{t} (1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2}\|\partial_x^m w(\cdot,\tau)\|_{L^1}\,d\tau \\
&\quad + C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\|\partial_x^m w(\cdot,\tau)\|_{L^2}^2\,d\tau\right)^{1/2}
\end{aligned} \tag{3.29}$$

holds for any $l = 0, 1, \dots, k$.

Proof. From (3.3) and Corollary 2.3, we have

$$\left\| \partial_x^l \int_0^t U[\partial_x w(\tau)](x,t,\tau)\,d\tau \right\|_{L^2(\mathbb{R}_x)} \le C\sum_{n=0}^{l+1}(1+t)^{-(l+1-n)/2}\|\partial_x^n I(\cdot,t)\|_{L^2}, \tag{3.30}$$

where we put

$$I(x,t) = \int_0^t\int_{\mathbb{R}} G(x-y,t-\tau)\eta_2(y,\tau)w(y,\tau)\,dy\,d\tau. \tag{3.31}$$

From Lemma 2.1, we have

$$\begin{aligned}
\|I(\cdot,t)\|_{L^2} &\le C\int_0^t \|e^{(t-\tau)\Delta}[(\eta_2 w)(\tau)](\cdot)\|_{L^2}\,d\tau \\
&\le C\int_0^t \{(1+t-\tau)^{-1/4}\|w(\cdot,\tau)\|_{L^1} + e^{-(t-\tau)}\|w(\cdot,\tau)\|_{L^2}\}\,d\tau \\
&\le C\int_0^t (1+t-\tau)^{-1/4}\|w(\cdot,\tau)\|_{L^1}\,d\tau + C\left(\int_0^t e^{-(t-\tau)}\|w(\cdot,\tau)\|_{L^2}^2\,d\tau\right)^{1/2},
\end{aligned} \tag{3.32}$$

because $t \ge t-\tau$ for $0 \le \tau \le t$. In the following, let $1 \le n \le l+1$. It follows that

$$\begin{aligned}
\|\partial_x^n I(\cdot,t)\|_{L^2} &\le \|(i\xi)^n \hat{I}(\xi,t)\|_{L^2(|\xi|\le 1)} + \|(i\xi)^n \hat{I}(\xi,t)\|_{L^2(|\xi|\ge 1)} \\
&\equiv I_1 + I_2.
\end{aligned} \tag{3.33}$$

First, we evaluate I_1. From (3.31), we have

$$\begin{aligned} I_1 &\le \int_0^{t/2} \|(i\xi)^n e^{-(t-\tau)|\xi|^2}\mathcal{F}[\eta_2 w](\xi,\tau)\|_{L^2(|\xi|\le 1)}\,d\tau \\ &\quad + \int_{t/2}^{t} \|(i\xi)^n e^{-(t-\tau)|\xi|^2}\mathcal{F}[\eta_2 w](\xi,\tau)\|_{L^2(|\xi|\le 1)}\,d\tau \\ &\equiv I_{1,1} + I_{1,2}. \end{aligned}$$

Since

$$\int_{|\xi|\le 1} |\xi|^j e^{-2(t-\tau)|\xi|^2}\,d\xi \le C(1+t-\tau)^{-(j+1)/2},$$

for $j \ge 0$, we have

$$\begin{aligned} I_{1,1} &\le C\int_0^{t/2} \sup_{|\xi|\le 1} |\mathcal{F}[\eta_2 w](\,\cdot\,,\tau)| \left(\int_{|\xi|\le 1} |\xi|^{2n} e^{-2(t-\tau)|\xi|^2}\,d\xi\right)^{1/2} d\tau \\ &\le C\int_0^{t/2} (1+t-\tau)^{-(2n+1)/4}\|w(\,\cdot\,,\tau)\|_{L^1}\,d\tau. \end{aligned} \tag{3.34}$$

While, we have from Corollary 2.3

$$\begin{aligned} I_{1,2} &\le C\int_{t/2}^{t} \sup_{|\xi|\le 1} |(i\xi)^{n-1}\mathcal{F}[\eta_2 w](\,\cdot\,,\tau)| \left(\int_{|\xi|\le 1} |\xi|^{2} e^{-2(t-\tau)|\xi|^2}\,d\xi\right)^{1/2} d\tau \\ &\le C\int_{t/2}^{t} (1+t-\tau)^{-3/4}\|\partial_x^{n-1}(\eta_2 w)(\,\cdot\,,\tau)\|_{L^1}\,d\tau \\ &\le C\int_{t/2}^{t} (1+t-\tau)^{-3/4}\sum_{m=0}^{n-1}(1+\tau)^{-(n-1-m)/2}\|\partial_x^m w(\,\cdot\,,\tau)\|_{L^1}\,d\tau. \end{aligned} \tag{3.35}$$

Therefore, we get from (3.34) and (3.35)

$$\begin{aligned} I_1 &\le C\int_0^{t/2} (1+t-\tau)^{-(2n+1)/4}\|w(\,\cdot\,,\tau)\|_{L^1}\,d\tau \\ &\quad + C\int_{t/2}^{t} (1+t-\tau)^{-3/4}\sum_{m=0}^{n-1}(1+\tau)^{-(n-1-m)/2}\|\partial_x^m w(\,\cdot\,,\tau)\|_{L^1}\,d\tau. \end{aligned} \tag{3.36}$$

Next we evaluate I_2. For $|\xi| \ge 1$, by using the Schwarz inequality, we have

$$\begin{aligned} |(i\xi)^n \hat{I}(\xi,t)| &\le C\int_0^t |\xi| e^{-(t-\tau)|\xi|^2}|(i\xi)^{n-1}\mathcal{F}[\eta_2 w](\xi,\tau)|\,d\tau \\ &\le C\left(\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^{n-1}\mathcal{F}[\eta_2 w](\xi,\tau)|^2\,d\tau\right)^{1/2}. \end{aligned}$$

Therefore we find from (3.33) and Corollary 2.3

$$
\begin{aligned}
I_2 &\le C\left(\int_0^t e^{-(t-\tau)} \int_{|\xi|\ge 1} |(i\xi)^{n-1}\mathcal{F}[\eta_2 w](\xi,\tau)|^2\, d\xi\, d\tau\right)^{1/2} \\
&\le C\left(\int_0^t e^{-(t-\tau)} \|\partial_x^{n-1}(\eta_2 w)(\,\cdot\,,\tau)\|_{L^2}^2\, d\tau\right)^{1/2} \\
&\le C\left(\int_0^t e^{-(t-\tau)} \sum_{m=0}^{n-1}(1+\tau)^{-(n-1-m)}\|\partial_x^m w(\,\cdot\,,\tau)\|_{L^2}^2\, d\tau\right)^{1/2}.
\end{aligned} \tag{3.37}
$$

Summarizing (3.30), (3.32), (3.33), (3.36) and (3.37), we obtain (3.29). This completes the proof. □

4. Proof of Theorem 1.1

In order to prove our result, we introduce the following auxiliary problem:

$$
v_t = v_{xx} - (b\chi v)_x - \left(\frac{c}{3}\chi^3\right)_x, \quad t > 0,\ x \in \mathbb{R}, \tag{4.1}
$$

$$
v(x,0) = 0. \tag{4.2}
$$

We derive the decay estimate for the solution $v(x,t)$ to the above problem.

Lemma 4.1. *Let $l \ge 0$ be an integer. Then we have*

$$
\|\partial_x^l v(\,\cdot\,,t)\|_{L^2} \le C|\delta|^3(1+t)^{-(3/4+l/2)}\log(2+t), \quad t \ge 0. \tag{4.3}
$$

Proof. By the Duhamel principle, we see from Lemma 3.1 that

$$
v(x,t) = \int_0^t U\left[\partial_x\left(-\frac{c}{3}\chi^3(\tau)\right)\right](x,t,\tau)\, d\tau. \tag{4.4}
$$

From Lemma 3.5, we get

$$
\begin{aligned}
\|\partial_x^l v(\,\cdot\,,t)\|_{L^2} &\le C\int_0^{t/2}(1+t-\tau)^{-(3/4+l/2)}\|\chi^3(\,\cdot\,,\tau)\|_{L^1}\, d\tau \\
&\quad + C\sum_{m=0}^{l}\int_{t/2}^t (1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2}\|\partial_x^m(\chi^3(\,\cdot\,,\tau))\|_{L^1}\, d\tau \\
&\quad + C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\|\partial_x^m(\chi^3(\,\cdot\,,\tau))\|_{L^2}^2\, d\tau\right)^{1/2} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{4.5}
$$

It follows from Lemma 2.2 that for $p \in [1, \infty]$

$$\begin{aligned}\|\partial_x^m(\chi^3(\cdot,\tau))\|_{L^p} &\le C\sum_{n=0}^{m}\sum_{s=0}^{m-n}\|\partial_x^n\chi(\cdot,\tau)\|_{L^\infty}\|\partial_x^s\chi(\cdot,\tau)\|_{L^\infty}\|\partial_x^{m-n-s}\chi(\cdot,\tau)\|_{L^p}\\ &\le C|\delta|^3\sum_{n=0}^{m}\sum_{s=0}^{m-n}(1+\tau)^{-(1+n)/2}(1+\tau)^{-(1+s)/2}(1+\tau)^{-(1-(1/p)+m-n-s)/2}\\ &\le C|\delta|^3(1+\tau)^{-(3-(1/p)+m)/2}.\end{aligned} \tag{4.6}$$

Using (4.6), we obtain

$$\begin{aligned}I_1 &\le C|\delta|^3\int_0^{t/2}(1+t-\tau)^{-(3/4+l/2)}(1+\tau)^{-1}\,d\tau\\ &\le C|\delta|^3(1+t)^{-(3/4+l/2)}\log(2+t),\end{aligned} \tag{4.7}$$

$$\begin{aligned}I_2 &\le C|\delta|^3\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-(1+l/2)}\,d\tau\\ &\le C|\delta|^3(1+t)^{-(3/4+l/2)},\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}I_3 &\le C|\delta|^3\left(\int_0^{t}e^{-(t-\tau)}(1+\tau)^{-(5/2+l)}\,d\tau\right)^{1/2}\\ &\le C|\delta|^3(1+t)^{-(5/4+l/2)}.\end{aligned} \tag{4.9}$$

This completes the proof. □

Our first step to prove Theorem 1.1 is the following.

Proposition 4.2. *Let $k \ge 1$ be an integer. Assume that $u_0 \in L^1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $E_{k,0}$ is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution $u(x,t)$ satisfying $u \in C^0([0,\infty); H^k)$ and $\partial_x u \in L^2(0,\infty; H^k)$. Moreover, if $u_0 \in L^1_1(\mathbb{R}) \cap H^k(\mathbb{R})$ and $E_{k,1}$ is small, then the estimate*

$$\|\partial_x^l(u(\cdot,t)-\chi(\cdot,t)-v(\cdot,t))\|_{L^2} \le CE_{k,1}(1+t)^{-(3/4+l/2)}, \quad t \ge 0$$

holds for $l \le k$. In particular,

$$\|u(\cdot,t)-\chi(\cdot,t)-v(\cdot,t)\|_{L^\infty} \le CE_{1,1}(1+t)^{-1}, \quad t \ge 0.$$

Here $\chi(x,t)$ is defined by (1.3)*, while $v(x,t)$ is the solution for the problem* (4.1) *and* (4.2).

Proof. We can show the local existence and uniqueness of the solution to (1.1) and (1.2) by standard argument. Using the argument of Theorem 8.2 in [7], we can continue the local solution globally in time. We shall prove only the decay estimate. We note that the smallness of $E_{k,1}$ implies that of $|\delta|$ by (1.5). We put

$$w(x,t) = u(x,t) - \chi(x,t) - v(x,t).$$

Then $w(x,t)$ satisfies

$$w_t = w_{xx} - (b\chi w)_x + (g(w,\chi,v))_x, \quad t > 0,\ x \in \mathbb{R}, \tag{4.10}$$

$$w(x,0) = w_0(x), \tag{4.11}$$

where we have set $w_0(x) = u_0(x) - \chi(x,0)$ and

$$\begin{aligned} g(w,\chi,v) = &-\frac{b}{2}(w+v)^2 \\ &- \frac{c}{3}[w^3 + v^3 + 3(w+v)(w+\chi)(\chi+v)]. \end{aligned} \tag{4.12}$$

Since $u_0(x),\ \chi(x,0) \in L^1_1(\mathbb{R}) \cap H^k(\mathbb{R})$, we have $w_0(x) \in L^1_1 \cap H^k$. Bisides by (1.5) and (1.7),

$$\int_{\mathbb{R}} w_0(x)\,dx = 0. \tag{4.13}$$

Now, we define $N(T)$ by

$$N(T) = \sup_{0 \le t \le T} \sum_{m=0}^{k} (1+t)^{3/4+m/2} \|\partial_x^m w(\cdot,t)\|_{L^2}. \tag{4.14}$$

For $0 \le l \le k-1$, we have from (4.14) and the Sobolev inequality

$$\|\partial_x^l w(\cdot,t)\|_{L^\infty} \le N(T)(1+t)^{-(1+l/2)}. \tag{4.15}$$

We shall show that for $0 \le l \le k$

$$\|\partial_x^l g(\cdot,t)\|_{L^1} \le C(1+t)^{-(3/2+l/2)}((|\delta|\log(2+t))^2 + N(T)^2), \tag{4.16}$$

$$\|\partial_x^l g(\cdot,t)\|_{L^2} \le C(1+t)^{-(7/4+l/2)}((|\delta|\log(2+t))^2 + N(T)^2). \tag{4.17}$$

We shall prove only (4.16), since we can prove (4.17) in a similar way. Here and later, $|\delta|$ and $N(T)$ are assumed to be small. We put $h_1(x,t) = w(x,t) + v(x,t)$, $h_2(x,t) = w(x,t) + \chi(x,t)$ and $h_3(x,t) = \chi(x,t) + v(x,t)$. Then, for $0 \le m \le k$, we have from (4.14), (4.3) and (2.4)

$$\|\partial_x^m h_1(\cdot\,,t)\|_{L^2} \le C(1+t)^{-(3/4+m/2)}(|\delta|\log(2+t)+N(T)), \tag{4.18}$$

$$\|\partial_x^m h_2(\cdot\,,t)\|_{L^2} \le C(1+t)^{-(1/4+m/2)}(|\delta|+N(T)), \tag{4.19}$$

$$\|\partial_x^m h_3(\cdot\,,t)\|_{L^\infty} \le C(1+t)^{-(1/2+m/2)}|\delta|. \tag{4.20}$$

Hence, we have from (4.18), (4.19), (4.20), (4.14), (4.15), (4.3) and (2.4)

$$\begin{aligned}\|\partial_x^l((w+v)^2(\cdot\,,t))\|_{L^1} &\le C\sum_{m=0}^{l}\|\partial_x^m h_1(\cdot\,,t)\|_{L^2}\|\partial_x^{l-m}h_1(\cdot\,,t)\|_{L^2}\\ &\le C(1+t)^{-(3/2+l/2)}((|\delta|\log(2+t))^2+N(T)^2),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\|\partial_x^l(w^3(\cdot\,,t))\|_{L^1} &\le C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m}\|\partial_x^m w(\cdot\,,t)\|_{L^\infty}\|\partial_x^n w(\cdot\,,t)\|_{L^2}\|\partial_x^{l-m-n}w(\cdot\,,t)\|_{L^2}\\ &\quad + C\|\partial_x^l w(\cdot\,,t)\|_{L^2}\|w(\cdot\,,t)\|_{L^2}\|w(\cdot\,,t)\|_{L^\infty}\\ &\le C(1+t)^{-(5/2+l/2)}N(T)^3,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\|\partial_x^l(v^3(\cdot\,,t))\|_{L^1} &\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m}\|\partial_x^m v(\cdot\,,t)\|_{L^\infty}\|\partial_x^n v(\cdot\,,t)\|_{L^2}\|\partial_x^{l-m-n}v(\cdot\,,t)\|_{L^2}\\ &\le C(1+t)^{-(5/2+l/2)}(|\delta|^3\log(2+t))^3\\ &\le C(1+t)^{-(3/2+l/2)}(|\delta|\log(2+t))^2\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\|\partial_x^l(h_1h_2h_3)(\cdot\,,t)\|_{L^1} &\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m}\|\partial_x^m h_1(\cdot\,,t)\|_{L^2}\|\partial_x^n h_2(\cdot\,,t)\|_{L^2}\|\partial_x^{l-m-n}h_3(\cdot\,,t)\|_{L^\infty}\\ &\le C(1+t)^{-(3/2+l/2)}(|\delta|\log(2+t)+N(T))(|\delta|+N(T))\\ &\le C(1+t)^{-(3/2+l/2)}((|\delta|\log(2+t))^2+N(T)^2).\end{aligned} \tag{4.24}$$

Summing up these estimates, we obtain (4.16) from (4.12).

Applying the Duhamel principle for the problem (4.10) and (4.11), we have

$$w(x,t) = U[w_0](x,t,0) + \int_0^t U[\partial_x g(w,\chi,v)(\tau)](x,t,\tau)\,d\tau, \quad t>0,\ x\in\mathbb{R}. \tag{4.25}$$

For $l \le k$, we have from (4.25), Corollary 3.4 and Lemma 3.5

(4.26)
$$\begin{aligned}\|\partial_x^l w(\cdot,t)\|_{L^2} \le{}& C(1+t)^{-(3/4+l/2)}E_{k,1} + C\int_0^{t/2}(1+t-\tau)^{-(3/4+l/2)}\|g(\cdot,\tau)\|_{L^1}\,d\tau\\ &+ C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2}\|\partial_x^m g(\cdot,\tau)\|_{L^1}\,d\tau\\ &+ C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\|\partial_x^m g(\cdot,\tau)\|_{L^2}^2\,d\tau\right)^{1/2}\\ \equiv{}& I_1+I_2+I_3+I_4.\end{aligned}$$

First we evaluate I_2. From (4.16), we have

$$\begin{aligned}I_2 &\le C\int_0^{t/2}(1+t-\tau)^{-(3/4+l/2)}(1+\tau)^{-3/2}((|\delta|\log(2+\tau))^2+N(T)^2)\,d\tau\\ &\le C(1+t)^{-(3/4+l/2)}(|\delta|^2+N(T)^2).\end{aligned}\tag{4.27}$$

Next we evaluate I_3. From (4.16), we have

$$\begin{aligned}I_3 &\le C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-(3+l)/2}((|\delta|\log(2+\tau))^2+N(T)^2)\,d\tau\\ &\le C(1+t)^{-(5/4+l/2)}((|\delta|\log(2+t))^2+N(T)^2).\end{aligned}\tag{4.28}$$

Finally we evaluate I_4. From (4.17), it follows that

$$\begin{aligned}I_4 &\le C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(7/2+l)}((|\delta|\log(2+\tau))^4+N(T)^4)\,d\tau\right)^{1/2}\\ &\le C(1+t)^{-(7/4+l/2)}((|\delta|\log(2+t))^2+N(T)^2).\end{aligned}\tag{4.29}$$

Since $|\delta| \le E_{k,1}$, if $E_{k,1}$ is small, then we obtain the inequality

$$(1+t)^{3/4+l/2}\|\partial_x^l w(\cdot,t)\|_{L^2} \le C(E_{k,1}+N(T)^2).\tag{4.30}$$

Therefore, (4.30) gives the desired estimate $N(T) \le CE_{k,1}$. This completes the proof. □

To prove Theorem 1.1, it is sufficent to show Proposition 4.3 below by virtue of Proposition 4.2. Although the similer estimate was shown by Lemma 3 in [5], but we need to modify the proof of it in order to avoid the logarithmic term in the right-hand side.

Proposition 4.3. *Assume that* $|\delta| \leq 1$. *Then the estimate*

$$\|v(\,\cdot\,,t) - V(\,\cdot\,,t)\|_{L^\infty} \leq C|\delta|^3(1+t)^{-1}, \quad t \geq 1 \tag{4.31}$$

holds. Here, $v(x,t)$ *is the solution for the problem* (4.1) *and* (4.2), *while* $V(x,t)$ *is defined by* (1.10).

Proof. It follows from (4.4) and (3.3) that

$$\begin{aligned} v(x,t) &= -\frac{c}{3}\int_0^t U[\partial_x\chi^3(\tau)](x,t,\tau)\,d\tau \\ &= -\frac{c}{3}\int_{t/2}^t\int_{\mathbb{R}} \partial_x(G(x-y,t-\tau)\eta_1(x,t))\eta_2(y,\tau)\chi^3(y,\tau)\,dy\,d\tau \\ &\quad -\frac{c}{3}\int_0^{t/2}\int_{\mathbb{R}} \partial_x(G(x-y,t-\tau)\eta_1(x,t))\eta_2(y,\tau)\chi^3(y,\tau)\,dy\,d\tau \\ &\equiv I_1 + I_2. \end{aligned} \tag{4.32}$$

First we evaluate I_1. By the integration by parts with respects to y, we have

$$\begin{aligned} I_1 &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t\int_{\mathbb{R}}\left(\partial_x G(x-y,t-\tau) + \frac{b}{2}\chi(x,t)G(x-y,t-\tau)\right) \\ &\qquad\qquad \times \eta_2(y,\tau)\chi^3(y,\tau)\,dy\,d\tau \\ &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t\int_{\mathbb{R}} G(x-y,t-\tau)\Big(\partial_y(\eta_2(y,\tau)\chi^3(y,\tau)) \\ &\qquad\qquad + \frac{b}{2}\eta_2(y,\tau)\chi(x,t)\chi^3(y,\tau)\Big)\,dy\,d\tau. \end{aligned}$$

Therefore, we get from Lemma 2.2, (2.8) and (2.9)

$$\begin{aligned} \|I_1(\,\cdot\,,t)\|_{L^\infty} &\leq C\int_{t/2}^t \|G(\,\cdot\,,t-\tau)\|_{L^1}\{\|\chi(\,\cdot\,,\tau)\|_{L^\infty}^4 + \|\chi(\,\cdot\,,\tau)\|_{L^\infty}^2\|\partial_x\chi(\,\cdot\,,\tau)\|_{L^\infty} \\ &\qquad + \|\chi(\,\cdot\,,t)\|_{L^\infty}\|\chi(\,\cdot\,,\tau)\|_{L^\infty}^3\}\,d\tau \\ &\leq C|\delta|^3\int_{t/2}^t (1+\tau)^{-2}\,d\tau \\ &\leq C|\delta|^3(1+t)^{-1}. \end{aligned} \tag{4.33}$$

Next we evaluate I_2. If we put

$$\Lambda(x,t,y,\tau) \equiv -\frac{c}{3}\partial_x(G(x-y,t-\tau)\eta_1(x,t)), \tag{4.34}$$

then we have

$$I_2 = \int_0^{t/2}\int_{\mathbb{R}} \Lambda(x,t,y,\tau)\eta_2(y,\tau)\chi^3(y,\tau)\,dy\,d\tau. \tag{4.35}$$

Spliting the y-integral at $y = 0$ and making the integration by parts, we have

$$\begin{aligned}
I_2 &= \int_0^{t/2}\int_0^{\infty} \partial_y\Lambda(x,t,y,\tau)\int_y^{\infty} \eta_2(\xi,\tau)\chi^3(\xi,\tau)\,d\xi\,dy\,d\tau \\
&\quad - \int_0^{t/2}\int_{-\infty}^{0} \partial_y\Lambda(x,t,y,\tau)\int_{-\infty}^{y} \eta_2(\xi,\tau)\chi^3(\xi,\tau)\,d\xi\,dy\,d\tau \\
&\quad + \int_0^{t/2} \Lambda(x,t,0,\tau)\int_{\mathbb{R}} \eta_2(\xi,\tau)\chi^3(\xi,\tau)\,d\xi\,d\tau \\
&\equiv I_3 + I_4 + I_5.
\end{aligned} \tag{4.36}$$

First we consider I_3. From (4.34) and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{0\le\tau\le t/2}\ \sup_{x\in\mathbb{R}}\sup_{y\in\mathbb{R}} |\partial_y\Lambda(x,t,y,\tau)| \\
&\le C \sup_{0\le\tau\le t/2} (\|\partial_x^2 G(\,\cdot\,,t-\tau)\|_{L^\infty} + \|\chi(\,\cdot\,,t)\|_{L^\infty}\|\partial_x G(\,\cdot\,,t-\tau)\|_{L^\infty}) \\
&\le C \sup_{0\le\tau\le t/2} (t-\tau)^{-3/2} \le Ct^{-3/2}.
\end{aligned} \tag{4.37}$$

From (4.36) and (4.37), we have

$$\|I_3(\,\cdot\,,t)\|_{L^\infty} \le Ct^{-3/2}\int_0^{t/2}\int_0^{\infty}\int_y^{\infty} |\chi(\xi,\tau)|^3\,d\xi\,dy\,d\tau.$$

Then, by the integration by parts with respect to y, it follows from Lemma 2.2 and (2.3) that

$$\begin{aligned}
\|I_3(\,\cdot\,,t)\|_{L^\infty} &\le Ct^{-3/2}\int_0^{t/2}\int_0^{\infty} y|\chi(y,\tau)|^3\,dy\,d\tau \\
&\le C|\delta|^3 t^{-3/2}\int_0^{t/2}(1+\tau)^{-1}\int_{\mathbb{R}} \frac{|y|}{\sqrt{1+\tau}}e^{-y^2/(4(1+\tau))}\,dy\,d\tau \\
&\le C|\delta|^3(1+t)^{-1}
\end{aligned} \tag{4.38}$$

for $t \ge 1$. Similarly, we have

$$\|I_4(\,\cdot\,,t)\|_{L^\infty} \le C|\delta|^3(1+t)^{-1}. \tag{4.39}$$

Next we consider I_5. From (2.6), (2.7), (1.3) and (1.13), we have

$$\int_{\mathbb{R}} \eta_2(\xi, \tau)\chi^3(\xi, \tau)\, d\xi = d(1+\tau)^{-1},$$

hence it follows from (4.34) and (4.36) that

$$\begin{aligned}
I_5 &= d \int_0^{t/2} \Lambda(x, t, 0, \tau)(1+\tau)^{-1}\, d\tau \\
&= -\frac{cd}{3}\eta_1(x, t) \int_0^{t/2} (1+\tau)^{-1} \Big((\partial_x G(x, t-\tau) - \partial_x G(x, t+1)) \\
&\qquad\qquad + \frac{b}{2}\chi(x, t)(G(x, t-\tau) - G(x, t+1)) \Big)\, d\tau \\
&\quad - \frac{cd}{3}\eta_1(x, t)\Big(\partial_x G(x, t+1) + \frac{b}{2}\chi(x, t)G(x, t+1)\Big) \log\Big(\frac{2+t}{2}\Big) \\
&\equiv I_{5,1} + I_{5,2}.
\end{aligned} \tag{4.40}$$

In order to evaluate $I_{5,1}$, we shall use

$$|\partial_x^l G(x, t-\tau) - \partial_x^l G(x, t+1)| \le C(t-\tau)^{-(3+l)/2}(1+\tau) \tag{4.41}$$

for $l = 0, 1$ and $0 \le \tau \le t/2$. This estimate can be shown by observing that

$$\partial_x^l G(x, t-\tau) - \partial_x^l G(x, t+1) = -(1+\tau)\int_0^1 (\partial_t \partial_x^l G)(x, 1+t-\theta(1+\tau))\, d\theta$$

and by recalling (2.5). Since $|d| \le C|\delta|^3$ by (1.13), we have from (4.41)

$$\begin{aligned}
\|I_{5,1}(\cdot, t)\|_{L^\infty} &\le C|\delta|^3 \int_0^{t/2} (1+\tau)^{-1}\{(t-\tau)^{-2}(1+\tau) + (1+t)^{-1/2}(t-\tau)^{-3/2}(1+\tau)\}\, d\tau \\
&\le C|\delta|^3 \int_0^{t/2} (t-\tau)^{-2}\, d\tau \le C|\delta|^3(1+t)^{-1}.
\end{aligned} \tag{4.42}$$

Finally, we deal with $I_{5,2}$. From (4.40), (2.2), (1.3) and (2.6), it follows that

$$\begin{aligned}
I_{5,2} = -\frac{cd}{12\sqrt{\pi}}\eta_*\Big(\frac{x}{\sqrt{1+t}}\Big)\Big(b\chi_*\Big(\frac{x}{\sqrt{1+t}}\Big) - \frac{x}{\sqrt{1+t}}\Big)e^{-x^2/(4(1+t))}(1+t)^{-1} \\
\times (\log(t+2) - \log 2).
\end{aligned}$$

We have from (1.10) and (1.11),

$$\|I_{5,2}(\cdot, t) - V(\cdot, t)\|_{L^\infty} \le C|\delta|^3(1+t)^{-1}. \tag{4.43}$$

Summarizing (4.32), (4.33), (4.36), (4.38), (4.39), (4.40), (4.42) and (4.43), we obtain (4.31). This completes the proof. □

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