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STRONGLY SEMIPRIME RINGS AND NONSINGULAR QUASI-INJECTIVE MODULES

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Following Handelman [8] we call a ring R is a right strongly semiprime ring provided if I is a two-sided ideal of R and is essential as a right ideal, then it contains a finite subset whose right annihilator is zero.

In this paper, we first show that a ring R is a right strongly semiprime ring if and only if

(1) $Q(R)$ is a direct sum of simple rings, and

(2) $eQ(R)eR=eQ(R)$ for all idempotents e in $Q(R)$ where $Q(R)$ denotes the maximal ring of right quotients of R.

Using these conditions (1) and (2), we shall investigate the following conditions:

(a) Every nonsingular quasi-injective right R-module is injective.

(b) Any finite direct sum of nonsingular quasi-injective right R-modules is quasi-injective.

(c) Any direct sum of nonsingular quasi-injective right R-modules is quasi-injective.

(d) Any direct product of nonsingular quasi-injective right R-modules is quasi-injective.

It is shown that the conditions (a), (b) and (d) are equivalent; indeed, the rings satisfying one of these conditions are determined as rings R such that $R/G(R)$ is a right strongly semiprime ring, where $G(R)$ denotes the right Goldie torsion submodule of R. A ring R satisfying the condition (c) is also characterized as a ring R such that $R/G(R)$ is a semiprime right Goldie ring.

1. Preliminaries and notations

Throughout this paper all rings considered have identity and all modules are unitary.

Let R be a ring. $Q(R)$ denotes its maximal ring of right quotients. Let M be a right R-module. By $E_R(M)$, nM, $Z(M)$ and $G(M)$ we denotes its injective hull, the direct product of n-copies, its singular submodule and its Goldie torsion submodule, respectively. (Note that $Z(M/Z(M))=G(M)/Z(M)$.) For a given two right R-modules N and M, we adopt the symbol $N \lesssim M$ to denote the fact that N is isomorphic to a submodule of M, and use the symbol $N \subseteq_e M$ to indicate N to be an essential submodule of M.

Now, for a nonsingular right R-module M, the following statements hold:

(1) $MG(R)=0$; so M become a right $R/G(R)$-module by usual way,

(2) M is also nonsingular as a right $R/G(R)$-module, and

(3) M is R-injective (R-quasi-injective) if and only if M is $R/G(R)$-injective ($R/G(R)$-quasi-injective).

Noting that $R/G(R)$ is a right nonsingular ring, we conclude from [4, Theorem 2.2] that any nonsingular injective right R-module has a unique right $Q(R/G(R))$-module structure compatible with the R-module structure. So, for a nonsingular right R-module M, we have $M \subseteq_e MQ(R/G(R)) \subseteq_e E_R(M)$.

It is well known (e.g. [4, Theorem 3.2]) that every finitely generated nonsingular right module over a right self-injective regular ring is both projective and injective. Therefore, if M is a finitely generated nonsingular injective right R-module, then M is both $Q(R/G(R))$-projective and $Q(R/G(R))$-injective.

For a subset S of a ring R, $(0:S)_R^r((0:S)_R^l)$ denotes the right (left) annihilator of S in R.

Lemma 1.1. *Let R be a ring and set $\bar{R}=R/G(R)$ and $Q=Q(\bar{R})$. If M is a nonsingular right Q-module, then the following statements hold*:

(a) *M is nonsingular as a right R-module.* (*Of course, M becomes a right R-module by a natural way.*)

(b) *M is Q-quasi-injective if and only if M is R-quasi-injective.*

Proof. (a) Let x be an element in M such that $(0:x)_R^r \subseteq_e R$. Inasmuch as $G(R) \subseteq (0:x)_R^r \subseteq_e R$, we see from [4, Proposition 1.28] that $(0:x)_{\bar{R}}^r \subseteq_e \bar{R}$. Hence it follows $(0:x)_Q^r \subseteq_e Q$, whence $x=0$.

(b) Clearly $M \subseteq_e E_{\bar{R}}(M)$ as a right Q-module. It is also easily seen that $M \subseteq_e E_Q(M)$ as a right $\bar{R}$-module. As a reuslt we get $E_{\bar{R}}(M)=E_Q(M)$, whence $E_R(M)=E_Q(M)$. On the other hand we see that $End_R(E_R(M))=End_{\bar{R}}(E_{\bar{R}}(M))=End_Q(E_{\bar{R}}(M))$ and $End_R(E_Q(M))=End_{\bar{R}}(E_Q(M))=End_Q(E_Q(M))$; consequently $End_R(E_R(M))=End_Q(E_Q(M))$, where $End_*(\#)$ denotes the endomorphism ring of a right $*$-module $\#$. The proof is now easily done by applying the well known fact that a module is quasi-injective if and only if it is a fully invariant submodule of its injective hull.

The following lemma is frequently used in this paper.

Lemma 1.2. *If M is a quasi-injective right R-module such that $R \lesssim nM$ for some positive integer n, then M is injective.*

Proof. By virtue of Harada [9, Proposition 2.4], nM is also quasi-injective.

Hence we can easily see from $R \subsetneq nM$ that nM is injective, whence so is M.

2. Strongly semiprime rings

We recall some definitions introduced by Handelman and Lawrence [7] and Handelman [8]. An right ideal I of a ring R is *insulated* if there exists a finite set $\subseteq I$ whose right annihilator in R is zero. For a non-zero element a in R, a finite set $\{r_1, \cdots, r_n\} \subseteq R$ is a *right insulator of a* if the right annihilator of $\{ar_1, \cdots, ar_n\}$ is zero. A ring R is said to be a right strongly prime ring provided every non-zero ideal of R is insulated as a right ideal, and said to be a right strongly semiprime ring if every ideal I of R with $I \subseteq_e R$ as a right ideal is insulated as a right ideal. As is easily seen, a ring R is right strongly prime if and only if every non-zero element in R has a right insulator.

The notion 'insulated' coincides with 'cofaithful' in Beachy-Blair [1] and is connected with 'finite intersection property on annihilator right ideals' in Zermanowitz [14]. The class of right strongly prime rings is just that of right absolutely torsion-free rings in the sense of Rubin [11]. For details of strongly prime rings and strongly semiprime rings, the reader is refered to [1], [6], [7], [8] and [11].

DEFINITION. For an element a in a ring R, we call a finite set $\{r_1, \cdots, r_n; b\} \subseteq R$ is a right semi-insulator of a when $RaR \cap RbR = 0$ and the right annihilator of $\{ar_1, \cdots, ar_n\} \cup bR$ is zero.

Proposition 2.1. *If R is a ring such that every element in R has a right semi-insulator, then R is a semiprime right nonsingular ring.*

Proof. Let $a \in R$. Then there exists a finite set $\{r_1, \cdots, r_n; b\} \subseteq R$ satisfying $RaR \cap RbR = 0$ and $[\bigcap_{i=1}^{n} (0: ar_i)_R^r] \cap (0: bR)_R^r = 0$. If $a \in Z(R)$ and $a \neq 0$, then $ar_i \in Z(R)$ for each i and $0 \neq ar \in \bigcap_{i=1}^{n} (0: ar_i)_R^r$ for some $r \in R$. But it follows from $bRar = 0$ that $ar = 0$, a contradiction. If $aRa = 0$, then $a = 0$ because $a \in [\bigcap_{i=1}^{n} (0: ar_i)_R^r] \cap (0: bR)_R^r = 0$. Thus R is a semiprime right nonsingular ring.

Lemma 2.2. *Let R be a semiprime ring.*

(a) *If I is an ideal of R and J is a right ideal of R such that $I \cap J = 0$, then $I \cap RJ = 0$ and moreover $Q(R)IQ(R) \cap Q(R)JQ(R) = 0$.*

(b) *For ideals I and J of R, $I \subseteq_e J$ as a right ideal if and only if $I \subseteq_e J$ as a left ideal.*

(c) *If $\{I_\lambda \mid \lambda \in \Lambda\}$ is an independent family of ideals of R, then so is $\{Q(R)I_\lambda Q(R) \mid \lambda \in \Lambda\}$.*

Proof. (b) and (c) easily follow from (a).

(a). Set $Q=Q(R)$. Since $I\cap J=0$, we see $JI=0$ and it follows $(IQJ\cap R)^2=0$. Hence $IQJ=0$, from which we have $(QIQ\cap QJQ\cap R)^2=0$ and therefore $QIQ\cap QJQ=0$.

NOTE. Let I and J be ideals of a semiprime ring R. When we use '$I\subseteq_e J$' instead of '$I\subseteq_e J$ as a right ideal' or '$I\subseteq_e J$ as a left ideal', no confusion arisies by Lemma 2.2(b).

Proposition 2.3. *The following conditions are equivalent for a semiprime right nonsingular ring R:*

(a) *$Q(R)$ is a direct sum of prime rings.*

(b) *The set of all central idempotents of $Q(R)$ is a finite set.*

(c) *R contains no infinite direct sums of ideals.*

(d) *Every ideal of R is essentially cyclic generated,* i.e., *if I is an ideal of R, then there exists a in I such that $RaR\subseteq_e I$.*

Proof. Set $Q=Q(R)$. (a)$\Rightarrow$(b) is clear.

(b)$\Rightarrow$(c). Suppose that R contains an infinite independent set $\{I_\lambda\,|\,\lambda\in\Lambda\}$ of non-zero ideals. Lemma 2.2 (c) says that $\{QI_\lambda Q\,|\,\lambda\in\Lambda\}$ is independent and so is $\{E_Q(QI_\lambda Q)\,|\,\lambda\in\Lambda\}$. However, inasmuch as each $E_Q(QI_\lambda Q)$ is an ideal of Q, each $E_Q(QI_\lambda Q)$ is generated by a central idempotent in Q by [5, Corollary 1.10]. This contradicts (b).

(c)$\Rightarrow$(d). Let I be a non-zero ideal of R. For $0\neq a_1\in I$, if Ra_1R is not essential in I, we can take $0\neq a_2$ in I such that $\{Ra_1R, Ra_2R\}$ is independent by Lemma 2.2(a). Similarly when $Ra_1R\oplus Ra_2R$ is not essential in I, then there exists a_3 in I such that $\{Ra_1R, Ra_2R, Ra_3R\}$ is independent. Continuing this manner, by (c), we must reach to n such that $\{Ra_1R, \cdots, Ra_nR\}$ is independent and $Ra_1R\oplus\cdots\oplus Ra_nR\subseteq_e I$. Here we claim $R(a_1+\cdots+a_n)R\subseteq_e I$. From Lemma 2.2(c), $\{Qa_1Q, \cdots, Qa_nQ\}$ is independent. This implies $a_1Q\oplus\cdots\oplus a_nQ=(a_1+\cdots+a_n)Q$ since Q is a regular ring. Hence we see $(Ra_1R\oplus\cdots\oplus Ra_nR)Q=R(a_1+\cdots+a_n)Q$, which shows $R(a_1+\cdots+a_n)R\subseteq_e Ra_1R\oplus\cdots\oplus Ra_nR$. Therefore surely $R(a_1+\cdots+a_n)R\subseteq_e I$.

(d)$\Rightarrow$(a). It is easily seen from (d) that Q is a direct sum of indecomposable rings, say $Q=Q_1\oplus\cdots\oplus Q_n$. To show that each Q_i is prime, let X be an ideal of Q_i. Then $E_{Q_i}(X)$ is generated by a central idempotent in Q_i by again [5, Corollary 1.10]. So, $X\subseteq_e Q_i$ as a right Q_i-module from which we see that Q_i is a prime ring.

REMARK. The equivalence of (a) and (b) is due to J. Kado (see [10, Proof of Proposition 3.2]).

Lemma 2.4 ([8]). *If R is a right strongly semiprime ring, then*

(a) *R is a semiprime right nonsingular ring, and*

(b) *$Q(R)$ is a direct sum of simple rings.*

Proof. (a). Let I be an ideal of R such that $I^2=0$. Clearly $I^2=0$ implies $(0: I)^l_R \subseteq_e R$ as a right ideal. So $(0: I)^l_R$ is insulated as a right ideal. Inasmuch as $(0: I)^l_R I=0$, it follows $I=0$. Hence R is a semiprime ring. Since R is semiprime, using Lemma 2.2(a), there exists an ideal $K \subseteq R$ such that $Z(R) \oplus K \subseteq_e R$. Since $Z(R) \oplus K$ is insulated as a right ideal, there exists a finite set $\{z_1, \cdots, z_n\} \subseteq Z(R)$ and $\{k_1, \cdots, k_n\} \subseteq K$ such that $\bigcap_{i=1}^{n}(0: z_i+k_i)^r_R = (\bigcap_{i=1}^{n}(0: z_i)^r_R) \cap (\bigcap_{i=1}^{n}(0: k_i)^r_R)$. Let $a \in Z(R)$ and suppose $a \neq 0$. Then $0 \neq ar \in \bigcap_{i=1}^{n}(0: z_i)^r_R$ for some r in R. But, since each $k_j ar=0$, we infer $ar=0$, a contradiction. Thus $Z(R)=0$.

(b). Inasumuch as every non-zero essential ideal of R is insulated, clearly, R contains no infinite direct sums of non-zero ideals. Hence, by Proposition 2.3, $Q(R)$ is a direct sum of prime rings, say $Q(R)=Q_1 \oplus \cdots \oplus Q_n$. In order to show that each Q_i is simple, let X_i be a non-zero ideal of Q_i, $i=1, \cdots, n$. Since Q_i is a prime right self-injective regular ring, we see $X_i \subseteq_e Q_i$ by [5, Proposition 1.10]. As a result, $(X_1 \oplus \cdots \oplus X_n) \cap R \subseteq_e R$. So $(X_1 \oplus \cdots \oplus X_n) \cap R$ is insulated as a right ideal, whence $R \lesssim k((X_1 \oplus \cdots \oplus X_n) \cap R) \subseteq k(X_1 \oplus \cdots \oplus X_n)$ for some positive integer k. Since $X_1 \oplus \cdots \oplus X_n$ is an ideal of Q, it is Q-quasi-injective and so is by Lemma 1.1, R-quasi-injective. Therefore we see that $X_1 \oplus \cdots \oplus X_n$ is R-injective, whence $Q(R)=X_1 \oplus \cdots \oplus X_n$. Therefore $Q_i=X_i$, $i=1, \cdots, n$.

Theorem 2.5. *For a given ring, R, the following conditions are equivalent:*

(a) *R is a right strongly semiprime ring.*

(b) (1) *$Q(R)$ is a direct sum of simple rings, and*

(2) *$Q(R)eR=Q(R)eQ(R)$, or equivalently, $eQ(R)eR=eQ(R)$ for all idempotents e in $Q(R)$.*

(c) (1) *R contains no infinite direct sums of ideals,*

(2) *every element of R has a right semi-insulator.*

(d) *$Q(R)I=Q(R)$ for any essential right ideal I of R.*

(e) *There exists a ring extension S of R with the same identity satisfying $SI=S$ for any essential right ideal I of R.*

Proof. Set $Q=Q(R)$. (a)$\Rightarrow$(b). According to Lemma 2.4, Q is a direct sum of simple rings. So every ideal of Q is a direct summand. Let $e=e^2 \in Q$ and take an ideal T of Q such that $QeQ \oplus T=Q$. Since $(QeR \cap R) \oplus (T \cap R)$ is essential in R, it is insulated as a right ideal, hence there exists a positive integer k such that $R \lesssim k((QeR \cap R) \oplus (T \cap R))$ as a right R-module. Since $QeR \oplus T$ is a left ideal of Q, $QeR \oplus T$ is Q-quasi-injective and so is R-quasi-injective (Lemma 1.1). Hence Lemma 1.2 says that $QeR \oplus T$ is R-injective, whence we have $QeR=QeQ$.

(b)$\Rightarrow$(c). In order to show R to be semiprime, let $a\in R$ such that $aRa=0$. Since Q is a direct sum of simple rings, clearly it is a right nonsingular ring; whence it is a regular ring. Thus $Qa=Qe$ for some $e=e^2$ in Q. Since $QaR=QeR=QeQ=QaQ$, we see $0=QaRa=QaQa$, from which we have $a=0$. (1) now follows from Proposition 2.3. Let us write $Q=Q_1\oplus\cdots\oplus Q_n$, where each Q_i is simple, and let $1=e_1+\cdots+e_n$ in this decomposition. $\{e_1, \cdots, e_n\}$ is a set of non-zero central orthogonal idempotents. Now, to show (2), let $a\in R$. Then $QaR=QaQ=\sum_{i\in I}\oplus Q_i$ for some $I\subseteq\{1, \cdots, n\}$. Without loss of generality, we can assume $I=\{1, \cdots, s\}$. Let us express $e_1+\cdots+e_s$ in QeR as $e_1+\cdots+e_s=\sum_{i=1}^{t} q_i ar_i$, where $q_i\in Q$ and $r_i\in R$. We can take r in R satisfying $0\neq e_m r\in R$, $m=s+1,\cdots,n$. Put $b=r(e_{s+1}+\cdots+e_n)$. Here we claim that $\{r_1,\cdots,r_t; b\}$ is right semi-insulator for a. $RaR\cap RbR=0$ is obvious. If x is in $[\bigcap_{i=1}^{t}(0: ar_i)_R^r]\cap(0: bR)_R^r$, then $(e_1+\cdots+e_s)x=0$. Further, inasmuch as $Qe_m rQ=Q_m$ for $m=s+1, \cdots, n$, we infer $QbR=Q_{s+1}\oplus\cdots\oplus Q_n$; whence $(e_{s+1}+\cdots+e_n)x=0$. Therefore $x=0$ as required.

(c)$\Rightarrow$(a). Proposition 2.1 says that R is a semiprime right nonsingular ring. If I is an essential ideal of R, then there exists a in I such that $RaR\subseteq_e I$ $(\subseteq_e R)$ by Proposition 2.3. Let $\{r_1, \cdots, r_n; b\}$ be a right semi-insulator of a. Since $RaR\subseteq_e R$ and $RaR\cap RbR=0$, we see $b=0$. Consequently $\bigcap_{i=1}^{n}(0: ar_i)=0$. Therefore I is insulated as a right ideal.

(b)$\Rightarrow$(d). If I is an essential right ideal of R, then $QI\subseteq_e Q$ as a right R-module. As is seen in the proof of (b)$\Rightarrow$(c), it follows from (1) that Q is regular. Therefore (2) easily implies $QI=QIQ$. As a result $QI=QIQ<\oplus Q$ and hence $QI=Q$.

(d)$\Rightarrow$(e)$\Rightarrow$(a) is obvious.

Corollary 2.6. *A ring R is a right strongly prime ring if and only if $Q(R)$ is simple and $Q(R)eR=Q(R)eQ(R)$ for all idempotents e in $Q(R)$.*

Corollary 2.7. *The following conditions are equivalent for a given ring R.*

(a) *R is a semiprime right Goldie ring.*

(b) *R is a right finite dimensional right strongly semiprime ring.*

(c) *$IQ(R)=Q(R)I=Q(R)$ for every essential right ideal I of R.*

Proof. (a)$\Rightarrow$(b). Since every essential ideal of R contains a regular element, clearly R is a right strongly semiprime ring.

(b)$\Rightarrow$(a) follows from Lemma 2.4, and (b)$\Leftrightarrow$(c) follows from Theorem 2.5 and [12, Theorem 1.6].

Corollary 2.8 ([8, Corollary 16]). *A regular right strongly semiprime ring R*

is a direct sum of simple rings. Therefore R is also a left strongly semiprime ring.

Proof. Inasmuch as R contains no infinite direct sums of ideals, it is sufficient to show that R contains no proper essential ideals. Let I be an essential ideal of R. Then $QI=Q$ by Theorem 2.5, whence it follows from regularity of R that $1=qe$ for some $q\in Q$ and $e=e^2\in I$. Then, clearly, $1=e$. So $I=R$.

3. Nonsingular quasi-injective modules

Lemma 3.1 ([1]). *If R is a right strongly prime ring, then every nonsingular quasi-injective right R-module is injective.*

Proof. Let M $(\neq 0)$ be a nonsingular quasi-injective right R-module and let $0\neq x\in M$. Since $xQ(R)$ is $Q(R)$-projective there exists e in $Q(R)$ and an isomorphism ψ: $xQ(R)\approx eQ(R)$ with $\psi(x)=e$. We can take r in R such that $0\neq er\in R$. Then $R\subsetneq n(erR)$ for some positive integer n, since er has a right insulator. Inasmuch as $R\subsetneq n(erR)\approx n(xrR)\subseteq nM$, M is injective by Lemma 1.2.

Lemma 3.2. *Let R be a right self-injective regular ring such that every nonsingular quasi-injective right R-module is injective. Then R is a direct sum of simple rings.*

Proof. According as every ideal of R is a nonsingular quasi-injective right R-module, every ideal of R is a direct summand. Hence R contains no infinite direct sums of ideals. Hence by Proposition 2.3, R is written as a direct sum of prime rings, say $R=R_1\oplus\cdots\oplus R_n$. Since R_i is prime and every ideal of R_i is a direct summand, R_i must be simple, $i=1,\cdots,n$.

Proposition 3.3. *If R is a right nonsingular ring, then the following conditions are equivalent:*

(a) *$Q(R)$ is a direct sum of simple rings.*

(b) *$E_R(M)=MQ(R)$ for all nonsingular quasi-injective right R-module M.*

Proof. Set $Q=Q(R)$. (a)$\Rightarrow$(b). If M is a nonsingular quasi-injective right R-module, then MQ is nonsingular Q-quasi-injective. Hence, by Lemma 3.1, MQ is Q-injective; whence MQ is R-injective.

(b)$\Rightarrow$(a). If M is a nonsingular quasi-injective right Q-module, then M is nonsingular R-quasi-injective (Lemma 1.1). Hence $M=MQ=E_R(M)=E_Q(M)$, which shows that M is Q-injective. Thus, by Lemma 3.2, we conclude that Q is a direct sum of simple rings.

We are now in a proposition to show our main theorem.

Theorem 3.4. *For a given ring R, the following conditions are equivalent:*

(a) *$R/G(R)$ is a right strongly semiprime ring.*

(b) *Every nonsingular quasi-injective right R-module is injective.*

(c) *Any finite direct sum of nonsingular quasi-injective right R-module is also quasi-injective.*

(d) *Any direct product of nonsingular quasi-injective right R-module is quasi-injective.*

Proof. Set $\bar{R}=R/G(R)$ and $Q=Q(R/G(R))$. (b)$\Rightarrow$(d)$\Rightarrow$(c): Obvious.

(a)$\Rightarrow$(b). Since $\bar{R}$ is a right strongly semiprime ring, Theorem 2.5 says that Q is a direct sum of simple rings and $eQe\bar{R}=eQ$ for all idempotents e in Q. Now, let $M\,(\neq 0)$ be a nonsingular quasi-injective right R-module. In order to show M is injective, we may show $M=MQ$ by Proposition 3.3. Let $0\neq x\in M$. Since xQ is Q-projective, there exists an idempotent e in Q and an isomorphism $\psi: xQ\approx eQ$ with $\psi(x)=e$. Inasmuch as xQ is Q-injective, $E_R(M)=xQ\oplus Y$ for some submodule Y. Since M is quasi-injective, this yields $M=(xQ\cap M)\oplus(Y\cap M)$. As a result, $xQ\cap M$ is quasi-injective. Put $Z=\psi(xQ\cap M)$. Inasumuch as $xR\subseteq_e xQ\cap M\subseteq_e xQ$, we infer that $E_R(xQ\cap M)=xQ$; whence $E_R(Z)=eQ$. Observing $eQ=eQeR=End_Q(eQ)eR=End_R(eQ)eR\subseteq End_R(eQ)Z=Z$, we see $eQ=Z=\psi(xQ\cap M)$. Consequently $xQ=xQ\cap M$ and it follows $xQ\subseteq M$. Therefore $MQ=M$ as desired.

(c)$\Rightarrow$(a). In view of Theorem 2.5, it is enough to show that $eQeR=eQ$ for all idempotents e in Q and Q is a direct sum of simple rings.

Let $e=e^2\in Q$ and set $T=eQeR\oplus(1-e)Q(1-e)R$. Then T is a nonsingular quasi-injective right R-module because both $eQeR$ and $(1-e)Q(1-e)R$ are so. Since $R\subsetneq T$, it follows that T is injective; whence so is $eQeR$. Thus we get $eQeR=eQeQ=eQ$. Now, assume that Q can not be expressed as a direct sum of prime rings. Then, by Proposition 2.3, we see that there exist infinite orthogonal non-zero central idempotents $\{e_i\,|\,i=1,2,\cdots\}$ in Q. Since $\sum_{i=1}^{\infty} e_iQ$ is nonsingular Q-quasi-injective, it is also nonsingular R-quasi-injective (Lemma 1.1). Putting $T=(1-e_1)Q\times(\sum_{i=1}^{\infty} e_iQ)$, T is then a nonsingular quasi-injective right R-module, since both $(1-e_1)Q$ and $\sum_{i=1}^{\infty} e_iQ$ are so. As a result, it follows from $R\subsetneq T$ that T is injective and $\sum_{i=1}^{\infty} e_iQ\mathrel{<\!\!\!\sim}\oplus Q$, a contradiction. Hence Q must be written as a direct sum of prime rings, say $Q=Q_1\oplus\cdots\oplus Q_n$. Let X be a non-zero ideal of Q_i. Then X is a nonsingular quasi-injective right Q-module and hence it is nonsingular R-quasi-injective by Lemma 1.1. Take a non-zero idempotent e in X and consider $X\times(1-e)Q$. Since both X and $(1-e)Q$ are nonsingular quasi-injective right R-module, so is $X\times(1-e)Q$. Inasmuch as $R\subsetneq X\times(1-e)Q$, it follows that $X\times(1-e)Q$ is injective; whence $X\mathrel{<\!\!\!\sim}\oplus Q_i$. Since Q_i is a prime ring, this shows $X=Q_i$. Accordingly each Q_i is simple.

Combining Theorem 3.4 with Corollary 2.8, we have

Corollary 3.5. *If R is a regular ring, then the following conditions are equivalent:*

(a) *R is a direct sum of simple rings.*

(b) *Every nonsingular quasi-injective right R-module is injective.*

(b′) *Every nonsingular quasi-injective left R-module is injective.*

Corollary 3.6. *If R is a right strongly semiprime ring, then its right socle is a direct summand of R as a ring.*

Proof. By Theorem 3.4(b), we conclude that the right socle S of R is a direct summand of R as a right R-module. Since R is a semiprime ring and S is a two-sided ideal of R, it follows that S is a direct summand of R as a ring.

Boyle and Goodearl [3] showed that every nonsingular quasi-injective right module over a semiprime right Goldie ring is injective. However, according as every essential ideal of a semiprime right Goldie ring R has a regular element, R is a right and left strongly semiprime ring. Hence Theorem 3.4 guarantees the following result.

Corollary 3.7. *If R is a semiprime right Goldie ring, then every nonsingular quasi-injective right R-module is injective and, at the same time, every nonsingular quasi-injective left R-module is also injective.*

Finally we show the following result.

Theorem 3.8. *For a given ring R, the following conditions are equivalent:*

(a) *$R/G(R)$ is a semiprime right Goldie ring.*

(b) *Any direct sum of nonsingular quasi-injective right R-modules is quasi-injective.*

Proof. As is well known ([13]), the following conditions are equivalent:

(1) *$Q(R/G(R))$ is a semisimple artinian ring.*

(2) *$R/G(R)$ is right finite dimensional.*

(3) Any direct sum of nonsingular injective right R-modules is injective.

Convining this fact with Theorem 3.4 and Corollary 3.7, the proof is established.

REMARK. It seems to be also meaningful to study those rings whose nonsingular quasi-injective right modules are written as direct sums of indecomposable modules. Such rings were determined by Berry [2] as rings R such that $R/G(R)$ is right finite dimensional.

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