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EXISTENCE OF GREEN FUNCTION
AND BOUNDED HARMONIC FUNCTIONS
ON GALOIS COVERS OF RIEMANNIAN MANIFOLDS

A. MUHAMMED ULUDAĞ

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1. Existence of Green functions on transient covers

Let \( p : X \rightarrow M \) be a Galois covering of an orientable Riemannian manifold \( M \), with \( T \) as the deck transformation group. In the case where \( M \) is compact, it is known since Royden [14] that \( X \) and \( T \) share same properties. Namely, the canonical Brownian motion on \( X \) induces a Brownian motion on \( T \) and vice versa. Moreover, if one of them is transient, then so is the other one. It is also well known that the transience of the canonical Brownian motion on \( X \) is equivalent to the existence of Green functions on \( X \) (See [1] and references therein). However, much less is known in the case where the base \( M \) is a non-compact manifold. In this paper, we prove the following result in this direction:

**Theorem 1.1.** Let \( p : X \rightarrow M \) be a Galois covering of a Riemannian manifold \( M \) whose deck transformation group \( T \) is an extension of a finitely generated transient group. Then \( X \) is transient also, i.e., \( X \) carries Green functions.

**Question.** Is Theorem 1.1 valid for an arbitrary transient group \( T \)?

Before passing to the proof, let us note that if \( M \) is compact, then \( T \) is a finitely generated group. For such groups, there is a nice characterization of transience given by Guivarch [4], Lyons-Sullivan [8] and Varopoulos [17]: A finitely generated group is recurrent (i.e. not transient) if and only if it is a finite extension of one of the groups \{0\}, \( \mathbb{Z} \), or \( \mathbb{Z}^2 \).

In what follows, \( X \in P_G \) means that \( X \) has Green functions (equivalently, \( X \) is transient) and \( X \in O_G \) means that \( X \) do not carry Green functions (equivalently, \( X \) is recurrent).

Theorem 1.1 is based on the following generalization of a theorem of Kusunoki-Mori [5], which was originaly stated for Riemann surfaces (See also [15] III.1.2G). The proof for Riemannian manifolds given below is a mere repetition of their proof, which involves the concept of Royden compactification. Existence of the Royden com-
pactification for Riemannian manifolds and all the related results that we use in the proof have been established by Glasner-Katz [3].

**Theorem 1.2.** Let $X$ be a Riemannian manifold of class $O_G$, and let $Y$ be an open submanifold of $X$ with a smooth boundary $\partial Y$. Then the double $dbl$ $Y$ of $Y$ along $\partial Y$ is also of class $O_G$.

Recall that the double $dbl$ $Y$ of $Y$ is obtained by gluing together two copies of $Y$ along their boundaries, metric structure on $dbl$ $Y$ being the natural one.

**Notation.** For a non-compact orientable Riemannian manifold $X$, a continuous function $f : X \to \mathbb{R}$ with locally integrable first partial derivatives is called a Tonelli function. The set $\mathcal{M}(X)$ of all bounded Tonelli functions $f$ on $X$ with finite Dirichlet integrals $D(f)$ is an algebra, the Royden algebra of $X$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a bounded sequence of functions in $\mathcal{M}(X)$ converging uniformly to $f$ on compact subsets of $X$. We say that $f = BD$-$\lim f_n$ if $\lim D(f - f_n) = 0$. The Royden algebra is complete under this topology. There is a unique compact Hausdorff space $X^*$, which is called the Royden compactification of $X$, satisfying (i) $X$ is dense in $X^*$, (ii) Every $f \in \mathcal{M}(X)$ extends continuously to $X^*$, (iii) $\mathcal{M}(X)$ separates the points of $X^*$. The Royden boundary is the set $\nabla := X^* \setminus X$. The ideal of $\mathcal{M}(X)$ consisting of functions with compact support is denoted by $\mathcal{M}_0(X)$, and the closure of $\mathcal{M}_0(X)$ in the $BD$-topology is denoted by $\mathcal{M}_\Delta(X)$. The subset of $\nabla$ consisting of points $p$ with the property that $f(p) = 0$ for all $f \in \mathcal{M}_\Delta(X)$ is called the harmonic boundary of $X$, and denoted by $\Delta$. Existence of Green functions on $X$ implies that the harmonic boundary is nonempty, in fact, $X \in P_G \iff \Delta \neq \emptyset$.

Proof of Theorem 1.2. Let $X^*$ be the Royden compactification of $X$, $\Delta$ be its harmonic boundary, and let $\overline{Y}$ be the closure of $Y$ in $X^*$. Since $X \in O_G$, we know that $\Delta = \emptyset$. Hence, for all $y \in \overline{Y}$ one can find an $f_y \geq 0$ in $\mathcal{M}_\Delta(X)$ with $f_y(y) > 1$. The set $\overline{Y}$ being compact in $X^*$, one can choose a finite number of points $y_1, y_2, \ldots, y_n$ such that

$$\overline{Y} \subset \bigcup_{i=1}^n \{ y : y \in X^*, f_{y_i}(y) > 1 \}.$$

Thus, the function $f = \sum_{i=1}^n f_{y_i} \in \mathcal{M}_\Delta(X)$ satisfies $f > 1$ on $\overline{Y}$. Let $\{g_i\}_{i \in \mathbb{N}}$ be a sequence of functions in $\mathcal{M}_0(\mathbb{R})$ such that $f = BD$-$\lim g_i$. Put $\hat{f}, \hat{g}_i$ for the symmetric extensions of $f|Y$, $g_i|Y$ to $dbl Y$. Then one has $\hat{f} \in \mathcal{M}(dbl Y)$, $\hat{g}_i \in \mathcal{M}_0(dbl Y)$, and $\hat{f} = BD$-$\lim \hat{g}_i$. This implies that $\hat{f} \in \mathcal{M}_\Delta(dbl Y)$. Since $f > 1$ on $dbl Y$, $1 = (1/\hat{f}) \hat{f} \in \mathcal{M}_\Delta(dbl Y)$, which shows that the harmonic boundary of $dbl Y$ is empty, i.e., $dbl Y \in O_G$. \qed
Proof of Theorem 1.1. The proof will be achieved in two steps. First assume that \( T \) is finitely generated. Set \( H := p_*\pi_1(X) \triangleleft \pi_1(M) \). As \( T \) is finitely generated, one can choose a finite number of loops \( \gamma_1, \ldots, \gamma_n \) in \( M \) such that \([\gamma_1], \ldots, [\gamma_n] \in \pi_1(M)\) generate \( T = \pi_1(M)/H \) in the quotient. Let \( N \) be a relatively compact, connected open submanifold of \( M \) with smooth boundary \( \partial N \) such that \( N \) contains the loops \( \gamma_1, \ldots, \gamma_n \). Then the open submanifold \( Y := p^{-1}(N) \) of \( X \) is connected, and its boundary \( \partial Y = p^{-1}(\partial N) \) is smooth. Consider the manifold \( \text{dbl}Y \). It is easy to see that the action of \( T \) on \( Y \) passes to \( \text{dbl}Y \). Thus, \( \text{dbl}Y \) is a Galois covering of the double \( \text{dbl}N \) of \( N \), with \( T \) as the deck transformation group. The manifold \( \text{dbl}N \) being compact, \( \text{dbl}Y \) is of class \( P_G \), and Theorem 1.2 implies that \( X \) is of class \( P_G \).

Now assume that \( T \) is an extension of a finitely generated transient group, i.e., assume that there is an exact sequence \( 0 \to H \to T \to T' \to 0 \) where \( T' \) is finitely generated and transient. Consider the intermediate covering \( X/H \to M \). Since this covering has \( T' \) as the deck transformation group, \( X/H \) has Green functions by the first part of the proof. Since \( X \to X/H \) is a covering, \( X \) is of class \( P_G \), too. This completes the proof.

**Remark.** To illustrate the “doubling” procedure described above, let \( \exp : X := \mathbb{C} \to M := \mathbb{C}\setminus\{0\} \) be the usual covering. One can choose \( N \) to be the annulus \( \{ z : 1 < |z| < e \} \), so that \( Y \) is the strip \( \{ z : 0 < \text{Re} z < 1 \} \). After the doubling, one obtains a covering of the torus \( \text{dbl}N \) by the cylinder \( \text{dbl}Y \).

**Some Corollaries of Theorem 1.1**

1. **On commutator subgroups of Fuchsian groups.** According to a theorem of Myrberg [11], if a Riemann surface \( X \) is covered by the unit disc \( \Delta \), and \( G \) is the corresponding Fuchsian group acting on \( \Delta \), then \( X \in P_G \) if and only if \( G \) is of convergence type; that is,

\[
\sum_{g \in G} (1 - |g(z)|) < \infty
\]

for one, and hence for all \( z \in \Delta \) (see also [16], X.13). The following statement is an immediate corollary of Myrberg’s characterization and Theorem 1.1.

**Corollary 1.3.** Let \( X \) be a Riemann surface covered by the unit disc \( \Delta \), and let \( \pi_1(X) = G \subset \text{Aut}(\Delta) \) be its covering group. A normal subgroup \( H \triangleleft G \) is of convergence type if the quotient group \( G/H \) is an extension of a finitely generated transitive group.

Now we shall consider the particular case of abelian coverings; that is Galois coverings whose deck transformation groups are abelian. Rank of an abelian covering is defined to be the rank of its deck transformation group. Theorem 1.1 and the Varopou-
los’ characterization of finitely generated transitive groups imply that for $3 \leq r \leq \infty$, an abelian covering of rank $r$ of a Riemannian manifold is of class $P_G$. This latter assertion is a generalization of a theorem proved in 1953 by Mori [10] in the case where the base is a compact Riemann surface.

It has been shown by McKean-Sullivan and Lyons-McKean [9], [7] that the maximal abelian (hence, $\mathbb{Z}^2$-) covering of $\mathbb{C}\setminus\{0,1\}$ is of class $P_G$. Hence, we have a complete list of Riemann surfaces which do not have an abelian cover of class $P_G$: Since the deck transformation group of the maximal abelian cover is the abelianization of the fundamental group, the genus of such a surface should be $\leq 1$ and it cannot have too many punctures. Namely, these are the sphere $S^2 = \mathbb{P}_1^\mathbb{C}$, the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}\setminus\{0\}$, the tori $\mathbb{T}$, and the punctured tori $\mathbb{T}\setminus\{q\}$. The only non-trivial case is that of a punctured torus, so we describe its maximal abelian cover. If $p : \mathbb{C} \to \mathbb{T}$ is the universal covering of $\mathbb{T}$, then $\mathbb{C}\setminus p^{-1}(q)$ is the maximal abelian cover of $\mathbb{T}\setminus\{q\}$, which is easily seen to be not of class $P_G$. Also, note that $\mathbb{T}\setminus\{q\}$ is the only surface in the above list which is covered by the unit disc $\Delta$. So, we have the following consequence of Theorem 1.1:

**Corollary 1.4.** Let a Riemann surface $X$ be covered by the unit disc $\Delta$, and let $\pi_1(X) = G \subset Aut(\Delta)$ be its covering group. Then

(i) If $X \neq \mathbb{T}\setminus\{q\}$, then the commutator subgroup $[G,G]$ is of convergence type.

(ii) If $H$ is a subgroup of $G$ such that $[G,G] \subset H \subset G$, and the rank of the abelian group $G/\{q\}$, which is easily seen to be not of class $P_G$. Also, note that $\mathbb{T}\setminus\{q\}$ is the only surface in the above list which is covered by the unit disc $\Delta$. So, we have the following consequence of Theorem 1.1:

**2. Carathéodory hyperbolicity of metabelian covers.** A Riemann surface $X$ is called *Carathéodory hyperbolic* if bounded holomorphic functions separate the points of $X$. It is interesting to know when Carathéodory hyperbolic surfaces appear as “small” covers of Riemann surfaces. In [6] Lin and Zaidenberg shows that if a Riemann surface $R$ has an abelian cover $Y$ of class $P_G$, then $Y$ has a Carathéodory hyperbolic, abelian cover $X$, such that $X$ is a metabelian (i.e. two-step solvable) Galois covering of $R$. Hence, Theorem 1.1 implies the following corollary.

**Corollary 1.5.** If $R$ is not one of the surfaces $S^2$, $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, $\mathbb{T}$, $\mathbb{T}\setminus\{q\}$, then it admits a metabelian, Carathéodory hyperbolic Galois covering $X \to R$.

The converse of this corollary is also true, for it is obvious that the surfaces $S^2$, $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, $\mathbb{T}$ do not possess any cover carrying bounded analytic functions. For the surface $\mathbb{T}\setminus\{q\}$, recall that its maximal abelian cover is of class $O_G$. One of the results in [8] asserts that an abelian cover of an $O_G$-manifold is of class $O_{HB}$, that is it has no bounded non-constant harmonic functions (sometimes such a surface is also called a Liouville surface). This shows that a metabelian cover of $\mathbb{T}\setminus\{q\}$ is of class $O_{HB}$, so in particular it has no bounded analytic functions, and it cannot be Carathéodory.
hyperbolic.

2. Existence of bounded harmonic functions on finite covers of $P_G$-surfaces

An unwritten rule in the classification theory of Riemann surfaces states that “passage to covers produces more and more functions”. In this section we consider the following question: While passing to covers, exactly when the bounded harmonic functions appear?

Let $R$ be a Riemann surface of class $O_G$, and let $X$ be a rank $\geq 3$ abelian cover of $R$. Then, as we have noticed above, $X \in O_{HB}$ by a result in [8]. On the other hand, Theorem 1.1 implies that $X \in P_G$. So there are many Riemann surfaces $X \in P_G \cap O_{HB}$. Let us denote by $Z$ the maximal abelian (hence, $\mathbb{Z}^\infty$-) cover of $X$. Lyons and Sullivan [8] observed that $Z$ is of class, that is, it does carry a non-constant bounded harmonic function. The theorem below states that $X$ has a finite Galois cover $Y$ which is of class $P_{HB}$. However, by an argument due to V. Lin, $Y$ does not carry any non-constant bounded analytic functions (see Remark 3 below).

**Theorem 2.1.** Any Riemann surface $X$ of class $P_G$ has a finite cover $Y$ which is of class $P_{HB}$ and, moreover, $Y$ carries a Dirichlet finite bounded harmonic function.

**Proof of Theorem 2.1.** A $P_G$-surface of genus $g = 0$ already carries a Dirichlet finite non-constant bounded harmonic function (see [15], III.5G). Hence, setting $X = Y$ we are done. If $g \neq 0$ then there exists a closed analytic curve $\gamma$ on $X$ which does not divide the surface. We denote two sides of $\gamma$ by $\gamma^+$, $\gamma^-$, and we cut $X$ along $\gamma$. Let $\tilde{X}$ be a second copy of $X$, $\tilde{\gamma}$ be the copy of $\gamma$ in $\tilde{X}$ with corresponding sides $\tilde{\gamma}^+$, $\tilde{\gamma}^-$. Gluing $X$ to $\tilde{X}$ via natural identifications $\gamma^+ \mapsto \tilde{\gamma}^-$, $\gamma^- \mapsto \tilde{\gamma}^+$, we obtain a $\mathbb{Z}_2$-covering $Y$ of $X$.

**Claim.** The surface $Y$ is of class $P_{HB}$.

An immediate way to see this is to observe that the harmonic boundary of $Y$ consists of two points, which implies the existence of a Dirichlet finite non-constant bounded harmonic function on $Y$ (see [13], or [15], III.3F). However, we shall give a more elementary proof based on the following theorem:

**Theorem 2.2** (Bader-Parreau [2], Nevanlinna [12]). Let $Y_1$, $Y_2$ be two disjoint subsurfaces of $Y$ with analytic boundaries $\partial Y_1$, $\partial Y_2$. Assume that there exists two non-constant bounded harmonic functions $u_1$ on $Y_1$ and $u_2$ on $Y_2$ such that $u_1 \equiv 0$ on $\partial Y_1$ and $u_2 \equiv 0$ on $\partial Y_2$. Then $Y$ carries a non-constant bounded harmonic function. Moreover, if $u_1$, $u_2$ have finite Dirichlet integrals, then $Y$ carries a non-constant bounded harmonic function with finite Dirichlet integral.
Proof of Claim. The surface \( X \) being of class \( P_G \), the harmonic measure \( \omega \) of the ideal boundary of \( X \) with respect to \( \gamma \) is non-vanishing, that is, \( \omega \) is a bounded non-constant harmonic function with finite Dirichlet integral on \( X \setminus \gamma \) which vanishes on \( \gamma \) (See [16] X.1). Let \( \tilde{\omega} \) be the same function on \( \tilde{X} \setminus \tilde{\gamma} \). Setting \( Y_1 := X \setminus \gamma, Y_2 := \tilde{X} \setminus \tilde{\gamma}, u_1 := \omega, u_2 := \tilde{\omega} \), the hypotheses of Theorem 2.1 are satisfied, hence \( Y \) carries a bounded non-constant harmonic function with finite Dirichlet integral.

\[ \square \]

Remarks. 1. A sufficient condition for a Galois cover \( Y \) of a \( P_G \)-surface \( X \) to be of class \( P_{HB} \) is the compactness of the boundary of a fundamental region of the corresponding group action on \( Y \); this can be proved in the same way as in the proof above. Looking at the Royden boundary shows that this latter condition is valid for Riemannian manifolds, too.

2. It should be observed that a finite (even infinite) cover of a \( P_G \)-Riemann surface can be of class \( O_{HB} \). For example, if \( X \) is a rank-3 abelian cover of a compact Riemann surface \( K \) of genus \( g = 2 \), and \( Y \) is the maximal abelian (i.e. rank-4) cover of \( K \), then \( X \in P_G \) by Theorem 1.1. On the other hand, an abelian cover of a compact surface is of class \( O_{HB} \) by a theorem of Lyons-Sullivan [8], so \( Y \in O_{HB} \), but \( Y \) is a \( \mathbb{Z} \)-cover of \( X \).

3. In contrast with the possible existence of bounded non-constant harmonic functions as stated in Theorem 2.1, a finite cover \( Y \) of an \( O_{HB} \) surface \( X \) cannot carry non-constant bounded analytic functions. The proof goes as follows\(^1\): Let \( n \) be the degree of a finite covering \( p : Y \to X \), and let \( f \) be a bounded analytic function on \( Y \). Let \( x \in X \) and \( p^{-1}(x) = \{y_1, \ldots, y_n\} \). For \( j = 1, \ldots, n \) define \( a_j(x) := \sigma_j(f(y_1), \ldots, f(y_n)) \), where \( \sigma_j \) is the elementary symmetric polynomial of degree \( j \) in \( n \) variables. Then each \( a_j \) is a bounded analytic function on \( X \). The real parts of the \( a_j \)'s, being bounded harmonic functions, are constant, hence \( a_j = \text{const} \), which implies that \( f = \text{const} \).

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References


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