Morinaka, H.
Osaka J. Math.
30 (1993), 331–348

ON THE REPRESENTATION OF POTENTIALS BY A GREEN FUNCTION AND THE PROPORTIONALITY AXIOM ON P-HARMONIC SPACES

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(Received January 23, 1992)
(Revised September 2, 1992)

Introduction

Let $(X, \mathcal{U})$ be a P-harmonic space in the sense of Constantinescu-Cornea [5] with a countable base. If we assume the Doob convegence property, the proportionality axiom and the condition (A) (see [7]) on $(X, \mathcal{U})$, then $(X, \mathcal{U})$ has a Green function and every potential on X is represented by the Green function with a Radon measure on X [7]. In this paper we consider the following problem: if we assume the existence of a Green function $k(x, y)$ on $(X, \mathcal{U})$, what conditions derive the representation of all potentials by the function $k(x, y)$ with a Radon measure on X.

In 1979, A. Boukricha [3] proved that if any finite continuous potential with compact harmonic support is represented by the function $k(x, y)$ then the proportionality axiom is satisfied. Furthermore, U. Schirmeier [9] proved that if at least one bounded continuous strict potential is represented by $k(x, y)$ then the proportionality axiom is satisfied. In 1982, for a Brelot space H. Ben Saad [1] proved that if the function $k(x, y)$ is symmetric i.e. $k(x, y)=k(y, x)$ or if there exists an adapted cone P' of continuous potentials which are represented by $k(x, y)$ such that $P'-P'$ is uniformly dense in $C(K)$ for an arbitrary compact subset $K\subset X$ then every potential on X can be represented by the function $k(x, y)$ and also the proportionality axiom is satisfied.

We shall show, first of all in section 2 that for a general P-harmonic space with a countable base which admits a Green function $k(x, y)$, if the convex cone P_1 of all continuous potentials which are represented by $k(x, y)$ is adapted, inf-stable and separates points of X, we obtain the same conclusions as those of H. Ben Saad. In section 3, we shall show that if there exists a second P-harmonic space $(X, \mathcal{U}^*)$ which has the Green function $k^*(x, y)=k(y, x)$ then the convex cone P_1 possesses the above properties. In section 4, we show that the assumptions of A. Boukricha and U. Schirmeier for concluding the proportionality axiom coincide with the assumption that the convex cone P_1 possesses the above three properties.

I would like to express profound thanks to Prof. Ikegami for proposing these problems and pointing out some mistakes in the original version; further I thank him especially for proving Proposition 4.2 and for very helpful discussions. Let me also express my gratitude to the refree for valuable instructions and recomposing the contents.

1. Preliminaries

Let $(X, \mathcal{U})$ be a P-harmonic space (a harmonic space will be called *P-harmonic space* if for any point $x \in X$, there exists a potential p such that $p(x)>0$) in the sense of Constantinescu-Cornea with a countable base, where $\mathcal{U}$ denotes the sheaf of cone of hyperharmonic functions.

Throughout this article we assume that constant 1 is superharmonic and $(X, \mathcal{U})$ admits a Green function $k(x, y)$ which possesses the following properties:

1) $(x, y) \mapsto k(x, y)$: $X \times X \to \overline{\boldsymbol{R}_+}$ is lower semicontonuous and finite continuous outside the diagonal,
2) k_y: $x \mapsto k(x, y)$ is a potential such that $S(k_y) = \{y\}$ for every $y \in X$ where $S(k_y)$ is the harmonic support of the function k_y.

In this situation, we can easily show the next lemma by using the fact that X has no isolated point ([5], p.31).

Lemma 1.1. *For any point $y \in X$, there is at least one point $x \in X$ such that $x \neq y$ and $k(x, y) > 0$.*

We list some basic notations which are used in the text.

$\boldsymbol{N}$: the set of natural numbers

$\boldsymbol{R}_+$: the set of positive real numbers

$\overline{\boldsymbol{R}_+}$: $= \boldsymbol{R}_+ \cup \{+\infty\}$

$B(X)$: the set of all Borel numerical functions on X

$C(X)$: the space of all continuous real functions on X

S: the set of all superhamronic functions on X

P: the set of all potentials on X

A_b, A^+, A_K: the set of all functions in A which are bounded which are positive, which have compact support, respectively

1_A: the characteristic function of a set A

$M^+(X)$: the set of all positive Radon measures on X

$M_K^+(X)$: the set of all positive Radon measures with compact supports on X

$f_n \nearrow f$ (resp. $f_n \searrow f$) means f_n converges pontwise increasingly (resp. decreasingly) to f

$(f \vee g)(x)$: $= \sup(f(x), g(x))$

$(f \wedge g)(x)$: $= \inf(f(x), g(x))$

$\mu\wedge\nu\,(\mu, \nu\in M^+(X))$: the infimum of μ and ν with respect to the order relation $\mu\leq\nu$ ($\mu\leq\nu$ means $\mu(f)\leq\nu(f)$ for any $f\in C_K^+(X)$)

$\mu|_A$: the restriction of a measure μ on A

ε_x: the unit mass (Dirac measure) at $x\in X$

∂A: the boundary of A

2. The convex cone P_1

Let P_1 be the convex cone defined by

$$P_1 := \{k\mu\in P\cap C(X);\ \mu\in M^+(X)\}\ ,$$

where

$$k\mu(x) := \int k(x, y)\, d\mu(y), \quad x\in X\,.$$

Our purpose in this section is to prove that if P_1 possesses the following properties:

i) P_1 is adapted, i.e. there exists $p_0\in P_1$ such that $p_0>0$ and for any $p\in P_1$, there exists $q\in P_1$ such that $p=o(q)$,

ii) P_1 is inf-stable, i.e. $p_1, p_2\in P_1$ implies $p_1\wedge p_2\in P_1$,

iii) P_1 separates points of X, i.e. for every pair of points $x_1, x_2\in X$, $x_1\neq x_2$; there exist $p_1, p_2\in P_1$ such that

$$p_1(x_1)\,p_2(x_2)\neq p_1(x_2)\,p_2(x_1)$$

then every potential on $(X, \mathcal{U})$ can be represented by the Green function $k(x, y)$ with a unique positive Radon measure on X and $(X, \mathcal{U})$ satisfies the proportionality axiom (P):

(P): two potentials with the same one point harmonic support are proportional.

We begin with some fundamental lemmas.

Lemma 2.1. *If $k\mu\in P$, $\mu\in M^+(X)$, then*

$$S(k\mu) = \operatorname{Supp}\mu\,.$$

Proof. To prove $S(k\mu)\subset\operatorname{Supp}\mu$, we may assume $\operatorname{Supp}\mu\neq X$. Let V be a relatively compact open set such that $\bar V\subset X\backslash\operatorname{Supp}\mu$. For each $x\in V$,

$$\begin{aligned} H^V_{k\mu}(x) &= \int k\mu(z)\, d\mu^V_x(z) = \int\Big[\int k_y(z)\, d\mu(y)\Big]\, d\mu^V_x(z) \\ &= \int\Big[\int k_y(z)\, d\mu^V_x(z)\Big]\, d\mu(y)\,, \end{aligned}$$

where μ^V_x is the harmonic measure on V at x and $H^V_{k\mu}$ is the solution of the (generalized) Dirichlet problem of $k\mu$ on V. Since $y\in\operatorname{Supp}\mu$ k_y is a positive harmonic function on $X\backslash\text{Supp}\ \mu$. Hence

$$H^V_{k\mu} = k\mu \text{ on } V$$

and this means $k\mu$ is harmonic on $X\backslash\text{Supp}\ \mu$, that is $S(k\mu)\subset\text{Supp}\ \mu$. Assume now that $S(k\mu)\neq\text{Supp}\ \mu$. Then there is a point y in $\text{Supp}\ \mu\backslash S(k\mu)$. Let $f\in C^+_K(X), f\leq 1, f(y)=1$ and $f=0$ on $S(k\mu)$. Then

$$S(k(f\mu))\subset\text{Supp} f\cap S(k\mu) = \emptyset$$

by the above result and $k(f\mu)<k\mu$. Hence

$$k(f\mu) = 0\,.$$

On the other hand, by Lemma 1.1, for each point $y\in\text{Supp}\ \mu$, there is at least one point $x\in X$, $x\neq y$ such that $k(x, y)>0$. Since $k(x, y)$ is continuous outside the diagonal, there is a neighbourhood U_y of y such that $x\in U_y$ and $k(x, y')>0$ for each $y'\in U_y$. Then

$$k(f\mu)(x)\neq 0\,.$$

This is a contradiction; hence $S(k\mu)=\text{Supp}\ \mu$.

Lemma 2.2. ([4], Prop. 1, p.2) *Let E be a real vector space, E' be a subspace of E and V be a convex cone in E. If*

$$E'+V = E\,,$$

then a linear functional l on E' such that

$$l\geq 0 \text{ on } E'\cap V$$

can be extended to a linear functional $\tilde{l}$ on E such that

$$\tilde{l}\geq 0 \text{ on } V\,.$$

Let $\varphi\in C^+_b(X)$. For each $u\in\mathcal{U}^+$ we define

$$R^{\varphi}_u(x) := \frac{1}{\sup\varphi}\int_0^{\sup\varphi} R^{[\varphi>\alpha]}_u(x)\, d\alpha,\ x\in X$$

where $[\varphi>\alpha] := \{x\in X;\ \varphi(x)>\alpha\}$. R^{φ}_u is called the *reduced function* of u over φ (Mokobodzki's *réduit* [8]). It is known that R^{φ}_u is superharmonic and $S(R^{\varphi}_u)\subset\text{Supp}\ \varphi$ if u is superharmonic.

Now we shall prove the next proposition by using Choquet's representation theorem on an inf-stable function cone.

Proposition 2.1. *Let $\mu\in M^+_K(X)$ and $\varphi\in C^+_b(X)$ satisfy* $\text{Supp}\ \mu\cap\text{Supp}\ \varphi=\emptyset$. *Then there exists a unique $\mu^{\varphi}\in M^+(X)$ such that*

$$\int s\, d\mu^{\varphi} = \int R_s^{\varphi}\, d\mu \quad \textit{for all} \quad s\in S^+ ,$$

and Supp $\mu^{\varphi}\subset$ Supp φ. *If, in addition,* $\varphi\leq 1$ *on* X *and* $\varphi=1$ *on* ∂U *for a relatively compact open set* $U\subset X$ *such that* Supp $\mu\subset U$*, then*

$$k^*\mu = k^*\mu^{\varphi} \ \textit{on}\ X\backslash\bar{U} ,$$

where

$$k^*\mu(y) := \int k(x, y)\, d\mu(x),\ y\in X .$$

Proof. Obviously T: $s\mapsto\int R_s^{\varphi}\, d\mu$ is additive, positively homogeneous and increasing on S^+. Hence by the representation theorem of Choquet ([2], Prop. 1.4) we see that there exists a unique measure $\mu^{\varphi}\in M^+(X)$ such that $T(s)=\int s\, d\mu^{\varphi}$ for all $s\in S^+$. Since $R_s^{\varphi}=R_{s'}^{\varphi}$ if $s, s'\in S^+$ and $s=s'$ on Supp φ, we see that Supp $\mu^{\varphi}\subset$ Supp φ. To prove the last assertion of the proposition, let $y\in X\backslash\bar{U}$. Then for every $x\in\bar{U}$,

$$k_y(x)\geq R_{k_y}^{\varphi}(x) = \int_0^1 R_{k_y}^{[\varphi>\alpha]}(x)\, d\alpha\geq\int_0^1 R_{k_y}^{\partial U}(x)\, d\alpha = R_{k_y}^{\partial U}(x) = k_y(x) .$$

Therefore $R_{k_y}^{\varphi}=k_y$ on Supp μ, so that

$$\begin{aligned} k^*\mu(y) &= \int k_y(x)\, d\mu(x) = \int R_{k_y}^{\varphi}(x)\, d\mu(x) = \int k_y(z)\, d\mu^{\varphi}(z) \\ &= k^*\mu^{\varphi}(y),\ y\in X\backslash\bar{U} . \end{aligned}$$

Lemma 2.3. *Let* $\varphi\in C_b^+(X)$, $\varphi\leq 1$ *and* $\mu\in M_K^+(X)$ *satisfy* Supp $\mu\cap$ Supp $\varphi=\emptyset$. *Then* $k^*\mu^{\varphi}$ *is continuous on* X.

Proof. If $x\notin$ Supp φ, then the function $y\mapsto R_{k_y}^{\varphi}(x)$ is continuous on X (cf. [6], Prop. 2.1). Let $y_0\in X$ be any point. If $y_0\notin$ Supp μ, then there is a neighbourhood U_{y_0} of y_0 such that $k(x, y)$ is bounded on Supp $\mu\times\bar{U}_{y_0}$. Then $R_{k_y}^{\varphi}(x)$ is bounded for $x\in$ Supp μ and $y\in\bar{U}_{y_0}$. If $y_0\in$ Supp μ, then $y_0\notin$ Supp φ. Thus, for a relatively compact open neighbourhood U_{y_0} of y_0 such that $\bar{U}_{y_0}\cap$ Supp $\varphi=\emptyset$, $k(x, y)$ is bounded on Supp $\varphi\times\bar{U}_{y_0}$, since $1\in S^+$ and k_y is a potential and is harmonic on $X\backslash\bar{U}_{y_0}$. It follows that $R_{k_y}^{\varphi}(x)=\int_0^1 R_{k_y}^{[\varphi>\alpha]}(x)\, d\alpha$ is bounded for $x\in X$ and $y\in\bar{U}_{y_0}$. Hence, in any cace, $(R_{k_y}^{\varphi})_{y\in\bar{U}_{y_0}}$ is uniformly bounded on Supp μ. Thus, by Lebesgue's bounded convergence theorem

$$\lim_n k^*\mu^{\varphi}(y_n) = \lim_n \int R_{k_{y_n}}^{\varphi}(x)\, d\mu(x) = \int R_{k_{y_n}}^{\varphi}(x)\, d\mu(x) = k^*\mu^{\varphi}(y_0)$$

for any sequence $y_n\to y_0$.

REMARK 2.1. T. Ikegami assumed that every potetnial with compact support can be represented by the Green function in his paper [6]. But the continuity of the function

$$y \mapsto R^{\varphi}_{k_y}(x), \quad x \notin \mathrm{Supp}\ \mu$$

is shown without this hypothesis there.

Lemma 2.4. *For any $p \in P$ and $x \in X$*

$$p(x) = \sup \{R^{\varphi}_p(x);\ \varphi \in C^+_K(X),\ \varphi \leq 1,\ x \notin \mathrm{Supp}\ \varphi\}\ .$$

Proof. Since $R^{\varphi}_p(x) \leq p(x)$ for any $\varphi \in C^+_K(X)$ with $\varphi \leq 1$, we may assume $p(x) > 0$. Let h be a positive harmonic function on a neighbourhood U_0 of x such that $h(x)=1$. For any $\alpha > 0$, with $\alpha < p(x)$, there is a relatively compact open neighbourhood U of X such that $\bar{U} \subset U_0$ and $\alpha h < p$ on $\bar{U}$. Choose $\varphi \in C^+_K(X)$ such that $x \notin \mathrm{Supp}\ \varphi$, $\varphi \leq 1$ on X and $\varphi = 1$ on ∂U. Then $R^{\varphi}_p(\xi) = p(\xi) > \alpha h(\xi)$ on ∂U. Since R^{φ}_p is superharmonic, it follows that $R^{\varphi}_p > \alpha h$ on U, in particular $R^{\varphi}_p(x) \geq \alpha$.

Lemma 2.5. (cf. [1], Lemma 6) *If $\sigma, \nu \in M^+(X)$ and $k\sigma = k\nu \in P$, then $\sigma = \nu$.*

Proof. First assume $\sigma \wedge \nu = 0$. Then we can find increasing sequences $(K_n)_{n \in N}$ and $(L_n)_{n \in N}$ of compact sets such that $K_n \cap L_n = \emptyset$ for each n and $\sigma(K_n) \nearrow \sigma(X)$, $\nu(L_n) \nearrow \nu(X)$. Let $\sigma_n = \sigma|_{K_n}$, $\nu_n = \nu|_{L_n}$ $\sigma'_n = \sigma - \sigma_n$ and $\nu'_n = \nu - \nu_n$. Then

$$k\,\sigma_n - k\,\nu'_n = k\,\nu_n - k\sigma'_n \ \text{ on } X \backslash A\ ,$$

where $A = \{x \in X;\ k\sigma(x) = k\nu(x) = +\infty\}$. Set $p_n = R(k\sigma_n - k\nu'_n) = R(k\nu_n - k\sigma'_n)$ (see [5], pp. 39–40 for the definition of Rf). Then by [5], Proposition 4.1.5, we see that $p_n \in P$, $p_n \prec k\sigma_n$ and $p_n \prec k\nu_n$. Hence, in view of Lemma 2.1,

$$S(p_n) \subset S(k\,\sigma_n) \cap S(k\,\nu_n) = \mathrm{Supp}\ \sigma_n \cap \mathrm{Supp}\ \nu_n = \emptyset\ .$$

This implies $p_n = 0$, and that $k\,\sigma_n \leq k\,\nu'_n$ on $X \backslash A$. Since A is polar, it follows that $k\,\sigma_n \leq k\,\nu'_n$ on X. Letting $n \to \infty$, we conclude that $k\,\sigma = 0$, namely $\sigma = 0$. Similarly we obtain $\nu = 0$.

In the genreal case, $k\,\sigma = k\,\nu$ implies $k(\sigma - \nu)^+ = k(\nu - \sigma)^+$. By the above, we obtain $(\sigma - \nu)^+ = (\nu - \sigma)^+$, and hence $\sigma = \nu$.

Lemma 2.6. *Let $p \in P \cap C(X)$ and $p \prec k\,\mu$, $k\,\mu \in P_1$ then $p \in P_1$.*

Proof. $(X, \mathcal{U}^+)$ is a balayage space in the sense of Bliedtner-Hansen [2], and the map

$$V: f \mapsto k(f\mu), \quad f \in B^+(X)$$

satisfies

1) $V1 = k\mu$

2) $Vf \in P \cap C(X)$ and $S(Vf) \subset \operatorname{Supp} f$ for any $f \in B_b^+(X)$.

That is, V is the potential kernel associated with $k\mu$. Hence by [2] Prop. 7.11, there exists a $g \in B^+(X)$ such that $g \leq 1$ and $Vg = k(g\mu) = p$. Therefore $p \in P_1$.

We set

$$Q^* := \{k^*\mu \in C(X);\ \mu \in M_K^+(X) \text{ and } \int p\, d\mu < \infty, \text{ for every } p \in P\}$$

$$\tilde{Q}^* := \{f \in C(X);\ |f| \leq k^*\mu \text{ for some } k^*\mu \in Q^*\}.$$

REMARK 2.2. If $\nu \in M_K^+(X)$, $\varphi \in C_K^+(X)$ and $\operatorname{Supp}\nu \cap \operatorname{Supp}\varphi = \emptyset$, then $k^*\nu^\varphi \in Q^*$.

Lemma 2.7. $C_K(X) \subset \tilde{Q}^*$.

Proof. It is enough to show that for any function $f \in C_K^+(X)$ there eixsts a function $k^*\mu \in Q^*$ such that $f \leq k^*\mu$.

Let $y \in \operatorname{Supp} f$. Then there exists a point $x \in X$, $x \neq y$ and a neighbourhood U_y of y such that $x \notin \bar{U}_y$ and $k(x, y') > 0$ for each $y' \in U_y$. Let $y_1, y_2, \cdots, y_n$ be points in $\operatorname{Supp} f$ such that

$$\operatorname{Supp} f \subset U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_n}$$

and $x_1, x_2, \cdots, x_n \in X$ be the points corresponding to $y_1, y_2, \cdots, y_n$ which have the above property. Let $\varphi_i \in C_K^+(X)$, $\varphi_i \leq 1$, $\varphi_i = 1$ on $\bar{U}_{y_i}$ and $x_i \notin \operatorname{Supp}\varphi_i$ $(i = 1, 2, \cdots, n)$. We set

$$\mu' := \sum_{i=1}^{n} \varepsilon_{x_i}^{\varphi_i}$$

Then for each $y \in \operatorname{Supp} f$, there is an U_{y_i} such that $y \in U_{y_i}$ and

$$k^*\varepsilon_{x_i}^{\varphi_i}(y) = \int_0^1 R_{k_y}^{[\varphi_i > \alpha]}(x_i)\, d\alpha \geq \hat{R}_{k_y}^{\bar{U}_{y_i}}(x_i) = k_y(x_i) > 0$$

Therefore, $k^*\mu'(y) > 0$ for all $y \in \operatorname{Supp} f$. Hence $ck^*\mu' \geq f$ for some constant $c > 0$. Let $\mu = c\mu'$. Then $k^*\mu \in Q^*$ by Lemma 2.3 and Remark 2.2.

Proposition 2.2. *Let $p \in P$, $(k\mu_n)_{n \in N} \subset P_1$ such that $k\mu_n \nearrow p$ $(n \to \infty)$. Then there exists a $\mu \in M^+(X)$ such that $p = k\mu$.*

Proof. If $k^*\sigma, k^*\nu \in Q^*$ and $k^*\sigma \leq k^*\nu$, then

$$\int p\, d\sigma = \lim_n \int k\mu_n\, d\sigma = \lim_n \int k^*\sigma\, d\mu_n \leq \lim_n \int k^*\nu\, d\mu_n$$

$$= \lim_n \int k\mu_n\, d\nu = \int p\, d\nu.$$

Therefore the map

$$\ell: k^*\sigma \mapsto \int p \, d\sigma: \; Q^* \to \boldsymbol{R}_+$$

is additive, positively homogeneous, increasing functional and ℓ can be extended to a positive linear functional on Q^*-Q^*. This map will be denoted by ℓ as well.

Let $E:=\tilde{Q}^*$, $E':=Q^*-Q^*$, $V:=\tilde{Q}^* \cap C^+(X)$. Then for each $f \in E$, there is a function $k^*\mu \in Q^*$ such that $|f| \leq k^*\mu$. Hence

$$f+k^*\mu \in V, \quad -k^*\mu \in E'$$

and

$$f = f+k^*\mu - k^*\mu$$

so that

$$E \subset E'+V.$$

The converse inclusion relation is trivial. Therefore

$$E = E'+V.$$

Then by Lemma 2.2, there exists a positive linear functional $\tilde{\ell}$ on Q^* such that

$$\tilde{\ell} = \ell \text{ on } Q^*-Q^*.$$

By Lemma 2.7 $\mu:=\tilde{\ell}|_{C_K(X)}$ is a positive Radon measure on X.

For any $k^*\nu \in Q^*$, let $(U_n)_{n\in N}$ be an exhaustion of relatively compact open sets of X such that

$$\bar{U}_n \subset U_{n+1}, \quad \bigcup_{n\in N} U_n = X, \quad \text{Supp}\, \nu \subset U_1.$$

Let $(\psi_n)_{n\in N} \subset C_K^+(X)$, $\psi_n \leq 1$, $\psi_n = 1$ on $\bar{U}_{n-1}$ and $\psi_n = 0$ on $X \setminus U_n$. Then

$$\ell(k^*\nu) = \tilde{\ell}(\psi_n \, k^*\nu) + \tilde{\ell}((1-\psi_n)\, k^*\nu) \tag{1}$$

Choose $\varphi_n \in C^+(X)$, $n=1, 2, \cdots$, such that $\varphi_n \leq 1$, $\varphi_n = 1$ on ∂U_n and $\varphi_n = 0$ on $\bar{U}_{n-1}$. By Proposition 2.1,

$$(1-\psi_n)\, k^*\nu \leq k^*\nu = k^*\nu^{\varphi_{n-1}} \text{ on } X \setminus \bar{U}_{n-1}$$

and hence

$$(1-\psi_n)\, k^*\nu \leq k^*\nu^{\varphi_{n-1}} \text{ on } X.$$

By Lemma 2.3 and Remark 2.2.

$$k^*\nu^{\varphi_{n-1}} \in Q^*$$

and

$$\begin{aligned} 0 \leq \tilde{\ell}((1-\psi_n)\, k^*\nu) &\leq \tilde{\ell}(k^*\nu^{\varphi_{n-1}}) = \ell(k^*\nu^{\varphi_{n-1}}) \\ &= \int p \, d\nu^{\varphi_{n-1}} = \int R_p^{\varphi_{n-1}} \, d\nu \searrow 0 \; (n \to \infty) \end{aligned} \tag{2}$$

becasuse

$$\lim_n \int R_p^{\varphi_{n-1}} d\nu = \lim_n \int \left[\int_0^1 R_p^{[\varphi_{n-1}>\alpha]} d\alpha \right] d\nu \leq \lim_n \int R_p^{X \setminus U_{n-1}} d\nu = 0 .$$

Letting $n \to \infty$ and by (1), (2)

$$l(k^*\nu) = \lim_n \tilde{l}(\psi_n k^*\nu) = \lim_n \mu(\psi_n k^*\nu)$$

$$= \int \lim_n \psi_n k^*\nu \, d\mu = \int k^*\nu \, d\mu = \int k\mu \, d\nu .$$

Therefore

$$\int p \, d\nu = \int k\mu \, d\nu$$

for every $\nu \in M_K^+(X)$ such that $k^*\nu \in Q^*$.

Suppose there is a point $x \in X$ such that $p(x) \neq k\mu(x)$. Then by Lemma 2.4, there is a function $\varphi \in C_K^+(X)$ such that $\varphi \leq 1$, $x \notin \mathrm{Supp}\, \varphi$ and

$$\int p \, d\varepsilon_x^\varphi = R_p^\varphi(x) \neq R_{k\mu}^\varphi(x) = \int k\mu \, d\varepsilon_x^\varphi .$$

Since $\varepsilon_x^\varphi \in M_K^+(X)$ and $k^*\varepsilon_x^\varphi \in Q^*$, this is a contradiction. Hence $p = k\mu$.

Lemma 2.8. *Suppose that for each $\varphi \in C_K^+(X)$ there exists an increasing sequence $(k\mu_n)_{n \in N} \subset P_1$ such that*

$$\lim_n k\mu_n = R\varphi \text{ on } X$$

where

$$R\varphi := \inf \{u \in \mathcal{U};\ u \geq \varphi\} \text{ ([5], p. 39)} .$$

Then for any $p \in P$, there exists $\mu \in M^+(X)$ such that

$$p = k\mu .$$

Proof. Let $(\varphi_n)_{n \in N} \subset C_K^+(X)$ be an increasing sequence of functions such that

$$\lim_n \varphi_n = p \quad \text{on } X .$$

Then

$$R\varphi_n \nearrow p \quad (n \to \infty) \quad \text{on } X .$$

By the hypothesis of the lemma and Prop. 2.2, there exists a sequence $(\mu_n)_{n \in N} \subset M^+(X)$ such that

$$R\varphi_n = k\mu_n,\ n \in N .$$

Here $k\mu_n \in P_1$ $(n \in N)$ because $R\varphi_n$ is continuous on X ([5], Prop. 2.2.3). Therefore by Prop. 2.2, there is $\mu\in M^+(X)$ such that

$$p=k\mu\,.$$

In the following we assume that the cone P_1 possesses the properties i), ii) and iii) which we mentioned at the beginning of this section.

Lemma 2.9. *P_1 is linearly separating, i.e. for any $x_1, x_2\in X$ $x_1\neq x_2$ and $\alpha\geq 0$, there is a $p\in P_1$ such that*

$$\alpha\, p(x_1)\neq p(x_2)\,.$$

Proof. Suppose there exists $x_1, x_2\in X$, $x_1\neq x_2$ and $\beta\geq 0$ such that $\beta\, p(x_1)=p(x_2)$ for every $p\in P_1$. By the property iii) of P_1 there are $p_1, p_2\in P_1$ such that $p_1(x_1)\,p_2(x_2)\neq p_1(x_2)\,p_2(x_1)$. On the other hand

$$p_1(x_1)\,p_2(x_2)=\beta\, p_1(x_1)\,p_2(x_1)=p_1(x_2)\,p_2(x_1)\,,$$

which is a contradiction.

Lemma 2.10. *Let $p_0\in P_1, p_0>0$. For any $\varphi\in C_K^+(X)$ and $\varepsilon>0$ there exist $p, p'\in P_1$ such that*

$$0\leq p-p'\leq\varphi\leq p-p'+p_0\,\varepsilon \quad on\ X\,.$$

Proof. The assertion follows from the assumptions on P_1 and Lemma 2.9 (cf. [2], Prop. 1.2).

Lemma 2.11. *For each $\varphi\in C_K^+(X)$, there exists an increasing sequence $(g_n)_{n\in N}\subset P_1-P_1$ such that*

$$\lim_n g_n=\varphi \quad \text{on } X\,.$$

Proof. Let $p_0\in P_1$, $p_0>0$. For each $n\in N$, there exists an $f_n\in P_1-P_1$ such that

$$0\leq f_n\leq\varphi\leq f_n+p_0/n \quad \text{on } X$$

by Lemma 2.10. Since P_1 is inf-stable and

$$f_m\vee f_n=f_m+f_n-f_m\wedge f_n \quad (m, n\in N)\,,$$
$$g_n=\sup\,\{f_1, f_2, \cdots, f_n\}\in P_1-P_1\,.$$

Then $(g_n)_{n\in N}$ is increasing and

$$0\leq g_n\leq\varphi\leq g_n+p_0/n \quad \text{on } X,\quad n\in N\,.$$

Theorem 2.1. *Assume that p_1 possesses the properties* i), ii) *and* iii). *Then for any $p\in P$, there exists a unique positive Radon measure μ such that*

$$p = k\mu .$$

Proof. For any $\varphi \in C_K^+(X)$, by Lemma 2.11, there exists

$$(g_n)_{n\in N} \in P_1 - P_1,\ g_n \geq 0 \quad (n\in \boldsymbol{N})$$

such that

$$g_n \nearrow \varphi \quad (n\to\infty) \quad \text{on } X .$$

Then

$$Rg_n \nearrow R\varphi \quad (n\to\infty) \quad \text{on } X .$$

We set

$$g_n := k\mu_n - k\mu_n', \quad n\in \boldsymbol{N} .$$

Then

$$Rg_n < k\mu_n, \quad n\in \boldsymbol{N}$$

by [5], Proposition 4.1.5 and Theorem 5.1.1. Therefore, by Lemma 2.6 $Rg_n \in P_1$. Thus the hypothesis of Lemma 2.8 is fulfilled, and hence there exists $\mu \in M^+(X)$ such that

$$p = k\mu .$$

The uniqueness of representation follows from Lemma 2.5.

Theorem 2.2. *If we assume that P_1 possesses the properties* i), ii) *and* iii), *then $(X, \mathcal{U})$ satisfies the proportionality axiom* (P).

Proof. Let $p\in P$ such that $S(p)=\{y\}$ with $y\in X$. Then there exists $\mu\in M^+(X)$ such that $p=k\mu$ by Theorem 2.1. By Lemma 2.1, we see that Supp $\mu=\{y\}$, so that $\mu=\alpha\,\varepsilon_y$ for $\alpha\geq 0$. Hence $p=\alpha\, k_y$.

3. The case where there exists the dual P-harmonic structure

In this section, we shall prove the following theorem:

Theorem 3.1. *Assume that there exists a second P-harmonic space $(X, \mathcal{U}^*)$ such that $k_x^*: y \mapsto k(x,y)$ is a potential with harmonic support $\{x\}$ with respect $\mathcal{U}^*$. Then P_1 possesses the properties* i), ii) *and* iii) *in section* 2.

Hereafter we assume the existence of such $(X, \mathcal{U}^*)$. The reduced function of $u\in \mathcal{U}^{*+}$ over $\varphi \in C_b^+(X)$ with respect to $\mathcal{U}^*$ will be denoted by $R_u^{*\varphi}$.

Lemma 3.1. *Set $P_K=\{k\mu\in P_1;\ \mu\in M_K^+(X)\}$. For any $p\in P_K$, $\varepsilon>0$ and any compact set $K\subset X$, there exist $q\in P_K$ and a compact set $K'\subset X$ such that $q<\varepsilon$ on K and $p=q$ on $X\backslash K'$.*

Proof. Let $(U_n)_{n\in N}$ be an exhaustion of X by relatively compact open sets such that $\bar{U}_n \subset U_{n+1}$ and Supp $\mu \subset U_1$. Choose $\varphi_n \in C^+(X)$, $n \in \boldsymbol{N}$, such that $\varphi_n \leq 1$ on X, $\varphi_n = 1$ on $X \backslash U_n$, Supp $\varphi_{n+1} \cap U_n = \emptyset$ and $\varphi_n \geq \varphi_{n+1}$. Let $p = k\mu$, $\mu \in M_K^+(X)$. Applying Proposition 2.1 and Lemma 2.3 to the P-harmonic space $(X, \mathcal{U}^*)$, we find $\mu^{*\varphi_n} \in M^+(X)$ such that

$$\int s\, d\mu^{*\varphi_n} = \int R_s^{*\varphi_n}\, d\mu$$

for all positive superharmonic functions s with respect to $\mathcal{U}^*$, $k\mu^{*\varphi_n} \in C(X)$ and $k\mu^{*\varphi_n} = k\mu$ on $X \backslash \bar{U}_n$. Set $q_n = k\mu^{*\varphi_n}$. Since Supp $\mu^{*\varphi_n} = S(q_n) \subset \bar{U}_n \cup S(k\mu) = \bar{U}_n$, $q_n \in P_K$. Furthermore, since Supp $\varphi_n \cap U_{n-1} = \emptyset$, $q_n(x) = \int R_{k_x^*}^{*\varphi_n}\, d\mu \leq \int R_{k_x^*}^{*X|U_{n-1}}\, d\mu$ for all $x \in X$. For each $x \in X$, $R_{k_x^*}^{*X|U_{n-1}} \searrow 0$ uniformly on Supp μ as $n \to \infty$. Hence $q_n \searrow 0$ $(n \to \infty)$. By Dini's theorem, this convergence is uniform on K. Therefore, given $\varepsilon > 0$, there is $N \in \boldsymbol{N}$ such that $q_N < \varepsilon$ on K. Thus, $q = q_N$ and $K' = \bar{U}_N$ satisfy the required conditions.

Lemma 3.2. $P_1 = \{\sum_{n=1}^{\infty} p_n \in C(X);\, p_n \in P_K,\, n \in \boldsymbol{N}\}$.

Proof. It is easy to see that P_1 is included in the set of the right hand side. To porve the converse inclusion, let

$$p = \sum_{n=1}^{\infty} k\mu_n \in C(X) \quad \text{with} \quad \mu_n \in M_K^+(X).$$

Since p is a potential (cf. [5], Proposition 2.2), it is sufficient to show that $\mu = \sum_{n=1}^{\infty} \mu_n$ is a Radon measure, i.e., $\sum_{n=1}^{\infty} \mu_n(K) < \infty$ for any compact set $K \subset X$.

As in the proof of Lemma 2.7, we can find $\nu \in M_K^+(X)$ such that $k^*\nu \in C(X)$ and $k^*\nu \geq 1$ on K. Then

$$\sum_{n=1}^{\infty} \mu_n(K) \leq \sum_{n=1}^{\infty} \int k^*\nu\, d\mu_n = \sum_{n=1}^{\infty} \int k\mu_n\, d\nu = \int p\, d\nu < \infty.$$

Proof of Theorem 3.1:

i) By the proof of Lemma 2.7 considered with respect to $(X, \mathcal{U}^*)$, for any compact set $K \subset X$ there exists $\mu_K \in M_K^+(X)$ such that $k\mu_K \in P_K$ and $0 < k\mu_K \leq 1$ on K. Let $(K_n)_{n \in \boldsymbol{N}}$ be a compaat exhaustion of X and let $p_0 = \sum_{n=1}^{\infty} 2^{-n} k\mu_{K_n}$. Then $p_0 \in P_1$ by Lemma 3.2 and $p_0 > 0$ on X.

Next, let $p = \sum_{n=1}^{\infty} p_n \in P_1$ with $p_n \in P_K$. For each $k \in \boldsymbol{N}$ there is $m_K \in \boldsymbol{N}$ such that $\sum_{n=m_k+1}^{\infty} p_n < 2^{-k-1}$ on K_k. Let $\tilde{p}_k = \sum_{n=1}^{m_k} p_n$. Then $\tilde{p}_k \in P_K$, Hence by Lemma 3.1 there are $\tilde{q}_k \in P_K$ and a compact set $K_k' \subset X$ such that $\tilde{q}_k < 2^{-k-1}$ on K_k and $\tilde{q}_k = \tilde{p}_k$ on $X \backslash K_k'$. Set $q_k = \tilde{q}_k + \sum_{n=m_k+1}^{\infty} p_n$. Then $q_k \in P_1$ by Lemma 3.2., $q_k < 2^{-k}$ on K_k and $q_k = p$ on $X \backslash K_k'$. Then $q := \sum_{k=1}^{\infty} q_k \in P_1$. For any $\varepsilon > 0$, choosing

$m\in N$ with $m>1/\varepsilon$, we see that

$$q\geq \sum_{k=1}^{\infty} q_k = mp>\varepsilon^{-1}p \text{ on } X\setminus \bigcup_{k=1}^{m} K'_k ,$$

which means that $p=o(q)$. Therefore P_1 is adapted.

ii) Let $kv_1, kv_2\in P_1$ and set $q=kv_1\wedge kv_2$. Consider the map ℓ: $P_K^*=\{k^*\mu\in C(X);\ \mu\in M_K^+(X)\}\to \boldsymbol{R}_+$ defined by

$$\ell(k^*\mu)=\int q\,d\mu,\quad \mu\in M_K^+(X)\,.$$

We show that ℓ is increasing, i.e., $\int q\,d\sigma\leq\int q\,d\mu$ if $k^*\sigma\leq k^*\mu$. Set $\mu_1=\mu|_{[k\mu_1>k\mu_2]}$ and $\mu_2=\mu-\mu_1$. Since the set of all potentials for $\mathcal{U}^*$ has the Riesz decomposition property (cf. [5], Theorem 5.1.1), there exist potentials q_1^*, q_2^* for $\mathcal{U}^*$ such that

$$q_1^*\leq k^*\mu_1,\ q_2^*\leq k^*\mu_2 \quad\text{and}\quad q_1^*+q_2^*=k^*\sigma\,.$$

By Lemma 2.6 applied for $\mathcal{U}^*$, there exists $\sigma_1, \sigma_2\in M_K^+(X)$ such that $q_1^*=k^*\sigma_1$ and $q_2^*=k^*\sigma_2$. Then by Lemma 2.5 applied for $\mathcal{U}^*$, $\sigma_1+\sigma_1=\sigma$. Hence,

$$\begin{aligned}\int q\,d\sigma &= \int q\,d\sigma_1+\int q\,d\sigma_2\\ &\leq \int kv_2\,d\sigma_1+\int kv_1\,d\sigma_2=\int q_1^*\,dv_2+\int q_2^*\,dv_1\\ &\leq \int k^*\mu_1\,dv_2+\int k^*\mu_2\,dv_1=\int kv_2\,d\mu_1+\int kv_1\,d\mu_2\\ &= \int q\,d\mu_1+\int q\,d\mu_2=\int q\,d\mu\,.\end{aligned}$$

Then ℓ can be extended to a positive linear functional on $P_K^*-P_K^*$, which will be again denoted by ℓ. Let

$$\tilde{Q}_K^*=\{f\in C(X);\ |f|\leq k^*\mu \quad\text{for some}\quad k^*\mu\in P_K^*\}\,.$$

Then $C_K(X)\subset\tilde{Q}_K^*$ by Lemma 2.7, and by the same argument as in the proof of Proposition 2.2, we obtain a positive linear functional $\tilde{\ell}$ on $\tilde{Q}_K^*$ such that $\tilde{\ell}=\ell$ on $P_K^*-P_K^*$ and

$$\int q\,d\mu=\ell(k^*\mu)=\int kv\,d\mu \quad\text{for all}\quad k^*\mu\in P_K^*\,,$$

where $v=\tilde{\ell}|_{C_K(X)}$. Then, again by the same argument as in the last part of the proof of Proposition 2.2, we see that $kv=q\in P\cap C(X)$. Hence P_1 is inf-stable.

iii) Let $x_1, x_2\in X$, $x_1\neq x_2$. By Lemma 2.7 applied for $\mathcal{U}^*$, there is $p\in P_1$ such that $p(x_2)=1$. Since $k_{x_1}^*\neq p(x_1)\,k_{x_2}^*$, there is $y\in X$ such that $k_{x_1}^*(y)\neq p(x_1)$

$k^*_{x_2}(y)$. Hence, by Lemma 2.4 applied for $\mathcal{U}^*$, there exists $\varphi\in C^+_K(X)$ such that $\varphi\leq 1$ on X, $y\notin \mathrm{Supp}\,\varphi$ and

$$R^{*\varphi}_{k^*_{x_1}}(y) \neq p(x_1)\, R^{*\varphi}_{k^*_{x_2}}(y)\,.$$

Thus

$$p(x_2)\, k\varepsilon^{*\varphi}_y(x_1) \neq p(x_1)\, k\varepsilon^{*\varphi}_y(x_2)$$

and $k\varepsilon^{*\varphi}_y\in P_1$ by Remark 2.2 applied for $\mathcal{U}^*$. Hence P_1 separates points of X.

In view of Theorems 2.1 and 2.2, we obtain the following corollaries:

Corollary 3.1. *Under the same assumptions as in Theorem* 3.1, *for any* $p\in P$ *there exists a positive Radon measure* μ *such that* $p=k\mu$.

Corollary 3.2. *Under the same sasumptions as in Theorem* 3.1, *both* $(X, \mathcal{U})$ *and* $(X, \mathcal{U}^*)$ *satisfy the proportionality axiom.*

REMARK 3.1. Let $(X, \mathcal{U})$ be a P-Brelot space with a countable base which admits a Green function $k(x, y)$. If $k(x, y)$ is symmetric, then we can take $\mathcal{U}^*=\mathcal{U}$. Therefore, Corollary 3.1 and 3.2 are extensions of [1], Theorem 9 and 10, respectively. In case $k(x, y)$ is not necessarily symmetric, Theorem 3.1 shows that the existenec of the dual P-harmonic structure $\mathcal{U}^*$ is a sufficient condition for the existence of the convex cone P' considered in [1], p.47.

REMARK 3.2. T. Ikegami [6] proved that if any potential p with compact harmonic support is represented as $p=k\mu$ with a Radon measure μ, then there exists the dual P-harmonic structure $\mathcal{U}^*$. Thus, in view of Theorem 2.1 and 3.1, the condition that P_1 satisfies i), ii) and iii) in section 2 is equivalent to the existence of the dual P-harmonic space $(X, \mathcal{U}^*)$.

EXAMPLE 3.1. Let $X=\boldsymbol{R}^n\times \boldsymbol{R}$ $(n\in \boldsymbol{N})$ and consider the harmonic space $(X, \mathcal{U})$ induced by the heat equation

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}-\frac{\partial h}{\partial t}=0, \quad (x, t)\in \boldsymbol{R}^n\times \boldsymbol{R}\,.$$

It is well known that

$$W((x, t), (y, s))=\begin{cases} \dfrac{1}{[4\pi(t-s)]^{n/2}}\exp\left(-\dfrac{|x-y|^2}{4(t-s)}\right), & t>s \\ 0, \ t\leq s & \end{cases}$$

is a Green function for $(X, \mathcal{U})$ satisfying conditions in section 1, and furthermore, $W^*((x, t), (y, s))=W((y, s), (x, t))$ is a Green function for the harmonic space $(X, \mathcal{U}^*)$ induced by the adjoint heat equation

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} + \frac{\partial h}{\partial t} = 0 .$$

Thus we can deduce the possibility of the representation $p = W\mu$ for any potential p on $(X, \mathcal{U})$ from Corollary 3.1, and the proportionality axiom for $(X, \mathcal{U})$ from Corollary 3.2.

4. Some equivalent conditions

In this section, we are first concerned with the following three conditions:

(B) (cf. [3]) Any continuous potential p with compact harmonic support is represented as $p = k\mu$ with a Radon measure μ on X.

(S) (cf. [9]) At least one bounded continuous strict potential p_0 is represented as $p_0 = k\lambda$ with a Radon measure λ on X.

(P_1) P_1 possesses the properties i), ii) and iii) in section 2.

By the Theorem 2.1, condition (P_1) implies conditions (B) and (S). If we assume condition (B), then P_1 contains all continuous potentials with compact harmonic support, so that the conclusion of Lemma 2.10 holds by [5], Theorem 2.3.1. Hence the same arguments as in section 2 leads to the conclusion of Theorem 2.1, in particular, condition (S).

The implication (S)$\Rightarrow$(P_1) is essentially proved in [9]. In fact, [9], Satz 2 contains the following result:

Proposition 4.1. *If we assume condition* (S), *then* P_1 *coincides with the set of all continuous potentials.*

In view of [2], II, Proposition 5.2 and [5], Proposition 2.3.2, the st of all continuous potentials possesses the properties i), ii), iii) in section 2, so that we obtain the implication (S)$\Rightarrow$(P_1).

Thus we have shown

Theorem 4.1. *Under the assumptions in section* 1, *conditions* (B), (S) *and* (P_1) *are equivalent to each other.*

The proof of [9], Satz 2, or our Proposition 4.1, is based on the following results:

Lemma 4.1. ([9], Satz 1). *If we assume condition* (S), *then* $(X, \mathcal{U})$ *has the Doob convergence property.*

Lemma 4.2. ([9], Lemma 3). *Under the assumptions in section* 1, $(X, \mathcal{U})$ *possesses the property* (A) *of K. Janssen* [7].

Lemma 4.3. ([9], Lemma 4). *If we assume condition* (S), *then* $(X, \mathcal{U})$

satisfies the proportionality axiom.

The conclusion of proposition 4.1 then follows from [7], Theorem 3.4. By this observation, we can also state

Theorem 4.2. *Under the assumptions in section* 1, *the proportionality axiom together with the Doob convergence property is equivalent to any one of the conditions* (B), (S), (P_1).

REMARK 4.1. We do not know whether the proportionality axiom implies the Doob convergence property or not, under the assumptions in section 1.

In the rest of this section, we shall be concerned with the proof of Lemma 4.3. In the proof of [9], Lemma 4, the following result is used:

Proposttion 4.2. *Let p_0 be a bounded continuous strict potential on X and let V be the associated potential kernel (cf.* [2]). *Then for any $s \in S^+$ and any open set U in X, there exists $(g_n)_{n\in N} \subset B_b^+(X)$ such that $g_n=0$ on $X \setminus U$ and $V_{g_n} \nearrow R_s^U$ on X.*

This result is shown in [9], Bemerkung 2 (p. 73), by probabilistic arguments. Here we shall give a proof of this proposition which does not appeal to the probability theory.

Let $\boldsymbol{V}=(V_\alpha)_{\alpha>0}$ be the submarkov resollvent of kernels on X such that

$$Vf = \sup_{\alpha>0} V_\alpha f, \quad f \in B^+(X)$$

and the set E_V of all $\boldsymbol{V}$-excessive functions on X coincides with $\mathcal{U}^+$ (cf. [2], II, Theorem 7.8). For an open set U in X, define

$$W^U f = V(f 1_U), \quad f \in B(X) .$$

Lemma 4.4. *W^U is a bounded kernel and satisfies the complete maximum principle.*

Proof. Since $W^U 1 \leq V1 \leq p_0$, W^U is bounded. Next, suppose $W^U f \leq W^U g + a$ on $[f>0]$ for $a \in \boldsymbol{R}^+, f, g \in B(X)$. Then $V(f1_U) \leq V(g1_U)+a$ on $[f1_U>0]$. Since V satisfies the complete maximum principle ([2], II, Propostion 7.1), this inequality holds on X, and hence $W^U f \leq W^U g+a$ on X. Thus W^U satisfies the complete maximum principle.

By this lemma, there exists a (unique) submarkov resolvent of kernels $\boldsymbol{W}^U=(W_\alpha^U)_{\alpha\geq 0}$ on X such that $W_0^U = W^U$ ([2], II, Theorem 7.7).

Lemma 4.5. *Given $s \in S^+$, there exists $(f_n)_{n\in N} \subset B_b^+(X)$ such that $Vf_n \nearrow s$ ($n \to \infty$).*

Proof. Let $s_n = s \wedge n$ and $f_n = n(s_n - nV_n s_n)$. Then $Vf_n = nV_n s_n \nearrow s\ (n \to \infty)$.

Lemma 4.6. *For any $\alpha > 0$ and $f \in B_b^+(X)$, $\alpha W_\alpha^U Vf \leq Vf$ on X.*

Proof. Let $h = \alpha(Vf - \alpha W_\alpha^U Vf)$. Then $W^U h = \alpha W_\alpha^U Vf$. Hence $Vf \geq W^U h = V(h1_U)$ on $[h > 0]$. Then the complete maximum principle for V implies that $Vf \geq W^U h$ on X.

By Lemmas 4.5 and 4.6, we obtain

Corollary 4.1. *Any $s \in S^+$ is W^U-supermedian, i.e., $\alpha W_\alpha^U s \leq s$ for all $\alpha > 0$.*

For $s \in S^+$, let $\tilde{s}_U = \sup_{\alpha > 0} \alpha W_\alpha^U s$. By the above corollary, $\tilde{s}_U \leq s$.

Lemma 4.7. *$W^U \tilde{s}_U = W^U s$ for any $s \in S^+$.*

Proof. For simplicity, we omit super- or subscript U. Let $s_n = s \wedge n$. Since $1 \in S^+$, $s_n \in S^+$, so that $\beta W_\beta s_n \leq s_n$ for any $\beta > 0$ by the above corollary. Therefore $W_\beta s_n \leq \beta^{-1} s_n \to 0$ as $\beta \to \infty$. Thus, letting $\beta \to \infty$ in the resolvent equation $Ws_n = W_\beta s_n + \beta WW_\beta s_n$, we obtain

$$Ws_n = \lim_{\beta \to \infty} \beta WW_\beta s_n \leq W\tilde{s} .$$

Hence, $Ws = \sup_n Ws_n \leq W\tilde{s} \leq Ws$, which shows the lemma.

Lemma 4.8. *For any $x_0 \in U$, $\lim_{\alpha \to \infty} \alpha V_\alpha(x_0, X \setminus U) = 0$.*

Proof. Choose $f \in C_K^+(X)$ such that $f \leq 1$ on X, $f(x_0) = 1$ and $f = 0$ on $X \setminus U$. For any $\varepsilon > 0$, there exist continuous potentials p, q such that

$$0 \leq p - q \leq f \leq p - q + \varepsilon \text{ on } X$$

([5]), Theorem 2.3.1). Since $\alpha V_\alpha p \to p$, $\alpha V_\alpha q \to q$ and $\alpha V_\alpha 1 \to 1$ as $\alpha \to \infty$, it follows that $\alpha V_\alpha f \to f (\alpha \to \infty)$. On the other hand,

$$\alpha V_\alpha 1(x_0) \geq \alpha V_\alpha f(x_0) + \alpha V_\alpha 1_{X \setminus U}(x_0) .$$

Hence, $1 \geq 1 + \lim_{\alpha \to \infty} \alpha V_\alpha(x_0, X \setminus U)$, which implies the required assertion.

Lemma 4.9. *$\tilde{s}_U = s$ on U.*

Proof. Let $x_0 \in U$. Then, for any $m \in \boldsymbol{N}$,

$$\begin{aligned} s_m(x_0) &= \lim_{\alpha \to \infty} \alpha V_\alpha s_m(x_0) \\ &= \lim_{\alpha \to \infty} \{\alpha V_\alpha(s_m 1_U)(x_0) + \alpha V_\alpha(s_m 1_{X \setminus U})(x_0)\} . \end{aligned}$$

By Lemma 4.8,

$$\alpha V_\alpha(s_m 1_{X\setminus U})(x_0) \leq m\alpha V_\alpha(x_0, X\setminus U) \to 0 \quad (\alpha \to \infty).$$

Hence $s_m(x_0) = \lim_{\alpha\to\infty} \alpha V_\alpha(s_m 1_U)(x_0)$. By Lemma 4.7, we see that $V_\alpha(s_m 1_U) = V_\alpha(\tilde{s}_m 1_U)$, where $\tilde{s}_m = (\tilde{s}_m)_U$. Hence

$$s_m(x_0) = \lim_{\alpha\to\infty} \alpha V_\alpha(\tilde{s}_m 1_U)(x_0) \leq \tilde{s}_m(x_0) \leq \tilde{s}_U(x_0).$$

Letting $m\to\infty$, we have $s(x_0) \leq \tilde{s}_U(x_0)$. Since $\tilde{s}_U \leq s$, it follows that $s = \tilde{s}_U$ on U.

Proof. of Proposition 4.2. Let $(f_n)_{n\in N} \subset B_b^+(X)$ be the sequence given in Lemma 4.5, and set $h_n = n(Vf_n - nW_n^U Vf_n)$, $n\in \boldsymbol{N}$. Then $h_n \in B_b^+(X)$ and $V(h_n 1_U) = W^U h_n = nW_n^U Vf_n \nearrow \tilde{s}_U (n\to\infty)$ by Lemma 4.6. Hence $\tilde{s}_U \in \mathcal{U}^+$ and by lemma 4.9, we see that $\tilde{s}_U \geq R_s^U$. On the other hand, $V(h_n 1_U) \leq s = R_s^U$ on U. Since R_s^U is V-dominant ([2], II, Proposition 7.1), it follows that $V(h_n 1_V) \leq R_s^U$ on X, and hence $\tilde{s}_U \leq R_s^U$. Therefore, $\tilde{s}_U = R_s^U$, so that $g_n = h_n 1_U$, $n\in \boldsymbol{N}$, satisfy the required conditions.

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