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1. Introduction

In the previous paper [3] we considered singular perturbations of symbolic flows and showed the existence of poles of the zeta functions associated with perturbed symbolic flows. The purpose of the present paper is to remove some conditions required in the previous paper.

Our aim in studying poles of the zeta functions is to show the validity of the modified Lax-Phillips conjecture for obstacles consisting of several small balls. The modified Lax-Phillips conjecture is concerned with the distribution of poles of scattering matrices. About this conjecture, see Lax-Phillips [8, Epilogue] and Ikawa [5]. When we want to apply Theorem 1 of the previous paper to this conjecture, we have to require some additional conditions on the configuration of the centers of balls, that is, the conditions (A.2) and (A.3) of [3, Section 4]. As a consequence of the improvement of the theorem, we can show the validity of the modified Lax-Phillips conjecture for all obstacles consisting of small balls whose centers satisfy only (A.1) of [3].

Now we shall introduce notations for the statement of our main theorem. Let $L$ be an integer $\geq 2$, and let $A = [A(i,j)]_{i,j=1,2,\ldots,L}$ and $B = [B(i,j)]_{i,j=1,2,\ldots,L}$ be zero-one $L \times L$ matrices.

For $i,j \in \{1, 2, \ldots, L\}$, we denote $i \rightarrow_B j$ when there is a sequence $i, i_2, \ldots, i_p$ such that $B(i, i_1) = 1, B(i_{q+1}, i_q) = 1$ for $q = 1, 2, \ldots, p - 1$ and $B(j, i_p) = 1$.

We assume on $B$ the following:

There is $1 < K \leq L$ such that

(1.1) \quad $B(i,j) = 0$ \quad for all $j$ \quad if $i \geq K + 1$,

(1.2) \quad $i \rightarrow_B i$ \quad for all \quad $1 \leq i \leq K$,

(1.3) \quad $i \rightarrow_B j$ implies $j \rightarrow_B i$ \quad if \quad $i, j \leq K$.

We assume also the following relation between $A$ and $B$:

(1.2) \quad $B(i,j) = 1$ \quad implies \quad $A(i,j) = 1$.
Let $f_\varepsilon, h_\varepsilon$ be functions with parameter $\varepsilon \geq 0$ satisfying
\[ f_\varepsilon, h_\varepsilon \in \mathcal{F}(\Sigma_\lambda^+) \quad \text{for all} \quad 0 \leq \varepsilon \leq \varepsilon_1, \]
where $\varepsilon_1$ is a positive constant, and let $k \in \mathcal{F}(\Sigma_\lambda^+)$ satisfy
\[
\begin{cases}
  k(\xi) = 0 & \text{if } B(\xi_1, \xi_2) = 1 \\
  k(\xi) > 0 & \text{if } B(\xi_1, \xi_2) = 0.
\end{cases}
\]
We assume that
\[
|||f_\varepsilon - f_0|||_\varepsilon, |||h_\varepsilon - h_0|||_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
For $0 < \varepsilon \leq \varepsilon_1$, we define zeta function $Z_\varepsilon(s)$ by
\[
Z_\varepsilon(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_\xi^e = \xi} \exp(S_\sigma r_\varepsilon(\xi, s))\right)
\]
where
\[
r_\varepsilon(\xi, s) = -sf_\varepsilon(\xi) + h_\varepsilon(\xi) + k(\xi) \log \varepsilon
\]
and
\[
S_\sigma r_\varepsilon(\xi, s) = r_\varepsilon(\xi, s) + r_\varepsilon(\sigma_\mu^e, s) + \cdots + r_\varepsilon(\sigma_\lambda^e, s).
\]
Our main theorem is the following

**Theorem 1.** Suppose that (1.1)~(1.4) are satisfied, and that
\[
f_\varepsilon(\xi) > 0 \quad \text{for all} \quad \xi \in \Sigma_\lambda^+,
\]
\[
h_\varepsilon(\xi) \text{ is real for all} \quad \xi \in \Sigma_\lambda^+ \text{ such that } B(\xi_1, \xi_2) = 1.
\]
Then there exist $s_0 \in \mathcal{R}$, $D$ a neighborhood of $s_0$ in $\mathcal{C}$ and $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$, $Z_\varepsilon(s)$ is meromorphic in $D$ and has a pole $s_\varepsilon$ in $D$ with
\[
s_\varepsilon \to s_0 \quad \text{as} \quad \varepsilon \to 0.
\]

Compared with Theorem 1 of the previous paper, the present one requires neither the condition (1.2) nor (1.4) nor (1.10) in [3]. The removal of the additional conditions gives us the following result on the modified Lax-Phillips conjecture:

**Theorem 2.** Let $P_j$, $j=1, 2, \cdots, L$, be points in $\mathcal{R}^3$, and set for $\varepsilon > 0$
\[
\mathcal{O}_\varepsilon = \bigcup_{j=1}^{j=L} \mathcal{O}_{j,\varepsilon}, \quad \mathcal{O}_{j,\varepsilon} = \{x; \ |x - P_j| < \varepsilon\}.
\]
Suppose that
\[
(A.1) \quad \text{any triple of } P_j's \text{ does not lie on a straight line.}
\]
Then, the modified Lax-Phillips conjecture is valid for $\mathcal{O}_\varepsilon$ if $0 < \varepsilon \leq \varepsilon_0$. 
The amelioration of the theorem is done with the aid of the results of Adachi-Sunada [1] and of Pollicott [10], Haydn [2].

The main reason why we had to assume the additional conditions was to guarantee the Property P of Parry [9] for \( \mathcal{L}_s \). Indeed, this property was essentially used in [9] for the proof of meromorphic extension of the zeta function. But Pollicott [10] and Hadyn [2] proved Theorems on the meromorphic extension without using the Property P. If we use the argument of [10,2], it suffices to consider the spectrum of the Perron-Frobenius operator \( \mathcal{L}_{s,s} \). To get informations of the perturbed operator \( \mathcal{L}_{s,s} \), first we have to consider the unperturbed operator \( \mathcal{L}_s \). But we cannot apply the standard Perron-Frobenius theorem to the unperturbed operator because we do not assume the unperturbed system to be topological mixing. To overcome this difficulty, we decompose the unperturbed dynamical system into a direct sum of irreducible subsystems, and apply the generalized Perron-Frobenius theorem in [1] to each subsystem. In order to extract informations of the spectrum of perturbed operator from those of the unperturbed operator, we shall follow the argument done in [3].

2. Decomposition of \( \mathcal{L}_s' \)

Hereafter we shall use freely notations used in [3]. As in [3], we introduce an operator \( \mathcal{L}_s' \) in \( C(\Sigma^+_s) \) defined by

\[
\mathcal{L}_s' u(\xi) = \begin{cases} 
\sum_{\eta(\gamma_1, \tau_1)=1} \exp(r_0(\gamma; s))u(\gamma) & \text{for } \xi \in \Sigma(1), \\
0 & \text{for } \xi \in \Sigma(2),
\end{cases}
\]

where

\[
r_0(\xi; s) = -s f_0(\xi) + h_0(\xi),
\]

and \( \sum_{\eta(\gamma_1, \tau_1)=1} \) indicates the summation taken over all \( \eta \in \Sigma^+ \) such that \( \sigma A \eta = \xi \) and \( B(\gamma_1, \xi_1) = 1 \).

In this section we shall consider the spectrum of \( \mathcal{L}_s' \) in the space \( \mathcal{F}_\theta(\Sigma^+_s) \). To this end, as mentioned in the introduction, we consider the spectrum of the operator \( \mathcal{L}_s \) in the unperturbed dynamical system \( \Sigma^+_s \), and compare the spectrum of \( \mathcal{L}_s' \) with that of \( \mathcal{L}_s \).

2.1. On the decomposition of \( \mathcal{L}_s' \)

Let us say that \( i \) and \( j \) are equivalent when \( i \rightarrow j \). Then the conditions (1.2) and (1.3) on \( B \) imply that this gives an equivalent relation in \( \{1, 2, \cdots, K\} \). Therefore, by changing the numbering of the elements of \( \{1, 2, \cdots, K\} \), we may assume that the set \( \{1, 2, \cdots, K\} \) is decomposed into equivalents classes

\[
M_j = \{i_j, i_{j+1}, \cdots, i_{j+1-1}\} \quad (j = 1, 2, \cdots, l).
\]
We shall denote by $C_j$ the $(i_{j+1} - i_j) \times (i_{j+1} - i_j)$ matrix $[B(i, j)]_{i, j \in M_j}$. Note that each $C_j$ is irreducible. We set

$\Sigma^+_{C_j} = \{ \xi = (\xi_1, \xi_2, \ldots); \xi_i \in M_j $ and $B(\xi_i, \xi_{i+1}) = 1 \text{ for all } i \}$

and

$\Sigma^+_C = \{ \xi = (\xi_1, \xi_2, \ldots); 1 \leq \xi_i \leq K $ and $B(\xi_i, \xi_{i+1}) = 1 \text{ for all } i \}.$

Regarding $\Sigma^+_C$ and $\Sigma^+_C$ as subsets of $\Sigma^+_A$, we have a decomposition

\begin{equation}
C(\Sigma^+_C) = C(\Sigma^+_C) \oplus C(\Sigma^+_C) \oplus \cdots \oplus C(\Sigma^+_C) .
\end{equation}

For $u \in C(\Sigma^+_C)$ we denote by $[u]$ and $[u]$, the restrictions of $u$ to $\Sigma^+_C$ and $\Sigma^+_C$, respectively. Conversely, for functions in $\Sigma^+_C$ or in $\Sigma^+_C$, we shall often treat them as functions defined in $\Sigma^+_A$ by extending them by zero in the outside of $\Sigma^+_C$ or of $\Sigma^+_C$.

Let $\tilde{L}_s$ be the operator in $C(\Sigma^+_C)$ defined by

$\tilde{L}_s v(\xi) = \sum \exp(r_0(\eta; s))v(\eta)$ \quad for $v \in C(\Sigma^+_C),$

and let $\tilde{L}_{j,s}$ be the operators in $C(\Sigma^+_C)$ defined by

$\tilde{L}_{j,s} v(\xi) = \sum \exp(r_0(\eta; s))v(\eta)$ \quad for $v \in C(\Sigma^+_C),$

where $\sigma_C$ and $\sigma_{C_j}$ denote the restrictions of $\sigma_A$ to $\Sigma^+_C$ and $\Sigma^+_C$, respectively. Then $\tilde{L}_s$ has a decomposition

\begin{equation}
\tilde{L}_s = \tilde{L}_1,s \oplus \tilde{L}_2,s \oplus \cdots \oplus \tilde{L}_i,s .
\end{equation}

By using the notation introduced in the above, we have for all $u \in \Sigma^+_A$

$\tilde{L}_s[u] = \tilde{L}_1,s[u] \oplus \tilde{L}_2,s[u] \oplus \cdots \oplus \tilde{L}_i,s[u] .$

Note that the conditions (1.5) and (1.6) imply that $r_0$ is real valued in $\Sigma^+_C$ for $s \in R$. Thus, taking account of the indecomposability of $C_j$ we can apply Theorem 3.8 and Lemma 3.11 of [1] to $\tilde{L}_{j,s}$ and get the following

**Lemma 2.1.** For $s \in R$, $\tilde{L}_{j,s}$ has a decomposition

$\tilde{L}_{j,s} = \sum_{k=1}^{s_j} \lambda_{j,k,s} \tilde{E}_{j,k,s} + \tilde{S}_{j,s},$

with the following properties:

(i) $\tilde{L}_{j,s} \tilde{E}_{j,k,s} = \lambda_{j,k,s} \tilde{E}_{j,k,s} .

(ii) $\lambda_{j,1,s} > 0$ and $-\frac{d \lambda_{j,1,s}}{ds} > 0 .

(iii) $|\lambda_{j,k,s}| = \lambda_{j,1,s}$ and $\lambda_{j,k,s} = \lambda_{j,k',s}$ if $k \neq k'$. 

SINGULAR PERTURBATION OF SYMBOLIC FLOWS

where \( v_{j,s} \) is a \( \delta_{j,k} \) satisfying \( v_{j,s}(\rho_{j,s}) = 1 \),

the spectral radius of \( S_{j,s} < \lambda_{j,1,s} \).

Hereafter, we shall denote often \( \lambda_{j,s} \), \( \lambda_{j,s} \) as \( \tilde{\lambda}_{j,s} \). Note that we have for each \( j \)

Thus, by changing the numbering of \( \lambda_{j,s} \), if necessary, we may suppose that

for some \( s_0 \in \mathbb{R} \)

Then, by using the perturbation theory we have immediately the following

Lemma 2.2. There are a neighborhood of \( s_0 \) in \( \mathbb{C} \) and a constant \( \delta > 0 \) such that for all \( s \neq D \) we have a decomposition

with the following properties:

(i) \( \tilde{E}_{j,k,s}(\xi) = v_{j,k,s}(u_j) \tilde{p}_{j,k,s}(\xi) \),

(ii) \( \tilde{E}_{j,k,s}\tilde{E}_{j',k',s} = \delta_{j,j'}\delta_{k,k'}E_{j,k,s} \),

(iii) \( \tilde{E}_{j,k,s}\tilde{S}_s = \tilde{S}_s \tilde{E}_{j,k,s} = 0 \),

(iv) \( |\tilde{\lambda}_{j,s} - 1| < \delta \),

(v) \( |\tilde{\lambda}_{j,k,s} - 1| > 2\delta \), \( 1 - \delta < |\tilde{\lambda}_{j,k,s}| < 1 + \delta \) for \( k \geq 2 \),

(vi) \( \text{the spectral radius of } S_s < 1 - 2\delta \).

2.2. On eigenvalues of \( \mathcal{L}' \)

With the aid of the results of the previous subsection, we shall consider the decomposition of \( \mathcal{L}' \). First remark that for any positive integer \( m \) and for \( \xi \in \Sigma(1) \) we have an expression

\[
\mathcal{L}'u(\xi) = \sum_{\eta_1, \ldots, \eta_m} \exp(S_m r_0(\eta_m, \eta_{m-1}, \ldots, \eta_1, \xi; s)) u(\eta_m, \eta_{m-1}, \ldots, \eta_1, \xi),
\]

where the summation is taken over all \( \eta_1, \eta_2, \ldots, \eta_m \) satisfying \( B(\eta_1, \xi_j) = 1, B(\eta_2, \eta_1) = 1, \ldots, B(\eta_m, \eta_{m-1}) = 1 \). If \( \xi \in \Sigma_\xi \), all \( (\eta_m, \ldots, \eta_1, \xi) \) in the right hand side of (2.6) belong to \( \Sigma_\xi \). Thus we have

\[
\mathcal{L}'u(\xi) = \mathcal{L}''u(\xi) \quad \text{for all } \xi \in \Sigma_\xi.
\]
Lemma 2.3. For each pair $j, k$ in Lemma 2.2, there is a function $w_{j,k,s}(\xi) \in \mathcal{F}_d(\Sigma^*)$ satisfying

\begin{equation}
|\hat{w}_{j,k,s}^{(-m)}(S_{\xi} \rho_{\xi}(\eta_m, \ldots, \eta_2, l, \xi)) p_{j,k,s}(\eta_m, \ldots, l, \eta^{(l)})
- w_{j,k,s}(\xi)| \leq C \gamma_1^m \quad \text{for } m = 1, 2, \ldots,
\end{equation}

and

\[ L^*_d w_{j,k,s}(\xi) = \hat{w}_{j,k,s}(\xi) w_{j,k,s}(\xi). \]

Here $\gamma_1$ is a constant such that $0 \leq \gamma_1 < 1$.

Proof. Let $\xi \in \Sigma(1)$ be an element such that $\xi_1 \in C_\eta$. Then all the $\eta_j$ in the summation of the right hand side of (2.6) belong to $C_\eta$. Thus the argument in Section 2 of [3] can be applied and we see that (2.8) holds for all $\xi$ such that $\xi_1 \leq K$. It follows from (2.8) that

\[ L^*_d w_{j,k,s}(\xi) = \hat{w}_{j,k,s}(\xi) w_{j,k,s}(\xi) \quad \text{for all } \xi \in \Sigma^* \text{ such that } \xi_1 \leq K. \]

Define $w_{j,k,s}(\xi)$ for $\xi \in \Sigma(1)$ such that $\xi_1 \geq K+1$ by

\begin{equation}
\hat{w}_{j,k,s}(\xi) = (\hat{w}_{j,k,s})^{-1} \sum_{\eta_j \in \Sigma^*} \sum_{j \in \Sigma^*} \exp(r(\eta_j, \xi)) w_{j,k,s}(\eta_j, \xi).
\end{equation}

We have immediately

\[ L^*_d w_{j,k,s}(\xi) = \hat{w}_{j,k,s}(\xi) w_{j,k,s}(\xi) \quad \text{for all } \xi \in \Sigma^* . \]

Concerning the converging estimate (2.8) for $\xi_1 \geq K+1$, we use the following relation:

\begin{align*}
(\hat{w}_{j,k,s})^{-m} \sum_{\eta_m, \ldots, \eta_2, l} \exp(S_{\xi} \rho_{\xi}(\eta_m, \ldots, \eta_2, l, \xi)) p_{j,k,s}(\eta_m, \ldots, l, \eta^{(l)}) \\
= (\hat{w}_{j,k,s})^{-1} \sum_{\eta_m, \ldots, \eta_2, l} \exp(r(\eta_j, \xi)) \{(\hat{w}_{j,k,s})^{-m+1} \sum_{\eta_m, \ldots, \eta_2, l} \exp(S_{\xi} \rho_{\xi}(\eta_m, \ldots, \eta_2, l, \xi)) p_{j,k,s}(\eta_m, \ldots, l, \eta^{(l)})\}.
\end{align*}

By using the fact that (2.8) holds for $\xi$ satisfying $\xi_1 \leq K$, we see immediately from (2.9) that (2.8) holds for all $\xi \in \Sigma(1)$.

Remark that we have from (2.7)

\[ w_{j,k,s}(\xi) = p_{j,k,s}(\xi) \quad \text{for all } \xi \in \Sigma^*, \]

from which it follows that

\[ \nu_{j,k,s}(w_{j',k',s}) = \delta_{j,j'} \delta_{k,k'} . \]

Define $E_{j,k,s}'$ by

\begin{equation}
E_{j,k,s}'(\xi) = \nu_{j,k,s}(\hat{w}_{j',k',s}) w_{j,k,s}(\xi).
\end{equation}
Then, we have

\[(2.11)\quad E_{j, l} E_{j', k, s} = \delta_{j, j'} \delta_{k, k} E_{j, k, s},\]

and

\[(2.12)\quad L_{s} E_{j, k, s} = \tilde{\lambda}_{j, k, s} E_{j, k, s}.\]

Now use the following expression

\[L_{s}^{m} u(\xi) = \sum_{\eta_{1}, \ldots, \eta_{q}} \exp(S_{s}^{(\eta_{1}, \ldots, \eta_{q})}) L_{s}^{m} u(\eta_{1}, \ldots, \eta_{q}, \xi)\]

\[= \sum_{\eta_{1}, \ldots, \eta_{q}} \exp(S_{s}^{(\eta_{1}, \ldots, \eta_{q})}) \{L_{s}^{m} u(\eta_{1}, \ldots, \eta_{1}, \eta^{(1)})\}
+ \{L_{s}^{m} u(\eta_{1}, \ldots, \eta_{1}, \eta^{(1)}) - L_{s}^{m} u(\eta_{1}, \ldots, \eta_{1}, \eta^{(1)})\}\]

We get immediately

\[|\Pi| \leq C \theta^{t} \exp(q \bar{P}(R e r_{0})) \exp(p \bar{P}(R e r_{0})) ||u||_{0},\]

where we set \(\bar{P}(r_{0}) = \max P(r_{j})\) and \(C\) is a positive constant independent of \(p\) and \(q\).

By using (2.7) and Lemma 2.2 we have

\[L_{s}^{m} u(\eta_{1}, \ldots, \eta_{1}, \eta^{(1)}) = \sum_{j=1}^{k} \sum_{h=1}^{k} (\tilde{\lambda}_{j, k, s})^{h} \nu_{j, s}(u) \cdot p_{j, k, s}(\eta_{q}, \ldots, \eta_{1}, \eta^{(1)}) + S_{s}^{0} u(\eta_{q}, \ldots, \eta_{1}, \eta^{(1)}).\]

Applying the argument in Section 2 of [3] to the above expression we have

\[|I - \sum_{j=1}^{k} \sum_{h=1}^{k} (\tilde{\lambda}_{j, k, s})^{h} E_{j, k, s} u(\xi)| \leq C ||u||_{0} \{1 - 2\theta^{t} \exp(q \bar{P}(R e r_{0}))) + \gamma_{1}^{n}\}\]

Then, by exchanging \(D_{1}\) by a smaller neighborhood of \(s_{0}\) if necessary, we have

**Lemma 2.4.** There exist a neighborhood \(D_{1}\) of \(s_{0}\) in \(C\) and a positive constant \(\delta_{2}\) such that we have for all \(s \in D_{1}\)

\[(2.13)\quad 1 - \delta_{2}/2 \leq |\tilde{\lambda}_{j, k, s}| \leq 1 + \delta_{2}/2,\]

\[(2.14)\quad ||L_{s}^{m} u - \sum_{j=1}^{k} \sum_{h=1}^{k} (\tilde{\lambda}_{j, k, s})^{h} E_{j, k, s} u(\xi)||_{0} \leq C ||u||_{0} (1 - 2\delta_{2})^{n}.\]

**2.3. On the decomposition of \(L_{s}^{m}\).**

By using the same argument as in [3], we have the following two estimates concerning \(L_{s}^{m}\) for any \(u \in \mathcal{F}_{e}(\Sigma_{1})\)
Thus, by applying the theorem of Ionescu Turcia, Marinescu [6] to the pair of the spaces $C(\Sigma_J)$ and $\mathcal{F}_\alpha(\Sigma_J)$, we have from the above inequalities the following decomposition of $L'_{\tau_0}$ in $\mathcal{F}_\alpha(\Sigma_J)$

(2.15) \[ L'_{\tau_0} E'_j = \sum_{j=1}^{J} c_j E'_j + S' = E' + S', \]

where

\[ L'_{\tau_0} E'_j = c_j E'_j \quad \text{and} \quad |c_j| = 1 \quad \text{for all } j, \]

\[ E'_j E'_l = \delta_{j,l} E'_j \quad \text{for all } j, l, \]

\[ E'_j S' = S'E'_j = 0 \quad \text{for all } j, \]

the spectral radius of $S' < 1$.

We shall show that there is no eigenvalue of $E'$ besides $\lambda_{j,k,\tau_0}$. Suppose that $c$ such that $|c| = 1$ is an eigenvalue and $w \in \mathcal{F}_\alpha(\Sigma_J)$ is its associated eigenfunction. Note that $w \equiv 0$ implies that $\bar{w} = [w] \equiv 0$. Indeed, suppose that $\bar{w} \equiv 0$. Note that (2.10) gives us $E'_j, k, w = 0$ for all $j, k$. Then, the application of (2.14) to $w$ implies that for all $\xi \in \Sigma_J$

\[ |w(\xi)| = |c^{-m} L'_{\tau_0} w(\xi)| \leq C||w||_\infty (1 - 2\delta_2)^m. \]

By letting $m$ tend to the infinity, we have $w(\xi) = 0$. This implies that $w \equiv 0$. Thus our assertion is proved.

It is evident that $\bar{w} \equiv 0$ satisfies

\[ \bar{L'}_{\tau_0} \bar{w} = c \bar{w}. \]

Lemma 2.2 shows that $c$ must be one of $\lambda_{j,k,\tau_0}$'s. Therefore, the eigenvalues of $E'$ are $\lambda_{j,k,\tau_0}$'s. Then we have

\[ E' = \sum_{j=1}^{J} \sum_{k=1}^{K_j} \lambda_{j,k,\tau_0} E'_{j,k,\tau_0}, \]

\[ \dim \text{Range } E' = \sum_{j=1}^{J} k_j = \bar{k}. \]

The decomposition (2.15) shows that for all $u \in \mathcal{F}_\alpha(\Sigma_J)$ and $m$

\[ L'_{\tau_0} u = \sum_{j=1}^{J} \sum_{k=1}^{K_j} (\lambda_{j,k,\tau_0})^m E'_{j,k,\tau_0} u + S'm u, \]

and (2.14) implies that

\[ ||S'm u||_\infty \leq C||u||_\infty (1 - 2\delta_2)^m. \]
This shows that

the spectral radius of $S' \leq 1 - 2\delta_2$.

By means of perturbation theory, we see that there are a neighborhood $D_2 \subset D_1$ of $s_0$ and a constant $0 < \delta_3 \leq \delta_2$ such that for all $s \in D_2$

$$\mathcal{L}' = E' + S',$$

$$\dim \text{Range } E' = \bar{k},$$

all the eigenvalues of $E' \in \{\lambda; 1 - \frac{1}{3}\delta_3 \leq |\lambda| \leq 1 + \frac{1}{3}\delta_3\}$

the spectral radius of $S' < 1 - 3\delta_3$.

On the other hand, it is proved that $\tilde{\lambda}_{j,k,s}$ and $w_{j,k,s}$ are eigenpairs of $\mathcal{L}'$, which satisfy $1 - \frac{1}{3}\delta_3 \leq |\tilde{\lambda}_{j,k,s}| \leq 1 + \frac{1}{3}\delta_3$ and that $w_{j,k,s}$ are linearly independent. This fact shows that

$$\dim \text{Range } \sum_{j=1}^{k} \sum_{k=1}^{i_j} \tilde{\lambda}_{j,k,s} E'_{j,k,s} = \bar{k}.$$

Therefore it follows that

$$E' = \sum_{j=1}^{k} \sum_{k=1}^{i_j} \tilde{\lambda}_{j,k,s} E'_{j,k,s}.$$

Denote by $\mu^0_{l,i}, l = 1, 2, \ldots, l_0$ the distinct values of $\tilde{\lambda}_{j,k,s}$, $k = 1, 2, \ldots, k_j, j = 1, 2, \ldots, h$, and rename all the $\tilde{\lambda}_{j,k,s}$ such that $\tilde{\lambda}_{j,k,s} = \mu^0_{l,i}$ as $\mu_{(l,i),s}, i = 1, 2, \ldots, i_l$. Hereafter we denote by $F'_{(l,i),s}$ and $w_{(l,i),s}$ the corresponding $E'_{j,k,s}$ and $w_{j,k,s}$.

We set

$$F'_{0,s} = \sum_{j=1}^{k} \sum_{k=1}^{i_j} \tilde{\lambda}_{j,k,s} E'_{j,1,s}$$

and

$$F'_{l,s} = \sum_{i=1}^{i_l} \mu_{(l,i),s} F'_{(l,i),s}.$$

Then, summing up the argument in this subsection we have the following

**Proposition 2.5.** There are $s_0 \in R$, a neighborhood $D_2$ of $s_0$ in $C$ and a positive constant $\delta_3$ such that, for all $s \in D_2$, $\mathcal{L}'$ has a decomposition

$$\mathcal{L}' = \sum_{l=1}^{l_0} F'_{l,s} + S'$$

satisfying the following:

1. $F'_{l,s} S' = S' F'_{l,s} = 0$, for all $l = 0, 1, \ldots, l_0$.
2. $F'_{l,s} F'_{k,s} = F'_{k,s} F'_{l,s} = 0$ for all $l, k = 0, 1, \ldots, l_0$ such that $l \neq k$.
3. For $0 \leq l \leq l_0$, the dimension of the range of $F'_{l,s} = i_l$ for all $s \in D_2$ and the
eigenvalues of $F', s$ are $\mu_{(i,i), s}, i=1, 2, \cdots, i$, which satisfy
\[ |\mu_{(i,i), s} - \mu_i\delta_3| < \frac{1}{3} |\mu_i - \mu_i'| > \delta_3 (l \neq l') . \]

Especially, $\mu_0 = 1, i_0 = h$ and $\mu_{(0,j), s} = \lambda_j, (j = 1, 2, \cdots, h)$.

(4) the spectral radius of $S'_s < 3 - \delta_3$.

3. Spectrum of $\mathcal{L}_{t,s}$

Let $\mathcal{L}_{t,s}$ be the operator in $\Sigma^+_t$ defined by
\[ \mathcal{L}_{t,s} u(t) = \sum_{\sigma \in \Gamma^+} \exp(r_\sigma(t,s)) u(\eta) . \]

We shall show the existence of $s$ such that $\mathcal{L}_{t,s}$ has 1 as an eigenvalue.

Even though the following is a well known fact on perturbations of linear operators, we shall mention it in the form of lemma to make clear the argument below.

**Lemma 3.1.** Let $T$ be a bounded operator in Banach space $B$ with norm $|| \cdot ||$. Suppose that $\{ \lambda; |\lambda - \mu| = \alpha \} (\mu \in C, \alpha > 0)$ is contained in the resolvent set of $T$, and that the projection
\[ P = \frac{1}{2\pi \sqrt{-1}} \oint_{|\lambda - \mu| = \alpha} (\lambda - T)^{-1} d\lambda \]
is of finite rank $h$. Let $\{ w_1, w_2, \cdots, w_h \}$ is a basis of the range of $P$.

Then there is $a > 0$ such that
\[ ||T' - T|| \leq a \]
implies the following:

(i) $\{ \lambda; |\lambda - \mu| = \alpha \}$ is also contained in the resolvent set of $T'$.

(ii) The projection
\[ P' = \frac{1}{2\pi \sqrt{-1}} \oint_{|\lambda - \mu| = \alpha} (\lambda - T')^{-1} d\lambda \]
is of rank $h$ and
\[ w_j' = P'w_j (j = 1, 2, \cdots, h) \]
form a basis of the range of $P'$.

(iii) $||\sum_{j=1}^h a_j w_j - \sum_{j=1}^h a_j' w_j'|| \leq ||T' - T||$
implies that
\[ \sum_{j=1}^h |a_j' - a_j| \leq C_1 ||T' - T|| , \]
where $C_1$ is a constant depending on $\mu$ and $\alpha$ but independent of $T'$.

Suppose that Lemma 2.4 and Proposition 2.5 hold for the open disk $D_2 = \{ s; |s-s_0| < \alpha_0 \}$ ($\alpha_0 > 0$). Recall that $\tilde{\lambda}_{j,s}, j = 1, 2, \ldots, h$ are analytic in $D_2$, and satisfies

$$\tilde{\lambda}_{j,s_0} = 1, \quad -\frac{d}{ds} \tilde{\lambda}_{j,s} \bigg|_{s=s_0} > 0.$$ 

Thus, by exchanging $\alpha_0$ by a smaller one if necessary, we may assume the following:

$$|\tilde{\lambda}_{s,s_0} - 1| \leq \frac{\delta_3}{3} \quad \text{for all } s \in D_2,$$

(3.1) \hspace{1cm} $$|\tilde{\lambda}_{s,s} - 1| \geq \delta_1 |s-s_0| \quad \text{for all } s \in \{ s; |s-s_0| \leq \alpha_0 \} (\delta_1 > 0).$$

By the same argument as in [3, Section 3] we have

$$\|L_{s,s} - L_{s,s} \|_s \to 0 \quad \text{uniformly in } s \in D_2 \text{ as } \epsilon \to 0.$$ 

Therefore by applying Lemma 3.1 to the pair of operators $T = L_{s}, T' = L_{s,s}$ we have

**Lemma 3.2.** There are positive constants $\epsilon_0$ and $\delta_4$ such that for all $0 < \epsilon \leq \epsilon_0$ and $s \in D_2$ we have the following decomposition of $L_{s,s}$:

(3.2) \hspace{1cm} $$L_{s,s} = \sum_{l=0}^{l_0} C_{(l),s,s} + S_{s,s},$$

where

(3.3) \hspace{1cm} $$C_{(l),s,s} = C_{(k),s,s} = 0 \quad \text{if } l \neq k,$$

(3.4) \hspace{1cm} $$C_{(l),s,s} S_{s,s} = S_{s,s} C_{(l),s,s} = 0,$$

(3.5) \hspace{1cm} $$\text{the spectral radius of } S_{s,s} < 1-2\delta_3, \quad \dim \text{Range } C_{(1),s,s} = i, \quad \text{for all } 0 < \epsilon < \epsilon_0,$$

(3.6) \hspace{1cm} $$\sum_{\sigma^2 \epsilon \epsilon} \exp (\Re r_n(\epsilon, s)) \leq C(1 + \delta_3)^n \quad \text{for all } n.$$ 

Moreover, denoting the eigenvalues of $C_{(l),s,s}$ by $\lambda_{l,i}(\epsilon, s), i = 0, 1, \ldots, i, l = 1, 2, \ldots, h$ we have for all $0 < \epsilon \leq \epsilon_0$

(3.7) \hspace{1cm} $$|\lambda_{i,j}(\epsilon, s) - \mu^j| \leq \frac{2}{3} \delta_3 \quad \text{for all } s \in D_2, \quad l = 0, 1, \ldots, l_0,$$

(3.8) \hspace{1cm} $$|\lambda_{0,j}(\epsilon, s) - 1| > \delta_4 \quad \text{for all } s \in \{ s; |s-s_0| = \alpha_0 \}.$$ 

By using the decomposition (3.2) with the properties (3.6) and (3.7), we have the expression
Recall that \( w_{(i),s}, i=1,2, \ldots, i \) form a basis of \( \text{Range } L_s \). From the continuity of \( L_s \) on \( \varepsilon \),

\[
w_{(i),s} = \mathcal{P}_{(i),s,w_{(i),s}}, \quad i = 1, 2, \ldots, i
\]

are linearly independent for all \( 0<\varepsilon \leq \varepsilon_1 \) and \( s \in D_2 \). This fact implies that \( \{w_{(i),s}; i=1,2, \ldots, i\} \) is a basis of the range \( \mathcal{P}_{(i),s} \). It holds that

\[
w_{(i),s} \text{ is analytic in } s \in D_2,
\]

\[
w_{(i),s} \rightarrow w_{(i),s} \text{ uniformly in } s \in D_2 \text{ as } \varepsilon \rightarrow 0 .
\]

Therefore \( \mathcal{E}_{(i),s,w_{(i),s}} \) is a linear combination of \( w_{(i),s}, i=1,2, \ldots, i \), that is,

\[
\mathcal{E}_{(i),s,w_{(i),s}} = \sum_{i=1}^{i} a_{(i),s}(\varepsilon, s)w_{(i),s} .
\]

Applying Lemma 3.1 we have

\[
a_{(i),s} \text{ depends on } s \in D_2 \text{ analytically},
\]

\[
a_{(i),s}(\varepsilon, s) \rightarrow a_{(i),s}(0, s) \text{ uniformly in } s \in D_2 \text{ as } \varepsilon \rightarrow 0 .
\]

Let \( \mathcal{A}(\varepsilon, s) \) be the \( i \times i \) matrix defined by

\[
\mathcal{A}(\varepsilon, s) = [a_{(i),s}]_{i=1,2, \ldots, i} .
\]

It is evident from Lemma 3.1 that the eigenvalues of \( \mathcal{E}_{(i),s} \) in the space \( \mathcal{F}(\varepsilon, s) \) and those of \( \mathcal{A}(\varepsilon, s) \) in \( \mathcal{C}^i \) coincide including the multiplicities. Set

\[
f(\lambda, s; \varepsilon) = \det (\lambda I - \mathcal{A}(\varepsilon, s)) = \lambda^2 + \lambda^2 b_{1,1}(\varepsilon, s) + \lambda^2 b_{2,1}(\varepsilon, s) + \cdots + b_{i,i}(\varepsilon, s) .
\]

Then it follows from the properties of \( a_{(i),s} \) that

\[
b_{1,1}(\varepsilon, s) \text{ is analytic in } s \in D_2 ,
\]

\[
b_{1,1}(\varepsilon, s) \text{ is continuous in } \varepsilon \in [0, \varepsilon_0] \text{ uniformly for } s \in D_2 .
\]

**Lemma 3.3.** The eigenvalues of \( \mathcal{E}_{(i),s} \) are the roots of \( f(\lambda, s; \varepsilon)=0 \), which is the polynomial given by (3.9) whose coefficients satisfy (3.10) and (3.11). Moreover, for each \( 0<\varepsilon \leq \varepsilon_0 \), \( f_0(1, s; \varepsilon)=0 \) has exactly \( h \) zeros in \( \{s; |s-s_0| \leq \alpha_0\} \), which converge to \( s_0 \) when \( \varepsilon \) tends to zero.

**Proof.** Note that \( i_0=h \) and

\[
\mathcal{E}(\varepsilon) = \frac{1}{2\pi i} \int_{|z-w_0|=\delta_1} z(z-L_{\varepsilon})^{-1} dz
\]

and

\[
\mathcal{P}(\varepsilon) = \frac{1}{2\pi i} \int_{|z-w_0|=\delta_1} (z-L_{\varepsilon})^{-1} dz .
\]
Then (3.1) shows that

\[ f_0(1, s; 0) \neq 0 \text{ for all } 0 < |s - s_0| \leq \alpha_0, \]

\( s = s_0 \) is a zero point of \( h \)-th order of \( f_0(l, s_0). \)

Thus, \( f_0(1, s; 0) \) has exactly \( h \) zeros in \( \{s; |s - s_0| \leq \alpha_0\} \). On the other hand, (3.11) and (3.8) imply that the number of zero points of \( f_0(1, s; \varepsilon) \) in \( \{s; |s - s_0| \leq \alpha_0\} \) is invariant for all \( 0 \leq \varepsilon \leq \varepsilon_0 \). Since the dependency on \( \varepsilon \) of the zero points of \( f_0(1, s; \varepsilon) \) is continuous, they converge to those of \( f_0(1, s; 0) = 0 \), which are equal to \( s_0 \). Thus the assertion of the lemma is proved.

4. Proof of Theorems

In order to show Theorem 1, we apply Theorem 2 of [9, Section 4] or Theorem 4 of [2, Section 4] to \( \mathcal{L}_{*,*} \). By exchanging \( \varepsilon_0 \) by a smaller one if necessary we may assume that

\[ \theta(1 + \delta_0) < 1. \]

Then, the application of the theorems of [9,2] to \( \mathcal{L}_{*,*} \) assures that

\[ Z_\varepsilon(s) \text{ is meromorphic in } \Re s > s_0 + \alpha_0 \]

and is of the form

\[ Z_\varepsilon(s) = \exp(\phi(s, \varepsilon)) \prod_{i=0}^{l_0} f_i(1, s; \varepsilon)^{-1}, \]

where \( \phi(\cdot, \varepsilon) \) is holomorphic in \( \Re s > s_0 + \alpha_0 \). From Lemma 3.3, we have Theorem 1.

As to Theorem 2, follow the argument in [3, Section 4] by using Theorem 1 of the present paper instead of Theorem 1 of [3], and we have Theorem 2.

References


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