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# ORIENTATION REVERSING INVOLUTIONS ON CLOSED 3-MANIFOLDS

Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

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## 1. Introduction.

Let M be a closed connected oientable 3-manifold admitting an oreintation reversing involtuion  $\tau$  (i.e.  $\tau^2 = \text{identity}$  and  $\tau_*([M]) = -[M]$  for the fundamental class [M] of M).

By Smith theory, each component of the fixed point set of  $\tau$ , Fix $(\tau, M)$ , is a point or a closed surface and  $\chi(\text{Fix}(\tau, M)) \equiv 0 \pmod{2} (\chi(X)$  is the Euler characteristic of X). A. Kawauchi [5] proved that for any  $(M, \tau)$ , Tor  $H_1(M; \mathbb{Z}) \cong A \oplus A$  or  $\mathbb{Z}_2 \oplus A \oplus A$  for some abelian group A, and that  $\operatorname{rank}_{\mathbb{Z}_2} H_1(\operatorname{Fix}(\tau, M); \mathbb{Z}_2) \equiv 0 \pmod{2}$  if and only if Tor  $H_1(M; \mathbb{Z}) \cong A \oplus A$ . J. Hempel has proved in [3] that if Fix $(\tau, M)$  is empty or contains a closed orientable surface of positive genus, then the first Betti number of M is greater than zero. He has also shown in [4] that if  $\pi_1(M)$  is not isomorphic to {1} or and  $\mathbb{Z}_2$  is not virtually representable to  $\mathbb{Z}$ , then Fix $(\tau, M)$  consists of a 2-sphere or two points, or contains a projective plane.

The auther gave a characterization of  $Fix(\tau, M)$  when M is a rational homoogy 3-sphere in [6] and, for a general M, an inequality on the first Betti numbers of M and  $Fix(\tau, M)$  in [7]. In this paper we give a complete characterization of the topological type of  $Fix(\tau, M)$  for a general M.

NOTATIONS. For a space X, let  $\beta_i(X)$  denote the *i*<sup>th</sup> Betti number and  $\beta_i(X; \mathbf{Z}_2)$  the  $\mathbf{Z}_2$ -coefficient Betti number. For a group G, let  $\beta_1(G) = \operatorname{rank}_{\mathbf{Z}} H_1(G; \mathbf{Z})$  and  $\beta_1(G; \mathbf{Z}_2) = \operatorname{rank}_{\mathbf{Z}_2} H_1(G; \mathbf{Z}_2)$ .

First, we classify  $(M, \tau)$  into two types.

**Proposition 1.** For any  $(M, \tau)$ , one of the following holds: (1)  $M - \text{Fix}(\tau, M)$  consists of two components and  $\text{Fix}(\tau, M)$  is a closed orientable 2-manifold.

(2)  $M - \operatorname{Fix}(\tau, M)$  is connected.

For each type of  $(M, \tau)$ , we shall prove the following:

**Theorem 2.** For any  $(M, \tau)$  with  $M - Fix(\tau, M)$  disconnected, we have the following (1)–(3):

- (1) Tor  $H_1(M; \mathbb{Z}) \cong A \oplus A$  for some ablelian group A.
- (2)  $\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M)) \leq 1 + \beta_1(M).$
- (3)  $\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M)) \equiv 1 + \beta_1(M) \pmod{2}$ .

REMARK 1. (1) was proved by Kawauchi [5].

**Theorem 3.** Let G be an ableian group and E a closed orientable 2-manifold satisfying the following conditions (1)-(3):

- (1) Tor  $G \simeq A \oplus A$  for some ableian group A.
- (2)  $\beta_1(E)/2 + \beta_2(E) \le 1 + \beta_1(G).$
- (3)  $\beta_1(E)/2 + \beta_2(E) \equiv 1 + \beta_1(G) \pmod{2}$ .

Then there exists  $(M, \tau)$  such that  $M - \text{Fix}(\tau, M)$  is disconnected,  $H_1(M; \mathbb{Z}) \simeq G$  and  $\text{Fix}(\tau, M) = E$ .

**Theorem 4.** For any  $(M, \tau)$  with  $M - \text{Fix}(\tau, M)$  connected, we have the folloaing (1)-(7);

- (1) Tor  $H_1(M, \mathbb{Z}) \simeq A \oplus A$  or  $\mathbb{Z}_2 \oplus A \oplus A$  for some abelian group A.
- (2)  $\beta_1(\operatorname{Fix}(\tau, M); \mathbb{Z}_2) \equiv \beta_1(M; \mathbb{Z}_2) \beta_1(M) \pmod{2}.$
- (3)  $\sum_{i=0}^{2} \beta_{i}(\operatorname{Fix}(\tau, M); \mathbf{Z}_{2}) \leq 2 + 2\beta_{1}(M; \mathbf{Z}_{2}).$
- (4)  $\chi(\operatorname{Fix}(\tau, M))/2 2\beta_2(\operatorname{Fix}(\tau, M)) \ge 1 \beta_1(M).$
- (5)  $\chi(\text{Fix}(\tau, M))/2 \leq 1 + \beta_1(M).$
- (6)  $\chi(\text{Fix}(\tau, M))/2 \equiv 1 + \beta_1(M) \pmod{2}$ .

(7) Consider a direct sum decomposition of Tor  $H_1(M; \mathbb{Z})$  such that each factor is a cyclic group of prime power order. Let u be the number of  $\mathbb{Z}_2$  factors. Then the number of nonorientable surfaces of odd genera contained in  $Fix(\tau, M)$  is not greater than u.

REMARK 2. (1) and (2) were proved by Kawauchi [5]. (3) is obtained by Smith theory (cf. [1] p. 126).

**Theorem 5.** Let G be an abelian group and X be a disjoint union of points and closed surfaces. If G and X satisfy the following conditions (1)-(7):

- (1) Tor  $G \simeq A \oplus A$  or  $\mathbb{Z}_2 \oplus A \oplus A$  for some abelian group A.
- (2)  $\beta_1(X; \mathbf{Z}_2) \equiv \beta_1(G; \mathbf{Z}_2) \beta_1(G) \pmod{2}$ .
- (3)  $\sum_{i=0}^{2} \beta_i(X; \mathbf{Z}_2) \leq 2 + 2\beta_1(G; \mathbf{Z}_2).$
- (4)  $\chi(X)/2-2\beta_2(X) \ge 1-\beta_1(G).$
- (5)  $\chi(X)/2 \leq 1 + \beta_1(G)$ .
- (6)  $\chi(X)/2 \equiv 1 + \beta_1(G) \pmod{2}$ .

(7) Consider a direct sum decomposition of Tor G such that each factor is a cyclic group of prime power order. Let u be the number of  $Z_2$  factors. Then the number

of nonorientable surfaces of odd genera contained in X is not greater than u.

Then there exists  $(M, \tau)$  such that  $M - \operatorname{Fix}(\tau, M)$  is connected,  $H_1(M; \mathbb{Z}) \cong G$  and  $\operatorname{Fix}(\tau, M) = X$ .

Throughout this paper, we will work in the piecewise-linear category, and a surface is assumed to be compact and connected.

The author owes the idea of  $\beta_i^+$  to Prof. M. Sakuma. She is very grateful to him for suggesting her Lemma 6.

### 2. Proofs of Proposition 1 and Theorems 2 and 4.

Proof of Proposition 1. We will show that if  $M - \text{Fix}(\tau, M)$  is disconnected, then  $M - \text{Fix}(\tau, M)$  consists of two components and  $\text{Fix}(\tau, M)$  is a closed orientable 2-manifold.

Let  $C_1, C_2, \dots, C_r$  be the components of  $M - \mathring{N}(\operatorname{Fix}(\tau, M))$ , where  $\mathring{N}$ (Fix $(\tau, M)$ ) is the interior of a  $\tau$ -invariant regular neighborhood of Fix $(\tau, M)$ . Then the identifying space of  $C_1 \cup C_2 \cup \dots \cup C_r$  by the identifying map  $\tau |_{\bigcup \partial C_i}$  is homeomorphic to M. Since  $\tau^2 =$ identity and M is connected, we can see that r=2 and  $\tau(C_1) = C_2$ . Hence  $\tau(\partial C_1) = \partial C_2$  and for each component X of Fix $(\tau, M), \partial N(X)$  consists of 2-components. In general, if there exists an isolated point p in Fix $(\tau, M), N(p)$  is a ball, and if there exists a nonorientable surface F in Fix $(\tau, M), N(F)$  is a twisted I-bundle over F. Hence this X must be an orientable surface. This completes the proof.

Condider the homomorphism  $\tau_{*}^{(i)}$  on  $H_i(M; \mathbf{Q})$  induced by  $\tau$  (i=1, 2). We may regard  $\tau_{*}^{(i)}$  as a linear transformation of the vector space  $H_i(M; \mathbf{Q})$  over  $\mathbf{Q}$ . Since  $(\tau_{*}^{(i)})^2$ =identity, every eigenvalue of  $\tau_{*}^{(i)}$  is +1 or -1. Let  $B_i^+$  and  $B_i^-$  be the eigenspace of  $H_i(M; \mathbf{Q})$  corresponding to +1 and -1, respectively. Put  $\beta_i^+ = \dim B_i^+$  and  $\beta_i^- = \dim B_i^-$ . Clearly,  $\beta_1^+ + \beta_1^- = \beta_2^+ + \beta_2^- = \beta_1(M)$ . We have the folloaing lemma:

**Lemma 6.** For any  $(\tau, M)$ , we have

$$\chi(\operatorname{Fix}(\tau, M)) = 2(1 + \beta_1(M) - 2\beta_1^+).$$

Proof. Let  $\{a_1, a_2, \dots, a_{\beta_1}^+\}$  be a basis of  $B_1^+$  and  $\{b_1, b_2, \dots, b_{\beta_1}^-\}$  a basis of  $B_1^-$ . Then there exsists a basis  $\{a_1, a_2, \dots, a_{\beta_1}^+, \bar{b}_1, \bar{b}_2, \dots, \bar{b}_{\beta_1}^-\}$  of  $H_2(M; \mathbf{Q})$  such that  $\operatorname{Int}(a_i, a_j) = \operatorname{Int}(b_i, \bar{b}_j) = \delta_{ij}$  and  $\operatorname{Int}(a_i, \bar{b}_j) = \operatorname{Int}(b_i, a_j) = 0$   $(1 \le i \le \beta_1^+, 1 \le j \le \beta_1^-)$ , where  $\operatorname{Int}(x, y)$  is the intersection number of x and y, and  $\delta_{ij}$  is the Kronecker delta. Then we have

$$\begin{split} \operatorname{Int}(a_i, \, \tau_*(\boldsymbol{a}_j)) &= \langle [M], \, \varphi(a_i) \cup \varphi(\tau_*(\boldsymbol{a}_j)) \rangle \\ &= \langle \tau_*[M], \, \varphi(\tau_*(a_i)) \cup \varphi(\boldsymbol{a}_j) \rangle \\ &= \langle -[M], \, \varphi(a_i) \cup \varphi(\boldsymbol{a}_j) \rangle = -\delta_{ij} \end{split}$$

and

$$egin{aligned} &\operatorname{Int}(b_i,\, au_*(m{a}_j)) = \langle [M],\, arphi(b_i) \cup arphi( au_*(m{a}_j)) 
angle \ &= -\langle [M],\, (-arphi(b_i)) \cup arphi(m{a}_j) 
angle = 0 \ , \ & (arphi \ ext{is the Ponncaré dual map}). \end{aligned}$$

Hence

$$au_{st}(m{a}_i) = -m{a}_i$$
 .

By the same way, we have  $\tau_*(\bar{b}_i) = \bar{b}_i$ . Hence  $\{a_1, a_2, \dots, a_{\beta_1^+}\}$  is a basis of  $B_2^-$  and  $\{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_{\beta_1^-}\}$  is a basis of  $B_2^+$ . Therefore  $\beta_1^+ = \beta_2^-$  and  $\beta_1^- = \beta_2^+$ .

On the other hand, it is known that for a periodic transformation f on a compact ENR X,  $L(f) = \chi(\text{Fix}(f, X))$ , where L(f) is the Lefschetz number of f (cf. [2], p. 261). For our  $(M, \tau)$ ,

$$L(\tau) = \sum_{i=0}^{3} (-1)^{i} \operatorname{Trace} \tau_{*}^{(i)}$$
  
= 1-(\beta\_{1}^{+}-\beta\_{1}^{-})+(\beta\_{2}^{+}-\beta\_{2}^{-})-(-1)  
= 2(1+\beta\_{1}(M)-2\beta\_{1}^{+}).

Hence we have

$$\chi(\text{Fix}(\tau, M)) = 2(1 + \beta_1(M) - 2\beta_1^+).$$

This completes the proof.

Proof of Theorem 2. (1) holds from a theorem of Kawauchi [5], since for any closed orientable surface E,  $\beta_1(E; \mathbb{Z}_2) \equiv 0 \pmod{2}$ .

(3) holds from Lemma 6. Since

$$egin{aligned} \chi( ext{Fix}( au,M)) &= -eta_1( ext{Fix}( au,M)) + 2eta_2( ext{Fix}( au,M)) \ &= 2(1 + eta_1(M) - 2eta_1^+) \ , \end{aligned}$$

we have

$$\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M))$$
  
$$\equiv -\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M)) \equiv \beta_1(M) + 1 \pmod{2}.$$

We will prove (2). Let  $M_1$  and  $M_2$  be components of  $M - \mathring{N}(\operatorname{Fix}(\tau, M))$ . Then  $\partial M_1$  is homeomorphic to  $\operatorname{Fix}(\tau, M)$ . We identify  $\partial M_1$  with  $\operatorname{Fix}(\tau, M)$ . Let I (resp. J) be the homomorphism from  $H_1(\operatorname{Fix}(\tau, M); \mathbf{Q})$  to  $H_1(M; \mathbf{Q})$ (resp.  $H_1(M_1; \mathbf{Q})$ ) induced by the inclusion map. We show Ker  $I = \operatorname{Ker} J$ .

Ker  $I \supset$  Ker J is trivial. Let x=[C] be an element of Ker I. Then there exists a 2-chain D in M such that  $\partial D = C$ . Put  $D_i = D \cap M_i$  (i=1, 2). By a tiny collapsing of  $D_1 + \tau(D_2)$ , we may obtain a 2-chain D' in  $M_1$  with  $\partial D' = C$ .

Therefore we obtained dim Im I=dim Im J. Note that for any orientable 3-manifold M with boundary, dim Im (incl.<sub>\*</sub>:  $H_1(\partial M; \mathbf{Q}) \rightarrow H_1(M; \mathbf{Q})) =$ dim  $H_1(\partial M; \mathbf{Q})/2$ . Hence dim Im I=dim  $H_1(\text{Fix}(\tau, M); \mathbf{Q})/2$ . On the other hand, for any  $x \in \text{Im } I$ ,  $\tau_*(x) = x$ . Hence Im  $I \subset B_1^+$ . Thus we obtain that

$$\beta_1(\operatorname{Fix}(\tau, M))/2 = \dim H_1(\operatorname{Fix}(\tau, M); \boldsymbol{Q})/2 \leq \beta_1^+$$

Therefore by Lemma 6,

$$2\beta_2(\operatorname{Fix}(\tau, M)) - \beta_1(\operatorname{Fix}(\tau, M)) = (\operatorname{Fix}(\tau, M))$$
  
= 2(1+ $\beta_1(M) - 2\beta_1^+$ )  
 $\leq 2(1+\beta_1(M) - \beta_1(\operatorname{Fix}(\tau, M)).$ 

Hence

$$\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M)) \leq \beta_1(M) + 1$$
.

This completes the proof.

Proof of Theorem 4. (1) and (2) are proved by Kawauchi [5]. By Smith theory  $\sum_{i} \beta_{i}(\operatorname{Fix}(\tau, M); \mathbb{Z}_{2}) \leq \sum_{j} \beta_{j}(M; \mathbb{Z}_{2})$  (cf [1], p. 126), and for a 3-manifold  $M, \sum_{j} \beta_{j}(M; \mathbb{Z}_{2}) = 2\beta_{1}(M; \mathbb{Z}_{2}) + 2$ . Hence (3) holds.

Recall that  $\beta_1^+$  is a non negative integer. Hence by Lemma 6,  $0 \le 2\beta_1^+ = 1 + \beta_1(M) - (\chi(\operatorname{Fix}(\tau, M))/2)$  and  $\chi(\operatorname{Fix}(\tau, M))/2 = 1 + \beta_1(M) - 2\beta_1^+ \equiv 1 + \beta_1(M)$  (mod 2). Therefore (5) and (6) hold.

Note that  $M - \text{Fix}(\tau, M)$  is connected and for any orientable surface E contained in  $\text{Fix}(\tau, M), \tau(E) = E$  and  $[E] \subset B_2^+$ . Hence

$$egin{aligned} eta_2( ext{Fix}( au, M)) &\leq eta_2^+ \ &= eta_1(M) - eta_1^+ \ &= eta_1(M) - \{(1 + eta_1(M))/2 - m{\chi}( ext{Fix}( au, M))/4\} \end{aligned}$$

and

$$\chi(\operatorname{Fix}(\tau, M))/2 - 2\beta_2(\operatorname{Fix}(\tau, M)) \ge 1 - \beta_1(M).$$

Therefore (4) holds.

For (7), consider a  $\tau$ -invariant regular neighborhood N of the unoin of nonorientable surfaces of odd genera contained in Fix( $\tau$ , M) and let  $M' = M - \mathring{N}$ ( $\mathring{N}$  is the interoir of N). Then the Mayer-Vietoris exact sequence for M = $M' \cup N$  and  $\partial N = \partial M' = M' \cap N$  is as follows:

$$\cdots \to H_1(\partial N; \mathbf{Z}) \xrightarrow{I} H_1(M'; \mathbf{Z}) \oplus H_1(N; \mathbf{Z}) \xrightarrow{J} H_1(M; \mathbf{Z}) \to \cdots,$$

where  $I=(i_{1*}, i_{2*}), i_1: \partial N \rightarrow M'$  and  $i_2: \partial N \rightarrow N$  are inclusion maps. Since the image of  $i_{2*}$ ;  $H_2(\partial N; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})$  is torsion free, we see that  $J|_{\text{Tor } H_1(N; \mathbb{Z})}$ : Tor  $H_1(N; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$  is injective. Hence (7) holds.

#### 3. Basic manifolds and operations.

For proofs of Theorems 3 and 5, we construct eight basic manifolds with involutions and then introduce six additive operations on manifolds with involutions. For this purpose, we difine the *data* of  $(M, \tau)$  as follows: Suppose  $Fix(\tau, M)$  consists of m orientable surfaces  $E_1, E_2, \dots, E_m, n$  nonorientable surfaces  $F_1, F_2, \dots, F_n$  and p points, and that the number of nonorientable surfaces of odd genera contained in  $F_1, F_2, \dots, F_n$  is s. Then the data of  $(M, \tau)$  is defined to be

$$[\beta_1(M), s, r, p; g_1, g_2, \cdots, g_m; c_{,1} c_2, \cdots, c_n],$$

where  $r = (\beta_1(M; \mathbb{Z}_2) - \beta_1(M) - s)/2$ ,  $g_i = \beta_1(E_i)/2$ , the genus of  $E_i$   $(i=1, 2, \dots, m)$ and  $c_j = \beta_1(F_j; \mathbb{Z}_2)$ , the nonorientable genus of  $F_j$   $(j=1, 2, \dots, n)$ .

Now we consider eight basic manifolds with involutions.

(1)  $A_1 = (S^3, \tau); S^3$  is the 3-sphere.  $A_1$  has the data [0, 0, 0, 2; ; ].

 $\tau$  is defined as follows: We regard  $S^3$  as  $\mathbb{R}^3 \cup \{\infty\}$ . Then  $\tau: S^3 \to S^3$  is an involution defined by  $\tau(x, y, z) = (-x, -y, -z) ((x, y, z) \in \mathbb{R}^3)$  and  $\tau(\infty) = \infty$ .

(2)  $A_2 = (P^3, \tau)$ ;  $P^3$  is the 3-dimensional projective space.  $A_2$  has the data [0, 1, 0, 1; ; 1].

 $\tau$  is defined as follows: We regard  $P^3$  as  $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \le 1\}/((x, y, z) \sim (-x, -y, -z) (x^2 + y^2 + z^2 = 1))$ . Then  $\tau: P^3 \to P^3$  is an involution defined by  $\tau(x, y, z) = (-x, -y, -z)$ .

(3)  $A_3 = (S^2 \times S^1, \tau_1); A_3$  has the data [1, 0, 0, 0; 1; ].

 $\tau_1$  is defined as follows: Consider an orientation reversing involution  $\tau'$  on  $S^2$  such that  $\operatorname{Fix}(\tau', S^2)$  is a circle. Then  $\tau_1: S^2 \times S^1 \to S^2 \times S^1$  is an involution defined by  $\tau_1 = \tau' \times \operatorname{identity}$ .

(4)  $A_4 = (S^2 \times S^1, \tau_2); A_4$  has the data [1, 0, 0, 0; ;2].

 $au_2$  is defined as follows: Consider an orientation reversing involution au'on  $S^2$  as in (3). Regard  $S^2 \times S^1$  as the identifying space of  $S^2 \times I$  with the identifying map from  $S^2 \times \{1\}$  to  $S^2 \times \{0\} : (x, 1) \sim (\tau'(x), 0)$ , where I is the unit interval [0, 1]. Then  $S^2 \times S^1$  has an orientation reversing involution  $\tau_2$  extending  $\tau'$  with Fix $(\tau_2, S^1 \times S^2)$  a Klein bottle.

(5)  $A_5 = (N_1, \tau); A_5$  has the data [0, 0, 1, 2; ; 2] and  $H_1(N_1; \mathbb{Z}) \cong \mathbb{Z}_{2q} \oplus \mathbb{Z}_{2q}$  $(q \in \mathbb{N}).$ 

 $(N_1, \tau_1)$  is defined as follows: Consider  $(V, \tau)$  such that V is a solid torus with Fix $(\tau, V)$  two points. Let K be a closed curve in V such that [K]=bgenerates  $H_1(V; \mathbb{Z})$  with  $K \cap \tau(K) = \phi$  (see Figure 1). Let  $V_1$  and  $V_2$  be solid tori. Attach them to  $V - \mathring{N}(K \cup \tau(K))$  as follows:  $\partial V_1$  is identified with  $\partial N(K)$ 



## Figure 1

so that a meridian of  $\partial V_1$  is a curve C on  $\partial N(K)$  with  $[C] = qc - b \in H_1(V - \mathring{N}(K \cup \tau(K)); \mathbb{Z})$ .  $\partial V_2$  is identified with  $\partial N(\tau(K))$  so that a meridian of  $\partial V_1$  is a curve C' on  $\partial N(\tau(K))$  with  $[C'] = qc' - b \in H_1(V - \mathring{N}(K \cup \tau(K)); \mathbb{Z})$  (c and c' are generators of  $H_1(V - \mathring{N}(K \cup \tau(K)); \mathbb{Z})$  as indicated in Figure 1). We denote the resulting manifold by  $M_1$ . Then  $M_1$  has an orientation reversing involution and

$$H_1(M_1; \mathbf{Z}) \cong \langle b, c, c' : qc - b = 0, qc' + b = 0 \rangle.$$

Let  $F=\partial M_1=\partial V$  and  $M_2$  a quotient space of  $F\times I$  by the identifying map of  $F\times\{1\}: (x, 1)\sim(\tau'(x), 1)$ , where  $\tau'=\tau|_F$ . Then  $M_2$  has an involution  $\tau''$ , induced by  $\tau'$ . Fix $(\tau'', M_2)$  consists of a Klein bottle  $F\times\{1\}/\sim$ .

Let  $N_1 = M_1 \cup M_2$  where *h* is the identity map of the boundary *F*. Then  $N_1$  has an orientation reversing involution  $\tau$  such that  $Fix(\tau, N_1)$  consists of two points and a Klein bottle.

To compute  $H_1(N_1; \mathbb{Z})$ , we choose generators of  $H_1(M_2; \mathbb{Z}) \cong H_1(F \times \{1\} / \sim; \mathbb{Z})$  represented by curves as indicated in Figure 1. Then we have

$$H_1(M_2; \mathbf{Z}) \cong \langle x, y: 2x+2y = 0 \rangle.$$

We can check that

$$\begin{aligned} H_1(N_1; \mathbf{Z}) &\simeq \langle b, c, c', x, y; qc - b = 0, qc' + b = 0, 2x + 2y = 0, \\ & 2x = c + c', x + y = b \rangle \\ &\simeq \langle c, x; 2qc = 0, 2qx = 0 \rangle. \\ &\simeq \mathbf{Z}_{2q} \oplus \mathbf{Z}_{2q}. \end{aligned}$$

(6)  $A_6(g) = (N_2, \tau); A_6(g)$  has the data [1, 0, g, 2g+2; g; ] and Tor  $H_1(N_2; \mathbb{Z})$  $\simeq \bigoplus_{i=1}^{g} (\mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i}) (g, q_1, q_2, \cdots, q_g \in \mathbb{N}).$ 

Let F be a closed orientable surface of genus g. There exists an orientation

preserving involution  $\alpha$  on F such that the fixed point set consists of 2g+2 points. Consider  $F \times I$  and  $\tau': F \times I \rightarrow F \times I$ ,  $\tau'(x, t) = (\alpha(x), 1-t) \ (x \in F, t \in I)$ . Then  $\tau'$  is an orientation reversing involution on  $F \times I$ , and  $\operatorname{Fix}(\tau', F \times I)$  consists of 2g+2 points. Let  $K_1, K_2, \dots, K_g$  be closed curves in  $F \times I$  as indicated in Figure 2. These curves satisfy the following:





1.  $K_1, K_2, \dots, K_g, \tau(K_1), \tau(K_2), \dots, \tau(K_g)$  are mutually disjoint.

2. Let  $[K_i] = b_i \in H_1(F \times I; \mathbb{Z})$   $(i=1, 2, \dots, g)$ , then  $\{b_1, b_2, \dots, b_g\}$  is a basis of  $H_1(F \times I; \mathbb{Z})$ .

Consider 2g solid tori and attach them to  $F \times I - \bigcup_{i=1}^{s} \mathring{N}(K_i \cup \tau(K_i))$  as in (5) so that the resulting manifold  $M_1$  has

$$H_1(M_1; \mathbf{Z}) \cong \langle b_1, b_2, \cdots, b_g, c_1, c_1', c_2, c_2', \cdots, c_g, c_g'; q_i c_i - b_i = 0, q_i c_i' + b_i = 0 \quad (i = 1, 2, \cdots, g) \rangle.$$

Consider the identifying space of  $M_1$  by the identifying map of  $F \times \{1\}$  to  $F \times \{0\}$ ;  $(x, 1) \sim (\alpha(x), 0)$ , and denote the resulting manifold by  $N_2$ . Then this manifold has an orientation reversing involution  $\tau$  induced by  $\tau'$  and the fixed point set consists of 2g+2 points and  $F \times \{1\}$  (an orientable surface of genus g).

To compute  $H_1(N_2; \mathbb{Z})$ , we choose generators of  $H_1(N_2; \mathbb{Z})$  represented by curves as indicated in Figure 2. Then we have

$$\begin{array}{l} H_1(N_2; \, Z) \\ \simeq \langle a_i, \, b_i, \, c_i, \, c_i', \, t; \, q_i c_i - b_i = 0, \, q_i \, c_i' + b_i = 0, \, a_i - (c_i + c_i') = a_i, \\ b_i = -b_i \quad (i = 1, \, 2, \, \cdots, \, g) \rangle \\ \simeq \langle a_i, \, c_i, \, t: \, 2q_i a_i = 0, \, 2q_i c_i = 0 \quad (i = 1, \, 2, \, \cdots, \, g) \rangle \\ \simeq \stackrel{g}{=} \underbrace{(Z_{2q_i} \oplus Z_{2q_i}) \oplus Z}_{i=1} . \end{array}$$

(7)  $A_7(g) = (N_3, \tau); A_7(g)$  has the data  $[g, 0, 0, 0; g; ] (g \in \mathbb{N})$  and  $H_1(N_3; \mathbb{Z})$  is a free abelian group.

Consider a handle body V of genus g. Let  $N_3$  be the double of V and  $\tau: N_3 \rightarrow N_3$  a map interchanging the copies of V. Then  $\tau$  is an orientation revers-

ing involution on  $N_3$ , Fix $(\tau, N_3)$  consists of  $\partial V$  (a colsed orientable surface of genus g), and clearly  $H_1(N_3; \mathbb{Z})$  is a free ablelian group of rank g.

(8) 
$$A_8(n) = (N_4, \tau); (A_8(n) \text{ has the data } [1, 0, n, 0; ; 2, 2, ..., 2] \text{ and}$$
  
Tor  $H_1(N_4; \mathbb{Z}) \cong \bigoplus_{i=1}^n (\mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i}) (n, q_1, q_2, ..., q_n \in \mathbb{N}).$ 

Consider *n* manifolds with involutions  $(M_1, \tau_1), (M_2, \tau_2), \dots, (M_n, \tau_n)$  of type  $A_5$  such that Tor  $H_1(M_i; \mathbb{Z}) \cong \mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i}$   $(i=1, 2, \dots, n)$ . Note that Fix $(\tau, M_i)$  contains two points  $(i=1, 2, \dots, n)$ . Let  $B_1(B'_n, \text{ resp.})$  be a  $\tau$ -invariant ball in  $M_1(M_n, \text{ resp.})$  containing a fixed point, and  $B_i$  and  $B'_i$  disjoint  $\tau_i$ -invariant balls in  $M_i$  such that each balls contain a fixed point  $(i=2, 3, \dots, n-1)$ . Let  $M = (M_1 - \mathring{B}_1) \cup (\bigcup_{i=1}^{n-1} (M_i - \mathring{B}_i \cup \mathring{B}'_i)) \cup (M_n - \mathring{B}'_n)$ , where  $\partial B'_i$  is identified with  $\partial B_{i-1}$  so that the identifying map commutes with  $\tau_{i-1}$  and  $\tau_i$   $(i=2, 3, \dots, n)$ . Then M is the connected sum of  $M_1, M_2, \dots, M_{n-1}$  and  $M_n$  with an orientation reversing involution  $\tau$  extending  $\tau_1, \tau_2, \dots, \tau_{n-1}$  and  $\tau_n$ . Let K be a  $\tau$ -invariant closed curve in M as indicated in Figure 3.



Let  $\tau' = \tau|_{\partial N(K)}$  and M' the identifying space of  $\partial N(K) \times [0, 1]$  by the identifying map of  $\partial N(K) \times \{1\}$ ;  $(x, 1) \sim (\tau'(x), 1)$ , and let  $N_4 = (M - \mathring{N}(K)) \cup_k M'$ , where h is the identify map of  $\partial N(K)$ . Then  $N_4$  has an orientation reversing involution and its fixed point set consists of n+1 Klein bottles.

To compute  $H_1(N_4; \mathbb{Z})$ , we choose generators of  $H_1(M - \mathring{N}(K); \mathbb{Z})$  and  $H_1(M'; \mathbb{Z})$  represented by curves as indicated in Figure 3. Then we have

$$\begin{split} H_1(N_4; \mathbf{Z}) \\ &\simeq \langle a_i, b_i, c_i, c_i', x_i, y_i, u, v, d, e: \\ &q_i c_i - b_i = 0, q_i c_i' + b_i = 0, a_i = c_i + c_i' + d, e = \sum_{j=1}^n b_j, 2x_i + 2y_i = 0, \\ &2u + 2v = 0, 2x_i = a_i, x_i + y_i = b_i, 2u = d, u + v = e \quad (i = 1, 2, \dots, n) \rangle \\ &\simeq \langle x_i, z_i, c_i, u: z_i = x_i - u, 2q_i z_i = 0, 2q_i c_i = 0 \quad (i = 1, 2, \dots, n) \rangle \\ &\simeq \langle z_i, c_i, u: 2q_i z_i = 0, 2q_i c_i = 0 \quad (i = 1, 2, \dots, n) \rangle \\ &\simeq \bigoplus_{i=1}^n (\mathbf{Z}_{2q_i} \oplus \mathbf{Z}_{2q_i}) \oplus \mathbf{Z}. \end{split}$$

We defined eight types of basic manifolds with involutions as follows:

$(M, \tau)$	data
$A_1$	[0, 0, 0, 2 ; ; ]
$A_2$	[0, 1, 0, 1; ; 1]
$A_{\mathfrak{z}}$	[1, 0, 0, 0; 1; ]
$A_4$	[1, 0, 0, 0; 2]
$A_{5}$	[0, 0, 1, 2; ; 2]
$A_{6}(g)$	[1, 0, g, 2g+2; g; ]
$A_{7}(g)$	[g, 0, 0, 0; g; ]
$A_{8}(n)$	$[1, 0, n, 0; 2, 2, \dots, 2]$
	$\underbrace{\overbrace{n+1}}_{n+1}$

REMARK 3. We have  $(M, \tau)$  of type  $A_5$ ,  $A_6(g)$  or  $A_8(n)$  such that Tor  $H_1(M; \mathbb{Z}) \simeq \mathbb{Z}_{2q} \oplus \mathbb{Z}_{2q}$ ,  $\bigoplus_{i=1}^{s} (\mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i})$  or  $\bigoplus_{i=1}^{n} (\mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i})$  for any  $q \neq 0$ , any  $q_i \neq 0$   $(i=1, 2, \dots, g)$  or any  $q_i \neq 0$   $(i=1, 2, \dots, n)$ , respectively.

Now we difine six operations.

Operation 1. Consider  $(M', \tau')$  and some closed orientable 3-manifold N(which may not have involutions). Let B be a 3-ball contained in M' with  $B \cap \tau'(B) = \phi$ , and B' a 3-ball contained in N. Let  $M = (N - \mathring{B}') \cup (M' - (\mathring{B} \cup \tau'(\mathring{B})) \cup (-(N - \mathring{B}')))$  where  $\partial B$  is identified with  $\partial B'$  and  $\partial(\tau'(B))$  with  $\partial B'$  (in -N). Then M has an orientation reversing involution  $\tau$  extending  $\tau'$  with  $\operatorname{Fix}(\tau, M) = \operatorname{Fix}(\tau', M')$ . Note that  $H_1(M; \mathbb{Z}) \cong H_1(M' \mathbb{Z}) \oplus H_1(N; \mathbb{Z})$ .

Operation 2. Consider  $(M_i, \tau_i)$  such that  $\operatorname{Fix}(\tau_i, M_i)$  contains an isolated point  $P_i$  (i=1, 2). Let  $B_i$  be a  $\tau_i$ -invariant 3-ball in  $M_i$  such that  $B_i \cap \operatorname{Fix}(\tau_i, M_i)$  $=P_i$  (i=1, 2). Let M be an identifying space  $(M_1 - \mathring{B}_1) \cup {}_{h}(M_2 - \mathring{B}_2)$  where the

identifying map  $h: \partial B_2 \to \partial B_1$  commutes with  $\tau_1$  and  $\tau_2$ . Then M is the connected sum of  $M_1$  and  $M_2$  with an orientation reversing involution  $\tau$  extending  $\tau_1$  and  $\tau_2$ . Suppose that  $(M_i, \tau_i)$  has a data as follows;

$$\begin{array}{ll} (M_1, \tau_1); \, [\beta_1, \, s_1, \, r_1, \, p_1, \, g_1, \, g_2, \, \cdots, g_{m'}; \, c_1, \, c_2, \, \cdots, \, c_{n'}] \\ (M_2, \, \tau_2); \, [\beta_2, \, s_2, \, r_2, \, p_2; \, g_{m'+1}, \, g_{m'+2}, \, \cdots, g_m; \, c_{n'+1}, \, c_{n'+2}, \, \cdots, \, c_n] & (m' \le m, \, n' \le n) \end{array}$$

then  $(M, \tau)$  has the data

$$[*, *, *, p_1+p_2-2; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

(\* means the sum of numbers which are in the same column. For example, in the first column, \* means  $\beta_1 + \beta_2$ .) Note that  $H_1(M; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})$ .

Operation 3. Consider  $(M_i, \tau_i)$  such that  $\operatorname{Fix}(\tau_i, M_i)$  contains a surface  $F_i$  (i=1, 2). Let  $B_i \subset M_i$  be a  $\tau_i$ -invariant 3-ball such that  $B_i \cap \operatorname{Fix}(M_i, \tau_i)$  is a 2-disk on  $F_i$  (i=1, 2). Let  $M=(M_1-\mathring{B}_1)\cup_k(M_2-\mathring{B}_2)$ , where the identifying map  $h: \partial B_2 \to \partial B_1$  commutes with  $\tau_1$  and  $\tau_2$ . Then M is the connected sum of  $M_1$  and  $M_2$  with an orientation reversing involution  $\tau$  extending  $\tau_1$  and  $\tau_2$ . Suppose that  $(M_i, \tau_i)$  has a data as in the definition of Operation 2 (i=1, 2). If  $F_1$  and  $F_2$  are orientable with genera  $g_j$  and  $g_k$ , respectively, where  $j \leq m' < k \leq m$ , then  $(M, \tau)$  has the data

$$[*, *, *, *; g_1, g_2, \dots, g_{j-1}, g_j + g_k, g_{j+1}, \dots, \check{g}_k, \dots, g_m; c_1, c_2, \dots, c_n]$$

(\* means removing the specified element). And if  $F_1$  and  $F_2$  are nonorientable with nonorientable genera  $c_j$  and  $c_k$ , respectively, where  $j \le n' < k \le n$ , then  $(M, \tau)$  has the data

$$[*, *, *, *; g_1, g_2, \dots, g_m; c_1, c_2, \dots, c_{j-1}, c_j + c_k, c_{j+1}, \dots, \check{c}_k, \dots, c_n]$$

Note that  $H_1(M; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z})$ .

Operation 3'. Consider  $(M_1, \tau_1), (M_2, \tau_2), \dots, (M_n, \tau_n)$   $(n \ge 2)$  and  $(M', \tau')$ such that  $\operatorname{Fix}(\tau_i, M_i)$  consists of a surface  $F_i$  and certain points  $(i=1, 2, \dots, n)$ , and such that  $\operatorname{Fix}(\tau', M')$  consists of *n* surfaces  $E_1, E_2, \dots, E_n$  and certain points. Let  $B_i$  be a  $\tau_i$ -invariant 3-ball in  $M_i$  such that  $B_i \cap \operatorname{Fix}(\tau_i, M_i)$  is a disk on  $F_i$  $(i=1, 2, \dots, n)$ , and let  $C_i$  be a  $\tau'$ -invariant 3-ball in M' such that  $C_i \cap \operatorname{Fix}(\tau', M')$ is a disk on  $E_i$   $(i=1, 2, \dots, n)$ . We consider an operation similar to Operation 3 with attaching homeomorphism  $h_i$ ;  $\partial B_i \rightarrow \partial C_i$   $(i=1, 2, \dots, n)$ . Then we can obtain  $(M, \tau)$  such that M is the connected sum of  $M_1, M_2, \dots, M_n$  and M', and such that  $\operatorname{Fix}(\tau, M)$  consists of the connected sum of  $F_i$  and  $E_i$   $(i=1, 2, \dots, n)$  and  $(M', \tau')$ have data as follows;

$$\begin{array}{ll} (M_i, \, \tau_i) \colon [\beta_i, \, s_i, \, r_i, \, p_i; \quad ; c_i] & (i = 1, \, 2, \, \cdots, \, n) \\ (M', \, \tau') \colon [\beta', \, s', \, r', \, p'; \quad ; c_1', \, c_2', \, \cdots, \, c_n'] \, . \end{array}$$

Then  $(M, \tau)$  has the data

$$[\sum_{i=1}^{n} \beta_{i} + \beta', \sum_{i=1}^{n} s_{i} + s', \sum_{i=1}^{n} r_{i} + r', \sum_{i=1}^{n} p_{i} + p'; ; c_{1} + c_{1}', c_{2} + c_{2}', \cdots, c_{n} + c_{n}']$$
  
Note that  $H_{1}(M; \mathbb{Z}) \cong \bigoplus_{i=1}^{n} (M_{i}; \mathbb{Z}) \oplus H_{1}(M'; \mathbb{Z}).$ 

Operation 4. Consider  $(M_1, \tau_1)$  and  $(M_2, \tau_2)$ . Let  $B_i$  be a 3-ball in  $M_i$ with  $B_i \cap \tau_i(B_i) = \phi$  (i=i, 2). Let  $M = (M_1 - (\mathring{B}_1 \cup \tau_1(\mathring{B}_1)) \cup (M_2 - (\mathring{B}_2 \cup \tau_2(\mathring{B}_2)))$ , where  $\partial B_1$  is identified with  $\partial B_2$  and  $\partial \tau_1(B_1)$  identified with  $\partial \tau_2(B_2)$  so that the identifying map commutes with  $\tau_1$  and  $\tau_2$ . Then M has an orientation reversing involution  $\tau$  extending  $\tau_1$  and  $\tau_2$  with  $\operatorname{Fix}(\tau, M) = \operatorname{Fix}(\tau_1, M_1) \cup \operatorname{Fix}(\tau_2, M_2)$ . Suppose that  $(M_i, \tau_i)$  (i=1, 2) has a data as in the definition of Operation 2, then  $(M, \tau)$  has the data

$$[\beta_1+\beta_2+1, *, *, *; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

Note that  $H_1(M; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}) \oplus \mathbb{Z}$ .

Operation 5. Consider  $(M_i, \tau_i)$  such that  $\operatorname{Fix}(\tau_i, M_i)$  contains two isolated points  $p_{i_1}$  and  $p_{i_2}$  (i=1, 2). Let  $B_{i_j}$  be a  $\tau_i$ -invariant 3-ball in  $M_i$  containing  $p_{i_j}$  (i=1, 2, j=1, 2). Let  $M=(M_1-(\mathring{B}_{1_1}\cup\mathring{B}_{1_2}))\cup(M_2-(\mathring{B}_{2_1}\cup\mathring{B}_{2_2}))$ , where  $\partial B_{1_j}$ is identified with  $\partial B_{2_j}$  (j=1, 2) so that the identifying map commutes with  $\tau_1$ and  $\tau_2$ . Then M has an orientation reversing involution  $\tau$  extending  $\tau_1$  and  $\tau_2$ . Suppose that  $(M_i, \tau_i)$  (i=1, 2) has the data as in the definition of Operation 2, then  $(M, \tau)$  has the data

$$[\beta_1 + \beta_2 + 1, *, *, p_1 + p_2 - 4; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

Note that  $H_1(M; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}) \oplus \mathbb{Z}$ .

## 4. Constructions.

Proof of Theorem 3. Let  $E_1, E_2, \dots, E_m$  be the components of E. Consider a handlebody  $V_1$  such that  $\partial V_1 = E_1$ . Let  $V_2, V_3, \dots, V_m$  be murtually disjoint handlebodies contained in  $\mathring{V}_1$  such that the natural homomorphism  $H_1(V_i; \mathbb{Z}) \to H_1(V_1; \mathbb{Z})$  is trivial. Let  $M_1 = V_1 - \bigcup_{i=2}^m \mathring{V}_i$ . Then the double of  $M_1, DM_1$ , has an orientation reversing involtuion  $\tau$  interchanging the copies of  $M_1$  with  $\operatorname{Fix}(\tau, DM_1) = \bigcup_{i=1}^m V_i$ . Note that  $H_1(DM_1; \mathbb{Z})$  is a free abelian group of rank  $m + \sum_{i=1}^m g(E_i) - 1$ .

By (2) and (3), we can see that rank $G-(m+\sum_{i=1}^{m}g(E_{i})-1)$  is a nonnegative even integer. Hence by (1), we can consider that  $H_{1}(DM_{1}; \mathbb{Z})$  is a direct summand of G with  $G/H_{1}(DM_{1}; \mathbb{Z})\cong B\oplus B$  for some abelian group B. Let  $M_{2}$  be a closed orientable manifold with  $H_{1}(M_{2}; \mathbb{Z})\cong B$ . Put  $M=M_{2} \# DM_{1} \# (-M_{2})$ by using Operation 1. Then we can see that M is the required manifold. This completes the proof.

**Lemma 7.** Let  $t, s, r, p, m, n, g_1, g_2, \dots, g_m, c_1, c_2, \dots, c_n$  be nonnegative integers satisfying the following conditions:

- (1)  $s \leq n, g_i > 0$   $(i=1, 2, \dots, m), c_j > 1$   $(j=1, 2, \dots, n)$
- (2) s, p,  $c_{s+1}$ ,  $c_{s+2}$ ,  $\cdots$ ,  $c_n$  are even, and  $c_1$ ,  $c_2$ ,  $\cdots$ ,  $c_s$  are odd.

(3) 
$$m+n+\sum_{i=1}^{m}g_i+p+\sum_{j=1}^{n}c_j \leq 2s+4r+2t+2$$
.

(4) 
$$m+n-\sum_{i=1}^{m}g_i+(p+\sum_{j=1}^{n}c_j)/2\geq 1-t.$$

(5) 
$$m+n-\sum_{i=1}^{m}g_i+(p-\sum_{j=1}^{n}c_j)/2 \le 1+t.$$

(6) 
$$m+n-\sum_{i=1}^{m}g_i+(p-\sum_{j=1}^{n}c_j)/2\equiv 1+t \pmod{2}.$$

Then there exists  $(M, \tau)$  which has a data  $[t, s, r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$ .

Proof. We consider the following six cases (where  $\sum g_i = \sum_{i=1}^{m} g_i$  and  $\sum c_j = \sum_{i=1}^{n} c_j$ ):

Case 1) 
$$2 \sum g_i + \sum c_j \leq s + 2r$$
.

Case 2)  $2 \sum g_i + \sum c_j > s + 2r$  and  $r \ge n$ .

Case 3)  $n > r \ge s/2$  and  $p \ge 2$ .

- Case 4) s/2 > r and  $p \ge 2$ .
- Case 5) n > r, p=0 and  $r+1 \ge s/2$ .
- Case 6) n > r, p = 0 and r + 1 < s/2.

Case 1) 
$$2 \sum g_i + \sum c_j \leq s + 2r$$
.

We prepare  $|1-n+\sum g_i+((\sum c_j-p)/2|$  copies of  $A_1$ , s copies of  $A_2$ ,  $(\sum c_j-s)/2$  copies of  $A_5$  and  $A_6(g_1)$ ,  $A_6(g_2)$ ,  $\cdots$ ,  $A_6(g_m)$ . Now we have the following data

$$\begin{array}{rcl} A_1 &: [0, 0, 0, 2 ;; ;] & (|1 - n + \sum g_i + (\sum c_j - p)/2| \text{ times}), \\ A_2 &: [0, 1, 0, 1 ;; ;] & (s \text{ times}), \\ A_5 &: [0, 0, 1, 2 ;; ;2] & ((\sum c_j - s)/2 \text{ times}), \\ A_6(g_i): [1, 0, g_i, 2g_i + 2; g_i; ] & (i = 1, 2, \cdots, m). \end{array}$$

We denote by  $X \stackrel{i}{\longrightarrow} Y$  the result of Operation *i* on the manifolds with involutions X and Y, and by  $X(\stackrel{i}{\longrightarrow} Y)^n$ ,  $X \stackrel{i}{\longrightarrow} Y \stackrel{i}{\longrightarrow} Y \stackrel{i}{\longrightarrow} Y$  (*n* copies of Y). Then we apply Operation 3 as indicated in Figure 4 and obtain  $B_1^i$  (*j*=1,

$$B_{1}^{j} = A_{5}(\frac{3}{2} A_{5})^{(c_{j}-1)/2-1} \frac{3}{2} A_{2} \qquad (j = 1, 2, \dots, s)$$
  

$$B_{1}^{j} = A_{5}(\frac{3}{2} A_{5})^{c_{j}/2-1} \qquad (j = s+1, s+2, \dots, n)$$
  
Figure 4

2, ..., n) with data  $[0, 0, (c_j-1)/2, c_j; ; c_j]$  or  $[0, 0, c_j/2, c_j; ; c_j]$  according to whether  $j \leq s$  or  $j \geq s+1$ .

Applying Operation 2 as indicated in Figure 5, we obtain  $B_2$  with data

$$B_{2} = B_{1}^{1} \stackrel{2}{-} B_{1}^{2} \stackrel{2}{-} B_{1}^{3} \stackrel{2}{-} \cdots$$

$$\cdots \stackrel{2}{-} B_{1}^{n} \stackrel{2}{-} A_{6}(g_{1}) \stackrel{2}{-} A_{6}(g_{2}) \stackrel{2}{-} \cdots \stackrel{2}{-} A_{6}(g_{m})$$
Figure 5

$$[m, s, \sum g_i + (\sum c_j - s)/2, \sum c_j + \sum_{i=1}^{m} (2g_i + 2) - 2(m + n - 1);$$
  
$$g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n],$$

If  $1-n+\sum g_i+(\sum c_j-p)/2\geq 0$ , then we apply Operation 5 as indictaed in Figure 6 and obtain  $B_3$  with data

$$B_{3} = B_{2}(\frac{k}{2}A_{1})^{|\mathbf{n}-1-\sum g_{i}+((p-\sum c_{j})/2)|}$$

$$k = \begin{cases} 4 & if \quad n-1-\sum g_{i}+((p-\sum c_{j})/2) \ge 0\\ 5 & if \quad n-1-\sum g_{i}+((p-\sum c_{j})/2) \le 0\\ & \text{Figure 6} \end{cases}$$

$$[1+m-n+\sum g_i+(\sum c_j-p)/2, s, \sum g_i+(\sum c_j-s)/2, p;g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

By (4),  $1+m-n+\sum g_i+(\sum c_j-p)/2 \le t$ . By (6),  $t-(1+m-n+\sum g_i+(c_j-p)/2)$  is even. By Assumption of Case 1),  $\sum g_i+(\sum c_j-s)/2 \le r$ . Hence we can obtain a manifold with involution with the required data by Operation 1.

If  $1-n+\sum g_i+(\sum c_j-p)/2 \le 0$ , then we apply Operation 4 as indicated in Figure 6 and obtain a manifold with data

$$\begin{bmatrix} -1 + m + n - \sum g_i + (p - \sum c_j)/2, s, \sum g_i + (\sum c_j - s)/2, p; \\ g_1, g_2, \dots, g_m; c_1, c_2, \dots, c_n \end{bmatrix}$$

By (5) and (6),  $t-(-1+m+n-\sum g_i+(p-\sum c_j)/2)$  is a nonnegative even integer. By Assumption of Case 1),  $\sum g_i+(\sum c_j-s)/2 \le r$ . Hence by Operation 1, we can obtain a manifold with the required data.

Case 2)  $2 \sum g_i + \sum c_j > s + 2r$  and  $r \ge n$ .

There exist integers m',  $g'_1, g'_2, \dots, g'_m, c'_1, c'_2, \dots, c'_n$  satisfying the following conditions;

1. 
$$0 \le m' \le m$$
,  
2.  $0 < g'_i \le g_i \ (i=1, 2, \dots, m')$ .  
3.  $1 < c'_j \le c_j \text{ and } c'_j \equiv c_j \ (\text{mod } 2) \ (j=1, 2, \dots, n)$ , and  
4.  $2\sum_{i=1}^{m'} g'_i + \sum_{j=1}^{n} c'_j = s + 2r$ .  
(Note that if  $m'=0, \ c'_j = 3 \ (j=1, 2, \dots, s)$  and  $c'_j = 2 \ (j=s+1, s+2, \dots, n)$ , then  
 $\sum_{j=1}^{n} c'_j = 3s + 2(n-s) = s + 2n \le s + 2r$ .)

We prepare the basic manifolds and apply Operations as indicated in Figure 7. Then we obtain  $B_6$  with the following data

$$[m+\sum g_{i}+(\sum c_{j}-s-2r)/2, s, r, 2-2n+s+2r;$$
  

$$g_{1}, g_{2}, \cdots, g_{m}, c_{1}, c_{2}, \cdots, c_{n}].$$
  

$$B_{4}^{i}=A_{6}(g_{i}')(\overset{3}{-}A_{3})^{g_{i}-g_{i}'} \quad (i=1, 2, \cdots, m')$$
  

$$B_{5}^{j}=A_{5}(\overset{3}{-}A_{5})^{(c_{j}'-1)/2-1}(\overset{3}{-}A_{3})^{(c_{j}-c_{j}')/2} \overset{3}{-}A_{2} \quad (j=1, 2, \cdots, s)$$
  

$$B_{5}^{j}=A_{5}(\overset{3}{-}A_{5})^{c_{j}'/2-1}(\overset{3}{-}A_{3})^{(c_{j}-c_{j}')/2} \quad (s+1\leq j \leq n)$$
  

$$B_{6}=B_{4}^{1} \overset{2}{-}B_{4}^{2} \overset{2}{-}\cdots \overset{2}{-}B_{4}^{m'} \overset{2}{-}B_{5}^{1} \overset{2}{-}B_{5}^{2} \overset{2}{-}\cdots$$
  

$$\cdots \overset{2}{-}B_{5}^{n} \overset{4}{-}A_{7}(g_{m'+1}) \overset{4}{-}A_{7}(g_{m'+2}) \overset{4}{-}\cdots \overset{4}{-}A_{7}(g_{m})$$
  

$$(if m'=n=0, B_{6}=A_{1} \overset{4}{-}A_{7}(g_{1}) \overset{4}{-}A_{7}(g_{2}) \overset{4}{-}\cdots \overset{4}{-}A_{7}(g_{m}))$$
  
Figure 7

If  $2-2n+s+2r-p \ge 0$ , then we apply Operation 5 as indicated in Figure 8 and obtain  $B_7$  with data

$$[1+m-n+\sum g_i+(\sum c_j-p)/2, s, r, p; g_1, g_2, \dots, g_m; c_1, c_2, \dots, c_n].$$

$$B_7 = B_6(\frac{k}{2}A_1)^{1(p-s)/2+n-r-1}$$

$$k = \begin{cases} 4 & \text{if } (p-s)/2+n-r-1 \ge 0\\ 5 & \text{if } (p-s)/2+n-r-1 \le 0 \end{cases}$$
Figure 8

By (4) and (6),  $t-(1+m-n+\sum g_i+(\sum c_j-p)/2)$  is a nonnegative even integer. Hence by Operation 1, we can obtain a manifold with the required data.

If  $2-2n+s+2r-p \le 0$ , then we apply Operation 4 as indicated in Figure

8 and obtain  $B_7$  aith data

 $[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2, s, r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$ 

By (3) and (6),  $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$  is a nonnegative even integer. Hence by Operation 1, we can obtain a manifold with the required data.

Case 3)  $n > r \ge s/2$  and  $p \ge 2$ .

First, consider the numbers  $g_1, g_2, \dots, g_m, c_1, c_2, \dots, c_{s/2}, c_{s+1}, \dots, c_{r+(s/2)}$ . Note that  $\sum_{j=1}^{s/2} c_j + \sum_{j=s+1}^{r+(s/2)} c_j \ge (s/2) + 2r$  and  $s/2 + (r+s/2-s) \le r$ . Hence, by the same way as in Case 2), we have  $B'_6$  with data

$$[m+\sum g_i + (\sum_{j=1}^{s/2} c_j + \sum_{j=s+1}^{r+(s/2)} c_j - s/2 - 2r)/2, s/2, r, 2 + (s/2);$$
  
$$g_1, g_2, \dots, g_m; c_1, c_1, \dots, c_{s/2}, c_{s+1}, c_{s+2}, \dots, c_{r+(s/2)}]$$

Applying Operations to  $B'_6$  and the basic manifolds as indicated in Figure 9, we have  $B_9$  with data



Figure 9

$$[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2, s, r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

By (3) and (6),  $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$  is a nonnegative even integer. By Operation 1, we can obtain a manifold with the required data.

Case 4) s/2 > r and  $p \ge 2$ . First, consider the numbers  $g_1, g_2, \dots, g_m, c_1, c_2, \dots, c_{2r}, c_{s+1}, c_{s+2}, \dots, c_n$ . Note that  $2r + (n-s) \ge r = 2r/2$ . By the same way as in Case 3), we have  $B'_9$  with data

$$\begin{bmatrix} -(1+2r+2r)+m+(2r+n-s)+\sum g_i+(p+\sum_{j=1}^{2r}c_j+\sum_{j=s+1}^{n}c_j)/2, \ 2r, r, p; \\ g_1, g_2, \cdots, g_m; \ c_1, \ c_2, \cdots, c_{2r}, \ c_{s+1}, \ c_{s+2}, \ldots, c_n \end{bmatrix}.$$

Applying Operations to  $B'_9$  and the basic manifolds as indicated in Figure 10, we obtain  $B_{11}$  with data

$$[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2, s, r, p; g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_n].$$

$$B_{10}^i = A_4(\frac{3}{4}A_4)^{(c_{2r+2i-1}-1)/2-1} \frac{3}{4}A_2 \frac{2}{4}A_2(\frac{3}{4}A_4)^{(c_{2r+2i}-1)/2} (1 \le i \le s/2 - r)$$

$$B_{11} = B_9' \frac{4}{4}B_{10}^1 \frac{4}{4}B_{10}^2 \frac{4}{4}\cdots B_{10}^{s/2-r}$$
Figure 10

By (4) and (7),  $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$  is a nonnegative even integer. By Operation 1, we can obtain a manifold with the required data.

Case 5) n > r, p = 0 and  $r + 1 \ge s/2$ .

Apply Operations to the basic manifolds as indicated in Figure 11. (In Figure 11,  $B_{13}$  is created by applying Operation 3' to  $B_{12}^1, B_{12}^2, \dots, B_{12}^{s/2}, B_{12}^{s+1}, B_{12}^{s+2}, \dots, B_{12}^{r+1+s/2}$  and  $A_8(r)$ .) We obtain  $B_{14}$  with data

$$[-(1+s+2r)+m+n+\sum g_i+\sum c_j/2, s, r, 0; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

$$B_{12}^{j} = A_{2} (\frac{3}{4})^{(c_{j}-3)/2} \quad (j=1, 2, \dots, s/2)$$

$$B_{12}^{j} = A_{2} (\frac{3}{4})^{(c_{j}-1)/2} \quad (j=s/2+1, s/2+2, \dots, s)$$

$$B_{12}^{j} = A_{4} (\frac{3}{4})^{(c_{j}-2)/2-1} \quad (j=s+1, s+2, \dots, r+1+s/2)$$

$$B_{12}^{j} = A_{4} (\frac{3}{4})^{c_{j}/2-1} \quad (j=r+1+s/2+1, r+1+s/2+2, \dots, n)$$

$$B_{13} = A_{8}(r) \frac{3'}{4} \{B_{12}^{1}, B_{12}^{2}, \dots, B_{12}^{s/2}, B_{12}^{s+1}, B_{12}^{s+2}, \dots, B_{12}^{r+1+s/2}\}$$

$$B_{14} = B_{13} \frac{4}{4} B_{12}^{j+1+s/2+1} \frac{4}{4} B_{12}^{r+1+s/2+2} \frac{4}{4} \dots \frac{4}{4} B_{12}^{n}$$

$$A_{7}(g_{1}) \frac{4}{4} A_{7}(g_{2}) \frac{4}{4} \dots \frac{4}{4} A_{7}(g_{m})$$

By (3) and (6),  $t-(-(1+s+2r)+m+n+\sum g_i+\sum c_j/2)$  is a nonnegative even integer. By Operation 1, we can obtain the required manifold.

Case 6) n > r, p = 0 and r + 1 < s/2.

First, consider the numbers  $g_1, g_2, \dots, g_m, c_1, c_2, \dots, c_{2(r+1)}, c_{s+1}, c_{s+2}, \dots, c_n$ . Since  $2(r+1)+(n-s) \ge r$  and  $r+1 \ge (2(r+1))/2$ , by the same way as in case 5) there exists  $B'_{14}$  with data

$$\begin{bmatrix} -(1+2(r+1)+2r)+m+(2(r+1)+n-s)+\sum g_i+(\sum_{j=1}^{2^{(r+1)}}c_j+\sum_{j=s+1}^n c_j)/2,\\ 2(r+1), r, 0; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_{2(r+1)}, c_{s+1}, c_{s+2}, \cdots, c_n \end{bmatrix}.$$

Applying Operations to  $B'_{14}$  and the basic manifolds as indicated in Figure 12, we have  $B_{16}$  with data

$$\begin{bmatrix} -(1+s+2r)+m+n+\sum g_i+\sum c_j/2, s, r, 0; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n \end{bmatrix}$$
  

$$B_{15}^i = A_4 (\frac{3}{2} A)_4^{(c_{2r+2+2i-1}-1)/2-1} \frac{3}{2} A_2 \frac{2}{2} A_2 (\frac{3}{2} A_4)^{(c_{2r+2+2i}-1)/2} (1 \le i \le s/2 - r - 1)$$
  

$$B_{16} = B_{14}^{\prime} \frac{4}{2} B_{15}^1 \frac{4}{2} B_{15}^2 \frac{4}{2} \cdots \frac{4}{2} B_{15}^{s/2-r-1}$$
  
Figure 12

By (3) and (6),  $t-(-(1+s+2r)+m+n+\sum g_i+\sum c_j/2)$  is a nonnegative even integer. By Operation 1, we can obtain the required manifold.

This completes the proof.

REMARK 4. In the proof of Lemma 7, we have constructed  $(M, \tau)$  with data stated in Lemma 7 such that Tor  $H_1(M; \mathbb{Z}) \cong (\bigoplus_{i=1}^{s} \mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i})$  for some nonzero integers  $q_1, q_2, \dots, q_r$ . Since  $(M, \tau)$  is obtained from the basic manifolds with involutions by the Operations, we see from Remark 3 that any given nonzero integers can be taken as  $q_1, q_2, \dots, q_r$ .

**Lemma 8.** Even if the numbers s and p in the assumption of Lemma 7 are odd, the same assertion of Lemma 7 holds.

Proof. We can check that  $t, s-1, r, p-1, m, n, g_1, g_2, \dots, g_m, c_1, c_2, \dots, c_{s-1}, c_s-1, c_{s+1}, \dots, c_n$  satisfy the assumption of Lemma 7. Hence there exists  $(M', \tau')$  with data  $[t, s-1, r, p-1; g_1, g_2, \dots, g_m; c_1, c_2, \dots, c_{s-1}, c_s-1, c_{s+1}, \dots, c_n]$ . Applying Operation 3 to  $(M', \tau')$  and  $A_2$   $(A_2$  has the data [0, 1, 0, 1; ; 1], we can obtain  $(M, \tau)$  with the data  $[t, s, r, p; g_1, g_2, \dots, g_m; c_1, c_2, \dots, c_n]$ .

This completes the proof.

Proof of Theorem 5. We may assume that X consists of p points, m closed

orientable surfaces  $E_1, E_2, \dots, E_m$  of genera  $g_1, g_2, \dots, g_m$  and n closed nonorientable surfaces  $F_1, F_2, \dots, F_n$  of nonorientable genera  $c_1, c_2, \dots, c_n$  such that  $g_i > 0$ or =0 according to whether  $1 \le i \le m'$  or  $m' \le i \le m$  for some m' and  $c_j$  is odd  $(\pm 1)$ , even or 1 according to whether  $1 \le j \le s, s+1 \le j \le n'$  or  $n'+1 \le j \le n$ for some s and n'. By conditions (1)-(7), we can see that the given abelian group G is isomorphic to  $(\bigoplus_{i=1}^{s+n-n'} Z_2) \oplus (\bigoplus_{i=1}^{r} (Z_{2q_i} \oplus Z_{2q_i})) \oplus (\bigoplus_{i=2}^{t} Z_i) \oplus B \oplus B$ , where  $r = (\beta_1(G; Z_2) - \beta_1(G) - s)/2$ , B is some abelian group of odd order, and  $t, q_1, q_2, \dots, q_r$  are some integers. We can check that the numbers t-m+m',  $s, r, p+n-n', m', n', g_1, g_2, \dots, g_{m'}, c_1, c_2, \dots, c_{n'}$  satisfy the assmption of Lemma 7 or 8.

Hence by Lemma 7 or 8 and Remark 4 there exists  $(M_1, \tau_1)$  with data  $[t-m+m', s, r, p+n-n'; g_1, g_2, \dots, g_{m'}; c_1, c_2, \dots, c_{n'}]$  and Tor  $H_1(M_1; \mathbb{Z}) \cong (\stackrel{\circ}{\bigoplus} \mathbb{Z}_2) \oplus (\stackrel{\circ}{\bigoplus}_{i=1}^{r} (\mathbb{Z}_{2q_i} \oplus \mathbb{Z}_{2q_i}))$ . Prepare n-n' copies of  $A_2$  (with data [0, 1, 0, 1; ; 1]) and m-m' copies  $A_7(0)$  (with data [0, 0, 0, 0; 0; ]). Applying Operation 2 n-n' times and Operation 4 m-m' times, we have a manifold  $M_2$  with data

$$[t, s+n-n', r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

Consider a manifold  $M_3$  with  $H_1(M_3; \mathbb{Z}) \cong B$  and apply Operation 1 to  $M_2$ and  $M_3$ . We denote the resulting manifold with involution by  $(M, \tau)$ . Then we can see that  $H_1(M; \mathbb{Z}) \cong G$  and  $\operatorname{Fix}(\tau, M) = X$ .

This completes the proof.

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