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ORIENTATION REVERSING INVOLUTIONS ON CLOSED 3-MANIFOLDS

Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

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1. Introduction.

Let M be a closed connected oientable 3-manifold admitting an oreintation reversing involtuion τ (i.e. $\tau^2=$ identity and $\tau_*([M])=-[M]$ for the fundamental class $[M]$ of M).

By Smith theory, each component of the fixed point set of τ, $\mathrm{Fix}(\tau, M)$, is a point or a closed surface and $\chi(\mathrm{Fix}(\tau, M))\equiv 0 \pmod 2$ ($\chi(X)$ is the Euler characteristic of X). A. Kawauchi [5] proved that for any (M, τ), $\mathrm{Tor}\, H_1(M; \boldsymbol{Z}) \cong A\oplus A$ or $\boldsymbol{Z}_2\oplus A\oplus A$ for some abelian group A, and that $\mathrm{rank}_{\boldsymbol{Z}_2} H_1(\mathrm{Fix}(\tau, M); \boldsymbol{Z}_2)\equiv 0 \pmod 2$ if and only if $\mathrm{Tor}\, H_1(M; \boldsymbol{Z})\cong A\oplus A$. J. Hempel has proved in [3] that if $\mathrm{Fix}(\tau, M)$ is empty or contains a closed orientable surface of positive genus, then the first Betti number of M is greater than zero. He has also shown in [4] that if $\pi_1(M)$ is not isomorphic to $\{1\}$ or and $\boldsymbol{Z}_2$ is not virtually representable to $\boldsymbol{Z}$, then $\mathrm{Fix}(\tau, M)$ consists of a 2-sphere or two points, or contains a projective plane.

The auther gave a characterization of $\mathrm{Fix}(\tau, M)$ when M is a rational homoogy 3-sphere in [6] and, for a general M, an inequality on the first Betti numbers of M and $\mathrm{Fix}(\tau, M)$ in [7]. In this paper we give a complete characterization of the topological type of $\mathrm{Fix}(\tau, M)$ for a general M.

NOTATIONS. For a space X, let $\beta_i(X)$ denote the i^{th} Betti number and $\beta_i(X; \boldsymbol{Z}_2)$ the $\boldsymbol{Z}_2$-coefficient Betti number. For a group G, let $\beta_1(G)=\mathrm{rank}_{\boldsymbol{Z}} H_1(G; \boldsymbol{Z})$ and $\beta_1(G; \boldsymbol{Z}_2)=\mathrm{rank}_{\boldsymbol{Z}_2} H_1(G; \boldsymbol{Z}_2)$.

First, we classify (M, τ) into two types.

Proposition 1. *For any (M, τ), one of the following holds:*
(1) *$M-\mathrm{Fix}(\tau, M)$ consists of two components and $\mathrm{Fix}(\tau, M)$ is a closed orientable 2-manifold.*
(2) *$M-\mathrm{Fix}(\tau, M)$ is connected.*

For each type of (M, τ), we shall prove the following:

Theorem 2. *For any* (M, τ) *with* $M-\mathrm{Fix}(\tau, M)$ *disconnected, we have the following* (1)–(3):

(1) $\mathrm{Tor}\, H_1(M; \boldsymbol{Z})\cong A\oplus A$ *for some ablelian group* A.

(2) $\beta_1(\mathrm{Fix}(\tau, M))/2+\beta_2(\mathrm{Fix}(\tau, M))\leq 1+\beta_1(M)$.

(3) $\beta_1(\mathrm{Fix}(\tau, M))/2+\beta_2(\mathrm{Fix}(\tau, M))\equiv 1+\beta_1(M)$ (mod 2).

Remark 1. (1) was proved by Kawauchi [5].

Theorem 3. *Let* G *be an ableian group and* E *a closed orientable* 2*-manifold satisfying the following conditions* (1)–(3):

(1) $\mathrm{Tor}\, G\cong A\oplus A$ *for some ableian group* A.

(2) $\beta_1(E)/2+\beta_2(E)\leq 1+\beta_1(G)$.

(3) $\beta_1(E)/2+\beta_2(E)\equiv 1+\beta_1(G)$ (mod 2).

Then there exists (M, τ) *such that* $M-\mathrm{Fix}(\tau, M)$ *is disconnected,* $H_1(M; \boldsymbol{Z})\cong G$ *and* $\mathrm{Fix}(\tau, M)=E$.

Theorem 4. *For any* (M, τ) *with* $M-\mathrm{Fix}(\tau, M)$ *connected, we have the folloaing* (1)–(7);

(1) $\mathrm{Tor}\, H_1(M, \boldsymbol{Z})\cong A\oplus A$ *or* $\boldsymbol{Z}_2\oplus A\oplus A$ *for some abelian group* A.

(2) $\beta_1(\mathrm{Fix}(\tau, M); \boldsymbol{Z}_2)\equiv\beta_1(M; \boldsymbol{Z}_2)-\beta_1(M)$ (mod 2).

(3) $\sum_{i=0}^{2}\beta_i(\mathrm{Fix}(\tau, M); \boldsymbol{Z}_2)\leq 2+2\beta_1(M; \boldsymbol{Z}_2)$.

(4) $\chi(\mathrm{Fix}(\tau, M))/2-2\beta_2(\mathrm{Fix}(\tau, M))\geq 1-\beta_1(M)$.

(5) $\chi(\mathrm{Fix}(\tau, M))/2\leq 1+\beta_1(M)$.

(6) $\chi(\mathrm{Fix}(\tau, M))/2\equiv 1+\beta_1(M)$ (mod 2).

(7) *Consider a direct sum decomposition of* $\mathrm{Tor}\, H_1(M; \boldsymbol{Z})$ *such that each factor is a cyclic group of prime power order. Let* u *be the number of* $\boldsymbol{Z}_2$ *factors. Then the number of nonorientable surfaces of odd genera contained in* $\mathrm{Fix}(\tau, M)$ *is not greater than* u.

Remark 2. (1) and (2) were proved by Kawauchi [5]. (3) is obtained by Smith theory (cf. [1] p. 126).

Theorem 5. *Let* G *be an abelian group and* X *be a disjoint union of points and closed surfaces. If* G *and* X *satisfy the following conditions* (1)–(7):

(1) $\mathrm{Tor}\, G\cong A\oplus A$ *or* $\boldsymbol{Z}_2\oplus A\oplus A$ *for some abelian group* A.

(2) $\beta_1(X; \boldsymbol{Z}_2)\equiv\beta_1(G; \boldsymbol{Z}_2)-\beta_1(G)$ (mod 2).

(3) $\sum_{i=0}^{2}\beta_i(X; \boldsymbol{Z}_2)\leq 2+2\beta_1(G; \boldsymbol{Z}_2)$.

(4) $\chi(X)/2-2\beta_2(X)\geq 1-\beta_1(G)$.

(5) $\chi(X)/2\leq 1+\beta_1(G)$.

(6) $\chi(X)/2\equiv 1+\beta_1(G)$ (mod 2).

(7) *Consider a direct sum decomposition of* $\mathrm{Tor}\, G$ *such that each factor is a cyclic group of prime power order. Let* u *be the number of* $\boldsymbol{Z}_2$ *factors. Then the number*

of nonorientable surfaces of odd genera contained in X *is not greater than* u.

Then there exists (M, τ) *such that* $M-\mathrm{Fix}(\tau, M)$ *is connected,* $H_1(M; \boldsymbol{Z})\cong G$ *and* $\mathrm{Fix}(\tau, M)=X$.

Throughout this paper, we will work in the piecewise-linear category, and a surface is assumed to be compact and connected.

The author owes the idea of β_i^+ to Prof. M. Sakuma. She is very grateful to him for suggesting her Lemma 6.

2. Proofs of Proposition 1 and Theorems 2 and 4.

Proof of Proposition 1. We will show that if $M-\mathrm{Fix}(\tau, M)$ is disconnected, then $M-\mathrm{Fix}(\tau, M)$ consists of two components and $\mathrm{Fix}(\tau, M)$ is a closed orientable 2-manifold.

Let $C_1, C_2, \cdots, C_r$ be the components of $M-\mathring{N}(\mathrm{Fix}(\tau, M))$, where $\mathring{N}(\mathrm{Fix}(\tau, M))$ is the interior of a τ-invariant regular neighborhood of $\mathrm{Fix}(\tau, M)$. Then the identifying space of $C_1\cup C_2\cup\cdots\cup C_r$ by the identifying map $\tau|_{\cup\partial C_i}$ is homeomorphic to M. Since $\tau^2=$identity and M is connected, we can see that $r=2$ and $\tau(C_1)=C_2$. Hence $\tau(\partial C_1)=\partial C_2$ and for each component X of $\mathrm{Fix}(\tau, M)$, $\partial N(X)$ consists of 2-components. In general, if there exists an isolated point p in $\mathrm{Fix}(\tau, M)$, $N(p)$ is a ball, and if there exists a nonorientable surface F in $\mathrm{Fix}(\tau, M)$, $N(F)$ is a twisted I-bundle over F. Hence this X must be an orientable surface. This completes the proof.

Condider the homomorphism $\tau_*^{(i)}$ on $H_i(M: \boldsymbol{Q})$ induced by τ $(i=1, 2)$. We may regard $\tau_*^{(i)}$ as a linear transformation of the vector space $H_i(M; \boldsymbol{Q})$ over $\boldsymbol{Q}$. Since $(\tau_*^{(i)})^2=$identity, every eigenvalue of $\tau_*^{(i)}$ is $+1$ or -1. Let B_i^+ and B_i^- be the eigenspace of $H_i(M; \boldsymbol{Q})$ corresponding to $+1$ and -1, respectively. Put $\beta_i^+=\dim B_i^+$ and $\beta_i^-=\dim B_i^-$. Clearly, $\beta_1^++\beta_1^-=\beta_2^++\beta_2^-=\beta_1(M)$. We have the folloaing lemma:

Lemma 6. *For any* (τ, M), *we have*

$$\chi(\mathrm{Fix}(\tau, M))=2(1+\beta_1(M)-2\beta_1^+)\,.$$

Proof. Let $\{a_1, a_2, \cdots, a_{\beta_1^+}\}$ be a basis of B_1^+ and $\{b_1, b_2, \cdots, b_{\beta_1^-}\}$ a basis of B_1^-. Then there exsists a basis $\{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{\beta_1^+}, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{\beta_1^-}\}$ of $H_2(M; \boldsymbol{Q})$ such that $\mathrm{Int}(a_i, \bar{a}_j)=\mathrm{Int}(b_i, \bar{b}_j)=\delta_{ij}$ and $\mathrm{Int}(a_i, \bar{b}_j)=\mathrm{Int}(b_i, \bar{a}_j)=0$ $(1\leq i\leq\beta_1^+, 1\leq j\leq\beta_1^-)$, where $\mathrm{Int}(x, y)$ is the intersection number of x and y, and δ_{ij} is the Kronecker delta. Then we have

$$\begin{aligned}\mathrm{Int}(a_i, \tau_*(\bar{a}_j)) &= \langle [M], \varphi(a_i)\cup\varphi(\tau_*(\bar{a}_j))\rangle\\ &= \langle \tau_*[M], \varphi(\tau_*(a_i))\cup\varphi(\bar{a}_j)\rangle\\ &= \langle -[M], \varphi(a_i)\cup\varphi(\bar{a}_j)\rangle = -\delta_{ij}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Int}(b_i, \tau_*(\bar{a}_j)) &= \langle [M], \varphi(b_i) \cup \varphi(\tau_*(\bar{a}_j))\rangle \\ &= -\langle [M], (-\varphi(b_i)) \cup \varphi(\bar{a}_j)\rangle = 0 .\end{aligned}$$

(φ is the Ponncaré dual map).

Hence

$$\tau_*(\bar{a}_i) = -\bar{a}_i .$$

By the same way, we have $\tau_*(\bar{b}_i)=\bar{b}_i$. Hence $\{\bar{a}_1, \bar{a}_2. \cdots, \bar{a}_{\beta_1^+}\}$ is a basis of B_2^- and $\{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{\beta_1^-}\}$ is a basis of B_2^+. Therefore $\beta_1^+=\beta_2^-$ and $\beta_1^-=\beta_2^+$.

On the other hand, it is known that for a periodic transformation f on a compact ENR X, $L(f)=\chi(\mathrm{Fix}(f, X))$, where $L(f)$ is the Lefschetz number of f (cf. [2], p. 261). For our (M, τ),

$$\begin{aligned}L(\tau) &= \textstyle\sum_{i=0}^3 (-1)^i \,\mathrm{Trace}\, \tau_*^{(i)} \\ &= 1-(\beta_1^+-\beta_1^-)+(\beta_2^+-\beta_2^-)-(-1) \\ &= 2(1+\beta_1(M)-2\beta_1^+) .\end{aligned}$$

Hence we have

$$\chi(\mathrm{Fix}(\tau, M)) = 2(1+\beta_1(M)-2\beta_1^+) .$$

This completes the proof.

Proof of Theorem 2. (1) holds from a theorem of Kawauchi [5], since for any closed orientable surface E, $\beta_1(E; \boldsymbol{Z}_2)\equiv 0 \pmod 2$.

(3) holds from Lemma 6. Since

$$\begin{aligned}\chi(\mathrm{Fix}(\tau, M)) &= -\beta_1(\mathrm{Fix}(\tau, M))+2\beta_2(\mathrm{Fix}(\tau, M)) \\ &= 2(1+\beta_1(M)-2\beta_1^+) ,\end{aligned}$$

we have

$$\begin{aligned}&\beta_1(\mathrm{Fix}(\tau, M))/2+\beta_2(\mathrm{Fix}(\tau, M)) \\ \equiv& -\beta_1(\mathrm{Fix}(\tau, M))/2+\beta_2(\mathrm{Fix}(\tau, M))\equiv \beta_1(M)+1 \qquad \pmod 2 .\end{aligned}$$

We will prove (2). Let M_1 and M_2 be components of $M-\mathring{N}(\mathrm{Fix}(\tau, M))$. Then ∂M_1 is homeomorphic to $\mathrm{Fix}(\tau, M)$. We identify ∂M_1 with $\mathrm{Fix}(\tau, M)$. Let I (resp. J) be the homomorphism from $H_1(\mathrm{Fix}(\tau, M); \boldsymbol{Q})$ to $H_1(M; \boldsymbol{Q})$ (resp. $H_1(M_1; \boldsymbol{Q})$) induced by the inclusion map. We show $\mathrm{Ker}\, I=\mathrm{Ker}\, J$.

$\mathrm{Ker}\, I \supset \mathrm{Ker}\, J$ is trivial. Let $x=[C]$ be an element of $\mathrm{Ker}\, I$. Then there exists a 2-chain D in M such that $\partial D=C$. Put $D_i=D\cap M_i$ ($i=1, 2$). By a tiny collapsing of $D_1+\tau(D_2)$, we may obtain a 2-chain D' in M_1 with $\partial D'=C$.

Therefore we obtained $\dim \operatorname{Im} I = \dim \operatorname{Im} J$. Note that for any orientable 3-manifold M with boundary, $\dim \operatorname{Im}(\text{incl.}_*: H_1(\partial M; \boldsymbol{Q}) \to H_1(M; \boldsymbol{Q})) = \dim H_1(\partial M; \boldsymbol{Q})/2$. Hence $\dim \operatorname{Im} I = \dim H_1(\operatorname{Fix}(\tau, M); \boldsymbol{Q})/2$. On the other hand, for any $x \in \operatorname{Im} I$, $\tau_*(x) = x$. Hence $\operatorname{Im} I \subset B_1^+$. Thus we obtain that

$$\beta_1(\operatorname{Fix}(\tau, M))/2 = \dim H_1(\operatorname{Fix}(\tau, M); \boldsymbol{Q})/2 \leq \beta_1^+ .$$

Therefore by Lemma 6,

$$\begin{aligned} 2\beta_2(\operatorname{Fix}(\tau, M)) - \beta_1(\operatorname{Fix}(\tau, M)) &= (\operatorname{Fix}(\tau, M)) \\ &= 2(1+\beta_1(M) - 2\beta_1^+) \\ &\leq 2(1+\beta_1(M) - \beta_1(\operatorname{Fix}(\tau, M)) . \end{aligned}$$

Hence

$$\beta_1(\operatorname{Fix}(\tau, M))/2 + \beta_2(\operatorname{Fix}(\tau, M)) \leq \beta_1(M) + 1 .$$

This completes the proof.

Proof of Theorem 4. (1) and (2) are proved by Kawauchi [5]. By Smith theory $\sum_i \beta_i(\operatorname{Fix}(\tau, M); \boldsymbol{Z}_2) \leq \sum_j \beta_j(M; \boldsymbol{Z}_2)$ (cf [1], p. 126), and for a 3-manifold M, $\sum_j \beta_j(M; \boldsymbol{Z}_2) = 2\beta_1(\mathrm{M}; \boldsymbol{Z}_2) + 2$. Hence (3) holds.

Recall that β_1^+ is a non negative integer. Hence by Lemma 6, $0 \leq 2\beta_1^+ = 1 + \beta_1(M) - (\chi(\operatorname{Fix}(\tau, M))/2)$ and $\chi(\operatorname{Fix}(\tau, M))/2 = 1 + \beta_1(M) - 2\beta_1^+ \equiv 1 + \beta_1(M)$ (mod 2). Therefore (5) and (6) hold.

Note that $M - \operatorname{Fix}(\tau, M)$ is connected and for any orientable surface E contained in $\operatorname{Fix}(\tau, M)$, $\tau(E) = E$ and $[E] \subset B_2^+$. Hence

$$\begin{aligned} \beta_2(\operatorname{Fix}(\tau, M)) &\leq \beta_2^+ \\ &= \beta_1(M) - \beta_1^+ \\ &= \beta_1(M) - \{(1+\beta_1(M))/2 - \chi(\operatorname{Fix}(\tau, M))/4\} \end{aligned}$$

and

$$\chi(\operatorname{Fix}(\tau, M))/2 - 2\beta_2(\operatorname{Fix}(\tau, M)) \geq 1 - \beta_1(M) .$$

Therefore (4) holds.

For (7), consider a τ-invariant regular neighborhood N of the unoin of non-orientable surfaces of odd genera contained in $\operatorname{Fix}(\tau, M)$ and let $M' = M - \mathring{N}$ ($\mathring{N}$ is the interoir of N). Then the Mayer-Vietoris exact sequence for $M = M' \cup N$ and $\partial N = \partial M' = M' \cap N$ is as follws:

$$\cdots \to H_1(\partial N; \boldsymbol{Z}) \xrightarrow{I} H_1(M'; \boldsymbol{Z}) \oplus H_1(N; \boldsymbol{Z}) \xrightarrow{J} H_1(M; \boldsymbol{Z}) \to \cdots ,$$

where $I = (i_{1*}, i_{2*})$, $i_1: \partial N \to M'$ and $i_2: \partial N \to N$ are inclusion maps. Since the image of i_{2*}; $H_2(\partial N; \boldsymbol{Z}) \to H_1(N; \boldsymbol{Z})$ is torsion free, we see that $J|_{\operatorname{Tor} H_1(N; \boldsymbol{Z})}$: $\operatorname{Tor} H_1(N; \boldsymbol{Z}) \to H_1(M; \boldsymbol{Z})$ is injective. Hence (7) holds.

3. Basic manifolds and operations.

For proofs of Theorems 3 and 5, we construct eight basic manifolds with involutions and then introduce six additive operations on manifolds with imvolutions. For this purpose, we difine the *data* of (M, τ) as follows: Suppose $\mathrm{Fix}(\tau, M)$ consists of m orientable surfaces $E_1, E_2, \cdots, E_m$, n nonorientable surfaces $F_1, F_2, \cdots, F_n$ and p points, and that the number of nonorientable surfaces of odd genera contained in $F_1, F_2, \cdots, F_n$ is s. Then the data of (M, τ) is defined to be

$$[\beta_1(M), s, r, p; g_1, g_2, \cdots, g_m; c_{,1} c_2, \cdots, c_n],$$

where $r=(\beta_1(M; \boldsymbol{Z}_2)-\beta_1(M)-s)/2$, $g_i=\beta_1(E_i)/2$, the genus of E_i $(i=1, 2, \cdots, m)$ and $c_j=\beta_1(F_j; \boldsymbol{Z}_2)$, the nonorientable genus of F_j $(j=1, 2, \cdots, n)$.

Now we consider eight basic manifolds with involutions.

(1) $A_1=(S^3, \tau)$; S^3 is the 3-sphere. A_1 has the data $[0, 0, 0, 2; \ ; \ ]$.

τ is defined as follows: We regard S^3 as $\boldsymbol{R}^3 \cup \{\infty\}$. Then $\tau: S^3 \to S^3$ is an involution defined by $\tau(x, y, z)=(-x, -y, -z)$ $((x, y, z) \in \boldsymbol{R}^3)$ and $\tau(\infty)=\infty$.

(2) $A_2=(P^3, \tau)$; P^3 is the 3-dimensional projective space. A_2 has the data $[0, 1, 0, 1; \ ; 1]$.

τ is defined as follows: We regard P^3 as $\{(x, y, z) \in \boldsymbol{R}^3 \mid x^2+y^2+z^2 \leq 1\}/((x, y, z) \sim (-x, -y, -z)\ (x^2+y^2+z^2=1))$. Then $\tau: P^3 \to P^3$ is an involution defined by $\tau(x, y, z)=(-x, -y, -z)$.

(3) $A_3=(S^2 \times S^1, \tau_1)$; A_3 has the data $[1, 0, 0, 0; 1; \ ]$.

τ_1 is defined as follows: Consider an orientation reversing involution τ' on S^2 such that $\mathrm{Fix}(\tau', S^2)$ is a circle. Then $\tau_1: S^2 \times S^1 \to S^2 \times S^1$ is an involution defined by $\tau_1=\tau' \times \mathrm{identity}$.

(4) $A_4=(S^2 \times S^1, \tau_{2,})$; A_4 has the data $[1, 0, 0, 0; \ ; 2]$.

τ_2 is defined as follows: Consider an orientation reversing involution τ' on S^2 as in (3). Regard $S^2 \times S^1$ as the identifying space of $S^2 \times I$ with the identifying map from $S^2 \times \{1\}$ to $S^2 \times \{0\}$: $(x, 1) \sim (\tau'(x), 0)$, where I is the unit interval $[0, 1]$. Then $S^2 \times S^1$ has an orientation reversing involution τ_2 extending τ' with $\mathrm{Fix}(\tau_2, S^1 \times S^2)$ a Klein bottle.

(5) $A_5=(N_1, \tau)$; A_5 has the data $[0, 0, 1, 2; \ ; 2]$ and $H_1(N_1; \boldsymbol{Z}) \cong \boldsymbol{Z}_{2q} \oplus \boldsymbol{Z}_{2q}$ $(q \in \boldsymbol{N})$.

(N_1, τ_1) is defined as follows: Consider (V, τ) such that V is a solid torus with $\mathrm{Fix}(\tau, V)$ two points. Let K be a closed curve in V such that $[K]=b$ generates $H_1(V; \boldsymbol{Z})$ with $K \cap \tau(K)=\phi$ (see Figure 1). Let V_1 and V_2 be solid tori. Attach them to $V-\mathring{N}(K \cup \tau(K))$ as follows: ∂V_1 is identified with $\partial N(K)$

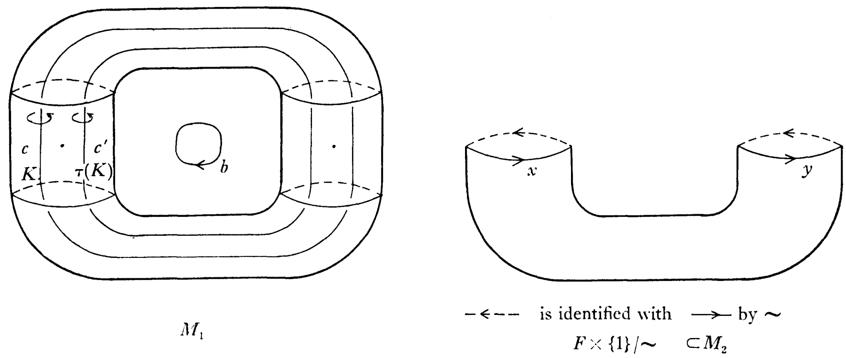

Figure 1

so that a meridian of ∂V_1 is a curve C on $\partial N(K)$ with $[C]=qc-b\in H_1(V-\mathring{N}(K\cup\tau(K)); \boldsymbol{Z})$. ∂V_2 is identified with $\partial N(\tau(K))$ so that a meridian of ∂V_1 is a curve C' on $\partial N(\tau(K))$ with $[C']=qc'-b\in H_1(V-\mathring{N}(K\cup\tau(K)); \boldsymbol{Z})$ (c and c' are generators of $H_1(V-\mathring{N}(K\cup\tau(K)); \boldsymbol{Z})$ as indicated in Figure 1). We denote the resulting manifold by M_1. Then M_1 has an orientation reversing involution and

$$H_1(M_1; \boldsymbol{Z})\cong\langle b, c, c'\colon qc-b=0, qc'+b=0\rangle .$$

Let $F=\partial M_1=\partial V$ and M_2 a quotient space of $F\times I$ by the identifying map of $F\times\{1\}: (x, 1)\sim(\tau'(x), 1)$, where $\tau'=\tau|_F$. Then M_2 has an involution τ'', induced by τ'. $\mathrm{Fix}(\tau'', M_2)$ consists of a Klein bottle $F\times\{1\}/\sim$.

Let $N_1=M_1\cup_h M_2$ where h is the identity map of the boundary F. Then N_1 has an orientation reversing involution τ such that $\mathrm{Fix}(\tau, N_1)$ consists of two points and a Klein bottle.

To compute $H_1(N_1; \boldsymbol{Z})$, we choose generators of $H_1(M_2; \boldsymbol{Z})\cong H_1(F\times\{1\}/\sim; \boldsymbol{Z})$ represented by curves as indicated in Figure 1. Then we have

$$H_1(M_2; \boldsymbol{Z})\cong\langle x, y\colon 2x+2y=0\rangle .$$

We can check that

$$\begin{aligned} H_1(N_1; \boldsymbol{Z})\cong\langle b, c, c', x, y\colon & qc-b=0, qc'+b=0, 2x+2y=0, \\ & 2x=c+c', x+y=b\rangle \\ \cong\langle c, x\colon & 2qc=0, 2qx=0\rangle . \\ \cong \boldsymbol{Z}_{2q} & \oplus\boldsymbol{Z}_{2q} . \end{aligned}$$

(6) $A_6(g)=(N_2, \tau)$; $A_6(g)$ has the data $[1, 0, g, 2g+2; g;]$ and $\mathrm{Tor}\, H_1(N_2; \boldsymbol{Z})\cong\bigoplus_{i=1}^{g}(\boldsymbol{Z}_{2q_i}\oplus\boldsymbol{Z}_{2q_i})$ $(g, q_1, q_2, \cdots, q_g\in\boldsymbol{N})$.

Let F be a closed orientable surface of genus g. There exists an orientation preserving involution α on F such that the fixed point set consists of $2g+2$ points. Consider $F\times I$ and τ': $F\times I\to F\times I$, $\tau'(x, t)=(\alpha(x), 1-t)$ $(x\in F, t\in I)$. Then τ' is an orientation reversing involution on $F\times I$, and $\mathrm{Fix}(\tau', F\times I)$ consists of $2g+2$ points. Let $K_1, K_2, \cdots, K_g$ be closed curves in $F\times I$ as indicated in Figure 2. These curves satisfy the following:

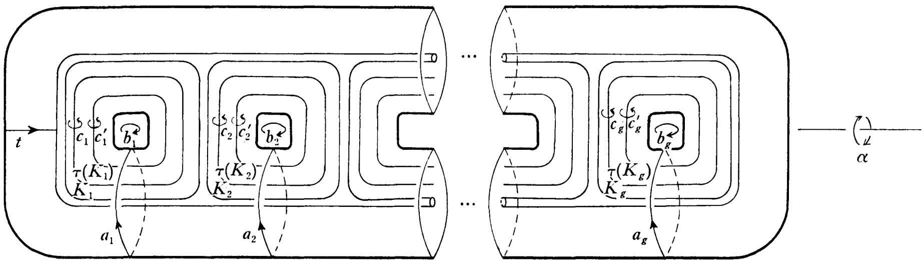

Figure 2

1. $K_1, K_2, \cdots, K_g, \tau(K_1), \tau(K_2), \cdots, \tau(K_g)$ are mutually disjoint.
2. Let $[K_i]=b_i\in H_1(F\times I; \boldsymbol{Z})$ $(i=1, 2, \cdots, g)$, then $\{b_1, b_2, \cdots, b_g\}$ is a basis of $H_1(F\times I; \boldsymbol{Z})$.

Consider $2g$ solid tori and attach them to $F\times I-\bigcup_{i=1}^{g}\mathring{N}(K_i\cup\tau(K_i))$ as in (5) so that the resulting manifold M_1 has

$$H_1(M_1; \boldsymbol{Z})\cong\langle b_1, b_2, \cdots, b_g, c_1, c_1', c_2, c_2', \cdots, c_g, c_g': \\ q_i c_i-b_i = 0,\ q_i c_i'+b_i = 0 \quad (i=1, 2, \cdots, g)\rangle .$$

Consider the identifying space of M_1 by the identifying map of $F\times\{1\}$ to $F\times\{0\}$; $(x, 1)\sim(\alpha(x), 0)$, and denote the resulting manifold by N_2. Then this manifold has an orientation reversing involution τ induced by τ' and the fixed point set consists of $2g+2$ points and $F\times\{1\}$ (an orientable surface of genus g).

To compute $H_1(N_2; \boldsymbol{Z})$, we choose generators of $H_1(N_2; \boldsymbol{Z})$ represented by curves as indicated in Figure 2. Then we have

$$\begin{aligned} &H_1(N_2; \boldsymbol{Z}) \\ &\cong\langle a_i, b_i, c_i, c_i', t;\ q_i c_i-b_i = 0,\ q_i c_i'+b_i = 0,\ a_i-(c_i+c_i') = a_i, \\ &\qquad\qquad\qquad\qquad b_i=-b_i \quad (i=1, 2, \cdots, g)\rangle \\ &\cong\langle a_i, c_i, t\colon 2q_i a_i = 0,\ 2q_i c_i = 0 \quad (i=1, 2, \cdots, g)\rangle \\ &\cong\bigoplus_{i=1}^{g}(\boldsymbol{Z}_{2q_i}\oplus\boldsymbol{Z}_{2q_i})\oplus\boldsymbol{Z}\,. \end{aligned}$$

(7) $A_7(g)=(N_3, \tau)$; $A_7(g)$ has the data $[g, 0, 0, 0; g; \ ]$ $(g\in\boldsymbol{N})$ and $H_1(N_3; \boldsymbol{Z})$ is a free abelian group.

Consider a handle body V of genus g. Let N_3 be the double of V and τ: $N_3\to N_3$ a map interchanging the copies of V. Then τ is an orientation reversing involution on N_3, $\mathrm{Fix}(\tau, N_3)$ consists of ∂V (a colsed orientable surface of genus g), and clearly $H_1(N_3; \boldsymbol{Z})$ is a free ablelian group of rank g.

(8) $A_8(n)=(N_4, \tau)$; ($A_8(n)$ has the data $[1, 0, n, 0; ; \underbrace{2, 2, \cdots, 2}_{n+1}]$ and $\mathrm{Tor}\, H_1(N_4; \boldsymbol{Z}) \cong \bigoplus_{i=1}^{n} (\boldsymbol{Z}_{2q_i} \oplus \boldsymbol{Z}_{2q_i})$ $(n, q_1, q_2, \cdots, q_n \in \boldsymbol{N})$.

Consider n manifolds with involutions $(M_1, \tau_1), (M_2, \tau_2), \cdots, (M_n, \tau_n)$ of type A_5 such that $\mathrm{Tor}\, H_1(M_i; \boldsymbol{Z}) \cong \boldsymbol{Z}_{2q_i} \oplus \boldsymbol{Z}_{2q_i}$ $(i=1, 2, \cdots, n)$. Note that $\mathrm{Fix}(\tau, M_i)$ contains two points $(i=1, 2, \cdots, n)$. Let B_1 (B'_n, resp.) be a τ-invariant ball in M_1 (M_n, resp.) containing a fixed point, and B_i and B'_i disjoint τ_i-invariant balls in M_i such that each balls contain a fixed point $(i=2, 3, \cdots, n-1)$. Let $M=(M_1-\mathring{B}_1) \cup (\bigcup_{i=1}^{n-1} (M_i - \mathring{B}_i \cup \mathring{B}'_i)) \cup (M_n - \mathring{B}'_n)$, where $\partial B'_i$ is identified with ∂B_{i-1} so that the identifying map commutes with τ_{i-1} and τ_i $(i=2, 3, \cdots, n)$. Then M is the connected sum of $M_1, M_2, \cdots, M_{n-1}$ and M_n with an orientation reversing involution τ extending $\tau_1, \tau_2, \cdots, \tau_{n-1}$ and τ_n. Let K be a τ-invariant closed curve in M as indicated in Figure 3.

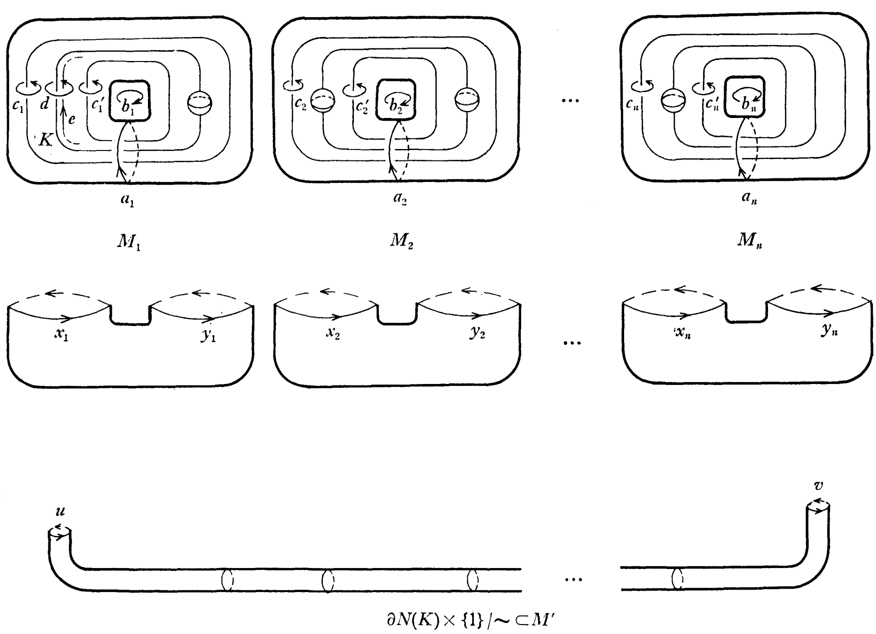

Figure 3

Let $\tau'=\tau|_{\partial N(K)}$ and M' the identifying space of $\partial N(K) \times [0, 1]$ by the identifying map of $\partial N(K) \times \{1\}$; $(x, 1) \sim (\tau'(x), 1)$, and let $N_4=(M-\mathring{N}(K)) \cup_h M'$, where h is the identify map of $\partial N(K)$. Then N_4 has an orientation reversing involution and its fixed point set consisits of $n+1$ Klein bottles.

To compute $H_1(N_4; \boldsymbol{Z})$, we choose generators of $H_1(M-\mathring{N}(K); \boldsymbol{Z})$ and $H_1(M'; \boldsymbol{Z})$ represented by curves as indicated in Figure 3. Then we have

$$\begin{aligned}
&H_1(N_4; \boldsymbol{Z}) \\
&\cong\langle a_i, b_i, c_i, c_i', x_i, y_i, u, v, d, e: \\
&\quad q_i c_i - b_i = 0,\ q_i c_i' + b_i = 0,\ a_i = c_i + c_i' + d,\ e = \sum_{j=1}^{n} b_j,\ 2x_i + 2y_i = 0, \\
&\quad 2u + 2v = 0,\ 2x_i = a_i,\ x_i + y_i = b_i,\ 2u = d,\ u + v = e \quad (i = 1, 2, \cdots, n)\rangle \\
&\cong\langle x_i, z_i, c_i, u: z_i = x_i - u,\ 2q_i z_i = 0,\ 2q_i c_i = 0 \quad (i = 1, 2, \cdots, n)\rangle \\
&\cong\langle z_i, c_i, u: 2q_i z_i = 0,\ 2q_i c_i = 0 \quad (i = 1, 2, \cdots, n)\rangle \\
&\cong \bigoplus_{i=1}^{n} (\boldsymbol{Z}_{2q_i} \oplus \boldsymbol{Z}_{2q_i}) \oplus \boldsymbol{Z}\,.
\end{aligned}$$

We defined eight types of basic manifolds with involutions as follows:

(M, τ)	data
A_1	$[0, 0, 0, \ 2 \ ; \ ; \ ]$
A_2	$[0, 1, 0, \ 1 \ ; \ ; 1]$
A_3	$[1, 0, 0, \ 0 \ ; 1; \ ]$
A_4	$[1, 0, 0, \ 0 \ ; \ ; 2]$
A_5	$[0, 0, 1, \ 2 \ ; \ ; 2]$
$A_6(g)$	$[1, 0, g, 2g+2; g; \ ]$
$A_7(g)$	$[g, 0, 0, \ 0 \ ; g; \ ]$
$A_8(n)$	$[1, 0, n, \ 0 \ ; \ ; \underbrace{2, 2, \cdots, 2}_{n+1}]$

REMARK 3. We have (M, τ) of type A_5, $A_6(g)$ or $A_8(n)$ such that $\mathrm{Tor}\, H_1(M; \boldsymbol{Z}) \cong \boldsymbol{Z}_{2q} \oplus \boldsymbol{Z}_{2q}$, $\bigoplus_{i=1}^{g} (\boldsymbol{Z}_{2q_i} \oplus \boldsymbol{Z}_{2q_i})$ or $\bigoplus_{i=1}^{n} (\boldsymbol{Z}_{2q_i} \oplus \boldsymbol{Z}_{2q_i})$ for any $q \neq 0$, any $q_i \neq 0$ $(i=1, 2, \cdots, g)$ or any $q_i \neq 0$ $(i=1, 2, \cdots, n)$, respectively.

Now we difine six operations.

Operation 1. Consider (M', τ') and some closed orientable 3-manifold N (which may not have involutions). Let B be a 3-ball contained in M' with $B \cap \tau'(B) = \phi$, and B' a 3-ball contained in N. Let $M = (N - \mathring{B}') \cup (M' - (\mathring{B} \cup \tau'(\mathring{B}))) \cup (-(N - \mathring{B}'))$ where ∂B is identified with $\partial B'$ and $\partial(\tau'(B))$ with $\partial B'$ (in $-N$). Then M has an orientation reversing involution τ extending τ' with $\mathrm{Fix}(\tau, M) = \mathrm{Fix}(\tau', M')$. Note that $H_1(M; \boldsymbol{Z}) \cong H_1(M'\ \boldsymbol{Z}) \oplus H_1(N; \boldsymbol{Z}) \oplus H_1(N; \boldsymbol{Z})$.

Operation 2. Consider (M_i, τ_i) such that $\mathrm{Fix}(\tau_i, M_i)$ contains an isolated point P_i $(i=1, 2)$. Let B_i be a τ_i-invariant 3-ball in M_i such that $B_i \cap \mathrm{Fix}(\tau_i, M_i) = P_i$ $(i=1, 2)$. Let M be an identifying space $(M_1 - \mathring{B}_1) \cup_h (M_2 - \mathring{B}_2)$ where the identifying map $h: \partial B_2 \to \partial B_1$ commutes with τ_1 and τ_2. Then M is the connected sum of M_1 and M_2 with an orientation reversing involution τ extending τ_1 and τ_2. Suppose that (M_i, τ_i) has a data as follows;

(M_1, τ_1); $[\beta_1, s_1, r_1, p_1, g_1, g_2, \cdots, g_{m'}; c_1, c_2, \cdots, c_{n'}]$
(M_2, τ_2): $[\beta_2, s_2, r_2, p_2; g_{m'+1}, g_{m'+2}, \cdots, g_m; c_{n'+1}, c_{n'+2}, \cdots, c_n]$ $\quad (m' \leq m, n' \leq n)$

then (M, τ) has the data

$$[*, *, *, p_1+p_2-2; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

($*$ means the sum of numbers which are in the same column. For example, in the first column, $*$ means $\beta_1+\beta_2$.) Note that $H_1(M; \boldsymbol{Z}) \cong H_1(M_1; \boldsymbol{Z}) \oplus H_1(M_2; \boldsymbol{Z})$.

Operation 3. Consider (M_i, τ_i) such that $\mathrm{Fix}(\tau_i, M_i)$ contains a surface F_i $(i=1, 2)$. Let $B_i \subset M_i$ be a τ_i-invariant 3-ball such that $B_i \cap \mathrm{Fix}(M_i, \tau_i)$ is a 2-disk on F_i $(i=1, 2)$. Let $M=(M_1-\mathring{B}_1) \cup_h (M_2-\mathring{B}_2)$, where the identifying map $h: \partial B_2 \to \partial B_1$ commutes with τ_1 and τ_2. Then M is the connected sum of M_1 and M_2 with an orientation reversing involution τ extending τ_1 and τ_2. Suppose that (M_i, τ_i) has a data as in the definition of Operation 2 $(i=1, 2)$. If F_1 and F_2 are orientable with genera g_j and g_k, respectively, where $j \leq m' < k \leq m$, then (M, τ) has the data

$$[*, *, *, *; g_1, g_2, \cdots, g_{j-1}, g_j+g_k, g_{j+1}, \cdots, \check{g}_k, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

($\check{}$ means removing the specified element). And if F_1 and F_2 are nonorientable with nonorientable genera c_j and c_k, respectively, where $j \leq n' < k \leq n$, then (M, τ) has the data

$$[*, *, *, *; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_{j-1}, c_j+c_k, c_{j+1}, \cdots, \check{c}_k, \cdots, c_n]$$

Note that $H_1(M; \boldsymbol{Z}) \cong H_1(M_1; \boldsymbol{Z}) \oplus H_1(M_2; \boldsymbol{Z})$.

Operation 3'. Consider $(M_1, \tau_1), (M_2, \tau_2), \cdots, (M_n, \tau_n)$ $(n \geq 2)$ and (M', τ') such that $\mathrm{Fix}(\tau_i, M_i)$ consists of a surface F_i and certain points $(i=1, 2, \cdots, n)$, and such that $\mathrm{Fix}(\tau', M')$ consists of n surfaces $E_1, E_2, \cdots, E_n$ and certain points. Let B_i be a τ_i-invariant 3-ball in M_i such that $B_i \cap \mathrm{Fix}(\tau_i, M_i)$ is a disk on F_i $(i=1, 2, \cdots, n)$, and let C_i be a τ'-invariant 3-ball in M' such that $C_i \cap \mathrm{Fix}(\tau', M')$ is a disk on E_i $(i=1, 2, \cdots, n)$. We consider an operation similar to Operation 3 with attaching homeomorphism h_i; $\partial B_i \to \partial C_i$ $(i=1, 2, \cdots, n)$. Then we can obtain (M, τ) such that M is the connected sum of $M_1, M_2, \cdots, M_n$ and M', and such that $\mathrm{Fix}(\tau, M)$ consists of the connected sum of F_i and E_i $(i=1, 2, \cdots, n)$ and certain points. Suppose that (M_i, τ_i) $(i=1, 2, \cdots, n)$ and (M', τ') have data as follows;

$$(M_i, \tau_i):\ [\beta_i, s_i, r_i, p_i;\quad ; c_i] \qquad (i=1, 2, \cdots, n)$$
$$(M', \tau'):\ [\beta', s', r', p';\quad ; c_1', c_2', \cdots, c_n'] .$$

Then (M, τ) has the data

$$[\sum_{i=1}^{n} \beta_i+\beta', \sum_{i=1}^{n} s_i+s', \sum_{i=1}^{n} r_i+r', \sum_{i=1}^{n} p_i+p';\quad ; c_1+c_1', c_2+c_2', \cdots, c_n+c_n']$$

Note that $H_1(M; \boldsymbol{Z})\cong\bigoplus_{i=1}^{n}(M_i; \boldsymbol{Z})\oplus H_1(M'; \boldsymbol{Z})$.

Operation 4. Consider (M_1, τ_1) and (M_2, τ_2). Let B_i be a 3-ball in M_i with $B_i\cap\tau_i(B_i)=\phi$ $(i=i, 2)$. Let $M=(M_1-(\mathring{B}_1\cup\tau_1(\mathring{B}_1))\cup(M_2-(\mathring{B}_2\cup\tau_2(\mathring{B}_2))$, where ∂B_1 is identified with ∂B_2 and $\partial\tau_1(B_1)$ identified with $\partial\tau_2(B_2)$ so that the identifying map commutes with τ_1 and τ_2. Then M has an orientation reversing involution τ extending τ_1 and τ_2 with $\mathrm{Fix}(\tau, M)=\mathrm{Fix}(\tau_1, M_1)\cup\mathrm{Fix}(\tau_2, M_2)$. Suppose that (M_i, τ_i) $(i=1, 2)$ has a data as in the definition of Operation 2, then (M, τ) has the data

$$[\beta_1+\beta_2+1, *, *, *; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n] .$$

Note that $H_1(M; \boldsymbol{Z})\cong H_1(M_1; \boldsymbol{Z})\oplus H_1(M_2; \boldsymbol{Z})\oplus\boldsymbol{Z}$.

Operation 5. Consider (M_i, τ_i) such that $\mathrm{Fix}(\tau_i, M_i)$ contains two isolated points p_{i_1} and p_{i_2} $(i=1, 2)$. Let B_{i_j} be a τ_i-invariant 3-ball in M_i containing p_{i_j} $(i=1, 2, j=1, 2)$. Let $M=(M_1-(\mathring{B}_{1_1}\cup\mathring{B}_{1_2}))\cup(M_2-(\mathring{B}_{2_1}\cup\mathring{B}_{2_2}))$, where ∂B_{1_j} is identified with ∂B_{2_j} $(j=1, 2)$ so that the identifying map commutes with τ_1 and τ_2. Then M has an orientation reversing involution τ extending τ_1 and τ_2. Suppose that (M_i, τ_i) $(i=1, 2)$ has the data as in the definition of Operation 2, then (M, τ) has the data

$$[\beta_1+\beta_2+1, *, *, p_1+p_2-4; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$$

Note that $H_1(M; \boldsymbol{Z})\cong H_1(M_1; \boldsymbol{Z})\oplus H_1(M_2; \boldsymbol{Z})\oplus\boldsymbol{Z}$.

4. Constructions.

Proof of Theorem 3. Let $E_1, E_2, \cdots, E_m$ be the components of E. Consider a handlebody V_1 such that $\partial V_1=E_1$. Let $V_2, V_3, \cdots, V_m$ be murtually disjoint handlebodies contained in $\mathring{V}_1$ such that the natural homomorphism $H_1(V_i; \boldsymbol{Z})\rightarrow H_1(V_1; \boldsymbol{Z})$ is trivial. Let $M_1=V_1-\bigcup_{i=2}^{m}\mathring{V}_i$. Then the double of M_1, DM_1, has an orientation reversing involtuion τ interchanging the copies of M_1 with $\mathrm{Fix}(\tau, DM_1)=\bigcup_{i=1}^{m}V_i$. Note that $H_1(DM_1; \boldsymbol{Z})$ is a free abelian group of rank $m+\sum_{i=1}^{m}g(E_i)-1$.

By (2) and (3), we can see that $\mathrm{rank}G-(m+\sum_{i=1}^{m}g(E_i)-1)$ is a nonnegative even integer. Hence by (1), we can consider that $H_1(DM_1;\boldsymbol{Z})$ is a direct summand of G with $G/H_1(DM_1;\boldsymbol{Z})\cong B\oplus B$ for some abelian group B. Let M_2 be a closed orientable manifold with $H_1(M_2;\boldsymbol{Z})\cong B$. Put $M=M_2\# DM_1\#(-M_2)$ by using Operation 1. Then we can see that M is the required manifold. This completes the proof.

Lemma 7. *Let $t, s, r, p, m, n, g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_n$ be nonnegative integers satisfying the following conditoins:*

(1) $s\leq n, g_i>0\ (i=1, 2, \cdots, m), c_j>1\ (j=1, 2, \cdots, n)$

(2) $s, p, c_{s+1}, c_{s+2}, \cdots, c_n$ *are even, and* $c_1, c_2, \cdots, c_s$ *are odd.*

(3) $m+n+\sum_{i=1}^{m}g_i+p+\sum_{j=1}^{n}c_j\leq 2s+4r+2t+2.$

(4) $m+n-\sum_{i=1}^{m}g_i+(p+\sum_{j=1}^{n}c_j)/2\geq 1-t.$

(5) $m+n-\sum_{i=1}^{m}g_i+(p-\sum_{j=1}^{n}c_j)/2\leq 1+t.$

(6) $m+n-\sum_{i=1}^{m}g_i+(p-\sum_{i=1}c_j)/2\equiv 1+t \pmod 2.$

Then there exists (M, τ) which has a data $[t, s, r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$.

Proof. We consider the following six cases (where $\sum g_i=\sum_{i=1}^{m}g_i$ and $\sum c_j=\sum_{j=1}^{n}c_j$):

Case 1) $2\sum g_i+\sum c_j\leq s+2r$.
Case 2) $2\sum g_i+\sum c_j>s+2r$ and $r\geq n$.
Case 3) $n>r\geq s/2$ and $p\geq 2$.
Case 4) $s/2>r$ and $p\geq 2$.
Case 5) $n>r, p=0$ and $r+1\geq s/2$.
Case 6) $n>r, p=0$ and $r+1<s/2$.

Case 1) $2\sum g_i+\sum c_j\leq s+2r$.

We prepare $|1-n+\sum g_i+((\sum c_j-p)/2|$ copies of A_1, s copies of A_2, $(\sum c_j-s)/2$ copies of A_5 and $A_6(g_1), A_6(g_2), \cdots, A_6(g_m)$. Now we have the following data

A_1	: $[0, 0, 0, \quad 2 \quad ; \quad ; \quad]$	$(\lvert 1-n+\sum g_i+(\sum c_j-p)/2\rvert$ times),
A_2	: $[0, 1, 0, \quad 1 \quad ; \quad ; \quad]$	(s times),
A_5	: $[0, 0, 1, \quad 2 \quad ; \quad ; 2]$	$((\sum c_j-s)/2$ times),
$A_6(g_i)$	: $[1, 0, g_i, 2g_i+2; g_i; \quad]$	$(i=1, 2, \cdots, m)$.

We denote by $X\overset{i}{—}Y$ the result of Operation i on the manifolds with involutions X and Y, and by $X(\overset{i}{—}Y)^n$, $X\overset{i}{—}Y\overset{i}{—}Y\overset{i}{—}\cdots\overset{i}{—}Y$ (n copies of Y). Then we apply Operation 3 as indicated in Figure 4 and obtain B_1^j ($j=1$,

$$B_1^j = A_5(\overset{3}{\longrightarrow} A_5)^{(c_j-1)/2-1} \overset{3}{\longrightarrow} A_2 \qquad (j=1, 2, \cdots, s)$$
$$B_1^j = A_5(\overset{3}{\longrightarrow} A_5)^{c_j/2-1} \qquad (j=s+1, s+2, \cdots, n)$$

Figure 4

$2, \cdots, n)$ with data $[0, 0, (c_j-1)/2, c_j; \ ; c_j]$ or $[0, 0, c_j/2, c_j; \ ; c_j]$ according to whether $j \leq s$ or $j \geq s+1$.

Applying Operation 2 as indicated in Figure 5, we obtain B_2 with data

$$B_2 = B_1^1 \overset{2}{\longrightarrow} B_1^2 \overset{2}{\longrightarrow} B_1^3 \overset{2}{\longrightarrow} \cdots$$
$$\cdots \overset{2}{\longrightarrow} B_1^n \overset{2}{\longrightarrow} A_6(g_1) \overset{2}{\longrightarrow} A_6(g_2) \overset{2}{\longrightarrow} \cdots \overset{2}{\longrightarrow} A_6(g_m)$$

Figure 5

$$[m, s, \sum g_i + (\sum c_j - s)/2, \sum c_j + \sum_{i=1}^{m} (2g_i+2) - 2(m+n-1);$$
$$g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n],$$

If $1-n+\sum g_i + (\sum c_j - p)/2 \geq 0$, then we apply Operation 5 as indictaed in Figure 6 and obtain B_3 with data

$$B_3 = B_2(\overset{k}{\longrightarrow} A_1)^{|n-1-\sum g_i + ((p-\sum c_j)/2)|}$$
$$k = \begin{cases} 4 & \textit{if} \ \ n-1-\sum g_i + ((p-\sum c_j)/2) \geq 0 \\ 5 & \textit{if} \ \ n-1-\sum g_i + ((p-\sum c_j)/2) \leq 0 \end{cases}$$

Figure 6

$$[1+m-n+\sum g_i + (\sum c_j - p)/2, s, \sum g_i + (\sum c_j - s)/2, p;$$
$$g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

By (4), $1+m-n+\sum g_i + (\sum c_j - p)/2 \leq t$. By (6), $t-(1+m-n+\sum g_i + (c_j - p)/2)$ is even. By Assumption of Case 1), $\sum g_i + (\sum c_j - s)/2 \leq r$. Hence we can obtain a manifold with involution with the required data by Operation 1.

If $1-n+\sum g_i + (\sum c_j - p)/2 \leq 0$, then we apply Operation 4 as indicated in Figure 6 and obtain a manifold with data

$$[-1+m+n-\sum g_i + (p-\sum c_j)/2, s, \sum g_i + (\sum c_j - s)/2, p:$$
$$g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

By (5) and (6), $t-(-1+m+n-\sum g_i + (p-\sum c_j)/2)$ is a nonnegative even integer. By Assumption of Case 1), $\sum g_i + (\sum c_j - s)/2 \leq r$. Hence by Operation 1, we can obtain a manifold with the required data.

Case 2) $2\sum g_i + \sum c_j > s+2r$ and $r \geq n$.

There exist integers $m', g'_1, g'_2, \cdots, g'_m, c'_1, c'_2, \cdots, c'_n$ satisfying the following conditions;

1. $0 \leq m' \leq m$,
2. $0 < g'_i \leq g_i$ $(i=1, 2, \cdots, m')$.
3, $1 < c'_j \leq c_j$ and $c'_j \equiv c_j \pmod 2$ $(j=1, 2, \cdots, n)$, and
4. $2\sum_{i=1}^{m'} g'_i + \sum_{j=1}^{n} c'_j = s+2r$.

(Note that if $m'=0$, $c'_j=3$ $(j=1, 2, \cdots, s)$ and $c'_j=2$ $(j=s+1, s+2, \cdots, n)$, then $\sum_{j=1}^{n} c'_j = 3s+2(n-s) = s+2n \leq s+2r$.)

We prepare the basic manifolds and apply Operations as indicated in Figure 7. Then we obtain B_6 with the following data

$$[m+\sum g_i+(\sum c_j-s-2r)/2,\ s,\ r,\ 2-2n+s+2r;\quad g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_n].$$

$$B_4^i = A_6(g'_i)(\overset{3}{\text{---}} A_3)^{g_i-g'_i} \qquad (i=1, 2, \cdots, m')$$

$$B_5^j = A_5(\overset{3}{\text{---}} A_5)^{(c'_j-1)/2-1}(\overset{3}{\text{---}} A_3)^{(c_j-c'_j)/2} \overset{3}{\text{---}} A_2 \qquad (j=1, 2, \cdots, s)$$

$$B_5^j = A_5(\overset{3}{\text{---}} A_5)^{c'_j/2-1}(\overset{3}{\text{---}} A_3)^{(c_j-c'_j)/2} \qquad (s+1 \leq j \leq n)$$

$$B_6 = B_4^1 \overset{2}{\text{---}} B_4^2 \overset{2}{\text{---}} \cdots \overset{2}{\text{---}} B_4^{m'} \overset{2}{\text{---}} B_5^1 \overset{2}{\text{---}} B_5^2 \overset{2}{\text{---}} \cdots$$

$$\cdots \overset{2}{\text{---}} B_5^n \overset{4}{\text{---}} A_7(g_{m'+1}) \overset{4}{\text{---}} A_7(g_{m'+2}) \overset{4}{\text{---}} \cdots \overset{4}{\text{---}} A_7(g_m)$$

$$(\text{if } m'=n=0,\ B_6 = A_1 \overset{4}{\text{---}} A_7(g_1) \overset{4}{\text{---}} A_7(g_2) \overset{4}{\text{---}} \cdots \overset{4}{\text{---}} A_7(g_m))$$

Figure 7

If $2-2n+s+2r-p \geq 0$, then we apply Operation 5 as indicated in Figure 8 and obtain B_7 with data

$$[1+m-n+\sum g_i+(\sum c_j-p)/2,\ s,\ r,\ p;\ g_1, g_2, \cdots, g_m;\ c_1, c_2, \cdots, c_n].$$

$$B_7 = B_6(\overset{k}{\text{---}} A_1)^{|(p-s)/2+n-r-1|}$$

$$k = \begin{cases} 4 & \text{if } (p-s)/2+n-r-1 \geq 0 \\ 5 & \text{if } (p-s)/2+n-r-1 \leq 0 \end{cases}$$

Figure 8

By (4) and (6), $t-(1+m-n+\sum g_i+(\sum c_j-p)/2)$ is a nonnegative even integer. Hence by Operation 1, we can obtain a manifold with the required data.

If $2-2n+s+2r-p \leq 0$, then we apply Operation 4 as indicated in Figure

8 and obtain B_7 aith data

$$[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2,\ s,\ r,\ p;\ g_1, g_2, \cdots, g_m;\ c_1, c_2, \cdots, c_n]\,.$$

By (3) and (6), $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$ is a nonnegative even integer. Hence by Operation 1, we can obtain a manifold with the required data.

Case 3) $n>r\geq s/2$ and $p\geq 2$.

First, consider the numbers $g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_{s/2}, c_{s+1}, \cdots, c_{r+(s/2)}$. Note that $\sum_{j=1}^{s/2} c_j+\sum_{j=s+1}^{r+(s/2)} c_j\geq(s/2)+2r$ and $s/2+(r+s/2-s)\leq r$. Hence, by the same way as in Case 2), we have B_6' with data

$$[m+\sum g_i+(\sum_{j=1}^{s/2} c_j+\sum_{j=s+1}^{r+(s/2)} c_j-s/2-2r)/2,\ s/2,\ r,\ 2+(s/2);$$
$$g_1, g_2, \cdots, g_m;\ c_1, c_1, \cdots, c_{s/2}, c_{s+1}, c_{s+2}, \cdots, c_{r+(s/2)}]$$

Applying Operations to B_6' and the basic manifolds as indicated in Figure 9, we have B_9 with data

$$B_8^j=A_4(\overset{3}{\text{---}}A_4)^{(c_j-1)/2-1}\overset{3}{\text{---}}A_2 \qquad (s/2+1\leq j\leq s)$$

$$B_8^j=A_4(\overset{3}{\text{---}}A_4)^{c_j/2-1} \qquad (r+s/2+1\leq j\leq n)$$

$$B_9=B_8^{r+s/2+1}\overset{4}{\text{---}}B_8^{r+s/2+2}\overset{4}{\text{---}}\cdots\overset{4}{\text{---}}B_8^n(\overset{4}{\text{---}}A_1)^{(p-2)/2}\text{---}B_6'$$

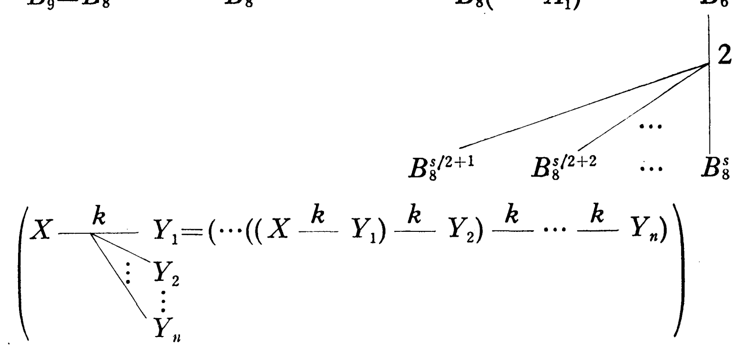

Figure 9

$$[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2,\ s,\ r,\ p;\ g_1, g_2, \cdots, g_m;\ c_1, c_2, \cdots, c_n]$$

By (3) and (6), $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$ is a nonnegative even integer. By Operation 1, we can obtain a manifold with the required data.

Case 4) $s/2>r$ and $p\geq 2$.

First, consider the numbers $g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_{2r}, c_{s+1}, c_{s+2}, \cdots, c_n$.

Note that $2r+(n-s)\geq r=2r/2$. By the same way as in Case 3), we have B_9' with data

$$[-(1+2r+2r)+m+(2r+n-s)+\sum g_i+(p+\sum_{j=1}^{2r}c_j+\sum_{j=s+1}^{n}c_j)/2,\ 2r,\ r,\ p;$$
$$g_1, g_2, \cdots, g_m;\ c_1, c_2, \cdots, c_{2r}, c_{s+1}, c_{s+2}, \ldots, c_n]\,.$$

Applying Operations to B_9' and the basic manifolds as indicated in Figure 10, we obtain B_{11} with data

$$[-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2,\ s,\ r,\ p;\ g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_n]\,.$$

$$B_{10}^i=A_4(\overset{3}{\text{---}}\ A_4)^{(c_{2r+2i-1}-1)/2-1}\ \overset{3}{\text{---}}\ A_2\ \overset{2}{\text{---}}\ A_2(\overset{3}{\text{---}}\ A_4)^{(c_{2r+2i}-1)/2}$$
$$(1\leq i\leq s/2-r)$$

$$B_{11}=B_9'\ \overset{4}{\text{---}}\ B_{10}^1\ \overset{4}{\text{---}}\ B_{10}^2\ \overset{4}{\text{---}}\cdots B_{10}^{s/2-r}$$

Figure 10

By (4) and (7), $t-(-(1+s+2r)+m+n+\sum g_i+(p+\sum c_j)/2)$ is a nonnegative even integer. By Operation 1, we can obtain a manifold with the required data.

Case 5) $n>r$, $p=0$ and $r+1\geq s/2$.

Apply Operations to the basic manifolds as indicated in Figure 11. (In Figure 11, B_{13} is created by applying Operation $3'$ to $B_{12}^1, B_{12}^2, \cdots, B_{12}^{s/2}, B_{12}^{s+1}, B_{12}^{s+2}, \cdots, B_{12}^{r+1+s/2}$ and $A_8(r)$.) We obtain B_{14} with data

$$[-(1+s+2r)+m+n+\sum g_i+\sum c_j/2,\ s,\ r,\ 0;\ g_1, g_2, \cdots, g_m;\ c_1, c_2, \cdots, c_n]\,.$$

$$B_{12}^j=A_2(\overset{3}{\text{---}}\ A_4)^{(c_j-3)/2}\qquad (j=1, 2, \cdots, s/2)$$

$$B_{12}^j=A_2(\overset{3}{\text{---}}\ A_4)^{(c_j-1)/2}\qquad (j=s/2+1, s/2+2, \cdots, s)$$

$$B_{12}^j=A_4(\overset{3}{\text{---}}\ A_4)^{(c_j-2)/2-1}\qquad (j=s+1, s+2, \cdots, r+1+s/2)$$

$$B_{12}^j=A_4(\overset{3}{\text{---}}\ A_4)^{c_j/2-1}\qquad (j=r+1+s/2+1, r+1+s/2+2, \cdots, n)$$

$$B_{13}=A_8(r)\ \overset{3'}{\text{---}}\ \{B_{12}^1, B_{12}^2, \cdots, B_{12}^{s/2}, B_{12}^{s+1}, B_{12}^{s+2}, \cdots, B_{12}^{r+1+s/2}\}$$

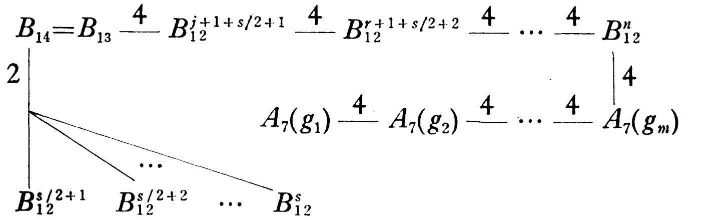

Figure 11

By (3) and (6), $t-(-(1+s+2r)+m+n+\sum g_i+\sum c_j/2)$ is a nonnegative even integer. By Operation 1, we can obtain the required manifold.

Case 6) $n>r, p=0$ and $r+1<s/2$.

First, consider the numbers $g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_{2(r+1)}, c_{s+1}, c_{s+2}, \cdots, c_n$. Since $2(r+1)+(n-s)\geq r$ and $r+1\geq(2(r+1))/2$, by the same way as in case 5) there exists B'_{14} with data

$$[-(1+2(r+1)+2r)+m+(2(r+1)+n-s)+\sum g_i+(\sum_{j=1}^{2(r+1)} c_j+\sum_{j=s+1}^{n} c_j)/2,$$
$$2(r+1), r, 0; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_{2(r+1)}, c_{s+1}, c_{s+2}, \cdots, c_n] .$$

Applying Operations to B'_{14} and the basic manifolds as indicated in Figure 12, we have B_{16} with data

$$[-(1+s+2r)+m+n+\sum g_i+\sum c_j/2, s, r, 0; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n] .$$

$$B^i_{15}=A_4(\overset{3}{\text{---}} A)_4^{(c_{2r+2+2i-1}-1)/2-1} \overset{3}{\text{---}} A_2 \overset{2}{\text{---}} A_2(\overset{3}{\text{---}} A_4)^{(c_{2r+2+2i}-1)/2}$$
$$(1\leq i\leq s/2-r-1)$$
$$B_{16}=B'_{14} \overset{4}{\text{---}} B^1_{15} \overset{4}{\text{---}} B^2_{15} \overset{4}{\text{---}} \cdots \overset{4}{\text{---}} B^{s/2-r-1}_{15}$$

Figure 12

By (3) and (6), $t-(-(1+s+2r)+m+n+\sum g_i+\sum c_j/2)$ is a nonnegative even integer. By Operation 1, we can obtain the required manifold.

This completes the proof.

REMARK 4. In the proof of Lemma 7, we have constructed (M, τ) with data stated in Lemma 7 such that $\mathrm{Tor}\, H_1(M; \boldsymbol{Z})\cong(\overset{s}{\oplus}\boldsymbol{Z}_2)\oplus(\bigoplus_{i=1}^{r}(\boldsymbol{Z}_{2q_i}\oplus\boldsymbol{Z}_{2q_i}))$ for some nonzero integers $q_1, q_2, \cdots, q_r$. Since (M, τ) is obtained from the basic manifolds with involutions by the Operations, we see from Remark 3 that any given nonzero integers can be taken as $q_1, q_2, \cdots, q_r$.

Lemma 8. *Even if the numbers s and p in the assumption of Lemma 7 are odd, the same assertion of Lemma 7 holds.*

Proof. We can check that $t, s-1, r, p-1, m, n, g_1, g_2, \cdots, g_m, c_1, c_2, \cdots, c_{s-1}, c_s-1, c_{s+1}, \cdots, c_n$ satisfy the assumption of Lemma 7. Hence there exists (M', τ') with data $[t, s-1, r, p-1; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_{s-1}, c_s-1, c_{s+1}, \cdots, c_n]$. Applying Operation 3 to (M', τ') and A_2 (A_2 has the data $[0, 1, 0, 1;\ \ ; 1]$), we can obtain (M, τ) with the data $[t, s, r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n]$.

This completes the proof.

Proof of Theorem 5. We may assume that X consists of p points, m closed orientable surfaces $E_1, E_2, \cdots, E_m$ of genera $g_1, g_2, \cdots, g_m$ and n closed nonorientable surfaces $F_1, F_2, \cdots, F_n$ of nonorientable genera $c_1, c_2, \cdots, c_n$ such that $g_i>0$ or $=0$ according to whether $1\leq i\leq m'$ or $m'\leq i\leq m$ for some m' and c_j is odd ($\neq 1$), even or 1 according to whether $1\leq j\leq s$, $s+1\leq j\leq n'$ or $n'+1\leq j\leq n$ for some s and n'. By conditions (1)–(7), we can see that the given abelian group G is isomorphic to $(\overset{s+n-n'}{\oplus} \boldsymbol{Z}_2)\oplus(\overset{r}{\underset{i=1}{\oplus}}(\boldsymbol{Z}_{2q_i}\oplus\boldsymbol{Z}_{2q_i}))\oplus(\overset{t}{\oplus}\boldsymbol{Z})\oplus B\oplus B$, where $r=(\beta_1(G; \boldsymbol{Z}_2)-\beta_1(G)-s)/2$, B is some abelian group of odd order, and $t, q_1, q_2, \cdots, q_r$ are some integers. We can check that the numbers $t-m+m'$, $s, r, p+n-n', m', n', g_1, g_2, \cdots, g_{m'}, c_1, c_2, \cdots, c_{n'}$ satisfy the assmption of Lemma 7 or 8.

Hence by Lemma 7 or 8 and Remark 4 there exists (M_1, τ_1) with data $[t-m+m', s, r, p+n-n'; g_1, g_2, \cdots, g_{m'}; c_1, c_2, \cdots, c_{n'}]$ and $\mathrm{Tor}\, H_1(M_1; \boldsymbol{Z})\cong (\overset{s}{\oplus}\boldsymbol{Z}_2)\oplus(\overset{r}{\underset{i=1}{\oplus}}(\boldsymbol{Z}_{2q_i}\oplus\boldsymbol{Z}_{2q_i}))$. Prepare $n-n'$ copies of A_2 (with data [0, 1, 0, 1; ; 1]) and $m-m'$ copies $A_7(0)$ (with data [0, 0, 0, 0; 0;]). Applying Operation 2 $n-n'$ times and Operation 4 $m-m'$ times, we have a manifold M_2 with data

$$[t, s+n-n', r, p; g_1, g_2, \cdots, g_m; c_1, c_2, \cdots, c_n].$$

Consider a manifold M_3 with $H_1(M_3; \boldsymbol{Z})\cong B$ and apply Operation 1 to M_2 and M_3. We denote the resulting manifold with involution by (M, τ). Then we can see that $H_1(M; \boldsymbol{Z})\cong G$ and $\mathrm{Fix}(\tau, M)=X$.

This completes the proof.

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