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Osaka University
1. Introduction

This paper is concerned with the stabilizability on the retarded functional differential equation

$$\frac{d}{dt} u(t) = A_0 u(t) + \int_{-h}^{0} a(s) A_1 u(t+s) \, ds + \Phi_0 f(t) ,$$

where

$$u(0) = g^0, \quad u(s) = g^0(s), \quad s \in [-h, 0)$$

in a Hilbert space $H$. Here, $A_0$ is the operator associated with a sesquilinear form $a(u, v)$ which is defined in $V \times V$ and satisfies Gårding’s inequality

$$\text{Re} \, a(u, u) \geq c_0 \| u \|^2 - c_1 | u |^2, \quad c_0 > 0, \quad c_1 \geq 0$$

where $V$ is another Hilbert space such that $V \subset H \subset V^*$. The notations $| \cdot |$, $\| \cdot \|$ denote the norms of $H, V$ respectively as usual. $A_1$ is a bounded linear operator from $V$ to $V^*$ such that it maps $D(A_0)$ into $H$, and $u(\cdot)$ is a real valued Hölder continuous function in $[-h, 0]$. $\Phi_0$ is a bounded linear operator from some Banach space $U$ to $H$.

We will establish a necessary and sufficient condition in order that the initial value problem (1.1), (1.2) is stabilizable in the sense that for any $g \in Z = H \times L^2(-h, 0; V)$ there exists $f \in L^2(0, \infty; U)$ such that for the solution $u$ of (1.1), (1.2) we have

$$\int_0^\infty \{ | u(t) |^2 + \int_{-h}^{0} | u(t+s) |^2 \, ds \} \, dt < \infty .$$

Our result is analogous to a recent result by G. Da Prato and A. Lunardi [1] for an integrrodifferential parabolic equation of Volterra type

$$\frac{d}{dt} u(t) = A u(t) + \int_{t}^{t} K(t-s) u(s) \, ds + \Phi f(s), \quad t \geq 0, \quad u(0) = u_0$$

in a Banach space $X$, where $A$ is a not necessarily densely defined closed linear operator generating an analytic semigroup, $K$ is a measurable function with values in $L(D(A), X)$ and $\Phi$ is a bounded linear operator from some Banach space
Y to X. Under the assumption that $F(\lambda)$ which is the analytic continuation of $(\lambda - A - \hat{K}(\lambda))^{-1}$ where $\hat{K}$ is the Laplace transform of $K$ has only a finite number of singularities in the half plane $\text{Re} \lambda \geq -\omega, \omega \geq 0$, and all these singularities are poles of $F(\lambda)$ such that the coefficients of the negative powers in the Laurent expansions of $F(\lambda)$ around them are all operators of finite rank, they established a necessary and sufficient condition in order that for every $u_0 \in \overline{D(A)}$ there exists $f$ satisfying $\sup \|e^{st} f(t)\|_F < \infty$ such that the solution $u$ of (1.3) satisfies $\sup \|e^{st} u(t)\|_F < \infty$.

When we investigate the equation (1.1), it is natural and usual to consider the equivalent enlarged system for the unknown functions $u(t)$ and $u_t$, where $u_t(s) = u(t+s)$ for $s \in [-h, 0)$, since it enables us to express the solution with the aid of the solution semigroup (cf. [2], [3], [7]). Since we necessarily consider the adjoint equation in the study of a stabilization problem, it is convenient to consider the original equation in $V^*$ as in [11] so that the enlarged system is an equation in the space $Z = H \times L^2(-h, 0; V)$:

$$\frac{d}{dt} x(t) = Ax(t) + \Phi f(t), x(0) = g = (g^0, g')$$

where $A$ is the infinitesimal generator of the associated solution semigroup $S(t)$ and $\Phi$ is the operator defined by $\Phi f = (\Phi_0 f, 0)$. Thus, we are led to studying the stabilizability in the sense stated above.

We assume that

$$\sigma_+ = \sigma(A) \cap \{ \lambda : \text{Re} \lambda > 0 \}$$

consists entirely of a finite number of eigenvalues of $A$ with generalized eigenspaces of finite dimension and

$$\sup \{ \text{Re} \lambda : \lambda \in \sigma(A) \setminus \sigma_+ \} < 0.$$ 

Let $P$ be the spectral projection corresponding to $\sigma_+$:

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - A)^{-1} d\lambda,$$

where $\Gamma$ is a rectifiable Jordan curve surrounding $\sigma_+$ inside but no other point of $\sigma(A)$. In our case the exponential decay of $\|S(t)(I-P)\|$ is not so evident as the corresponding fact for Volterra equations of [1]. We will show that owing to the absence of a discrete delay term in (1.1) $S(t)$ is Hölder continuous in $(3h, \infty)$ in the operator norm, and so eventually norm continuous. Hence the aforesaid exponential decay of $\|S(t)(I-P)\|$ follows from Theorem 1.20 of [6], and we can proceed as in [1] to establish the desired result.

We note here that it is shown in G. Di Blasio, K. Kunisch and E. Sinestrari [3] that $S(t)$ is differentiable in $(h, \infty)$, hence eventually norm continuous, if
a(·)∈W¹,∞(−h, 0) for a general equation of the form (1.1) in a Hilbert space. However, if there exists a discrete delay term, this is not the case as the following counter example shows. For the equation

\[ \frac{d}{dt} u(t) = A_0 u(t) + A_0 u(t-1) \quad (1.4) \]

we have

\[ S(t) (g^\delta, 0) = (W(t) g^\delta, W(t+\cdot) g^\delta) \]

where

\[ W(t) = \sum_{j=0}^{\infty} \frac{1}{j!} A_1 e^{t - DA_0 (t-j)^2}, \quad t \in [n, n+1], \quad n = 1, 2, \ldots \]

is the fundamental solution of (1.4) which is not norm continuous at \( t = 1, 2, \ldots \).

2. Assumptions and main theorem

Let \( H \) and \( V \) be complex Hilbert spaces such that \( V \) is a dense subspace of \( H \) and the imbedding of \( V \) into \( H \) is continuous. The norms of \( H \) and \( V \) are denoted by \( |\cdot| \) and \( ||\cdot|| \) respectively. Identifying the antidual of \( H \) with \( H \) we may consider \( V \subset H \subset V^* \). Let \( a(u,v) \) be a bounded sesquilinear form defined in \( V \times V \) satisfying Gårding’s inequality

\[ \text{Re} \ a(u, u) \geq c_0 ||u||^2 - c_1 |u|^2 \]  

(2.1)

where \( c_0 \) and \( c_1 \) are constants such that \( c_0 > 0 \) and \( c_1 \geq 0 \). Let \( A_0 \) be the operator associated with this sesquilinear form:

\[ (A_0 u, v) = -a(u, v), \quad u, v \in V. \]  

(2.2)

The operator \( A_0 \) is a bounded linear form \( V \) to \( V^* \). The realization of \( A_0 \) in \( H \) which is the restriction of \( A_0 \) to \( D(A_0) = \{ u \in V : A_0 u \in H \} \)

is also denoted by \( A_0 \). It is known that \( A_0 \) generates an analytic semigroup in both of \( H \) and \( V^* \). Let \( A_1 \) be a bounded linear operator from \( V \) to \( V^* \) such that \( A_1 \) maps \( D(A_0) \) endowed with the graph norm of \( A_0 \) to \( H \) continuously. Let \( a(\cdot) \) be a real valued Hölder continuous function on the interval \([-h, 0]\), where \( h \) is a fixed positive number. Let \( U \) be a complex Banach space and \( \Phi_0 \) be a bounded linear operator from \( U \) to \( H \). We are interested in the stabilization of the initial value problem of the retarded functional differential equation

\[ \frac{d}{dt} u(t) = A_0 u(t) + \int_{-h}^{0} a(s) A_1 u(t+s) \, ds + \Phi_0 f(t), \quad (2.3) \]

\[ u(0) = g^\delta, \quad u(s) = g^\delta(s) \text{ a.e. } s \in [-h, 0), \quad (2.4) \]
where \( g=(g^0, g^1) \in \mathbb{Z}=H \times L^2(-h, 0; V), f \in L^2(0, T; U) \). Applying Theorem 4.1 of [2] to (2.3) considered as an equation in \( V^* \) we see that there exists a solution semigroup \( S(t) \) associated with (2.3), (2.4):

\[
S(t)g = \begin{pmatrix} u(t;g) \\ u_t(t; g) \end{pmatrix}, \quad t \geq 0, \quad g = (g^0, g^1) \in \mathbb{Z},
\]

where \( u(t; g) \) is the solution of (2.3), (2.4) with \( f(t) \equiv 0 \), and \( u_t(s; g) = u(t+s; g) \), \( s \in [-h, 0] \). \( S(t) \) is a \( C_0 \)-semigroup in \( Z \) whose infinitesimal generator is denoted by \( A \).

In view of Theorem 4.2 of [2] \( A \) is characterized as

\[
D(A) = \{(\phi^0, \phi^1): \phi^1 \in W^{1,2}(-h, 0; V), \phi^0 = \phi^1(0), \quad A_0 \phi^0 + \int_{-h}^0 a(s) A_1 \phi^1(s) \, ds \in H \}, \\
A(\phi^0, \phi^1) = (A_0 \phi^0 + \int_{-h}^0 a(s) A_1 \phi^1(s) \, ds, \phi^1).
\]

We assume

\[(2.5) \quad \sigma(A) \cap \{\lambda: \Re \lambda = 0\} = \emptyset.\]

Set

\[(2.6) \quad \sigma_+ = \sigma(A) \cap \{\lambda: \Re \lambda \geq 0\}, \quad \sigma_- = \sigma(A) \cap \{\lambda: \Re \lambda < 0\}.\]

We assume also that \( \sigma_+ \) is a finite set and \( \sup \{\Re \lambda: \lambda \in \sigma_-\} < 0 \), that is,

\[(2.7) \quad \sigma_+ = \{\lambda_1, \ldots, \lambda_N\},\]

\[(2.8) \quad -\omega_0 = \sup \{\Re \lambda: \lambda \in \sigma_-\} < 0,\]

and for each \( j=1, \ldots, N \), the spectral projection

\[(2.9) \quad P_{\lambda_j} = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_j}} (\lambda - A)^{-1} \, d\lambda
\]

is an operator of finite rank, where \( \Gamma_{\lambda_j} \) is a small circle centered at \( \lambda_j \) such that it surrounds no point of \( \sigma(A) \) except \( \lambda_j \). As is well known \( \lambda_j \) is an eigenvalue of \( A \) and \( Z_{\lambda_j} = \text{Im} P_{\lambda_j} \) is the generalization of eigenspace for \( \lambda_j \). It is also well known that \( \lambda_j \) is a pole of \( (\lambda - A)^{-1} \) whose order we denote by \( m_j \).

We consider also the adjoint problem

\[
(2.10) \quad \frac{d}{dt} \psi(t) = A_0^* \psi(t) + \int_{-h}^0 a(s) A_1^* \psi(t+s) \, ds,
\]

\[
(2.11) \quad \psi(0) = \phi^0, \quad \psi(s) = \phi^1(s) \quad s \in [-h, 0],
\]

where \( A_0^*, A_1^* \in B(V; V^*) \) are adjoint operators of \( A_0, A_1 \) and \( \phi^0 \in H, \phi^1 \in L^2(-h, 0; V) \). The solution semigroup associated with this problem is denoted
by \( S_r(t) = e^{A_r t} \). Just as in [7] it can be shown that \( \lambda_i, \ldots, \lambda_N \) are eigenvalues of \( A_r \). The spectral projection and the generalized eigenspace for \( \lambda_j \) are denoted by \( P_{\lambda_j}^r \) and \( Z_{\lambda_j}^r \) respectively.

The structural operator \( F \) is defined by

\[
F = (\[Fg\]_0, [Fg]_1), \quad g = (g^0, g^1) \in Z, \\
[Fg]_0 = g^0, \quad [Fg]_1(s) = \int_{-h}^s a(\tau) A_1 g^1(\tau - s) \, d\tau.
\]

Here and in what follows we denote the first and second components of the element \( \varphi \) of \( Z \) by \( \varphi_0 \) and \( \varphi_1 \) respectively. \( F \) is a bounded linear operator from \( Z \) to its adjoint \( Z^* = H \times L^2(\mathbb{R}^\times, \mathbb{C}^n) \). Putting \( x(t) = (u(t), u_t) \) where \( u_t(s) = u(t+s), \, s \in [-h, 0) \), and \( g = (g^0, g^1) \) the problem (2.3), (2.4) is transformed to the problem

\[
\begin{align*}
\frac{d}{dt}x(t) &= Ax(t) + \Phi f(t), \\
x(0) &= g
\end{align*}
\]

in \( Z \), where \( \Phi \) is the operator defined by \( \Phi f = (\Phi_0 f, 0) \). The mild solution of (2.12), (2.13) is defined by

\[
x(t) = S(t) g + \int_0^t S(t-s) \Phi f(s) \, ds.
\]

We call the first component of the right hand side of (2.14) the mild solution of (2.3), (2.4). Following [3], [7] we set

\[
\Delta(\lambda) = \lambda - A_0 - \int_{-h}^0 e^{sA} A_1 \, ds, \\
\Delta_r(\lambda) = \lambda - A_0^* - \int_{-h}^0 e^{sA} A_1^* \, ds, \\
\Delta^{(i)}(\lambda) = (d/d\lambda)^i \Delta_r(\lambda) = \delta_{ii} - \int_{-h}^0 s^i e^{sA} A_1^* \, ds, \quad i = 1, 2, \ldots,
\]

where \( \delta_{ii} \) is the Kronecker symbol, i.e., \( \delta_{ii} = 1, \delta_{ii} = 0 \) if \( i \neq 1 \).

The main theorem of this paper is

**Theorem.** The following statements are equivalent:

(i) For any \( g \in Z \) there exists an \( f \in L^2(0, \infty; U) \) such that the mild solution \( u \) of (2.3), (2.4) satisfies

\[
\int_0^\infty \{ |u(t)|^2 + \int_{-h}^0 |u(t+s)|^2 \, ds \} \, dt < \infty.
\]

(ii) For each \( j = 1, \ldots, N \)
\[ \Phi_n^0 \phi_n^0 = 0, \ n = 0, \ldots, m_j - 1 \]

implies

\[ \phi_n^0 = 0, \ n = 0, \ldots, m_j - 1 \]

for any elements \( \phi_0^0, \ldots, \phi_{m_j-1}^0 \in V \) such that

\[ \sum_{n=0}^{m_j-1} \frac{(-1)^{i-n} \Delta_{i-n}(\lambda_j)}{(i-n)!} \phi_n^0 = 0, \ n = 0, \ldots, m_j - 1. \]

**Remark 1.** If \( m_j = 1 \), the statement (ii) of the theorem reduces to

\[ \Phi^0 \phi_0^0 = 0, \ \Delta_\gamma(\lambda_j) \phi_0^0 = 0 \implies \phi_0^0 = 0. \]

**Remark 2.** Set

\[ \rho(\Delta) = \{ \lambda : \Delta(\lambda) \text{ is an isomorphism from } V \text{ to } V^* \}, \]

\[ \sigma(\Delta) = \mathcal{C} \setminus \rho(\Delta). \]

According to Riemann-Lebesgue's lemma \( \int_{-\infty}^{\infty} e^{\lambda s} a(s) ds \) tends to 0 as \( |\text{Im} \lambda| \to \infty \) uniformly in \( \{ \lambda : \text{Re} \lambda \geq c \} \) for any real number \( c \). Hence

\[ \Delta(\lambda) = \{ I - \int_{-\infty}^{\infty} e^{\lambda s} a(s) ds A_1(\lambda - A_0)^{-1} \} (\lambda - A_0) \]

has a bounded inverse if \( \text{Re} \lambda \geq c \) and \( |\text{Im} \lambda| \) is sufficiently large. Consequently \( \sigma(\Delta) \cap \{ \lambda : \text{Re} \lambda \geq c \} \) is bounded.

Suppose that the imbedding of \( V \) to \( H \) is compact and \( A_1 = A_0 \). Then in view of Theorem 1 of [4]

\[ \sigma(\Delta) = \sigma(A) = \{ \lambda : m(\lambda) = 0 \} \cup \{ \lambda : m(\lambda) \neq 0, \lambda / m(\lambda) \in \sigma(A_0) \}, \]

where

\[ m(\lambda) = 1 + \int_{-\infty}^{\infty} e^{\lambda s} a(s) ds. \]

If \( \sigma(\Delta) \cap \{ \lambda : \text{Re} \lambda = 0 \} \) is empty and \( m(\lambda) \neq 0 \) for \( \text{Re} \lambda > 0 \), then the assumptions of the theorem are satisfied.

### 3. Stabilizability of functional differential equations

In this section we consider the stabilizability of the equation

\[ \frac{d}{dt} x(t) = Ax(t) + \Phi f(t), \]

(3.1)

\[ x(0) = g \]

(3.2)

in a general Banach space \( X \), where \( A \) is the infinitesimal generator of a \( C_0- \)
semigroup $S(t)$ and $\Phi_0$ is a bounded linear operator from some Banach space $U$ to $X$. We assume that $A$ satisfies (2.5), (2.7), (2.8) and the spectral projection $P_j$ defined by (2.9) is of finite rank for each $j=1, \ldots, N$. Hence $\lambda_j$ is a pole of $(\lambda-A)^{-1}$ whose order is denoted by $m_j$. We assume also that $S(t)$ is eventually norm continuous, i.e. $S(t)$ is continuous in the operator norm in the interval $(t', \infty)$ for some $t' \geq 0$. The Laurent expansion of $(\lambda-A)^{-1}$ around $\lambda_j$ is

$$
(\lambda-A)^{-1} = \sum_{n=0}^{m_j-1} \frac{Q_{X_j}^n}{(\lambda-\lambda_j)^{n+1}} + R_0(\lambda),
$$

where $Q_{X_j}^0 = P_{X_j}, Q_{X_j} = (A-\lambda_j)P_{X_j}$, and $R_0(\lambda)$ is the holomorphic part of $(\lambda-A)^{-1}$ at $\lambda = \lambda_j$. It is known that $Q_{X_j}^n = (A-\lambda_j)^n P_{X_j}$, and by assumption $Q_{X_j}^n = 0$. Put

$$
P = \sum_{j=1}^N P_{X_j}, j = 1, \ldots, N
$$

(3.4)

$$
X_+ = \text{Im } P, \quad X_- = \text{Im } (I-P),
$$

(3.5)

$$
S_+(t) = S(t)|_{X_+}, \quad S_-(t) = S(t)|_{X_-}
$$

(3.6)

where $S(t)|_{X_+}, S(t)|_{X_-}$ are the restrictions of $S(t)$ to $X_+, X_-$, respectively. Since both $\text{Im } P$ and $\text{Im } (I-P)$ are closed and invariant under $S(t)$, we can see that

$$
A_+ = A|_{X_+}, \quad A_- = A|_{D(\lambda) \cap X_-},
$$

(3.7)

$$
S_+(t) = e^{tA_+}, \quad S_-(t) = e^{tA_-},
$$

(3.8)

$$
\sigma(A_+) = \sigma_+, \quad \sigma(A_-) = \sigma_-
$$

(3.9)

$S(t)P$ is extended to the whole real line so that

$$
S(t)P = \sum_{j=1}^N \sum_{n=0}^{m_j-1} \frac{1}{n!} e^{\gamma^j t^n} Q_{X_j}^n, \quad -\infty < t < \infty.
$$

(3.10)

**Lemma 3.1.** For any $\omega \in (0, \omega_0)$ there exists a constant $M$ such that

$$
||S(t) (I-P)|| \leq M ||I-P|| e^{-\omega t}, \quad t \geq 0.
$$

**Proof.** For each eventually norm continuous semigroup $s(A) = \omega(A)$ is valid where $s(A) = \sup \{\text{Re } \lambda: \lambda \in \sigma(A)\}$, $\omega(A) = \inf \{\omega: ||S(t)|| \leq Me^{-\omega t} \text{ for some } M \text{ and any } t \geq 0\}$ (see e.g. [6: p. 109]). Therefore for any $\omega > s(A)$, there exists $M$ such that $||S(t)|| \leq Me^{-\omega t}$. By assumption (2.8) $-\omega_0 = \sup \{\text{Re } \lambda: \lambda \in \sigma_-\} < 0$. If $S(t)$ is eventually norm continuous, then so is $S_-(t)$. Hence for $0 < \omega < \omega_0$ there exists a constant $M$ such that $||S_-(t)|| \leq Me^{-\omega t}$. Thus,

$$
||S(t) (I-P)|| = ||S_-(t) (I-P)|| \leq M ||I-P|| e^{-\omega t}.
$$

Now we consider the following initial value problem:
where $\phi \in L^2(0, \infty; X)$. Let $u(t)$ be the mild solution of the equation (3.11), i.e.,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\phi(s)\,ds.$$ \hfill (3.12)

The following Lemma is related to Proposition 1.1 of [1], and we mimic its proof.

**Lemma 3.2.** For the equation (3.11), the mild solution $u(t)$ belongs to $L^2(0, \infty; X)$ if and only if

$$S(t)Pu_0 + \int_0^t S(t-s)P\phi(s)\,ds = 0, \ t \geq 0.$$ \hfill (3.13)

**Proof.** We set $u(t) = v(t) + z(t)$, with $$v(t) = S(t)(I-P)u_0 + \int_0^t S(t-s)(I-P)\phi(s)\,ds - \int_0^t S(t-s)P\phi(s)\,ds,$$

$$z(t) = S(t)Pu_0 + \int_0^t S(t-s)P\phi(s)\,ds, \ t \geq 0.$$ With the aid of Lemma 3.1, (3.10) and Hausdorff-Young’s inequality it is easily seen that $v \in L^2(0, \infty; X)$. In view of (3.10) and Lemma 3.1 $z(t)$ is of the form

$$z(t) = \sum_{j=1}^{N} \sum_{n=0}^{m_j-1} e^{\lambda_j^* t} y_{j,n} y_{j,n}^* \in D(A).$$

Since $\text{Re} \lambda_j > 0$ for each $j=1, \ldots, N$ and the function $t \mapsto e^{\lambda_j^* t}$ $(j=1, \ldots, N, n=0, \ldots, m_j-1)$ are linearly independent, we see that $z \in L^2(0, \infty; X)$ if $z(t) \equiv 0$.

The following Proposition is concerned with the stabilizability of (3.1), (3.2).

**Proposition 3.1.** The following statements are equivalent:

(i) For any $g \in X$, there exists $f \in L^2(0, \infty; U)$ such that the mild solution of (3.1), (3.2) belongs to $L^2(0, \infty; X)$.

(ii) For each $j=1, \ldots, N$

$$\{x^* \in Z^*_{kj}; \Phi^*(A^* - \lambda_j)^k x^* = 0, k = 0, \ldots, m_j-1\} = \{0\},$$

where $Z^*_{kj}$ is the generalized eigenspace for $\lambda_j$ which is an eigenvalue of $A^*$.

**Proof.** In view of Lemma 3.2 the solution of (3.1), (3.2) belongs to $L^2(0, \infty; X)$ iff

$$S(t)Pg + \int_0^t S(t-s)Pf(s)\,ds \equiv 0.$$
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By virtue of (3.10) and \( \sum_{n=0}^{m_j-1} \sum_{k=0}^{m_j-1} = \sum_{n=0}^{m_j-1} \sum_{k=0}^{m_j-1} \) this is equivalent to

\[ Q_{\lambda_j}^{n} g + \int_0^\infty e^{-\lambda_j s} \sum_{k=0}^{m_j-1} \sum_{k=0}^{m_j-1} \frac{(-1)^{k-n}}{(k-n)!} Q_{\lambda_j}^{k} \Phi f(s) \, ds = 0, \]

\( j=1, \ldots, N, n=0, \ldots, m_j-1. \) Following the proof of Theorem 2.1 or 2.3 of [1] we see (i) holds iff

(iii) For \( 1 \leq j \leq N, \) if \( x^*_n \in X^*(n=0, \ldots, m_j-1) \) satisfies

\[ \Phi^* \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^{k+n})^* x^*_n = 0, \quad k = 0, \ldots, m_j-1, \]

then

\[ \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^{k+n})^* x^*_n = 0. \]

Suppose that (iii) is true, and \( x^* \in Z_{\lambda_j}^*, \Phi^*(A^* - \lambda_j)^k x^* = 0, k = 0, \ldots, m_j-1. \) Put

\[ x^*_n = \begin{cases} x^* & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \]

Then

\[ \Phi^* \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^{k+n})^* x^*_n = \Phi^*(Q_{\lambda_j}^k)^* x^* = \Phi^*(A^* - \lambda_j)^k (P_{\lambda_j})^* x^* = \Phi^*(A^* - \lambda_j)^k x^* = 0, \]

\( k = 0, \ldots, m_j-1. \) By (iii)

\[ x^* = (P_{\lambda_j})^* x^* = \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^n)^* x^*_n = 0. \]

This shows that (iii) implies (ii).

Conversely, suppose (ii) is true and (3.14) holds. Set

\[ x^* = \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^n)^* x^*_n. \]

Then \( x^* \in Z_{\lambda_j}^* \) and

\[ \Phi^*(A^* - \lambda_j)^k x^* = \Phi^* \sum_{n=0}^{m_j-1} (A^* - \lambda_j)^{k+n} (P_{\lambda_j})^* x^*_n = \Phi^* \sum_{n=0}^{m_j-1} (Q_{\lambda_j}^{k+n})^* x^*_n = 0, \]

for \( k = 0, \ldots, m_j-1. \) By (ii) \( x^* = 0. \) Hence (iii) is true.
4. Some inequalities on the fundamental solution of (2.3), (2.4)

In this section we establish the Hölder continuity results concerning the fundamental solution of the equation (2.3), (2.4) in a Banach space $X$, where $A_0$ is the infinitesimal generator of an analytic semigroup $T(t)$ and $A_1$ is a closed linear operator in $X$ with domain containing that of $A_0$. We may assume without loss of generality that $A_0$ has an everywhere defined bounded inverse.

By definition the fundamental solution $W(t)$ is a bounded linear operator valued function satisfying

$$\frac{d}{dt} W(t) = A_0 W(t) + \int_{-h}^{0} a(s) A_1 W(t+s) ds,$$
$$W(0) = I, \quad W(s) = 0 \quad s \in [-h, 0).$$

The main object of this section is to prove the following

**Proposition 4.1.** For $h < t < t' \leq nh$, $n > 1$, and $0 < \kappa < \rho$, we have

\begin{align*}
(4.1) \quad \|W(t') - W(t)\| \leq C_n \log \left( \frac{t'}{t} \right), \\
(4.2) \quad \|A_0(W(t') - W(t))\| \leq C_{n, \kappa} (t' - t)^\kappa (t - h)^{-\kappa}, \\
(4.3) \quad \|A_0 W(t') - W(t') A_0^{-1}\| \leq C_{n, \kappa} (t' - t)^\kappa
\end{align*}

where $C_n$ and $C_{n, \kappa}$ are constants dependent on $n$ and $n, \kappa$, respectively but not on $t$ and $t'$.

For the sake of simplicity we assume that $T(t)$ is uniformly bounded. Then

\begin{align*}
(4.4) \quad \|T(t)\| \leq K (t \geq 0), \quad \|A_0 T(t)\| \leq \frac{K}{t} (t > 0), \quad \|A_0^2 T(t)\| \leq \frac{K}{t^2} (t > 0)
\end{align*}

for some constant $K$ (e.g. [9]). Since $a(\cdot)$ is Hölder continuous of order $\rho$, we can set

\begin{align*}
(4.5) \quad |a(s)| \leq H_0, \quad |a(s) - a(\tau)| \leq H_1 (s - \tau)^\rho
\end{align*}

for some constants $H_0, H_1$. From (4.4) for $0 < s < t$

\begin{align*}
(4.6) \quad \|A_0^2 T(t) - A_0 T(s)\| = \| \int_s^t A_0 T(\tau) d\tau \| \leq K \frac{t - s}{ts}, \\
(4.7) \quad \|T(t) - T(s)\| = \| \int_s^t A_0 T(\tau) d\tau \| \leq K \log \frac{t}{s}.
\end{align*}

As is easily seen for any $t > 0$ and $0 < \alpha < 1$

\begin{align*}
(4.8) \quad \log (1 + t) \leq \frac{t^\alpha}{\alpha}.
\end{align*}

Combining this with (4.4) we get
\( (4.9) \quad \| T(t) - T(s) \| \leq \frac{K}{\alpha} \left( \frac{t-s}{s} \right)^{\alpha} \)

for \( 0 < s < t \) and \( 0 < \alpha < 1 \). Set

\[
B(t) = A_0 \int_0^t T(t-s) a(-s) \, ds, \quad 0 \leq t \leq h.
\]

**Lemma 4.1.** \( B(t) \) is strongly continuous in \([0, h]\), and hence uniformly bounded:

\[
\| B \|_\infty = \sup_{0 \leq t \leq h} \| B(t) \| < \infty.
\]

Furthermore, \( B(t) \) is Hölder continuous in \((0, h]\), and for each \( \kappa \in (0, \rho) \) there exists a constant \( C_\kappa \) such that

\[
\| B(t') - B(t) \| \leq C_\kappa (t'-t)^\kappa t^{-\kappa}
\]

for \( 0 < t < t' \leq h \).

**Proof.** Since

\[
B(t) = A_0 \int_0^t T(t-s) a(-s) \, ds
\]

it follows without difficulty from (4.4), (4.5) that \( B(t) \) is strongly continuous in \([0, h]\).

For \( 0 < t < t' \leq h \)

\[
B(t') - B(t) = A_0 \int_0^{t'} T(t'-s) a(-s) \, ds - A_0 \int_0^t T(t-s) a(-s) \, ds
\]

\[
= \int_0^{t'} A_0 T(t'-s) (a(-s) - a(-t')) \, ds - (I - T(t)) a(-t),
\]

it follows from (4.4), (4.5), (4.6), (4.7), and (4.8) that

\[
\left\| \int_0^{t'} A_0 T(t'-s) (a(-s) - a(-t')) \, ds \right\| \leq \int_0^{t'} KH_1 (t'-s)^{\rho-1} \, ds = \frac{KH_1}{\rho} (t'-t)^\rho,
\]

\[
\left\| \int_0^t (A_0 T(t'-s) - A_0 T(t-s)) (a(-s) - a(-t)) \, ds \right\| \leq \int_0^t KH_1 \frac{t'-t}{(t'-s)(t-s)} (t-s)^\rho \, ds
\]

\[
\leq KH_1 (t'-t)^\kappa \int_0^t (t-s)^{\rho-\kappa} \, ds
\]

\[
\leq \frac{KH_1}{\rho - \kappa} (t'-t)^\kappa t^{\rho-\kappa}
\]
for $0 < \kappa < \rho$,
\[
||\{T(t') - T(t' - t)\} (a(-t) - a(-t'))|| \leq 2KH(t' - t)^\kappa,
\]
\[
||\{T(t') - T(t)\} a(-t)|| \leq KH_0 \log \frac{t'}{t} \leq KH_0 \frac{1}{\kappa} (t' - t)^\kappa t^\kappa,
\]
\[
||\{T(t) - I\} (a(-t') - a(-t))|| \leq (K + 1) H(t' - t)^\kappa.
\]

Combining these we obtain the desired inequality.

Set
\[
V(t) = \begin{cases} A_0 \{W(t) - T(t)\}, & t \in (0, h] \\ A_0 W(t), & t \in (nh, (n + 1)h], n = 1, 2, \ldots \end{cases}
\]

Then, the integral equation to be satisfied by $V(t)$ in $[nh, (n + 1)h]$ is
\[
(4.10) \quad V(t) = V_0(t) + \int_{nh}^{t} B(t - \tau) A_1 \left( A_0 W(\tau) d\tau \right),
\]

where
\[
V_0(t) = \int_{0}^{t} B(t - \tau) A_1 T(\tau) d\tau
\]
in $[0, h]$, and
\[
V_0(t) = A_0 T(t) + \int_{0}^{t - h} T(t - h - \tau) B(h) A_1 W(\tau) d\tau
\]
\[
+ \int_{nh}^{t} B(t - \tau) A_1 W(\tau) d\tau
\]
in $[nh, (n + 1)h]$, $n = 1, 2, \ldots$. As is stated in [10] $V(t)$ is bounded in each of the interval $[nh, (n + 1)h]$, $n = 1, 2, \ldots$. Using this and by (4.10) we get

**Proposition 4.2.** Let $W(t)$ be the fundamental solution of equation (2.3), (2.4) then for any natural number $n$ there exists a constant $C_n$ such that
\[
||W(t)|| \leq C_n, ||A_0 W(t)|| \leq \frac{C_n}{t}, ||A_1 W(t)|| \leq \frac{C_n}{t}
\]
\[
||\int_{s}^{t} A_0 W(\tau) d\tau|| \leq C_n, ||\int_{s}^{t} A_1 W(\tau) d\tau|| \leq C_n
\]
for $0 \leq s < t \leq nh$.

**Proof of Proposition 4.1.** For $nh \leq t < t' \leq (n + 1)h$
\[
V_0(t') - V_0(t) = (A_0 T(t') - A_0 T(t))
\]
\[
+ \int_{t - h}^{t'} T(t' - h - \tau) B(h) A_1 W(\tau) d\tau
\]
\[
+ \int_{t - h}^{t'} (T(t' - h - \tau) - T(t' - h) - T(t - h) + T(t - h)) B(h) A_1 W(\tau) d\tau
\]
From (4.6)
\[ ||I_1|| \leq K \frac{t'-t}{t'-t} \leq K \left( \frac{t'-t}{t'-t} \right)^{\kappa} \cdot \]

In view of (4.4) and Proposition 4.2
\[ ||I_2|| \leq \int_{t-h}^{t-t} K ||B(h)|| C_{\kappa} \tau^{-1} d\tau = K C_{\kappa} ||B(h)|| \log \frac{t'-h}{t-h} \]
\[ \leq \kappa C_{\kappa} ||B(h)|| (t'-t)^{t-h}, \]

In what follows throughout the proof let \( \mu \) be any number such that \( \kappa < \mu < \rho \). In view of (4.9)
\[ ||T(t'-h-\tau)-T(t-h-\tau)|| \leq K \left( \frac{t'-t}{t-h} \right)^{\mu}, \]
\[ ||T(t'-h)-T(t-h)|| \leq K \left( \frac{t'-t}{t-h} \right)^{\mu} \leq K \left( \frac{t'-t}{t-h} \right)^{\mu}. \]

Hence
\[ (4.12) \quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h)|| \leq 2K \left( \frac{t'-t}{t-h-\tau} \right)^{\mu}. \]

Similarly
\[ ||T(t'-h-\tau)-T(t'-h)|| \leq K \left( \frac{\tau}{t'-h-\tau} \right)^{\mu} \leq K \left( \frac{\tau}{t-h-\tau} \right)^{\mu}, \]
\[ ||T(t-h-\tau)-T(t-h)|| \leq K \left( \frac{\tau}{t-h-\tau} \right)^{\mu}, \]

which imply
\[ (4.13) \quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h)|| \leq 2K \left( \frac{\tau}{t-h-\tau} \right)^{\mu}. \]

Combining (4.12) and (4.13) we get
\[ (4.14) \quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h)|| \leq 2K \left( \frac{t'-t}{t-h-\tau} \right)^{\mu}. \]

In view of (4.14), Proposition 4.2, and Lemma 4.1
\[ ||I_3|| \leq \int_{t-t}^{t-t} \frac{2K}{\mu} \left( \frac{t'-t}{t-h-\tau} \right)^{\mu} ||B(h)|| \frac{C_{\kappa}}{\tau} d\tau. \]
\[ = \frac{2K}{\mu} C_n ||B(h)|| (t'-t)^{\kappa} \int_0^{t-h} (t-h-\tau)^{-\mu} \tau^{n-\kappa-1} d\tau \]
\[ = 2K C_n ||B(h)|| B(1-\mu, \mu-\kappa) \mu^{-\kappa} (t'-t)^{\kappa} (t-h)^{-\kappa}, \]

where \( B(\cdot, \cdot) \) is the Beta function.

From \((4.6)\), Proposition 4.2, and Lemma 4.1, we have

\[ ||I_4|| \leq \frac{KC_n}{\kappa} ||B(h)|| (t'-t)^{\kappa}. \]

We estimate \( I_5 \) in case of \( n=1 \) and \( n>1 \) separately. First, we consider the case \( n=1 \).

\[ I_5 = \int_{t'-h}^{t} \left( B(t'-\tau) - B(t-\tau) - B(h) + B(t-t'+h) \right) A_1 W(\tau) d\tau \]
\[ + (B(h) - B(t-t'+h)) \int_{t'-h}^{h} A_1 W(\tau) d\tau \]
\[ = I_{51} + I_{52}. \]

Noting \( t'-h<\tau<h<t, 0<t-\tau<t'-\tau<h, 0<t-t'+h<h, \) and using Lemma 4.1 we get analogously to \((4.14)\)

\[ ||B(t'-\tau) - B(t-\tau) - B(h) + B(t-t'+h)|| \leq 2C_{\mu} (t'-t)^{\kappa} (t-\tau)^{-\mu} (t'-t+\tau)^{\mu-\kappa}. \]

Hence

\[ ||I_{51}|| \leq \int_{t'-h}^{h} 2C_{\mu} (t'-t)^{\kappa} (t-\tau)^{-\mu} (t'-t+\tau)^{\mu-\kappa} C_1 \tau^{-1} d\tau \]
\[ \leq 2C_{\mu} C_1 (t'-t)^{\kappa} \int_{t'-h}^{t} (t-\tau)^{-\mu} (t'-t+\tau)^{\mu-\kappa-1} d\tau \]
\[ = 2C_{\mu} C_1 B(1-\mu, \mu-\kappa) (t'-t)^{\kappa} (t'-t+\tau)^{-\kappa} \]
\[ \leq 2C_{\mu} C_1 B(1-\mu, \mu-\kappa) (t'-t)^{\kappa} (t-h)^{-\kappa}. \]

As is easily seen

\[ ||I_{52}|| \leq C_{\kappa} C_1 (t'-t)^{\kappa} (t-t'+h)^{-\kappa}. \]

Therefore

\[ ||I_5|| \leq \{ 2C_{\mu} C_1 B(1-\mu, \mu-\kappa) + C_{\kappa} C_1 \} (t'-t)^{\kappa} (t-h)^{-\kappa}. \]

In case \( n>1 \) we have

\[ ||I_5|| \leq \int_{t'-h}^{t} \frac{C_{\kappa} (t'-t)^{\kappa}}{n-1} C_n \frac{d\tau}{h} \leq \frac{C_{\kappa} C_n (t'-t)^{\kappa}}{(n-1)h} \int_{t'-h}^{t} (t-\tau)^{-\kappa} d\tau \]
\[ \leq \frac{C_{\kappa} C_n (t'-t)^{\kappa}}{(n-1)(1-\kappa)h^{\kappa}} \]
\[ \leq \frac{C_{\kappa} C_n}{(n-1)(1-\kappa)} (t'-t)^{\kappa}. \]
Therefore
\[ ||I_0|| \leq \text{const.} \ (t' - t)^x (t - h)^{-x}. \]

From Lemma 4.1 and Proposition 4.2
\[ ||I_0|| \leq \int_{t-h}^{t'} K \ |B| \ \frac{C_n}{\tau} \ d\tau = C_n ||B||_\infty \ \log \frac{t'-h}{t-h} \leq C_n ||B||_\infty \ \left( \frac{t'-t}{t-h} \right)^x. \]

Therefore for \( nh \leq t < t' \leq (n+1) h \), we have
\[ ||V_0(t') - V_0(t)|| \leq \text{const.} \ (t' - t)^x (t - h)^{-x}, \]
from which it follows that
\[ (4.15) \quad ||V(t') - V(t)|| \leq \text{const.} \ (t' - t)^x (t - h)^{-x}. \]

It is not difficult to show that for \( n > 1 \)
\[ V(nh+0) = A_0 T(nh) + \int_0^{(n-1)h} T(nh-h-\tau) B(h) A_1 W(\tau) \ d\tau + \int_{(n-1)h}^{nh} B(nh-\tau) A_1 W(\tau) \ d\tau \]
\[ = V(nh-0), \]
where we used \( A_1 W(\tau) = V(\tau) \) in the second integral. Hence \( V(t) \) is continuous at \( t = nh, n=1, 2, \ldots \). Using this and (4.15) we have for \( (n-1)h < t < nh < t' \leq (n+1)h, n>1, \)
\[ ||V(t') - V(t)|| \leq \text{const.} \left( \frac{t' - nh}{nh - h} \right)^x \leq \text{const.} \left( \frac{t' - t}{t-h} \right)^x \]
\[ ||V(nh) - V(t)|| \leq \text{const.} \left( \frac{nh - t}{t-h} \right)^x \leq \text{const.} \left( \frac{t' - t}{t-h} \right)^x. \]

Thus (4.15) holds for \( h < t < t' < nh, t' - t < h, n>1 \) with const. independent on \( t \) and \( t' \). From the formular (4.11)
\[ V(t') - V(t) = V_0(t') - V_0(t) \]
\[ + \int_{nh}^{t'} B(t'-\tau) A_1 A_0^{-1} V(\tau) \ d\tau - \int_{nh}^t B(t-\tau) A_1 A_0^{-1} V(\tau) \ d\tau \]
\[ = V_0(t') - V_0(t) \]
\[ + \int_{t}^{t'} B(t'-\tau) A_1 A_0^{-1} V(\tau) \ d\tau \]
\[ + \int_{nh}^{t} (B(t'-\tau) - B(t-\tau)) A_1 A_0^{-1} V(\tau) \ d\tau. \]

Noting \( V(t) = A_0 W(t) \), for \( t>0 \) we have established that there exists a constant \( C_{n,x}>0 \) such that
\[ ||A_0 W(t') - A_0 W(t)|| \leq C_{x,n}(t' - t)^\kappa (t - h)^{-\kappa}. \]

for \( h < t < t' \leq nh, \ t' - t < h \) for \( n < 1 \). By the same way, we get
\[ ||A_0(W(t') - W(t)) A_0^{-1}|| \leq C_{x,n}(t' - t)^\kappa. \]

It follows from (4.5), Proposition 4.2, and
\[ \frac{d}{dt} W(t) = A_0 W(t) + \int_{-h}^{0} (a(s) - a(-h)) A_1 W(t + s) ds + a(-h) \int_{-h}^{0} A_1 W(t + s) ds \]

that \( ||dW(t)/dt|| \leq C_n/t, \ h < t < nh \). Hence the proof of Proposition 4.1 is complete.

5. \textbf{Hölder continuity of the solution semigroup}

Since in our case the solution of (2.3), (2.4) is represented by
\[ u(t) = W(t) g^0 + \int_{-h}^{0} W(t + s) [F^1 g^0](s) ds, \]

the solution semigroup \( S(t) \) can be expressed in terms of \( W(t) \).

Applying Proposition 4.1 to the equation (2.3) in the space \( V^* \) and noting that \( A_0 + c_1 \) is an isomorphism from \( V \) to \( V^* \) we get
\[ ||W(t') - W(t)||_{\mathcal{B}(V^*)} \leq C_n(t' - t), \]
\[ ||W(t') - W(t)||_{\mathcal{B}(V^*)} \leq C_{x,n}(t' - t)^\kappa (t - h)^{-\kappa}, \]
\[ ||W(t') - W(t)||_{\mathcal{B}(V^*)} \leq C_{x,n}(t' - t)^\kappa, \]

for \( h < t < t' \leq nh, \ n = 1, \ldots \), and \( 0 < \kappa < \rho \).

Similarly applying the same Proposition to (2.3) in \( H \)
\[ ||W(t') - W(t)||_{\mathcal{B}(H)} \leq C_n(t' - t) \]
for \( h < t < t' \leq nh \). Using \( (V, V^*)_{h,2} = H \) and the well known interpolation inequality we get from (5.2) and (5.3)
\[ ||W(t') - W(t)||_{\mathcal{B}(V^*)} \leq C_{x,n}(t' - t)^{(1 + \kappa)/2}(t - h)^{-\kappa/2}, \]
and from (5.3) and (5.4)
\[ ||W(t') - W(t)||_{\mathcal{B}(H)} \leq C_{x,n}(t' - t)^\kappa(t - h)^{-\kappa/2}. \]

\textbf{Proposition 5.1.} The solution semigroup \( S(t) \) for (2.3), (2.4) is Hölder continuous in \((3h, \infty)\) in the operator norm.

Proof. Let \( u(t) \) be the solution of (2.3), (2.4), and \( 3h < t < t' \). Using (5.1), (5.5), and (5.6)
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\[ |u(t') - u(t)| \leq |(W(t') - W(t))g^0| + \int_{-h}^{0} \int_{-h}^{0} |(W(t' + s - \sigma + \xi) - W(t + s - \sigma + \xi)) a(\xi) A_1 g^1(\sigma)| d\xi d\sigma \]

\[ \leq C_n(t' - t) g^0 \]

\[ + C_n, H_0 \int_{-h}^{0} \int_{-h}^{0} (t' - t)^{(\alpha+1)/2}(t' - \sigma + \xi - h)^{-\kappa/2} H_0 ||A||_{(V, W)} ||g^0(\sigma)|| d\xi d\sigma \]

\[ \leq C_n(t' - t) g^0 \]

\[ + C_n, H_0 |A|_{(V, W)} (t' - t)^{(\alpha+1)/2} \int_{-h}^{0} ||g^1(\sigma)|| d\sigma \]

\[ \leq \text{const.} \{ (t' - t) g^0 + (t' - t)^{(\alpha+1)/2} (\int_{-h}^{0} ||g^0(\sigma)||^2 d\sigma)^{1/2} \} . \]

With the aid of (5.3), (5.7) for \( s \in [-h, 0) \)

\[ ||u(t' + s) - u(t + s)|| = ||(W(t' + s) - W(t + s))g^0|| \]

\[ + \int_{-h}^{0} \int_{-h}^{0} (W(t' + s - \sigma + \xi) - W(t + s - \sigma + \xi)) a(\xi) A_1 g^1(\sigma)| d\xi d\sigma \]

\[ \leq C_n, (t' - t)^{\alpha} (t + s - h)^{-\kappa/2} g^0 \]

\[ + C_n, H_0 |A|_{(V, W)} (t' - t)^{\alpha} (t + s - h)^{-\kappa/2} \int_{-h}^{0} ||g^1(\sigma)|| d\sigma \]

\[ \leq D(t' - t)^{\alpha} \{ (t' + s - h)^{-\kappa/2} g^0 + (t + s - h)^{-\kappa/2} (\int_{-h}^{0} ||g^0(\sigma)||^2 d\sigma)^{1/2} \} . \]

where \( D = \max \{ C_n, H_0 h^{2/3}(1 - \kappa)^{-1} C_n, H_0 |A|_{(V, W)} \} \), hence

\[ \int_{-h}^{0} ||u(t' + s) - u(t + s)||^2 ds \leq 2D^2(t' - t)^{2\alpha} \int_{-h}^{0} (t + s - h)^{-\kappa} ds ||g^0||^2 \]

\[ + \int_{-h}^{0} (t + s - h)^{2(1 - \kappa)} ds \int_{-h}^{0} ||g^0(\sigma)||^2 d\sigma \]

\[ \leq \text{const.} (t' - t)^{2\alpha} ||(g^0, g^1)||^2 . \]

Consequently

\[ ||S(t') - S(t)||^2 = ||u(t') - u(t)||^2 + \int_{-h}^{0} ||u(t' + s) - u(t + s)||^2 ds \]

\[ \leq \text{const.} (t' - t)^{2\alpha} ||g||^2 . \]

Note that \( t' + s - \sigma + \xi > h \) since \( t > 3h \). Therefore for any \( T > 3h \) there exists a constant \( C_T \) such that

\[ ||S(t') - S(t)|| \leq C_T(t' - t)^{\alpha} . \]
Hence $S(t)$ is Hölder continuous on $(3h, \infty)$ in the operator norm.

**6. Proof of the main theorem**

In view of Proposition 5.1 the assumptions of Proposition 3.1 are satisfied for the problem (2.12), (2.13). Just as Theorems 4.2 and 8.1 of [7] it can be shown that $F^*$ maps $D(A_T)$ to $D(A^*)$ and $A^*F^*=F^*A_T$ on $D(A_T)$, and $F^*$ is an isomorphism from $Z^*_{\lambda_j}$ to $Z^*_{\lambda_j}$. Hence, the second statement of Proposition 3.1 is equivalent to

(ii)' For each $j=1, \ldots, N$

$$\{\phi \in Z^*_{\lambda_j}; \Phi^*_\phi [(A_T-\lambda_j)^n \phi]\n = 0, \quad n = 0, \ldots, m_j-1 \} = \{0\}.$$  

By the same manner as the proof of proposition 7.2 of [7] we have

$$Z^*_{\lambda_j} = \ker (\lambda_j - A_T)^n = \{ \phi^0_\phi, \exp (\lambda_j \cdot) \sum_{i=0}^{m_j-1} (-i)! \phi^i_\phi /i! : \sum_{i=n}^{m_j-1} (-1)^i (\lambda_j)^{-n} \phi^i_\phi / (i-n)! = 0, \quad n = 0, \ldots, m_j-1 \}.$$  

Using $A_T(\phi^0, \phi^1) = (A^*_\phi^0 + \int_{-h}^0 a(s) A^*_\phi^1(s) ds, \phi^1)$ we see that for the elements in the bracket of the right side of (6.1)

$$Z^*_{\lambda_j} = \ker (\lambda_j - A_T)^n = \{ \phi^0_\phi, \exp (\lambda_j \cdot) \sum_{i=0}^{m_j-1} (-i)! \phi^i_\phi /i! : \sum_{i=n}^{m_j-1} (-1)^i (\lambda_j)^{-n} \phi^i_\phi / (i-n)! = 0, \quad n = 0, \ldots, m_j-1 \}.$$  

Using $A_T(\phi^0, \phi^1) = (A^*_\phi^0 + \int_{-h}^0 a(s) A^*_\phi^1(s) ds, \phi^1)$ we see that for the elements

$$\begin{align*}
(\lambda_j - A_T)^n(\phi^0_\phi, \exp (\lambda_j \cdot) \sum_{i=0}^{m_j-1} (-i)! \phi^i_\phi /i!)
\end{align*}$$  

(6.2) $$(\lambda_j - A_T)^n(\phi^0_\phi, \exp (\lambda_j \cdot) \sum_{i=0}^{m_j-1} (-i)! \phi^i_\phi /i!)
$$  

for $n=0, \ldots, m_j-1$. Combining (6.1) and (6.2) we see that (ii)' is equivalent to the second statement of the theorem.

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References


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