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<th>Left serial rings over which every right module with homogeneous top is a direct sum of hollow modules</th>
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Osaka University
Let $R$ be a left and right artinian ring with identity, and $J$ the Jacobson radical of $R$. In [4], M. Harada has considered a left serial ring $R$ satisfying a condition (\(*,2\)) that every maximal submodule of a direct sum of any two hollow modules is also a direct sum of hollow modules, and characterized such a ring by the structure of $eR$ for each primitive idempotent $e$. Further it has been shown that the condition (\(*,2\)) is equivalent to saying that every factor module of $ej \oplus eR$ is a direct sum of hollow modules for every primitive idempotent $e$. Modifying this, we here consider the following condition on a projective indecomposable right module $eR$ over a ring $R$.

(A): Every factor module of $eR \oplus eR$ is a direct sum of hollow modules.

Clearly if $R$ is a ring of right local type, then all projective indecomposable right $R$-modules satisfy the condition (A), and as well known ([6]), $R$ is left serial. The purpose of this paper is to characterize left serial rings over which every projective indecomposable right module $eR$ satisfies the condition (A) (i.e. rings $R$ in the title (see Theorem 1 for the equivalence)) in terms of the structure of $eR$. Thus our result gives a generalization of rings of right local type.

In the first section we consider various conditions equivalent to (A) (Theorem 1). In particular, the condition 4) of Theorem 1 which is described in terms of homomorphisms between factor modules of $eR$ is frequently used later to check whether $eR$ satisfies (A) or not. We assume in the second and third section that $R$ is a left serial ring. In the second section we shall give some properties induced from the condition (A) to prepare the proof of the main theorem. In the third section we give the main theorem (Theorem 2). In the last section we give some examples.

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**Preliminaries**

Throughout this paper, $R$ is a left and right artinian ring with identity, and $J$ is the Jacobson radical of $R$. Since the property of $R$ that the condition (A) holds for all projective indecomposable right $R$-modules is Morita invariant, we may assume further that $R$ is a basic ring. $\pi(R)$ denotes the set of all primitive idempotents of $R$. All modules are finitely generated unitary right $R$-modules. A module $X$ is said to be hollow if the sum of any two proper submodules of $X$ is a proper submodule. By the assumption that $R$ is an artinian ring, a hollow module is precisely a local module, i.e., is isomorphic to a factor module of $eR$ for some $e$ in $\pi(R)$. For a module $X$, we put $\overline{X} := X/JX$, $T(X) := X/JX$, and denote by $X^{(n)}$ a direct sum of $n$ copies of $X$, and by $|X|$ the length of a composition series of $X$. For any $f$ and $g$ in $\pi(R)$, we put $T(fg) := fJg \setminus fg^{k+1}g$. For division rings $\Delta$ and $\Delta'$ with $\Delta \cong \Delta'$, we use the symbol $[\Delta : \Delta']$ to mean the dimension of $\Delta$ as a left (right) $\Delta'$-vector space. We say that $R$ is a left serial ring if as a left $R$-module, $R$ is a direct sum of uniserial submodules. In the second and the third sections we assume that $R$ is a left serial ring. In this case the following hold, which we use without any references. The first one follows from [6, Corollary 4.2], and the second one is clear from the definition of left serial rings.

**Lemma 1.** If $R$ is a left serial ring, then for every $e$ in $\pi(R)$ and for every natural number $j$, $ej^j$ is a direct sum of hollow modules, and $eR$ has a structure expressed by the following diagram:

$$
\begin{array}{c}
\begin{array}{c}
\vdots \\
A_{21} \cdots A_{2l_1} \\
A_{2l_1+1} \cdots A_{2l_2} \\
\vdots
\end{array}
\end{array}
\begin{array}{c}
\vdots
A_{11} \cdots A_{1r_1}
\end{array}
\begin{array}{c}
eR \\
eJ
\end{array}
$$

where each $A_{ik}$ is a hollow module, $ej^j = \bigoplus_{k=1}^{r_j} A_{ik}$ and $X \begin{array}{c}
\vdots
\end{array} X_i$ means $XJ = \bigoplus_{i=1}^{s_i} X_i$.

**Lemma 2.** Suppose that $R$ is a left serial ring. Let $e, f$ be in $\pi(R)$, and $a, b \in R$. If $a = eaf$ and $b = eaf$, then there exists either some $d \in eRe$ with $a = db$
or else some $d' \in \text{eRe}$ with $b = d'a$.

1. **The condition (A)**

   We use the following two lemmas from Sumioka [6] to prove Theorem 1.

   **Lemma 3** ([6, Lemma 1.3]). Let $M_i$ ($i=1, 2, 3$) and $T$ be submodules of a module $M$ such that $M = M_1 + (M_2 \oplus M_3)$ and $T = M_1 \cap (M_2 \oplus M_3)$, and $\pi_3: T \to M_3$ the restriction map of the projection $M_2 \oplus M_3 \to M$. Then $\pi_3$ is extended to a homomorphism $M_1 \to M_3$ if and only if $M = (M_1 + M_2) \oplus M_3$ for some submodule $M_1'$ of $M$.

   **Lemma 4** ([6, Lemma 2.1]). Let $S$ be a simple module and $L_1, \ldots, L_n$ hollow modules of length $\geq 2$ and $0 \to S \to \bigoplus_{i=1}^n L_i \to M \to 0$ an exact sequence with each $\alpha_i$ a monomorphism, where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$, and $n \geq 2$. Then $M$ is decomposable if and only if the identity map $1_S$ of $S_j$ is extended to a homomorphism: $\bigoplus_{i=1}^n L_i \to L_j$ for some $j$, $1 \leq j \leq n$, where $S_j := (\bigoplus_{i=1}^n L_i) \cap L_j$.

   Now we state the theorem in this section.

   **Theorem 1.** For $e$ in $\pi(\Lambda)$, the following four statements are equivalent.

   1) (A): Every factor module of $\text{eRe} \oplus \text{eRe}$ is a direct sum of hollow modules.

   2) Every factor module of $\text{eRe}^{(0)}$ is a direct sum of hollow modules for each natural number $n$.

   3) If $M$ is an $R$-module such that $M/M \cong \text{eRe}^{(0)}$ for some $n$, then $M \cong \bigoplus_{i=1}^n \text{eRe}/X_i$, where each $X_i$ is a submodule of $\text{eRe}$.

   4) Let $C_i$ and $D_i$ ($i=1, 2$) be submodules in $\text{eRe}$ such that $\text{eRe} \succeq C_i > D_i$. If $f: C_i/D_i \to C_i/D_i$ is an isomorphism, and $C_i/D_i$ is simple, then $f$ or $f^{-1}$ is extended to some homomorphism from $\text{eRe}/D_i$ to $\text{eRe}/D_i$ or one from $\text{eRe}/D_i$ to $\text{eRe}/D_i$, respectively.

   **Proof.** First, we introduce the following conditions 1') and 2') which are useful in proving the theorem:

   1') Let $S$ be simple, $X$, $Y$ be submodules of $\text{eRe}$, and assume that the following sequence is exact:

   $$0 \to S \to \text{eRe}/X \oplus \text{eRe}/Y \to M \to 0.$$ 

   Then $M \cong \text{eRe}/X \oplus \text{eRe}/Y'$ for some submodules $X'$ and $Y'$ of $\text{eRe}$.

   2') Let $n$ be any natural number, $S$ a simple module, and $X_i$ a submodule of $\text{eRe}$ for each $i$ ($1 \leq i \leq n$). Consider an exact sequence:

   $$0 \to S \to \bigoplus_{i=1}^n \text{eRe}/X_i \to M \to 0.$$
Then \( M \cong \bigoplus_{i=1}^{n} eR/Y_i \) for some submodules \( Y_i(1 \leq i \leq n) \) of \( eR \).

The proof proceeds as follows: 2) \( \Rightarrow \) 3) and 1) \( \Rightarrow \) 1') \( \Rightarrow \) 4) \( \Rightarrow \) 2') \( \Rightarrow \) 2) \( \Rightarrow \) 1).

2) \( \Rightarrow \) 3): It follows from \( M/MJ = eR^{(n)} \) that \( M \) is isomorphic to a factor module of \( eR^{(n)} \). So by 2), \( M \cong \bigoplus_{i=1}^{n} eR/X_i \) for some \( X_i < eR(1 \leq i \leq n) \).

3) \( \Rightarrow \) 2): Let \( M \) be a factor module of \( eR^{(n)} \). Then \( M/MJ = eR^{(k)} \) for some \( k \leq n \), so by 3), \( M \) is a direct sum of hollow modules.

1) \( \Rightarrow \) 1'): This is clear from the fact that the module \( M \) in 1') is an epimorphic image of \( eR \oplus eR^{(n)} \).

1') \( \Rightarrow \) 4): Let \( C_i \) and \( D_i \) be submodules with \( eR \cong C_i \cong D_i(i=1, 2) \). We assume that a homomorphism \( f: C_i/D_i \to C_j/D_j \) is an isomorphism and \( C_i/D_i \) is simple. To show that the assertion 4) holds, we may assume that \( C_i < eR(i=1, 2) \). We consider an exact sequence:

\[
0 \to C_1/D_1 \xrightarrow{(i_1, i_2)f} eR/D_1 \oplus eR/D_2 \to X \to 0 ,
\]

where \( i_j(j=1, 2) \) is the inclusion \( C_j/D_j \to eR/D_j \), and \( X \) is the cokernel of the homomorphism \( (i_1, i_2)f \). Then \( i_1 \) and \( i_2f \) are monomorphisms and not epimorphisms since \( C_i < eR(i=1, 2) \). By 1'), \( X = eR/X' \oplus eR/Y' \) for some \( X', Y' < eR \), so \( X \) is decomposable. Then by Lemma 4, \( (i_2f)(i_1)^{-1} = f \) (or \( i_1(i_2f)^{-1} = f^{-1} \)) is extended to some homomorphism \( eR/D_1 \to eR/D_2 \) (or \( eR/D_2 \to eR/D_1 \)).

4) \( \Rightarrow \) 2'): We shall show the assertion by induction on \( n \). When \( n=1 \), the assertion is trivial. In the case that \( n=2 \), consider an exact sequence:

\[
0 \to S \xrightarrow{\alpha} eR/X \oplus eR/Y \to M \to 0 ,
\]

where \( S \) is simple, \( \alpha(s) := \alpha_1(s) + \alpha_2(s)(\alpha_1(s) \in eR/X, \alpha_2(s) \in eR/Y) \) for any \( s \in S \), and \( X', Y' \leq eR \). Here we may assume that \( \alpha_1 \) and \( \alpha_2 \) are monomorphisms. If \( \text{im} \alpha \leq \text{rad}(eR/X \oplus eR/Y) \), then \( \text{im} \alpha \) is a direct summand of \( eR/X \oplus eR/Y \) since \( \text{im} \alpha \) is simple, thus the assertion holds. So we may assume that \( \alpha = \alpha' \leq \text{rad}(eR/X \oplus eR/Y) \) and \( |eR/X|, |eR/Y| \geq 2 \). Put \( T_i/X := \alpha_1(S) \) and \( T_i/Y := \alpha_2(S) \) for some \( T_i \) and \( T_i \leq eR \). Then \( T_i/X \) is simple and \( \alpha_2 \alpha_1^{-1}: T_i/X \to T_i/Y \) is an isomorphism. By 4), \( \alpha_2 \alpha_1^{-1}(\alpha_2 \alpha_1^{-1} = \alpha_1 \alpha_2^{-1}) \) is extended to a homomorphism: \( eR/X \to eR/Y \) (or \( eR/Y \to eR/X \)). Hence by Lemma 4, \( M \) is decomposable and \( M = eR/X' \oplus eR/Y' \) for some \( X', Y' \leq eR \). Finally we assume that the assertion holds for \( n=1(\geq 2) \). Suppose that the following sequence is exact:

\[
0 \to S \to \bigoplus_{i=1}^{n} (eR/X_i) \xrightarrow{\phi} M \to 0 ,
\]

where \( S \) is simple. Put \( L_i := \varphi(eR/X_i) \) for each \( i(1 \leq i \leq n) \). Then by [6, Section
[2], we have $M = L_1 + \left( \bigoplus_{i=2}^{n} L_i \right)$ and $T := L_1 \cap \left( \bigoplus_{i=2}^{n} L_i \right) \subset S$. Let $p : \bigoplus_{i=2}^{n} L_i \to L_n$ be the canonical projection. Then $p \mid_T$ (or $(p \mid_T)^{-1}$) is extended to a homomorphism: $L_1 \to L_n$ (or $L_n \to L_1$). In the case that $p \mid_T$ is extended to some $\Phi : L_1 \to L_n$, we put $L'_1 := \{ x - \Phi(x) \mid x \in L_1 \}$. Then by Lemma 3, we obtain $M = L_1 + \left( \bigoplus_{i=2}^{n} L_i \right) = (L_1 + \bigoplus_{i=2}^{n-1} L_i) \oplus L_n$. Put $M' := L_1 + \left( \bigoplus_{i=2}^{n-1} L_i \right)$ and $S' := L_1 \cap \left( \bigoplus_{i=2}^{n-1} L_i \right)$, then $S'$ is simple. Noting that $L_1$ and each $L_i (1 \leq i \leq n-1)$ are factor modules of $eR$, the following exact sequence implies that $M' \cong \bigoplus_{i=1}^{n} (eR/Y_i)$ for some $Y_i \leq eR$ by induction hypothesis:

$$0 \to S' \xrightarrow{\alpha'} L'_1 \oplus \bigoplus_{i=2}^{n-1} L_i \xrightarrow{\beta'} M' \to 0,$$

where $\alpha' := (\alpha'_1, \ldots, \alpha'_n)^T$, $\alpha'_i$ is the inclusion map and $\alpha' :\equiv p_i \mid_{S'}$ ($p_i$ is the canonical projection) ($2 \leq i \leq n-1$); and $\beta := (\beta_1, \ldots, \beta_{n-1})$, $-\beta_1$ and $\beta_i$ ($2 \leq i \leq n-1$) are the inclusion maps. Hence $M = M' \oplus L_n \cong \bigoplus_{i=1}^{n-1} (eR/Y_i) \oplus eR/X_n$. The remaining case is proved similarly.

2') $\Rightarrow$ 2): Put $D := \{ H \mid H \subseteq \bigoplus_{i=1}^{n} eR/X_i \text{ for some } n \text{ and for some } X_i \leq eR \}$. Then we have only to show that $D$ is closed under factor modules. Thus it suffices to verify the following for all $m \geq 1$, by induction on $m$: For each $H \subseteq D$ and for each $X \subseteq H$, $|X| = m$ implies $H/X \subseteq D$. When $m = 1$, this follows from 2'). Let $m \geq 2$. Take $0 \neq Y \subseteq X$. Then $|Y| < m$ and $|X/Y| < m$, which imply $H/X \subseteq (H/Y)/(X/Y) \subseteq D$ by induction hypothesis.

2) $\Rightarrow$ 1) is trivial. Q.E.D.

We shall characterize later the structure of $eR$ satisfying the condition (A), i.e., the conditions of the above theorem, using mainly 4) of it.

2. Properties of $eR$ with (A)

From now on we assume that $R$ is a left serial ring. Further throughout this section, we assume that $e$ is a fixed primitive idempotent, and $eR$ satisfies the condition (A). Then we have several properties similar to ones in [4] as follows, by using 4) of the theorem.

**Proposition 1** ([4, Proposition 1]). If $eR/eJ^i$ is uniserial and $eJ^i = A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_p}$ for some $i$, where $p \geq 3$ and each $A_{i_k} (1 \leq k \leq p)$ is a hollow module, then we have $A_{i_1} \cong A_{i_2} = \cdots = A_{i_p}$, and each $A_{i_k}$ is simple.

**Proposition 2** ([4, Proposition 5]). If $eR/eJ^i$ is uniserial and $eJ^i = A_{i_1} \oplus A_{i_2}$, where $A_{i_1}$ and $A_{i_2}$ are hollow modules, then each $A_{i_k}$ is uniserial.
We put $\Delta := eRe/eJe$ and $\Delta(A) := \{x \in \Delta \mid x' A \leq A, x' = x \text{ for some } x' \text{ in } eRe\}$, where $A$ is a hollow submodule of $eR$ and $x$ is the coset of $x$ in $\Delta$. Then $\Delta$ is a division ring and $\Delta(A)$ is a division subring of $\Delta$ (see [4]). In the case that $eJ^i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ip} (p \geq 2)$, we put $\Delta(A_{i1}) = \Delta_i$. Now we consider the case $p \geq 3$ in more detail. Since $eJ(eJ^i) = eJ^{i+1} = 0$, we have $x = xc$ for any $c \in eJ^i$ and any $x \in eRe$. Proposition 1 shows that each $A_{ik}(1 \leq k \leq p)$ is simple, so we may put $A_{i1} = aR$ and $a = eaf$ for some $f$ in $\pi(R)$. Then for any $b(\neq 0)$ in $A_{i1}$, $b = ebf$ since $A_{i1}$ is simple and $R$ is basic. Noting here that $R$ is left serial, there exists some $x \in eRe\setminus eJe$ with $b = xa = xa$. Here $x$ is in $\Delta_i$ since $xA_{i1} = xaR = bR = A_{i1}$, whence $b$ is in $\Delta_i a$. Thus $A_{i1} = \Delta_i a$. For each $k(1 \leq k \leq p)$, put $A_{ik} = \Delta_i aR = \Delta_i a$. Using this fact, we obtain

Lemma 5. Suppose $eJ^i = A_{i1} \oplus \cdots \oplus A_{ip} (p \geq 3)$, and let $A_{i1} = aR$. Put $L(eJ^i) := \text{the lattice of submodules of } eJ^i$ and $L(\Delta) := \text{the lattice of subspaces of } \Delta_i$. Then we have a bijection $\alpha : L(\Delta) \to L(eJ^i)$ defined by $\alpha(V) := Va$ for every $V \in L(\Delta)$. Further $\alpha$ preserves and reflects the linear independence, i.e., for any $\{V_i\} \subseteq L(\Delta)$, $\{V_i\}$ is independent if and only if $\alpha(\{V_i\})$.

Proof. Since $VaR = V\Delta_i a = Va$ for any $V \in L(\Delta)$, $\alpha$ is well defined. It is easy to show that $\alpha$ preserves and reflects the linear independence. To show that $\alpha$ is a surjection, let $T$ be any submodule $(\neq 0)$ of $eJ^i$. Then $T$ is expressed as $T = X_1 \oplus \cdots \oplus X_t$ with $X_k = A_{ik}(1 \leq k \leq t)$. So we have $X_k = \delta_k A_{i1} = \delta_k \Delta_i a$ for some $\delta_k$ in $\Delta$ by the consideration above. Hence $T = (\oplus \delta_k \Delta_i a)$ since $\alpha$ reflects the independence. Thus $\alpha$ is a surjection. $\alpha$ being an injection is immediate from the fact that $ya = 0(y \in \Delta)$ implies $y = 0$.

Q.E.D.

Lemma 5 implies the following (see [4] for $p = 2$).

Proposition 3 ([4, Proposition 2]). It holds $[\Delta : \Delta_i] = |eJ^i|$ except for the case that $eJ^i = A_{i1} \oplus B_i$ and $A_{i1} \neq B_i$, where $A_i$ and $B_i$ are hollow modules (in this exceptional case, we have $\Delta = \Delta_i$).

We consider the following condition $(\#)$ on $\Delta$ as a right $\Delta_i$-vector space.

$(\#)$ Let $V_1$ and $V_2$ be subspaces of $\Delta_i$ and $v_1$ and $v_2$ be elements of $\Delta$ satisfying $|V_1| \leq |V_2|$ and $v_1 \Delta_i \cap V_1 = 0 = v_2 \Delta_i \cap V_2$. Then there exists $x$ in $\Delta$ such that $xv_1 \leq V_2$ and $xv_1 \equiv v_2 (\mod V_2)$.

The following is immediate from Lemma 5.

Proposition 4. If $|eJ^i| \geq 3$, then the following are equivalent.

1. Let $eJ^i \geq T_1 \geq T_2$, $eJ^i \geq S_1 \geq S_2$, and $T_1/T_2 \to S_1/S_2$ be simple, and $f : T_1/T_2 \to S_1/S_2$ be an isomorphism. Then $f$ is extended to a homomorphism : $eR/T_2 \to eR/S_2$. 

By \cite{4, Lemma 5}, the following holds.

**Proposition 5.** Let \( \Delta \succeq \Delta_i \) be division rings. If \( \Delta \) and \( \Delta_i \) satisfy the condition (\#), then \( [\Delta : \Delta_i] \leq 2 \). In particular, if \( eR \) satisfies the condition (A), then \( [\Delta : \Delta_i] \leq 2 \).

### 3. The structure of \( eR \) with (A)

Also in this section, we assume that \( R \) is left serial. Using Propositions 1 and 2, \( eR \) with the condition (A) has one of the following structures.

- (a) \( eR \) is a uniserial module.
- (b\(_1\)) For some natural number \( i \), \( eR/ef^i \) is uniserial and \( ef^i = A_{i+1} \oplus A_{i+2} \), where \( A_{i+1} \) and \( A_{i+2} \) are uniserial modules which are not isomorphic to each other.
- (b\(_2\)) For some natural number \( i \), \( eR/ef^i \) is uniserial and \( ef^i = A_{1+1} \oplus A_{1+2} \), where \( A_{1+1} \) and \( A_{1+2} \) are uniserial modules.
- (c) For some natural number \( i \), \( eR/ef^i \) is uniserial and \( ef^i = A_{1+1} \oplus \cdots A_{1+p} \) (\( p \geq 3 \)), where \( A_{1+1} \simeq A_{1+2} \simeq \cdots A_{1+p} \) are simple modules.

Thus we can illustrate the structures (b\(_1\)), (b\(_2\)) and (c) as follows.

\[
\begin{array}{cccc}
(b_1) & eR & (b_2) & eR & (c) & eR \\
& ef & & ef & & ef \\
& \vdots & & \vdots & & \vdots \\
& ef^{i-1} & & ef^{i-1} & & ef^{i-1} \\
\end{array}
\]

\[
\begin{array}{cccc}
& A_i \not\simeq B_i & & A_i \simeq B_i & & A_{1+1} \simeq A_{1+2} \simeq \cdots A_{1+p} \\
& A_{i+1} & & A_{i+1} & & 0 & & 0 & \cdots & 0 \\
& \vdots & & \vdots & & \vdots \\
& 0 & & 0 & & p \geq 3 \\
\end{array}
\]

Now we state the main theorem.

**Theorem 2.** Let \( R \) be a left serial ring. The following are equivalent for each \( e \in \text{pi}(R) \).

1) \( eR \) satisfies the condition (A).
2) \( eR \) has one of the structures (a); (b\(_1\)); (b\(_2\)) with \( [\Delta : \Delta_i] = 2 \); and (c) with the condition (\#) for \( \Delta \) and \( \Delta_i \).

The proof of 1) \( \Rightarrow \) 2) is already done. So we show that each of the conditions in 2) implies the condition 1). This is immediate from Proposition 4 in the case of the structure (c), and in the case of (a) this follows from Lemma
2 and Theorem 1 (see case (i) in Lemma 8). For the proof of the other cases, we divide the argument to some lemmas.

**Lemma 6.** In the diagram \((b_1)\), the following statements hold.

1. For any \(j\) and \(k\) with \(i \leq j \leq k\) and \(k-j<i\), we have \(\tilde{A}_j \neq \tilde{B}_k\).
2. For any \(j\) and \(k\) with \(i \leq j < k\) and \(k-j<i\), we have \(\tilde{A}_j \neq \tilde{A}_k\).

**Proof.**

1) For \(j(i \leq j)\), we have \(\tilde{A}_j = \tilde{B}_k\) for some \(j, k\) with \(i \leq j < k\) and \(k-j<i\). Put \(A_j = a_jR, B_k = b_kR\) and \(a_j = e_a f, b_k = e_b f\) for some \(f\) in \(\pi(R)\). Then there exists \(d\) in \(T(e^{j-i})\) such that \(d a_j = b_k\). So \(e^{j-i} e \neq 0\), and there exists an epimorphism: \(eR \rightarrow e^{j-i} R\). This epimorphism induces an epimorphism: \(e^{j-i} \rightarrow e^{j-i} R\). Thus we have \(e^{j-i} e \neq e^{j-i} R\). But this is a contradiction since \(i \geq j = (k-j)\). We conclude that \(\tilde{A}_j \neq \tilde{B}_k\).

2) Suppose that \(\tilde{A}_j = \tilde{A}_k\) for some \(j, k\) with \(i \leq j < k\) and \(0 < k-j < i\). Then we may put \(A_j = a_jR\) and \(A_k = a_kR\). So there exists \(d\) in \(e^{j-i} e\) such that \(d a_j = a_k\), thus \(e^{j-i} e \neq 0\) and this yields the similar contradiction as in 1). Q.E.D.

**Lemma 7.** In the diagram \((b_2)\), we have that \(\tilde{A}_j \neq \tilde{B}_k\) for each \(j, k\) with \(i \leq j < k\).

**Proof.** Suppose that \(\tilde{A}_j = \tilde{A}_k\) in the case \(k-j<i\). Put \(A_j = a_jR, B_k = b_kR\) and \(a_j = e_a f, b_k = e_b f\) for some \(f\) in \(\pi(R)\). Then there exists \(d\) in \(T(e^{j-i})\) such that \(d a_j = b_k\), and this yields the contradiction similar to that in the proof of Lemma 6. Next suppose that \(\tilde{A}_j = \tilde{B}_k\) in the case \(k-j \geq i\). Then there exists \(a_j, b_k\) and \(d\) as above. It follows from \(k-j \geq i\) that \(d = d_1 + d_2\) for some \(d_1\) in \(A_i\) and \(d_2\) in \(B_i\). So \(b_k = da_j = d_1 a_j + d_2 a_j\) and \(b_k\) is in \(B_i\), thus \(b_k = d_2 a_j\), where \(d_2\) in \(T(B^{k-j})\). Then for \(b_j\) with \(B_j = b_j R\), we have that \(b_j = b_j g\) and \(d_2 b_j \neq 0\) is in \(T(B_k)\) by \(\tilde{A}_j = \tilde{B}_k\). There exists \(r\) in \(T(gR)\) such that \(d_2 b_j r - b_k\), and hence \(T(e^{j-i} g) \rightarrow T(a_j - b_j r) = b_k - b_k = 0\), a contradiction. Q.E.D.

Using Lemmas 6 and 7, we show the implication \(2) \Rightarrow 1)\) of Theorem 2 as the following two lemmas.

**Lemma 8.** Let the diagram \((b_1)\) be the structure of \(eR\). Then \(eR\) satisfies the condition (A).

**Proof.** Let \(C_j\) and \(D_j\) be submodules of \(eR\) such that \(eR \geq C_j \geq D_j\) and \(C_j/D_j\) is simple for \(j = 1, 2\), and \(f: C_j/D_j \rightarrow C_j/D_j\) be an isomorphism. We may assume that \(C_j = c_j R + D_j\) for some \(c_j\) in \(C_j\) satisfying \(f(c_j + D_j) = c_j + D_j\), and \(c_j g = c_j g\) for some \(g \in \pi(R)\).

(i) In the case where both \(c_1\) and \(c_2\) are in \(T(e^{j})\) for some \(t < i\), there exists a unit \(x\) in \(eRe\) such that \(x c_1 = c_2\). Then \(x t\) (the left side multiplication of \(x\)) induces \(f\).

(ii) Suppose that \(c_1\) is in \(T(e^{j})\) for some \(t < i\) and \(e^{j} \geq C_2 > D_2\). Then
there exists some $x$ in $eJ$ such that $xe_1=c_2$. So $xD_1=xC_1J=C_2J<D_2$,
whence $x_1$ induces $f$

iii) In the case that $eJ^J \geq C_j > D_j$ for each $j=1, 2$.

First, we show that for any $C, D$ such that $D < C \leq eJ$ and $C/D$ is simple,
there exists a unit $x$ in $eRe$ satisfying

$$(P): \quad xC = A_{t_i-1} \oplus B_s \quad \text{(or } xC = A_t \oplus B_{s-i})$$

For a module $X \leq eJ$, put $X^{(1)} = \pi_{A_i}(X)$, $X^{(2)} = \pi_{B_i}(X)$, $X^{(1)} = X \cap A_i$ and
$X^{(2)} = X \cap B_i$, where $\pi_{A_i}: eJ \rightarrow A_i$ and $\pi_{B_i}: eJ \rightarrow B_i$ are the canonical projections.
Then it is easy to see that $X^{(j)} \leq X^{(j)} (j=1, 2)$ and $X^{(1)} / X^{(2)} = X / X^{(2)}$. Now, if $D^{(j)} = D^{(j)}$, then $D^{(j)} = D^{(j)}$, and we can take $x = e$ for $x$ in $(P)$. Thus we may assume $D^{(j)} < D^{(j)}$. Then by the above, $D^{(j)} < D^{(j)}$ and $D^{(j)} / D^{(j)} \leq D^{(j)} / D^{(j)}$. Since $R$ is left serial, there exists some $\delta \in eJe$ (by Lemma 6) such that either $\delta D^{(j)} = D^{(j)}$ and $\delta D^{(j)} = D^{(j)}$ or $\delta D^{(j)} = D^{(j)}$ and $\delta D^{(j)} = D^{(j)}$. We may assume that the former holds. There exists a unique $s$ such that $D^{(j)} \leq eJ^s \leq D$. Then $D = (e+\delta) D^{(j)} + (e+\delta) D^{(j)} + eJ$. Noting that $u = e+\delta$ is a unit in $eRe$ and $u^{-1} eJ^s = eJ^s$, we have $u^{-1} D = D^{(j)} + u^{-1} eJ^s = D^{(j)} + eJ = D^{(j)} + D^{(j)}$. Put $C' = u^{-1}C$, $D' = u^{-1}D$. When $C' = C^{(1)}$, we can take $x = u^{-1}$. So suppose that $C' < C^{(2)}$. Then $C' = D^{(j)}$, $C'' = D^{(j)}$ and $C'' / C' = C'' / C'$ is simple
because so is $C'/D' \leq C/D$. By an argument similar to one for $D$, we can take some $\omega \in eJe$ such that $\omega C' = C^{(2)}$ and $\omega C' = C^{(2)}$. Hence putting $y = e+\omega$, we have $C = yC^{(1)} \oplus C^{(2)}$ and $y^{-1} C' = C^{(1)} \oplus C^{(2)}$. It follows from $C' = D^{(j)}$ and $C'' = D^{(j)}$ that $\omega D^{(j)} = D^{(j)}$. Then $D' = D^{(j)} \oplus D^{(j)} = yD^{(j)} \oplus D^{(j)}$, whence $y^{-1} D' = D^{(j)} \oplus D^{(j)}$. Consequently, we can take $x = y^{-1} u^{-1}$.

Next, consider the case that $C_j$ and $D_j (j=1, 2)$ have the following forms:
$C_j = A_{t_j-1} \oplus B_{s_j}$ (or $C_j = A_{t_j} \oplus B_{s_j-1}$) and $D_j = A_{t_j} \oplus B_{s_j}$. Considering the structure of $(b_i)$ and $C_j / D_j = C_s / D_s$, we see that the possible cases are the following.

$$(\alpha) \quad C_1 = A_{t_1-1} \oplus B_{s_1}, \quad D_1 = A_{t_1} \oplus B_{s_1}$$
$C_2 = A_{t_2} \oplus B_{s_2-1}, \quad D_2 = A_{t_2} \oplus B_{s_2} \quad (t_2-t_1 \geq i)$

$$(\beta) \quad C_1 = A_{t_1-1} \oplus B_{s_1}, \quad D_1 = A_{t_1} \oplus B_{s_1}$$
$C_2 = A_{t_2} \oplus B_{s_2}, \quad D_2 = A_{t_2} \oplus B_{s_2}$

$$(\gamma) \quad C_1 = A_{t_1-1} \oplus B_{s_1}, \quad D_1 = A_{t_1} \oplus B_{s_1}$$
$C_2 = A_{t_2} \oplus B_{s_2}, \quad D_2 = A_{t_2} \oplus B_{s_2} \quad (t_2-t_1 \geq i)$

In the case $(\alpha)$, we put $f(a_i + D_1) = b_1 + D_2$, $A_{t_1-1} = a_i R$ and $B_{s_2-1} = b_i R$. There exists some $d$ in $eJ^{s_2-t_1}$ such that $da_i = b_i$. Since $t_2-t_1 \geq i$, we have $d = d_1 + d_2$ for some $d_1$ in $A_i$ and $d_2$ in $B_i$, and $b_2 = da_1 = d_1 a_1 + d_2 a_1$. Then $b_2 = d_2 a_1$. Hence $d_2 B_{s_1} = 0$. Indeed if not, for some $x \in T(A_i)$ and some $y \in T(B_i)$, both $d_2 x$ and $d_2 y$ are in $T(B_{t_2-t_1})$ and non-zero. Thus, $A_i \sim B_i$, a contradiction. Thus $(d_2)_i$
induces \( f \). The similar argument works for the cases (\( \beta \)) and (\( \gamma \)).

Finally in the general case, there exist units \( x \) and \( y \) in \( eR \) for \( C_j \) and \( D_j \) \((j=1, 2)\) as in (P). Using the isomorphism \( f \), we put

\[
\begin{align*}
f' : (A_{t_1-1} \oplus B_{s_1})/(A_{t_1} \oplus B_{s_1}) & \rightarrow C_1/D_1 \\
& \rightarrow C_2/D_2 \\
& \rightarrow (A_{t_2-1} \oplus B_{s_2})/(A_{t_2} \oplus B_{s_2})
\end{align*}
\]

and apply the argument above. Q.E.D.

**Lemma 9.** Let the diagram (b_2) be the structure of \( eR \), and assume \([\Delta : \Delta, \gamma] = 2\). Then \( eR \) satisfies the condition (A).

**Proof.** Let \( C_j \) and \( D_j \) be submodules of \( eR \) such that \( eR \supseteq C_j > D_j \) and \( C_j/D_j \) are simple for \( j=1, 2 \), and \( f : C_j/D_j \rightarrow C_2/D_2 \) be an isomorphism. Then \( C_j = c_j R + D_j \) for some \( c_j \) in \( C_j \) \((j=1, 2)\), where we may assume that \( f(c_j + D_j) = c_2 + D_2 \) and \( c_1 = c_2 g \), \( c_2 = c_2 g \) for some \( g \in \pi(R) \).

The proof similar to that of Lemma 8 works in the following two cases:

(i) both \( c_1 \) and \( c_2 \) are in \( T(eJ) \) \((t < s)\).

(ii) \( c_1 \) is in \( T(eJ) \) \((t \leq s)\) and \( eJ \supseteq C_j > D_j \).

So we show only the following case:

(iii) \( eJ \supseteq C_j > D_j \) for both \( j=1, 2 \).

We have that for any \( C \) and \( D \) with \( eJ \supseteq C > D \), there exists a unit \( x \) in \( eR \) such that \( xC = A_{t-1} \oplus B_s > xD = A_t \oplus B_s \) \((or \ xC = A_{t} \oplus B_{s-1} > xD = A_t \oplus B_s)\) for some \( t, s \geq i \) \( (the \ proof \ is \ in \ Lemma \ 8) \). Further we have that for \( C = A_k \oplus B_r \) \((k, r \geq i)\), there exists a unit \( y \) in \( eR \) such that \( yC = A_k \oplus B_r \). So we may assume that \( C_j > D_j \) are of the following form:

\[
\begin{align*}
C_1 &= A_{t_1-1} \oplus B_{s_1}, \quad D_1 = A_{t_1} \oplus B_{s_1} \\
C_2 &= A_{t_2-1} \oplus B_{s_2}, \quad D_2 = A_{t_2} \oplus B_{s_2}.
\end{align*}
\]

It follows from \( C_1/D_1 \supseteq C_2/D_2 \) that \( t_1 - t_2 = t \).

(\( \alpha \)) In the case that \( t \leq \max(s_1, s_2) \). We may assume \( s_1 \geq s_2 \). Let \( C_1 = c_1 R + D_1 \) and \( c_1 R = A_{t-1} \). Then there exists a unit \( z \) in \( eR \) such that \( f(c_1 + D_1) = zc_1 + D_2 \). It follows from \( zD_1 \supseteq A_1 \oplus B_{s_1} \supseteq D_2 \) that \( z_i \) induces \( f \).

(\( \beta \)) In the case that \( t > \max(s_1, s_2) \). We may assume \( s_1 \geq s_2 \). Let \( B_{t-1} = bR \) and \( \delta B_i = A_i \). Then \( A_{t-1} = \delta b R \) and \( f(\delta b + D_1) = \delta w b + D_4 \) for some \( w \) in \( eR \) with \( w B_{t-1} = B_{t-1} \), i.e. \( w \) is in \( \Delta_i \). Since \([\Delta : \Delta, \gamma] = 2\), there exist \( \tilde{y}_1 \) and \( \tilde{y}_2 \) in \( \Delta_i \) such that \( \delta w = \tilde{y}_1 + \tilde{y}_2 \). So we have \( \delta w = \tilde{y}_1 + \tilde{y}_2 \delta + j \) for some \( j \) in \( eR \), whence \( \gamma_w b = (\delta w - \gamma_w j) b \equiv \delta w b \) \((\text{mod } D_4)\), since \( \gamma_j b \) is in \( B_{t-1} \leq D_4 \) and \( j b \) is in \( A_{t-1} \oplus B_{s_1} \leq D_4 \). Then \( f(\delta b + D_1) = \gamma_2(\delta b) + D_4 \) and \( \gamma_2 D_1 \leq A_1 \oplus B_{s_1} \leq D_4 \). So we have that \( (y_2)_t \) induces \( f \). Q.E.D.

**Remark.** If \( R \) is a finite dimensional algebra over a field, then \([\Delta : \Delta, \gamma] = [\Delta : \Delta, \gamma] \) holds. So for a primitive idempotent \( e \), if \( eR \) satisfies the condition
(A), it follows from $[\Delta: \Delta_j] \leq 2$ (by Proposition 5) that $[\Delta: \Delta_j] \leq 2$. Hence $eR$ never has the structure (c). Further suppose that $R$ is a finite dimensional algebra over an algebraically closed field. Then we have $[\Delta: \Delta_j]=[\Delta: \Delta_j]=1$. Hence $eR$ has the structure (a) or (b).

4. Examples

Here we give some examples of left serial rings having projective indecomposable modules with structures (b), (b), and (c) which satisfy the condition (A).

**Example 1.** Let $k$ be a field and put

$$R = \begin{pmatrix} k & k & k & k \\ 0 & k & k & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Then every projective indecomposable $R$-module satisfies the condition (A) and $eR$ has the structure (b). Note that $R$ is not of right local type (Cf. [6]).

**Example 2.** Let $K \leq L$ be fields with $[L: K]=2$. Put

$$R = \begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Then $eR$ has the structure (b) and satisfies the condition on the left dimension in Theorem 2. Also in this case every projective indecomposable module satisfies the condition (A) but $R$ is not of right local type (Cf. [6]).

**Example 3 (Asashiba [1]).** Let $F$ and $G$ be division rings and $M$ an $(F, G)$-bimodule having the dimension sequence $(3, 1, 2, 2, 1)$ (see Dowbor, Ringel and Simson [2]). The existence of such an $M$ follows from Schofield [5, section 13] and [2, Proposition 1]. Then $R:=\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ has exactly 5 non-isomorphic indecomposable modules and $[M: G]=3$, say $M_0=A_1 \oplus A_2 \oplus A_3$ with each $A_i \cong G$. Put $e_1:=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2:=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can identify $e_1 R = M_0$. Since the set $S:=\{e_2 R, e_1 R, e_1 R/A_1, e_1 R/(A_1 \oplus A_2), e_1 R/e_1 J\}$ consists of 5 non-isomorphic local modules, $S$ is a complete set of representatives of isomorphism classes of indecomposable $R$-modules. Thus $R$ is of right local type. Hence every projective indecomposable $R$-module satisfies the condition (A). In particular so does $e_1 R$. Further since $e_1 J$ is isomorphic to a direct sum of
three copies of a simple module, $e_1 R$ has the structure (c) and satisfies the condition (#).

References


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