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ON CONGRUENT AXIOMS IN LINEARLY ORDERED SPACES, I

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1. Introduction

In connection with the axioms of congruence of segments on a straight line given in Hilbert's Grundlagen der Geometrie, we will set up a group of axioms of congruence on a linearly ordered space and study their mutual dependency and independency.

In the following let L be a linearly ordered space, that is a set of points, in which for any pair of distinct points A and B either of the relations $A<B$ and $B<A$ holds, and for any three points A, B and C, if $A<B$ and $B<C$ then $A<C$.

When we write AB, it will be understood that A and B are distinct points of L such that $A<B$. AB will be called a *segment*. We write $AC\equiv AB+BC$ if and only if $A<B<C$.

The axioms we are going to study is the following:

Axiom E (UNIQUE EXISTENCE): $\forall AB\ \forall A'\ \exists_1 B'$: $AB=A'B'$, that is, *for any segment AB and for any point A′ there is one and only one point B′ such that*

$$AB=A'B'.$$

Axiom R (REFLEXIVITY): $AB=AB$.

Axiom S (SYMMETRICITY): $AB=A'B' \Rightarrow A'B'=AB$.

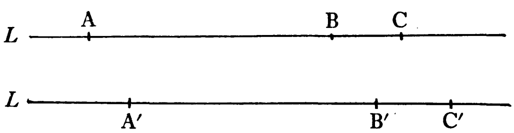
Axiom T (TRANSITIVITY): $AB=A'B',\ A'B'=A''B'' \Rightarrow AB=A''B''$.

Axiom A (ADDITIVITY):

$$AC\equiv AB+BC,\ A'C'\equiv A'B'+B'C',\ AB=A'B',\ BC=B'C' \Rightarrow AC=A'C'.$$

The following scheme will be used in application:

$$\left.\begin{array}{l} AC\equiv AB+BC\,, \\ A'C'\equiv A'B' \\ \qquad +B'C'\,, \\ AB=A'B'\,, \\ BC=B'C' \end{array}\right\} \overset{(A)}{\Longrightarrow} AC=A'C'\,.$$

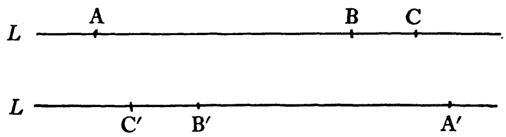


Axiom C (COMMUTATIVE ADDITION):

$$AC \equiv AB+BC,\ C'A' \equiv C'B'+B'A',\ AB=B'A',\ BC=C'B' \Rightarrow AC=C'A'.$$

In application we write:

$$\left.\begin{array}{l} AC\equiv AB+BC, \\ C'A'\equiv C'B' \\ \qquad +B'A', \\ AB=B'A', \\ BC=C'B' \end{array}\right\} \overset{(C)}{\Longrightarrow} AC=C'A'.$$



Axiom I (INTERCHANGING): $A<B<A'<B',\ AB=A'B' \Rightarrow AA'=BB'$.

Under the assumption of Axiom E we studied in this paper all the relationship between the remaining axioms R, S, T, A, C, I as to their mutual dependency and independency, and obtaind among others the following Main Theorem.

Main Theorem: *Under the assumption of Axiom* E,

I. *Axioms* T *and* C *are independent of each other, and Axioms* R, S, A *and* I *follow from them.* In symbol:

R, S, **T**, A, **C**, I

II. *Axioms* T *and* I *are independent of each other, and Axioms* R, S, A *and* C *follow from them.* In symbol:

R, S, **T**, A, C, **I**

III. *Axioms* S, A *and* I *are independent of one another, and Axioms* R, T *and* C *follow from them.* In symbol:

R, **S**, T, **A**, C, **I**

IV. *Axioms* S, A *and* C *are independent of one another, and Axioms* R, T *and* I *follow from them.* In symbol:

R, **S**, T, **A**, **C**, I

V. *Axioms* R, A *and* C *are independent of one another, and Axioms* S, T *and* I *follow from them.* In symbol:

R, S, T, **A**, **C**, I

2. Theorems

In the following we always assume the unique existence of Axiom E, if not otherwise stated.

To make proofs as clear as possible we introduce first some useful notations.

(a) $X \overset{(T)}{\Longrightarrow} Y$ means that Y follows from the left side X by the use of T.

(b) $A=B$ means that A coincides with B and $AB \equiv A'B'$ means that $A=A'$,

$B=B'$ at the same time.

(c) "$\exists_1 X$:" means that "there exist one and only one X such that."

Theorem 1. *If* T *is assumed, then* R$\Leftrightarrow$S.

Proof. (i) R$\Rightarrow$S.

$$\left.\begin{array}{l}\text{By Axiom E, } \exists_1 B': AB=A'B'. \\ \text{By Axiom E, } \exists_1 B'': A'B'=AB''.\end{array}\right\} \xrightarrow{(\mathrm{T})} AB=AB''. \qquad (1)$$

Now by Axiom R,

$$AB=AB. \qquad (2)$$

$$(1), (2) \xrightarrow{(\mathrm{E})^{1)}} B''=B.$$

(ii) R$\Leftarrow$S.

Assume $AB=A'B'$. Then by Axiom S, $A'B'=AB$. Hence by Axiom T, $AB=AB$.

Theorem 2. R, C$\Rightarrow$I.

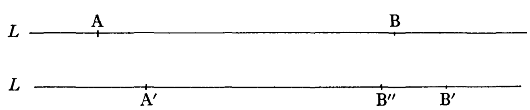
Proof. Let $A<B<A'<B'$ and $AB=A'B'$.
Then we have

$$\left.\begin{array}{l}AA'\equiv AB+BA', \\ BB'\equiv BA'+A'B', \\ AB=A'B', \\ BA'=BA' \text{ (by Axiom R)}\end{array}\right\} \xrightarrow{(\mathrm{C})} AA'=BB'.$$

Lemma 1. *Under the assumption of* T: $AB=A'B' \Rightarrow A'B'=A'B'$.
Especially: $AB=AB' \Rightarrow AB'=AB'$.

Proof. Let

$$AB=A'B' \qquad (1)$$



Then we have

$$\left.\begin{array}{r}\left.\begin{array}{r}AB=A'B'. \\ \text{By Axiom E, } \exists_1 B'': A'B'=A'B'' \text{ (2).}\end{array}\right\} \xrightarrow{(\mathrm{T})} AB=A'B''. \\ \text{By (1): } AB=A'B'.\end{array}\right\} \xrightarrow{(\mathrm{E})} B''=B'.$$

Therefore we have from (2) $A'B'=A'B'$.

1) If $AB=A'B'$ and $AB=A'B''$ then we have by Axiom E $B'=B''$. As a special case, if $AB=AB'$ and $AB=AB$ then $B'=B$.

Theorem 3. T, C $\Rightarrow$ A.

Proof. Let $AC \equiv AB + BC$, $A'C' \equiv A'B' + B'C'$,

$$AB = A'B', \tag{1}$$

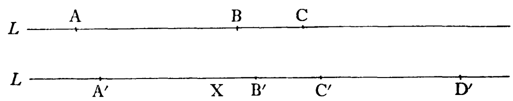
and

$$BC = B'C'. \tag{2}$$

Then by Axiom E, $\exists_1 D'$: $A'B' = C'D'$. (3)

Then we have first from (1) and (3) by using T

$$AB = C'D'. \tag{4}$$



Further

$$\left.\begin{array}{l} AC \equiv AB + BC, \\ B'D' \equiv B'C' + C'D', \\ (4):\ AB = C'D', \\ (2):\ BC = B'C' \end{array}\right\} \xRightarrow{(C)} AC = B'D'. \tag{5}$$

By Axiom E, $\exists_1 X$: $C'D' = A'X$. (6)

Then by (3) and (6) we have by using T

$$A'B' = A'X. \tag{7}$$

Since by Lemma 1

$$A'B' = A'B', \tag{8}$$

we have from (7) and (8) by the use of Axiom E $X = B'$.
Hence by (6)

$$C'D' = A'B'. \tag{9}$$

Then

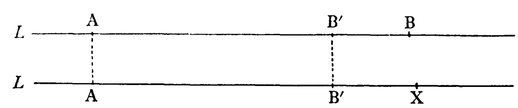
$$\left.\begin{array}{l} B'D' \equiv B'C' + C'D', \\ A'C' \equiv A'B' + B'C', \\ B'C' = B'C' \text{ (by Lemma 1)}, \\ (9):\ C'D' = A'B' \end{array}\right\} \xRightarrow{(C)} B'D' = A'C'. \tag{10}$$

From (5) and (10) we have finally by Axiom T $AC = A'C'$.

Theorem 4. T, A $\Rightarrow$ R.

Proof. Let AB be a given segment. Then by Axiom E, $\exists_1 B'$:

$$AB = AB'. \tag{1}$$



(i) Suppose first $A < B' < B$.
Then we have

$$\left.\begin{array}{l} AB \equiv AB' + B'B\,, \\ AX \equiv AB' + B'X\,, \\ AB' = AB' \text{ (by Lemma 1)}, \\ \text{by Axiom E, } \exists_1 X\colon B'B = B'X \end{array}\right\} \xrightarrow{(\mathrm{A})} AB = AX\,. \tag{2}$$

From (1) and (2) we would have by Axiom E $B'=X$, which is clearly a contradiction.

(ii) Next suppose $B<B'$.

Then we have

$$\left.\begin{array}{l} AB' \equiv AB + BB'\,, \\ AX \equiv AB' + B'X\,, \\ (1)\colon AB = AB'\,, \\ \text{by Axiom E, } \exists_1 X\colon BB' = B'X \end{array}\right\} \xrightarrow{(\mathrm{A})} AB' = AX\,. \tag{3}$$

Since by Lemma 1 $AB'=AB'$, we have from (3) $X=B'$, which is clearly a contradiction.

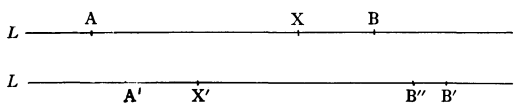
From (i) and (ii) we conclude $AB=AB$.

Theorem 5. T, C $\Rightarrow$ R.

This is an easy consequence of Theorem 3: T, C $\Rightarrow$ A and Theorem 4: T, A $\Rightarrow$ R. In the following an alternative proof will be given without an intermediation of Axiom A.

Lemma 2. *Under the assumption of Axiom* C

$$\left.\begin{array}{ll} AB = A'B'\,, & (1) \\ A < X < B\,, & \\ XB = A'X' & (2) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} A' < X' < B'\,, \\ AX = X'B'\,. \end{array}\right.$$



Proof.

$$\left.\begin{array}{ll} AB \equiv AX + XB, & \\ A'B'' \equiv A'X' + X'B''\,, & \\ \text{by Axiom E, } \exists_1 B''\colon AX = X'B'' & (3)\,, \\ \text{by (2)}\colon\ XB = A'X' & \end{array}\right\} \xrightarrow{(\mathrm{C})} AB = A'B''\,. \tag{4}$$

Then we have

$$(1), (4) \xrightarrow{(\mathrm{E})} B' = B''\,.$$

Thus $AX=X'B'$ from (3) and clearly $A'<X'<B''=B'$.

Proof of Theorem 5.

$$\text{By Axiom E,} \quad \exists_1 B'\colon\ AB=AB'. \tag{1}$$

(i) Suppose first $A<B'<B$.
By Lemma 2 there is an X such that

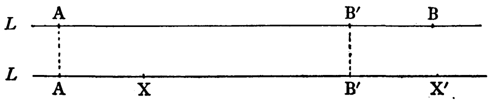
$$A<X<B', \tag{2}$$

$$B'B=AX, \tag{3}$$

$$AB'=XB'. \tag{4}$$

Now by Axiom E, $\exists_1 X'\colon\ AX=B'X'$ (5)
and by Lemma 1

$$XB'=XB' \tag{6}$$



Thus

$$\left.\begin{array}{l} AB'\equiv AX+XB', \\ XX'\equiv XB'+B'X', \\ (5)\colon\ AX=B'X', \\ (6)\colon\ XB'=XB' \end{array}\right\} \xrightarrow{(C)} AB'=XX'. \tag{7}$$

Then

$$(4), (7) \xrightarrow{(E)} X'=B',$$

which is a contradiction.

(ii) Next suppose $A<B<B'$.

$$\text{By Axiom E,} \quad \exists_1 X\colon\ BB'=AX. \tag{8}$$

$$\text{By Axiom E,} \quad \exists_1 B''\colon\ AB'=XB''. \tag{9}$$

Then

$$(1), (9) \xrightarrow{(T)} AB=XB''. \tag{10}$$

Hence

$$\left.\begin{array}{l} \left.\begin{array}{l} AB'\equiv AB+BB', \\ AB''\equiv AX+XB'', \\ (10)\colon\ AB=XB'', \\ (8)\colon\ BB'=AX \end{array}\right\} \xrightarrow{(C)} AB'=AB''. \\ \text{By Lemma 1} \quad AB'=AB' \end{array}\right\} \xrightarrow{(E)} B''=B'.$$

Consequently, we have from (9)

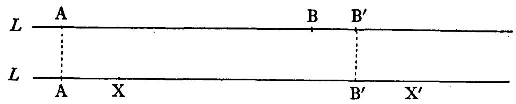
$$AB'=XB' \tag{11}$$

Now

$$\left.\begin{array}{l} AB' \equiv AX + XB' , \\ XX' \equiv XB' + B'X' , \\ \text{by Axiom E,} \quad \exists_1 X' \colon AX = B'X' , \\ \text{by Lemm 1,} \quad XB' = XB' \end{array}\right\} \overset{(C)}{\Longrightarrow} AB' = XX' \tag{12}$$

Then we have

$$(11), (12) \overset{(E)}{\Longrightarrow} B' = X' ,$$



which is a contradiction.

From (i) and (ii) we conclude $AB = AB$.

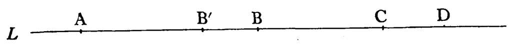
Theorem 6. T, I ⇒ R.

Proof.

$$\text{By Axiom E,} \quad \exists_1 B' \colon AB = AB' . \tag{1}$$

(i) Suppose first $A < B' < B$.

$$\text{By Axiom E,} \quad \exists_1 C \colon AB' = BC . \tag{2}$$



Then

$$(1), (2) \overset{(T)}{\Longrightarrow} AB = BC . \tag{3}$$

and

$$A < B' < B < C, \ (2) \overset{(I)}{\Longrightarrow} AB = B'C . \tag{4}$$

$$\text{By Axiom E,} \quad \exists_1 D \colon B'B = CD . \tag{5}$$

Then

$$B' < B < C < D, \ (5) \overset{(I)}{\Longrightarrow} B'C = BD , \tag{6}$$

and

$$(4), (6) \overset{(T)}{\Longrightarrow} AB = BD . \tag{7}$$

Hence

$$(3), (7) \overset{(E)}{\Longrightarrow} C = D .$$

which is a contradiction.

(ii) Next suppose $A < B < B'$.

$$\text{By Axiom E,} \quad \exists_1 C \colon AB = B'C . \tag{8}$$

$$A < B < B' < C, \ (8) \overset{(I)}{\Longrightarrow} AB' = BC . \tag{9}$$

$$(1), (9) \overset{(T)}{\Longrightarrow} AB = BC . \tag{10}$$

$$\text{By Axiom E,} \quad \exists_1 D\colon\ BB'=CD\,. \tag{11}$$

$$B<B'<C<D,\ (11) \overset{(\mathrm{I})}{\Longrightarrow} BC=B'D\,. \tag{12}$$

$$(10),\ (12) \overset{(\mathrm{T})}{\Longrightarrow} AB=B'D\,. \tag{13}$$

Hence

$$(8),\ (13) \overset{(\mathrm{E})}{\Longrightarrow} C=D\,,$$

which is a contradiction.

From (i) and (ii) we conclude $AB=AB$.

Lemma 3. $A<A'<B,\ AB=A'B' \overset{(\mathrm{I})}{\Longrightarrow} B<B',\ AA'=BB'$.

Proof.

$$\text{By Axiom E,} \quad \exists_1 B''\colon\ AA'=BB''\,. \tag{1}$$

$$A<A'<B<B'',\ (1) \overset{(\mathrm{I})}{\Longrightarrow} AB=A'B''\,. \tag{2}$$

Let $AB=A'B'$. Then we have

$$AB=A'B',\ (2) \overset{(\mathrm{E})}{\Longrightarrow} B'=B''\,.$$

Therefore we have from (1) $B<B'$ and $AA'=BB'$.

Theorem 7. T, I $\Rightarrow$ A.

Proof. (i) First let $A<B<C<A'<B'<C'$ and $AB=A'B'$, $BC=B'C'$.

$$\left.\begin{array}{l} A<B<A'<B',\ AB=A'B' \overset{(\mathrm{I})}{\Longrightarrow} AA'=BB', \\ B<C<B'<C',\ BC=B'C' \overset{(\mathrm{I})}{\Longrightarrow} BB'=CC' \end{array}\right\} \overset{(\mathrm{T})}{\Longrightarrow} \left.\begin{array}{l} AA'=CC', \\ A<C<A'<C' \end{array}\right\} \overset{(\text{Lem.3})}{\Longrightarrow} AC=A'C'.$$

(ii) Next let $A<B<C$, $A'<B'<C'$ with $AB=A'B'$, $BC=B'C'$, but let $C<A'$ fail to be true.

Take points A'', B'', C'' such that $A<B<C<A''<B''<C''$ and $A'<B'<C'<A''<B''<C''$ with $A'B'=A''B''$, $B'C'=B''C''$.

Then by (i)

$$AC=A''C'' \tag{1}$$

and

$$A'C'=A''C''\,. \tag{2}$$

Now, since T, I $\Rightarrow$ R by Theorem 6 and T, R $\Rightarrow$ S by Theorem 1, Axiom S holds by our assumption of T and I.

Therefore

$$A'C'=A''C'' \overset{(\mathrm{S})}{\Longrightarrow} A''C''=A'C'\,. \tag{3}$$

Hence

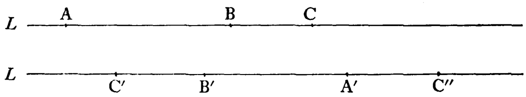
$$(1), (3) \overset{(T)}{\Longrightarrow} AC = A'C' .$$

Theorem 8. T, I $\Rightarrow$ C.

Proof. Notice that Axiom S is a consequence of our assumption of T and I as we have shown in the proof of Theorem 7 and that Axiom A is a consequence of T and I by Theorem 7.

Let $A<B<C$, $C'<B'<A'$ and let

$$AB = B'A' , \qquad (1)$$
$$BC = C'B' . \qquad (2)$$



$$\left.\begin{array}{l} AC \equiv AB + BC , \\ B'C'' \equiv B'A' + A'C'' , \\ (1):\ AB = B'A' , \\ \left.\begin{array}{l} (2):\ BC = C'B', \\ \text{by Axiom E,} \\ \quad \exists_1 C'':\ C'B' = A'C'' \end{array}\right\} \overset{(T)}{\Longrightarrow} BC = A'C'' \end{array}\right\} \overset{(A)}{\Longrightarrow} AC = B'C'' ,$$

$$C'<B'<A'<C'',\ C'B' = A'C'' \overset{(I)}{\Longrightarrow} C'A' = B'C'' \overset{(S)}{\Longrightarrow} B'C'' = C'A'$$

$$\overset{(T)}{\Longrightarrow} AC = C'A' .$$

Theorem 9. S, A $\Rightarrow$ R.

Proof.

$$\text{By Axiom E,} \quad \exists_1 B':\ AB = AB' . \qquad (1)$$

(i) Let $A<B<B'$.

$$\text{By Axiom E,} \quad \exists_1 X:\ BB' = B'X \qquad (2)$$

$$\left.\begin{array}{l} AB' \equiv AB + BB' , \\ AX \equiv AB' + B'X , \\ (1):\ AB = AB' , \\ (2):\ BB' = B'X \end{array}\right\} \overset{(A)}{\Longrightarrow} AB' = AX . \qquad (3)$$

$$(1) \overset{(S)}{\Longrightarrow} AB' = AB . \qquad (4)$$

$$(3), (4) \overset{(E)}{\Longrightarrow} X = B ,$$

which is a contradiction.

(ii) Let $A<B'<B$.

By Axiom S $AB' = AB$, $A<B'<B$ and the case (ii) reduces to that of (i). From (i) and (ii) we conclude $AB = AB$.

Lemma 4. *Under the assumption of Axioms* S, A *and* I, *if* $AB=A'B'$, *then*

1) $A<A' \Rightarrow B<B',\ AA'=BB'$.
2) $A'<A \Rightarrow B'<B,\ A'A=B'B$.
3) $A=A' \Rightarrow B=B'$.

Proof. 1) follows from Lemma 3.
2) reduces to 1) by Axiom S.
3) follows from Theorem 9 which asserts S, A $\Rightarrow$ R.

Lemma 5. $PQ=P'Q'$ (1), $P<X<Q$, $PX=P'X'$ (2) $\overset{(A)}{\Longrightarrow} P'<X'<Q'$, $XQ=X'Q'$.

Proof.

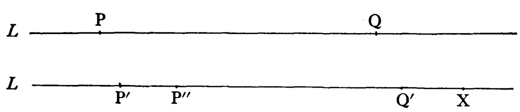
$$\left.\begin{array}{l} PQ\equiv PX+XQ\,, \\ P'Q''\equiv P'X'+X'Q''\,, \\ (2)\!:\ PX=P'X'\,, \\ \text{by Axiom E, } \exists_1 Q''\!:\ XQ=X'Q'' \end{array}\right\} \begin{array}{l} \overset{(A)}{\Longrightarrow} PQ=P'Q''\,, \\ (1)\!:\ PQ=P'Q' \end{array}\left.\vphantom{\begin{array}{l}a\\a\end{array}}\right\} \overset{(E)}{\Longrightarrow} Q''=Q'\,.$$

Lemma 6. $PQ=P'Q',\ PQ=P''Q',\ P<P',\ P<P'' \overset{(A,I)}{\Longrightarrow} P'=P''$.

Proof. We may assume without loss of generality that $P'<P''$.

$$PQ=P'Q' \overset{(\text{I or Lem. 3})}{\Longrightarrow} PP'=QQ'\,. \qquad (1)$$

$$PQ=P''Q' \overset{(\text{I or Lem. 3})}{\Longrightarrow} PP''=QQ'\,. \qquad (2)$$

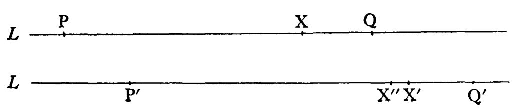


$$\left.\begin{array}{l} PP''\equiv PP'+P'P''\,, \\ QX\equiv QQ'+Q'X\,, \\ (1)\!:\ PP'=QQ'\,, \\ \text{by Axiom E, } \exists_1 X\!:\ P'P''=Q'X \end{array}\right\} \begin{array}{l} \overset{(A)}{\Longrightarrow} PP''=QX\,, \\ (2)\!:\ PP''=QQ' \end{array}\left.\vphantom{\begin{array}{l}a\\a\end{array}}\right\} \overset{(E)}{\Longrightarrow} X=Q'\,,$$

which is a contradiction.

Lemma 7. $\left.\begin{array}{l} PQ=P'Q',\ P<P', \\ P<X<Q,\ XQ=X'Q' \end{array}\right\} \overset{(S,A,I)}{\Longrightarrow} P'<X'<Q',\ PX=P'X'$.

Proof. By Lemma 3 we have first



$$PQ=P'Q',\ P<P' \overset{(I)}{\Longrightarrow} Q<Q'\,.$$

From

$$XQ=X'Q' \tag{1}$$

we have $X'Q'=XQ$ by Axiom S, and combined with $Q<Q'$ we obtain by Lemma 4

$$X<X'. \tag{2}$$

Now,

$$\begin{array}{lll} \text{by Axiom E,} & \exists_1 X'': PX=P'X'', & (3) \\ \text{by Lemma 5} & P'<X''<Q' \quad \text{and} \quad XQ=X''Q', & (4) \\ \text{by Lemma 4} & X<X''. & (5) \end{array}$$

Then (1), (4), (2) and (5) yield by Lemma 6 $X'=X''$. Consequently we have $P'<X'<Q'$ and $PX=P'X'$.

Theorem 10. S, A, I $\Rightarrow$ T.

Proof. Let $AB=A'B'$, $A'B'=A''B''$.

(i) The case where at least two of A, A' and A'' coincide:

(i)$_1$ $A=A'$. Since S, A $\Rightarrow$ R by Theorem 9 we have $B=B'$ and hence $AB=A''B''$.

(i)$_2$ $A'=A''$. The same as (i)$_1$.

(i)$_3$ $A=A''$. $$\left.\begin{array}{l} AB=A'B' \overset{(S)}{\Longrightarrow} A'B'=AB, \\ A'B'=A''B'',\ A=A'' \Longrightarrow A'B'=AB'' \end{array}\right\} \overset{(E)}{\Longrightarrow} B=B''.$$

Hence

$$AB=A''B''.$$

(ii) The case where A, A' and A'' are distinct: there are six cases to be considered.

$$\begin{array}{lll} \text{I. } A<A'<A'', & \text{II. } A<A''<A', & \text{III. } A'<A<A'', \\ \text{I}'.\ A''<A'<A, & \text{II}'.\ A''<A<A', & \text{III}'.\ A'<A''<A. \end{array}$$

Proof of Case I.

$$\left.\begin{array}{l} AB=A'B' \overset{(\text{I or Lem. 3})}{\Longrightarrow} AA'=BB', \\ A'B'=A''B'' \overset{(\text{I or Lem. 3})}{\Longrightarrow} A'A''=B'B'', \\ A<A'<A'' \overset{(\text{Lem. 4})}{\Longrightarrow} B<B'<B'' \end{array}\right\} \overset{(A)}{\Longrightarrow} AA''=BB'' \overset{(\text{I or Lem. 3})}{\Longrightarrow} AB=A''B''.$$

Proof of Case II.

$$AB=A'B' \overset{\text{(I or Lem. 3)}}{\Longrightarrow} AA'=BB' . \tag{1}$$

$$A'B'=A''B'' \overset{(S)}{\Longrightarrow} A''B''=A'B' \overset{\text{(I or Lem. 3)}}{\Longrightarrow} A''A'=B''B' . \tag{2}$$

$$\left.\begin{array}{l}(1),\\(2),\\A<A''<A'\end{array}\right\} \overset{\text{(Lem. 7)}}{\Longrightarrow} \left\{\begin{array}{l}B<B''<B' ,\\AA''=BB''\end{array}\right\} \overset{\text{(I or Lem. 3)}}{\Longrightarrow} AB=A''B'' .$$

Proof of Case III.

$$A'B'=A''B'' \overset{\text{(I or Lem. 3)}}{\Longrightarrow} A'A''=B'B'' . \tag{1$'$}$$

$$AB=A'B' \overset{(S)}{\Longrightarrow} A'B'=AB \overset{\text{(I or Lem. 3)}}{\Longrightarrow} A'A=B'B . \tag{2$'$}$$

$$\left.\begin{array}{l}(1)',\\(2)',\\A'<A<A''\end{array}\right\} \overset{\text{(Lem. 5)}}{\Longrightarrow} \left\{\begin{array}{l}B'<B<B'' ,\\AA''=BB''\end{array}\right\} \overset{\text{(I or Lem. 3)}}{\Longrightarrow} AB=A''B'' .$$

Proof of Case I$'$.

$$AB=A'B' \overset{(S)}{\Longrightarrow} A'B'=AB . \tag{1$''$}$$

$$A'B'=A''B'' \overset{(S)}{\Longrightarrow} A''B''=A'B' . \tag{2$''$}$$

$$A''<A'<A,\ (2)'',\ (1)'' \overset{\text{(Case I)}}{\Longrightarrow} A''B''=AB \overset{(S)}{\Longrightarrow} AB=A''B'' .$$

Similarly the proofs of II$'$ and III$'$ may be reduced to those of II and III respectively.

Theorem 11. S, A, I $\Rightarrow$ C.

Proof. S, A, I $\Rightarrow$ T by Theorem 10. Then by Theorem 8 T, I $\Rightarrow$ C.

Lemma 8. *Under the assumption of Axioms* R *and* C, *if* $AB=A'B'$, *then*

1) $A<A' \Rightarrow B<B'$.
2) $A=A' \Rightarrow B=B'$.
3) $A'<A \Rightarrow B'<B$.

Proof. 1) $A<A'$. $B<B'$ is clear if $B<A'$ or if $B=A'$. Let $A<A'<B$, and suppose either $B'<B$ or $B'=B$.

$$\text{By Axiom E, } \exists_1 X\colon AA'=BX . \tag{1}$$

$$\left.\begin{array}{r}\left.\begin{array}{l}A<A'<B<X ,\\(1)\colon AA'=BX\end{array}\right\} \overset{(I)}{\Longrightarrow} AB=A'X .\\ \text{By assumption } AB=A'B'\end{array}\right\} \overset{(E)}{\Longrightarrow} X=B' ,$$

which is a contradiction.

2) Clear.

3) $A'<A$. Suppose either $B<B'$ or $B=B'$.

$$\left.\begin{array}{l}\left.\begin{array}{l}A'B\equiv A'A+AB\,,\\ A'X\equiv A'B'+B'X\,.\\ \text{By Axiom E, } \exists_1 X:\ \ A'A=B'X\,.\\ \qquad\qquad\qquad\quad AB=A'B'\,.\end{array}\right\}\overset{(C)}{\Longrightarrow} A'B=A'X\,.\\ \qquad\qquad\qquad\text{By Axiom R}\quad A'B=A'B\,.\end{array}\right\}\overset{(E)}{\Longrightarrow} X=B\,.$$

which is a contradiction.

Theorem 12. R, A, C $\Rightarrow$ S.

Proof. Let

$$AB=A'B'\,. \tag{1}$$

Case I. $A'<A$.

$$\text{By Axiom E, } \exists_1 X:\quad A'A=B'X\,. \tag{2}$$

$$A'A=B'X \overset{(\text{I or Lem. 3})}{\Longrightarrow} A'B'=AX\,. \tag{3}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}A'B\equiv A'A+AB\,,\\ A'X\equiv A'B'+B'X\,,\\ (2):\ A'A=B'X,\quad (1):\ AB=A'B'\end{array}\right\}\overset{(C)}{\Longrightarrow} A'B=A'X\,.\\ \qquad\qquad\qquad\text{By Axiom R}\quad A'B=A'B\,.\end{array}\right\}\overset{(E)}{\Longrightarrow} X=B\,.$$

Hence from (3) $A'B'=AB$.

Case II. $A<A'$.

$$\text{By Axiom E, } \exists_1 B'':\quad A'B'=AB''\,. \tag{4}$$

Then we have from (4) by Case I

$$AB''=A'B'\,. \tag{5}$$

(i) Suppose first $A<B''<B$.

$$\text{By Axiom E, } \exists_1 X:\quad B''B=B'X\,. \tag{6}$$

From (5) and (6) we have by Axiom A $AB=A'X$. This, combined with (1), would yield by Axiom E $X=B'$, which is a contradiction.

(ii) Next suppose $B<B''$.

$$\text{By Axiom E, } \exists_1 X:\quad BB''=B'X\,.$$

On account of (1) we have then by Axiom A $AB''=A'X$, which, combined with (5), would yield by Axiom E $X=B'$, again a contradiction.

Corollary. $AB=A'B',\ A'<A \overset{(R,\,C)}{\Longrightarrow} A'B'=AB.$

Theorem 13. S, C, I $\Rightarrow$ R.

Proof.

$$\text{By Axiom E, } \exists_1 B'\text{:} \quad AB=AB'\,.$$

(i) First suppose $A<B'<B$.

By Lemma 2 there is an X such that

$$A<X<B'\,, \quad B'B=AX\,,$$
$$AB'=XB'\,. \tag{1}$$
$$\text{By Axiom E, } \exists_1 X'\text{:} \quad AX=B'X'\,.$$

Since $A<X<B'<X'$ we have by Axiom I

$$AB'=XX'\,. \tag{2}$$

From (1) and (2) we would have by Axiom E $B'=X$, which is a contradiction.

(ii) Next suppose $B<B'$.

Since we have from (1) by Axiom S $AB'=AB$, the argument of (i) gives again a contradiction.

Thus we conclude from (i) and (ii) $B'=B$ and then $AB=AB$ follows from (1).

3. Models

By a *model* of a geometry denoted for example by M(S, C) we mean a linearly ordered space L with congruent relations which satisfy among our group of seven Axioms E, R, S, T, A, C and I Axioms S and C alone besides Axiom E but not the remaining ones.

In the following models the space L is for the most part given by the real line $-\infty<x<\infty$ or by the half line $0\leqq x<\infty$. In these cases points denoted by A, B, A', X etc. will be those points of the real line having coordinates a, b, a', x etc. respectively. $A<B$ is defined by $a<b$, $|AB|$ denotes the distance $b-a$ of points A and B.

M(R): *A model of a geometry in which Axiom* R *alone holds besides Axiom* E.

Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:

If $A=A'$, then let $AB=A'B'$ if and only if $B=B'$.

If $A\neq A'$, then let $AB=A'B'$ if and only if $|A'B'|=1$.

This model satisfies Axioms E and R but fails to satisfy the remaining Axioms S, T, A, C, I.

M(S): *A model of a geometry in which Axiom* S *alone holds besides Axiom* E.
Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:
In case $A=A'$, let $AB=A'B'$
(i) if $|AB|=1$ and $|A'B'|=3$
or (ii) if $|AB|=3$ and $|A'B'|=1$
or (iii) if $|AB|$ and $|A'B'|$ are both different from 1 and 3, and $|AB| =|A'B'|$.

In case $A<A'$, let $AB=A'B'$ and $A'B'=AB$ if $2|AB|=|A'B'|$. This model satisfies Axioms E and S but fails to satisfy the remaining Axioms R, T, A, C, I.

M(T): *A model of a geometry in which Axiom* T *alone holds besides Axiom* E.
Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$: For any AB and for any A', let $AB=A'B'$ if and only if $|A'B'|=1$.
This model satisfies Axioms E and T but fails to satisfy the remaining Axioms R, S, A, C, I.

M(A): *A model of a geometry in which Axiom* A *alone holds besides Axiom* E.
Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:
(i) In case $A<A'$ or $A=A'$, then let $AB=A'B'$ if and only if $2|AB| =|A'B'|$.
(ii) In case $A'<A$, then let $AB=A'B'$ if and only if $|AB|=|A'B'|$.
This model satisfies Axioms E and A but fails to satisfy the remaining Axioms R, S, T, C, I.

M(I): *A model of a geometry in which Axiom* I *alone holds besides Axiom* E.
Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:
In case $A=A'$, let $AB=A'B'$ if and only if $2|AB|=|A'B'|$.
In case $A\neq A'$, let $AB=A'B'$ if and only if $|AB|=|A'B'|$.
This model satisfies Axioms E and I but fails to satisfy the remaining Axioms R, S, T, A, C.

M(A, C): *A model of a geometry in which Axioms* A *and* C *alone hold besides Axiom* E.
Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$: Let $AB=A'B'$ if and only if $2|AB|=|A'B'|$.
This model satisfies Axioms E, A and C but fails to satisfy the remaining Axioms R, S, T, I.

M(S, I): *A model of a geometry in which Axioms* S *and* I *alone hold besides Axiom* E.

Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:

In case $A=A'$, let $AB=A'B'$ if $|AB|=1$ and $|A'B'|=2$ or if $|AB|=2$ and $|A'B'|=1$ or if $|AB|$ and $|A'B'|$ are both different from 1 and 2, and $|AB|=|A'B'|$.

In case $A\neq A'$, let $AB=A'B'$ if $|AB|=|A'B'|$.

This model satisfies Axioms E, S and I but fails to satisfy the remaining Axioms R, T, A, C.

M(R, S, A): *A model of a geometry in which Axioms* R, S *and* A *alone hold besides Axiom* E.

Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:

In case $A=A'$, let $AB=A'B'$ if $B'=B$.

In case $A<A'$, let $AB=A'B'$ and $A'B'=AB$ if $2|AB|=|A'B'|$.

This model satisfies Axioms E, R, S and A but fails to satisfy the remaining Axioms T, C, I.

M(R, A, I): *A model of a geometry in which Axioms* R, A *and* I *alone hold besides Axiom* E.

Let L be the real line $-\infty<x<\infty$.

DEFINITION OF $AB=A'B'$:

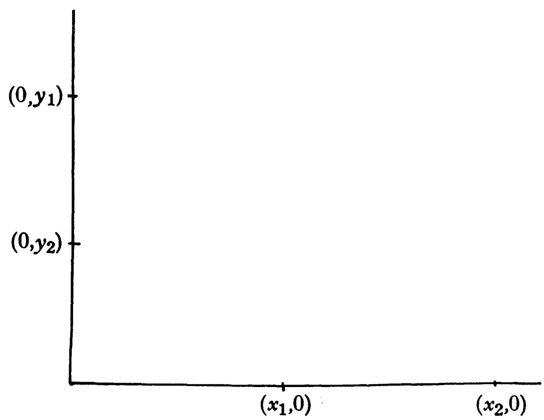
In case $A=A'$ or $A<A'$, let $AB=A'B'$ if $|AB|=|A'B'|$.

In case $A'<A$, let $AB=A'B'$ if $2|A'B'|=|AB|$.

This model satisfies Axioms E, R, A and I but fails to satisfy the remaining Axioms S, T, C.

M(R, S, T): *A model of a geometry in which Axioms* R, S *and* T *alone hold besides Axiom* E.

Let a point of the space L be defined as an ordered pair (x,y) of real numbers x and y such that either $x\geqq 0$ and $y=0$ or $x=0$ and $y\geqq 0$.



Definition of the linear order:

If $A=(x,y)$, $A'=(x',y')$, then let $A<A'$ if $x<x'$ or if $y>y'$.

DEFINITION OF $AB=A'B'$:

If $A=(x_1, y_1)$, $B=(x_2, y_2)$, $A'=(x_1', y_1')$, $B'=(x_2', y_2')$,
then let $AB=A'B'$ if $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}$.
This model satisfies Axioms E, R, S and T but fails to satisfy the remaining Axioms A, C, I.

M(A, C, I): *A model of a geometry in which Axioms* A, C, *and* I *alone hold besides Axiom* E.

Let L be the half real line $0\leqq x<\infty$, and let O denote the point with coordinate 0.

Definition of $AB=A'B'$:

In case $A=O$, let $AB=A'B'$ if $|AB|+1=|A'B'|$.

In case $O<A$, let $AB=A'B'$ if $|AB|=|A'B'|$.

This model satisfies Axioms E, A, C and I but fails to satisfy the remaining Axioms R, S, T.

M(R, S, I): *A model of a geometry in which Axioms* R, S *and* I *alone hold besides Axiom* E.

Let L be the half real line $0\leqq x<\infty$ with the origin O.

Definition of $AB=A'B'$: Let $f(x)=x^3$.

In case $A=O$ or $A'=O$, let $AB=A'B'$ if $f(b)-f(a)=f(b')-f(a')$.

In case $A\neq O$ and $A'\neq O$, let $AB=A'B'$ if $|AB|=|A'B'|$.

This model satisfies Axioms E, R, S and I but fails to satisfy the remaining Axioms T, A, C.

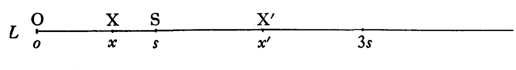
M(R, S, C, I): *A model of a geometry in which Axioms* R, S, C *and* I *alone hold besides Axiom* E.

Let L be the half real line $0\leqq x<\infty$.

For any $s>0$ make correspond to each x with $0\leqq x\leqq s$ an x' with $s\leqq x'\leqq 3s$ and vice versa, by the relation

$$\frac{2x+x'}{3}=s .$$

Call this correspondence σ a *skew symmetrization* with *centre s*.



It should be observed that for any pair of non negative numbers a and b there is one and only one skew symmetrization σ that interchanges a and b: $\sigma(a)=b$, $\sigma(b)=a$; indeed, if $a<b$, then we are only to set

$$\frac{2a+b}{3}=s .$$

DEFINITION OF $AB=A'B'$:

Let $AB=A'B'$, if there is a skew symmetrization σ such that $\sigma(a)=b'$, $\sigma(b)=a'$, where a, b, a' and b' are coordinates of A, B, A' and B' respectively.

Clearly Axiom E holds by the above observation. Likewise for Axioms R, S.

As for Axiom C, let $AB=B'A'$, $BC=C'B'$. Then there must be one and only one skew symmetrization σ with centre s that carries A to A', B to B' and C to C', hence $AC=C'A'$.

Axiom I follows then from Theorem 2.

$\neg$T: To show that Axiom T does not hold, let O, A_1, A_3, A_5, and A_7 be points with coordinates 0, 1, 3, 5 and 7 respectively. Then $OA_1=A_1A_3$, $A_1A_3=A_3A_7$ but $OA_1=A_3A_5$. Therefore $OA_1=A_3A_7$ fails to hold, as will be seen by a simple calculation.

$\neg$A: Axiom A does not hold, for otherwise T would follow by Theorem 10 which asserts S, A, I $\Rightarrow$ T.

REMARK. Instead of $0\leqq x<\infty$ in our M(R, S, C, I) we may take as L the real line $-\infty<x<\infty$.

In this case the skew symmetrization σ should be modified as follows, according as the centre s lies <0, $=0$ or >0, the range of symmetrization spreading along the whole line:

Case I: $s>0$.

(i) Points x with $0\leqq x\leqq s$ and x' with $s\leqq x'\leqq 3s$ interchange by the relation

$$\frac{2x+x'}{3}=s\,.$$

(ii) Points x with $x\leqq 0$ and x' with $x'\geqq 3s$ interchange by the relation $x+x'=3s$.

Case II: $s<0$.

(i) Points x with $s\leqq x\leqq 0$ and x' with $3s\leqq x'\leqq s$ interchange by the same relation

$$\frac{2x+x'}{3}=s$$

as above.

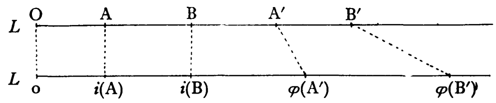
(ii) Points x with $x\geqq 0$ and x' with $x'\leqq 3s$ interchange by the same relation $x+x'=3s$ as above.

Case III: $s=0$. For any real numbers, points x and x' interchange by the relation $x'+x=0$.

M(C): *A model of a geometry in which Axiom* C *alone holds besides Axiom* E.

Let L and $\bar{L}$ be the half real lines $0\leqq x<\infty$ and let φ be a mapping of points X of L with coordinates x onto points $\bar{X}$ of $\bar{L}$ with coordinates $\bar{x}$ such that $\bar{x}=3x$ and let i be an identical mapping $\bar{x}=x$.

DEFINITION of $AB=A'B'$: Given AB and $A'B'$ on L, let $AB=A'B'$ if and only if $i(A)i(B)=\varphi(A')\varphi(B')$ on $\bar{L}$ in the sense of the Model M (R, S, C, I).



Verification that this gives an M(C) is easy.

M(R, S, T, A): *A model of a geometry in which Axioms* R, S, T *and* A *alone hold besides Axiom* E.

Let L be the real line $-\infty<x<\infty$.

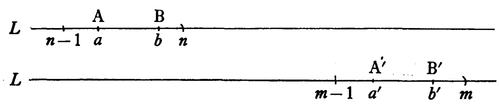
DEFINITION OF $AB=A'B'$:

For any integer n consider for a pair of real numbers x and y in $[n-1, n)$ with $x<y$ a function $d(x, y)$ defined by

$$d(x, y)=e^{1/(n-y)}-e^{1/(n-x)} .$$

In the following a, b, a', b' etc. denote the coordinates of points A, B, A', B' respectively as usual.

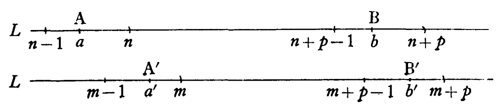
I. In case a, $b\in[n-1, n)$ and a', $b'\in[m-1, m)$, provided m, n denote arbitrary integers, let $AB=A'B'$ if $d(a, b)=d(a', b')$.



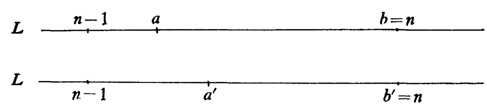
II. In case

$$a\in[n-1, n)\,, \qquad b\in[n+p-1, n+p)\,,$$
$$a'\in[m-1, m)\,, \quad b'\in[m+p-1, m+p)$$

for any natural number p, let $AB=A'B'$ if $d(n+p-1, b)=d(m+p-1, b')$.



Especially then, $AB=A'B'$ if $a\in[n-1, n)$, $b=n$ and $a'\in[n-1, n)$, $b'=n$ for any choice of a and a'.



E, R, S: Clearly Axioms E, R and S hold.

T: To see that Axiom T holds, let A, B, A', B', A'' and B'' be points with coordinates, a, b, a', b', a'' and b'' respectively such that $AB=A'B'$, $A'B'=A''B''$.

If $a, b\in[n-1, n)$ for some integer n, then by the definition of equality$=$, $a', b'\in[n'-1, n')$ and $a'', b''\in[n''-1, n'')$ for some integers n' and n''. Then we have $d(a, b)=d(a', b')$ and $d(a', b')=d(a'', b'')$, hence $d(a, b)=d(a'', b'')$, therefore $AB=A''B''$.

If $a\in[n-1, n)$, $b\in[m-1, m)$ for some integers n and m with $n<m$, then as before $a'\in[n'-1, n')$, $b'\in[m'-1, m')$, $a''\in[n''-1, n'')$, $b''\in[m''-1, m'')$. Then we have by the definition of $AB=A'B'$ and $A'B'=A''B''$, $d(m-1, b)=d(m'-1, b')$, $d(m'-1, b')=d(m''-1, b'')$, hence $d(m-1, b)=d(m''-1, b'')$, therefore $AB=A''B''$.

A: Similarly for Axiom A.

$\neg$C, $\neg$I: To see that Axioms C and I do not hold, let A, B, A' and B' be points with coordinates a, b, a' and b' respectively such that

$$a\in[n-1, n),\ b=n\,,\quad a'\in(n, n+1),\ b'=n+1\,.$$

Then by definition $AB=A'B'$ but not $AA'=BB'$, thus Axiom I does not hold. Axiom C fails to hold too.

Notice that this model M(R, S, T, A) is non-Archimedean.

M(S, C): *A model of a geometry in which Axioms* S *and* C *alone hold besides Axiom* E.

Let L be a linearly ordered space with points A_n^i, i, n ranging over all integers $0, \pm1, \pm2, \cdots$, with the order relation

(i) $A_m^i<A_n^i$, if $m<n$,

(ii) $A_m^i<A_n^j$, if $i<j$ (for any integers m, n.)

DEFINITION OF $AB=A'B'$: let $A_m^iA_n^j=A_{m'}^{i'}A_{n'}^{j'}$, if

(i) $j-i=j'-i'=0$ and $n-m=n'-m'>0$,

or (ii) $j-i=j'-i'$ is an even number>0 and $m-n=m'-n'$,

or (iii) $j-i=j'-i'$ is an odd number>0 and $m+n+m'+n'=-1$.

E, S: Axioms E and S evidently hold.

C: To see that Axiom C holds, let

$$A_m^iA_n^j\equiv A_m^iA_q^p+A_q^pA_n^j\,, \tag{1}$$
$$A_{m'}^{i'}A_{n'}^{j'}\equiv A_{m'}^{i'}A_{q'}^{p'}+A_{q'}^{p'}A_{n'}^{j'}\,, \tag{2}$$

and

$$A_m^iA_q^p=A_{q'}^{p'}A_{n'}^{j'}\,, \tag{3}$$
$$A_q^pA_n^j=A_{m'}^{i'}A_{q'}^{p'}\,. \tag{4}$$

Then by the definition (i), (ii), (iii) of $=$, we have first of all from (3) and (4)

$$p-i=j'-p', \tag{5}$$

$$j-p=p'-i', \tag{6}$$

whence

$$j-i=j'-i' \tag{7}$$

follows. Next we have to consider three cases:

(i) The case: $j-i=j'-i'=0$. We have from (1) and (2):

$$p=i=j \quad \text{and} \quad p'=i'=j'.$$

From (3) and (4) we have then

$$q-m=n'-q', \quad n-q=q'-m',$$

whence

$$m-n=m'-n',$$

which is evidently different from 0 because $A^i_m < A^i_n$.

Thus in this case we have

$$A^i_m A^j_n = A^{i'}_{m'} A^{j'}_{n'} \tag{*}$$

(ii) The case: $j-i=j'-i'$ is an even number >0.

Subcase 1): If $p-i$ is even, so is $j-p=(j-i)-(p-i)$ and we have from (3) and (4) by the definition of $=$,

$$m-q=q'-n',$$

$$q-n=m'-q',$$

whence

$$m-n=m'-n',$$

and (*) is proved.

Subcase 2): If $p-i$ is odd, so is $j-p=(j-i)-(p-i)$ and from (3) and (4) we obtain

$$m+q+q'+n'=-1,$$

$$q+n+m'+q'=-1,$$

whence

$$m-n=m'-n',$$

and (*) is again proved.

(iii) The case: $j-i=j'-i'$ is an odd number >0

Subcase 1): If $p-i$ is even, then $j-p=(j-i)-(p-i)$ is odd and we have from (3) and (4)

$$m-q=q'-n',$$

$$q+n+m'+q'=-1,$$

whence

$$m+n+m'+n' = -1\,,$$

and again (*) holds.

Subcase 2): If $p-i$ is odd, then $j-p$ is even and similarly as above we have (*).

The following examples show that Axioms R, T, A and I do not hold true.

$\neg$R: $A_0^1A_0^2=A_0^1A_{-1}^2$ but not $A_0^1A_0^2=A_0^1A_0^2$, so Axiom R fails to hold.

$\neg$T: $A_0^1A_0^2=A_0^1A_{-1}^2$, $A_0^1A_{-1}^2=A_1^1A_{-1}^2$ and $A_0^1A_0^2=A_1^1A_{-2}^2$ but not $A_0^1A_0^2=A_1^1A_{-1}^2$, so Axiom T fails to hold.

$\neg$A: $A_0^1A_{-1}^2=A_0^1A_0^2$, $A_{-1}^2A_0^2=A_0^2A_1^2$ and $A_0^1A_0^2=A_0^1A_{-1}^2$ but not $A_0^1A_0^2=A_0^1A_1^2$, so Axiom A fails to hold.

$\neg$I: $A_0^1A_1^1=A_{-1}^2A_0^2\,(A_0^1<A_1^1<A_{-1}^2<A_0^2)$ and $A_0^1A_{-1}^2=A_1^1A_{-1}^2$ but not $A_0^1A_{-1}^2=A_1^1A_0^2$, so Axiom I fails to hold.

A model M(R, C, I) will be given in the second part of this paper.

4. Proof of Main Theorem

I. T *and* C *are independent, and* T, C $\Rightarrow$ R, S, A, I.

Proof. (i) T, C $\Rightarrow$ A by Theorem 3.
(ii) T, A $\Rightarrow$ R by Theorem 4.
(iii) T, R $\Rightarrow$ S by Theorem 1.
(iv) R, C $\Rightarrow$ I by Theorem 2.

By Models M(T) and M(C) we see that T and C are independent.

II. T *and* I *are independent, and* T, I $\Rightarrow$ R, S, A, C.

Proof. (i) T, I $\Rightarrow$ R by Threoem 6.
(ii) T, R $\Rightarrow$ S by Theorem 1.
(iii) T, I $\Rightarrow$ A by Theorem 7.
(iv) T, I $\Rightarrow$ C by Theorem 8.

By Models M(T) and M(I) we see that T and I are independent.

III. S, A *and* I *are independent, and* S, A, I $\Rightarrow$ R, T, C.

Proof. (i) S, A $\Rightarrow$ R by Theorem 9.
(ii) S, A, I $\Rightarrow$ T by Theorem 10.
(iii) S, A, I $\Rightarrow$ C by Theorem 11.

1) M(R, S, A) shows that S and A do not yield I.
2) M(R, S, C, I) shows that S and I do not yield A.
3) M(A, C, I) shows that A and I do not yield S.

Hence S, A and I are independent.

IV. S, A *and* C *are independent, and* S, A, C $\Rightarrow$ R, T, I.

Proof. (i) S, A $\Rightarrow$ R by Theorem 9.
(ii) R, C $\Rightarrow$ I by Theorem 2.
(iii) S, A, I $\Rightarrow$ T by Theorem 10.

1) M(R, S, A) shows that S and A do not yield C.
2) M(R, S, C, I) shows that S and C do not yield A.
3) M(A, C, I) shows that A and C do not yield S.

Hence S, A and C are independent.

V. R, A *and* C *are independent, and* R, A, C $\Rightarrow$ S, T, I.

Proof. (i) R, C $\Rightarrow$ I by Theorem 2.
(ii) R, A, C $\Rightarrow$ S by Theorem 12.
(iii) S, A, I $\Rightarrow$ T by Theorem 10.

1) M(R, S, A) shows that R and A do not yield C.
2) M(R, S, C, I) shows that R and C do not yield A.
3) M(A, C, I) shows that A and C do not yield R.

Hence R, A and C are independent.

REMARK: By the use of our Theorems and Models it may easily be proved that there is no further theorem of the above type I–V.

5. Tables

[1] Baisic Theorems[2)]

T_1	**R**	S	**T**			
T_1	R	**S**	**T**			
T_2	**R**				**C**	I
T_3			**T**	A	**C**	
T_4	R		**T**	**A**		
T_5	R		**T**		**C**	
T_6	R		**T**			**I**
T_7			**T**	A		**I**
T_8			**T**		C	**I**
T_9	R	**S**		**A**		
T_{10}		**S**	T	**A**		**I**
T_{11}		**S**		**A**	C	**I**
T_{12}	**R**	S		**A**	**C**	
T_{13}	R	**S**			**C**	**I**

2) In the following tables **R**, S, **T** indicates for example that Axiom S follows from Axioms R, T and Axiom E. T_n means Theorem n.

[2] Models

M(R)	**R**	¬S	¬T	¬A	¬C	¬I
M(S)	¬R	**S**	¬T	¬A	¬C	¬I
M(T)	¬R	¬S	**T**	¬A	¬C	¬I
M(A)	¬R	¬S	¬T	**A**	¬C	¬I
M(C)	¬R	¬S	¬T	¬A	**C**	¬I
M(I)	¬R	¬S	¬T	¬A	¬C	**I**
M(A, C)	¬R	¬S	¬T	**A**	**C**	¬I
M(S, C)	¬R	**S**	¬T	¬A	**C**	¬I
M(S, I)	¬R	**S**	¬T	¬A	¬C	**I**
M(R, S, A)	**R**	**S**	—T	**A**	¬C	¬I
M(R, A, I)	**R**	¬S	¬T	**A**	¬C	**I**
M(R, S, T)	**R**	**S**	**T**	¬A	¬C	¬I
M(A, C, I)	¬R	¬S	¬T	**A**	**C**	**I**
M(R, S, I)	**R**	**S**	¬T	¬A	¬C	**I**
M(R, S, C, I)	**R**	**S**	¬T	¬A	**C**	**I**
M(R, S, T, A)	**R**	**S**	**T**	**A**	¬C	¬I
M(R, C, I)[3]	**R**	¬S	¬T	¬A	**C**	**I**

[3] Main Theorem[4]

I	R	S	**T**	A	**C**	I
II	R	S	**T**	A	C	**I**
III	R	**S**	T	**A**	C	**I**
IV	R	**S**	T	**A**	**C**	I
V	**R**	S	T	**A**	**C**	I

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[3] H. Terasaka: Shotō Kikagaku (in Japanese), Tokyo, 1952.

3) M(R, C, I) will be given in the second part of this paper.
4) For the notation, see Main Theorem, p. 270.