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ON CATEGORIES OF PROJECTIVE MODULES

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The authors have studied some structures in categories of completely indecomposable modules in [5], [6] and [7], respectively. Furthermore, one of the authors has given some characterization of semi-perfect modules, defined in [9], in terms of semi-T-nilpotent system in [6].

In this note, we shall work in the same frame and give generalizations of some results in [6], [9] and [11].

Let R be a ring with identity and $\mathfrak{M}_R$ the category of R-right modules. By $\mathfrak{A}$ (resp. $\mathfrak{A}_f$) we denote the full sub-additive category of $\mathfrak{M}_R$, whose objects consist of all R (resp. R-finitely generated)-projective modules and we denote the Jacobson radical of $\mathfrak{A}$ by $\mathfrak{J}$ or $J(\mathfrak{A})$, (see the definition in [3], [6] and [8]). Then we shall show, in the first section, that $\mathfrak{A}/J(\mathfrak{A})$ (resp. $\mathfrak{A}_f/J(\mathfrak{A}_f)$) is a C_3-completely reducible (resp. completely reducible artinian) abelian category if and only if R is a right (resp. semi-) perfect ring, defined in [1]. In the second section, we shall study a directsum of projective modules $P=\sum_{\alpha\in I}\oplus P_\alpha$, and show that $J(P)$ is small in P if and only if $J(P_\alpha)$ is small in P_α for all $\alpha\in I$ and $\{P_\alpha\}$ is a (elementwise) semi-T-nilpotent system with respect to the Jacobson radical if the cardinal $|I|$ is infinite (see the section 2 for the definition or [6] and [7]). We have immediately [6], Theorems 6 and 7 and [7], Theorem from this theorem. In the third section, we define a quasi-perfect module, which is a generalization of perfect modules defined in [9] and give analogous results to [9]. In the final section, we shall give another proof of [7], Theorem.

In this note, we always assume that a ring R has the identity and R-modules are unitary. We shall use terminologies of categories in [6], [3], [10] and [8]. Let $\mathfrak{B}$ be a full subcategory of $\mathfrak{M}_R$. We assume that Im., Ker. directsum etc. are considered in $\mathfrak{M}_R$ (not in $\mathfrak{B}$), unless otherwise stated, and for any object P, P' in $\mathfrak{M}_R$ we write $[P, P']_R$ or $[P, P']_{\mathfrak{M}_R}$ instead of $\mathrm{Hom}_R(P, P')$.

1. A right perfect ring

Let M be a right R-module, and N an R-submodule of M. N is called *small in* M if $Q+N=M$ implies $Q=M$ for $Q\subseteq M$. By $J(M)$ we denote the radical of M and hence $J(R)$ is the Jacobson radical of R. We denote $[M, M]_R$ by S_M. We shall make use of the definition of (semi-) perfect modules defined in [9].

Now, let $\mathfrak{A}$ be a full sub-additive category of $\mathfrak{M}_R$. We define a subfamily $\mathfrak{C}$ of morphisms in $\mathfrak{A}$ as follows: for any objects P, P' in $\mathfrak{A}$, $\mathfrak{C}\cap[P, P']_R =\{f \mid \in[P, P']_R, \operatorname{Im} f \text{ (in } \mathfrak{M}_R) \text{ is small in } P'\}$. Then we have

Lemma 1. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be as above. Then $\mathfrak{C}$ is an ideal in $\mathfrak{A}$.*

Proof. Let f, f' be in $\mathfrak{C}\cap[P, P']_R$. Then $\operatorname{Im}(f\pm f')\subseteq\operatorname{Im}f+\operatorname{Im}f'$. Hence, $f\pm f'\in\mathfrak{C}\cap[P, P']_R$. Let g be an element in $[P', P'']_R$ and $A=\operatorname{Im} f$. We shall show that $g(A)$ is small in P''. We assume $g(A)+N=P''$ for some N in $\mathfrak{M}_R$. Then for any p' in P' we have $g(p')=g(a)+n$, $(a\in A,\ n\in N)$. Hence, $p'-a\in g^{-1}(N)$ and $g(g^{-1}(N)+A)=g(P')$. On the other hand, since $g^{-1}(N)$ contains Ker g, $P'=A+g^{-1}(N)$. A is small in P' and hence, $P'=g^{-1}(N)$. Therefore, $N\supseteq g(g^{-1}(N))=g(P')\supseteq g(A)$ and $N=P''$. Hence, $gf\in\mathfrak{C}\cap[P, P'']_R$. It is clear that $fg'\in\mathfrak{C}$ for any g' in $[P'', P]_R$. Thus, $\mathfrak{C}$ is an ideal.

Corollary. *If every object P in $\mathfrak{A}$ is projective in $\mathfrak{M}_R$, then $\mathfrak{C}$ is equal to the Jacobson radical of $\mathfrak{A}$.*

Proof. Since $\mathfrak{C}\cap[P, P]_R$ is the Jacobson radical of $[P, P]_R$ by [12], Lemma 1, $\mathfrak{C}$ is the radical of $\mathfrak{A}$.

From now on, we shall denote the Jacobson radical of $\mathfrak{A}$ by $\mathfrak{J}$.

Proposition 1. *Let P be a projective R-module. Then $J(P)$ is small in P if and only if $[P, J(P)]_R=J(S_P)$.*

Proof. It is clear from the above corollary that $J(S_P)\subseteq[P, J(P)]_R$ for any projective R-module. Hence, if $J(P)$ is small, $J(S_P)=[P, J(P)]_R$. Conversely, we assume $J(S_P)=[P, J(P)]_R$ and $P=N+J(P)$ for some N in $\mathfrak{M}_R$. Then we have a diagram:

$$\begin{array}{ccc} J(P) & \xrightarrow{\nu} & J(P)/N\cap J(P)\to 0 \\ & & \wr\!\wr\, f \\ & & P/N \\ & & \uparrow \nu' \\ & & P \end{array}$$

where ν and ν' are canonical epimorphisms.
Since P is projecitve, we have h in $[P, J(P)]_R$ such that $\nu h=f\nu'$. Hence, $J(P)=h(P)+N\cap J(P)$ and $P=N+J(P)=N+h(P)$. On the other hand $h(P)$ is small in P, since h is in $J(S_P)$. Hence, $P=N$.

Let I be any well ordered set. By R_I we denote the ring of column finite matices of R over I. An ideal $\mathfrak{J}$ of a ring R is called *right T-nilpotent*, if for any set $\{a_i\}_{i=1}$ of elements a_i in $\mathfrak{J}$, there exists n so that $a_n a_{n-1} \cdots a_1 = 0$, ($n$ depends on $\{a_i\}$, *cf.* [1]).

Corollary 1 ([11], [13] and [14]). *Let I be an infinite set. Then $J(R)$ is right T-nilpotent if and only if $J(R_I)=J(R)_I$.*

Proof. Let $P=\sum_I \oplus R$. If $J(R)$ is T-nilpotent, then $J(P)=\sum \oplus J(R)$ is small by [9], Theorem 7,2. On the other hand R_I is equal to S_P. Hence, $J(S_P) = [P, J(R)]_R = J(R)_I$. Conversely, If $J(R_I)=J(R)_I$, $J(R)$ is small. Hence, $J(R)$ is T-nilpotent from the argument of [9], Theorem 7.4.

Corollary 2 ([6]). *Let P be a projective module. We assume P is a directsum of completely indecomposable modules. Then P is semi-prefect if and only if $[P, J(P)]_R = J(S_P)$.*

Proof. It is clear from [9], Theorem 5.1 and [6], Theorem 5.

Lemma 2. *If R has a family of mutually orthogonal non-zero idempotents $\{e_i\}_{i=1}^{\infty}$, then R_I is not regular in the sense of Von Neumann for any infinite set I.*[0]

Proof. We may sasume that (the cardinal of I)$=|I|=\aleph_0$. We denote a family of matrix units in R_I by e_{ij}. Put $B=\sum e_i e_{1i}$. If R_I is a regular ring, then there exists A in R_I so that $BAB=B$, say $A=\sum a_{ij} e_{ij}$. We may assume $a_{i1}=0$ if $i>t$ for a large t. Then $BAB=B$ implies that $\sum_{i=1}^{t} e_i a_{i1} e_j = e_j$ for all j. If $j>t$, then $e_j = e_j^2 = \sum_{i=1}^{t} e_j e_i a_{ij} e_j = 0$, which is a contradiction.

Corollary. *Let R be a regular ring in the sense of Von Neumann. Then R_I is regular for any set I if and only if R is artinian.*

Proof. If R is artinian, then it is clear that R_I is regular for any set I. We assume that there exists an infinite series of principal left ideals of R: $Ra_1 \supset Ra_2 \supset \cdots$. Since R is regular $Ra_n = Re'_n$ for some idempotent e'_n. Hence, R has an infinite set of non-zero mutually orthogonal idempotents $\{e_i\}$, which is a contradiction to Lemma 2. Therefore, R has the non zero socle, which is atrinian and hence, R is artinian, since R is equal to the socle.

Let $\mathfrak{A}$ be an additive category in $\mathfrak{M}_R$ and $\mathfrak{C}$ an ideal of $\mathfrak{A}$. Then we can define the factor category $\mathfrak{A}/\mathfrak{C}$ with respect to $\mathfrak{C}$. Let P and f be an object and a morphism in $\mathfrak{A}$, respectively. Then P is also an object in $\mathfrak{A}/\mathfrak{C}$, however we shall denote it by $\bar{P}$ if P is regarded as an object in $\mathfrak{A}/\mathfrak{C}$. Similary, $\bar{f}$ means a class of f in $\mathfrak{A}/\mathfrak{C}$.

Let $\{M_\alpha\}$ be a family of R-modules, We consider the full sub-additve category $\mathfrak{B}$ (resp. $\mathfrak{B}_f$) in $\mathfrak{M}_R$, whose objects consist of all directsums of M_α 's (resp. all

Added in proof. 0) It was obtained by M. Tsukerman; Siberian Math. J. 7 (1966).

dircetsums of finite number of M_α's), and of their isomorphic images. We call $\mathfrak{B}$ (resp. $\mathfrak{B}_f$) the *induced category from* $\{M_\alpha\}$.

Proposition 2. *Let $\mathfrak{A}$ be the induced additive category from a family of projective modules, and $\mathfrak{J}$ the radical of $\mathfrak{A}$. We assume $\mathfrak{A}/\mathfrak{J}$ is a spectral abelian category. Then*

1) *For every P in $\mathfrak{A}$, $J(P)$ is small in P.*

Furthermore, we assume $\mathfrak{A}/\mathfrak{J}$ is C_3-abelian.

2) *If P in $\mathfrak{A}$ is a directsum*[1] *of subobject P_α in $\mathfrak{A}$, then $\bar{P}=\sum\oplus\bar{P}_\alpha$ in $\mathfrak{A}/\mathfrak{J}$.*

3) *If $\bar{P}$ is a directsum of minimal objects in $\mathfrak{A}/\mathfrak{J}$, then P is semi-perfect.*

4) *If Q in $\mathfrak{A}$ is a finietly generated R-module, then Q is perfect.*

Proof. 1). Put $S_P=[P, P]_R$ and $J'(S_P)=[P, J(P)]_R$. We assume $J'(S_P) \neq J(S_P)$. Since $S_P/J(S_P)$ is a regular ring and $J'(S_P)$ is a tow-sided ideal in S_P, there exists non zero element e' in $J'(S_P)$ so that $e'\equiv e'^2$ (mod $\mathfrak{J}$). Herce, we obtain an idempotent e in $J'(S_P)$ so that $e\equiv e'$ (mod $\mathfrak{J}$) by [5], Lemma 2. Therefore, $eP\subset J(P)$, which is a contradiction. Thus, we obtain $J'(S_P)=J(S_P)$ and $J(P)$ is small in P by Proposition 1.

2). We shall show that $\sum_I\oplus\bar{P}_\alpha=\overline{\sum_I\oplus P_\alpha}$ in $\mathfrak{A}/\mathfrak{J}$. Let J be a finite subset of I, then $P_J=\sum_{\alpha\in J}\oplus P_\alpha$ is a direct summand of $P=P_I$. Hence, $\sum_J\bar{P}_\alpha=\bar{P}_J$ is a direct summand of $\bar{P}$, (use the method in the proof of Proposition 1 or see [5], Lemma 2). Therefore, $\cup\bar{P}_J=\sum\oplus\bar{P}_\alpha$ is a subobject of $\bar{P}$ by [10], p. 82, Proposition 1.2. Let $\bar{P}=\sum\oplus\bar{P}_\alpha\oplus\bar{Q}$ and $\bar{f}$ a projection of $\bar{P}$ to $\bar{Q}$. Then $fg\equiv 1$ (mod $\mathfrak{J}$) for some $g\in[Q, P]_R$. Since $\mathfrak{J}$ is the radical, fg is isomorphic as R-modules. $\bar{f}(\sum\oplus\bar{P}_\alpha)=0$ implies $f(\sum\oplus P_\alpha)\subset J(Q)$. Hence, $J(Q)\supset f(P)\supset fg(Q)=Q$. Therefore, $Q=0$.

3). We assume $\bar{P}=\sum\oplus\bar{P}'_\alpha$. Put $P'=\sum\oplus P'_\alpha$. Then $\bar{P}\approx\bar{P}'$ from 2). Therefore, $P\approx P'$ as R-modules, since $\mathfrak{J}$ is the radical. Furthermore, P'_α is semi-perfect and so is P by 1), (see [9], Theorem 5.2 and [5] , Theorem 5).

4). Let Q be a finitely generated R-projective module in $\mathfrak{A}$, and $S_Q=[Q, Q]_R$. Put $Q^*=\sum_{i=1}^{\infty}\oplus Q_i$; $Q_i\approx Q$ for all i. Since Q is finitely generated, S_{Q^*} is the ring $(S_Q)_\infty$ of column finite matrices with entries in S_Q. From the assumption $S_{Q^*}/J(S_{Q^*})$ is regular and hence $(S_Q/J(S_Q))_\infty$ is a regular ring. Therefore, $S_Q/J(S_Q)$ is an artinian ring by Corollary to Lemma 2. Thus, $\bar{Q}=\sum_{i=1}^{n}\oplus\bar{Q}_i'$ in $\mathfrak{A}/\mathfrak{J}$, where $Q_i's$ are minimal objects in $\mathfrak{A}/\mathfrak{J}$. Hence, $Q=\sum_{i=1}^{n}\oplus Q_i$ and $Q_i's$ are completely indecomposable by [5], Lemma 2. It is clear from the first half that Q is perfect.

Theorem 1. *Let $\mathfrak{A}$ be the full sub-additive category of all R-projective*

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1) Directsum is considered in $\mathfrak{M}_R$.

modules in $\mathfrak{M}_R$ *and* $\mathfrak{J}$ *the radical of* $\mathfrak{A}$. *Then the following statements are equivalent.*

1 $\mathfrak{A}/\mathfrak{J}$ *is a* C_3*-abelian completely reducible category.*

2 $\mathfrak{A}/\mathfrak{J}$ *is a* C_3*-spectral abelian category.*

3 *R is a right perfect ring.*

Proof. 1→2. It is clear. 2→3. Since R is a finitely generated R-module, R is right perfect from Proposition 2. 3→1. If R is right perfect, then every object P in $\mathfrak{A}$ is perfect by [1] or [9] and hence, P is a directsum of completely indecomposable modules. Furthermore, $\mathfrak{J}\cap[P, P]_R=[P, J(P)]_R$ is equal to the ideal defined in [5], §3, (see [6], §3). Hence, $\mathfrak{A}/\mathfrak{J}$ is a C_3-completely reducible abelian category by [5], Theorem 7.

Similarly to Theorem 1, we obtain

Theorem 2. *Let* $\{P_\alpha\}$ *be a family of finitely generated projective R-modules, and* $\mathfrak{A}_f$ *the induced category from* $\{P_\alpha\}$. *Then the following two conditions are equivalent.*

1 $\mathfrak{A}_f/\mathfrak{J}$ *is a completely reducible and artinian abelian category.*

2 *Every object in* $\mathfrak{A}_f$ *is semi-perfect.*

Especially, let $\mathfrak{A}_f'$ *be the full sub-category of all R-finitely generated projective modules. Then* $\mathfrak{A}_f'/\mathfrak{J}$ *is a completely reducible and artinian abelian category if and only if R is semi-perfect.*

REMARK. If we omit the assumption "*artinian*" in Theorem 2, then the thorem is not true in general. For example, let K be a field and $R=[P, P]_K$, where P is a K-vector space with infinite dimension. It is well known that R is self injective as a right R-module and R has the socle $S=\sum_{i=1}^{\infty}\oplus e_iR$. Let $\mathfrak{A}_f'$ be as above. Then $\mathfrak{A}_f'$ is a spectral abelian category from [12], Theorem 2, since R is a regular ring. First, we shall show that $R=\sum\oplus e_iR$[2] in $\mathfrak{A}_f'$. It is clear that $S_J=\sum_{i\in J} e_iR$ is in $\mathfrak{A}_f'$ for every finite set J and is a direct summand of R in $\mathfrak{A}_f'$ via the inclusion. Let $\{f_i\}$ be a set of R-homomorphisms f_i: $e_iR\to R$. Then $f=\sum f_i$ is in $[S, R]_R$. Since R is self-injective and a prime ring, we have a unique extension $g\in[R, R]_R$ of f. Therefore, $R=\sum e_iR$ in $\mathfrak{A}_f'$, since every object in $\mathfrak{A}_f'$ is a finitely generated R-module. Noting that $\mathfrak{A}_f'$ is spectral and $R=\sum e_iR$ in $\mathfrak{A}_f'$ even though $\mathfrak{A}_f'$ is not co-complete, we can easily show that $\mathfrak{A}_f'$ is completely reducible. However, R is not semi-perfect.

We have shown in Proposition 2 that $\overline{\sum\oplus P_\alpha}=\sum\oplus\bar{P}_\alpha$ in $\mathfrak{A}/\mathfrak{J}$ if $\mathfrak{A}/\mathfrak{J}$ is a C_3-abelian spectral category. However, as above this fact is not true if $\mathfrak{A}/\mathfrak{J}$ is not co-complete, since $\sum\oplus P_\alpha\notin\mathfrak{A}_f'$.

2) Directsum is considered in $\mathfrak{A}_f'$.

Proposition 3. *Let $\mathfrak{A}_f$ be the induced additive category from a family of semi-perfect modules. Then $\mathfrak{A}_f/\mathfrak{J}$ is an abelian spectral category.*

Proof. It is clear that every object in $\mathfrak{A}_f$ is semi-perfect from [9], Theorem 5.1. Therefore, $\mathfrak{A}_f/\mathfrak{J}$ is an abelian spectral category by [12], Theorem 2.

Corollary. *Let P and Q be semi-perfect modules and f an element in $[P, Q]_R$. Then we have decomposition $P=P_1\oplus P_2$, $Q=Q_1\oplus Q_2$ such that $f(P_2)$ is small in Q and $f\,|\,P_1$ gives an isomorphism of P_1 to Q_1. Furthermore, under those conditions, P_i and Q_i are unique up to isomorphism.*

Proof. Let $\mathfrak{A}_f$ be the induced category from P and Q. Put $\bar{P}_2'=\mathrm{Ker}\,\bar{f}$. Since $\mathfrak{A}_f/\mathfrak{J}$ is abelian spectral, $\bar{P}=\bar{P}_1'\oplus\bar{P}_2'$. Hence, we have $P=P_1\oplus P_2$ so that $\bar{P}_1=\bar{P}_1'$ by [5], Lemma 2. Then $\bar{f}_1=\overline{f\,|\,P_1}$ is monomorphic in $\mathfrak{A}_f/\mathfrak{J}$. Hence, there exists $g\in[Q, P]_R$ such that $\bar{g}\bar{f}_1$ is equal to the identity of P_1 modulo $\mathfrak{J}$. Hence, $Q=\mathrm{Ker}\,g\oplus\mathrm{Im}\,f_1$. Since $\bar{f}(\bar{P}_2)=0$, $f(P_2)$ is small in Q, If P_i, Q_i satisfy the above conditions, then $\bar{P}_2=\mathrm{Ker}\,\bar{f}$, $\bar{P}_1=\mathrm{Coim}\,\bar{f}$ and $\bar{Q}_1=\mathrm{Im}\,\bar{f}$, $\bar{Q}_2=\mathrm{Coker}\,\bar{f}$. Hence, they are unique up to isomorphism as R-modules.

2. Directsum of projective modules

It is known by [9], Corollary 5.3 that every semi-perfect module is a directsum of completely indecomposable projective modules. Thus, we shall study, in this section, a projective module which is a directsum of some submodules. First, we shall generalize the definition of T-nilpotent.

Let $\{M_\alpha\}_I$ be a family of R-modules M_α, $\mathfrak{A}$ the induced category from $\{M_\alpha\}$ and $\mathfrak{C}$ an ideal of A. We call $\{M_\alpha\}_I$ a (elementwise) *T-nilpotent* (resp. *semi-T-nilpotent*) *system with respect to* $\mathfrak{C}$ if the following conditions are satisfied: for any sequence $\{f_i\}_{i=1}^{\infty}$ of morphisms f_i in $\mathfrak{C}\cap[M_{\alpha_i}, M_{\alpha_{i+1}}]_R$ and any element x in M_{α_1}, there exists n, depending on x and $\{f_i\}$, such that $f_n f_{n-1}\cdots f_1(x)=0$, where M_i's are in $\{M_\alpha\}$, (resp. $\alpha_i\neq\alpha_j$ if $i\neq j$), (cf. [5], §3).

Let I be a well ordered set and put $M=\sum_I\oplus M_\alpha$, then $[M, M]_R=S_M$ is equal to the ring of column summable matrices, whose entries $a_{\sigma\tau}$ consist of elements in $[M_\tau, M_\sigma]_R$, namely for $f\in S_M$ and $x_\tau\in M_\tau$, $f=(b_{\sigma\tau})$ and $b_{\sigma\tau}(x_\tau)=0$ for almost all $\sigma\in I$. In this case $\sum_{\sigma\in I}b_{\sigma\tau}$ has a meaning and it is an element in $[M_\tau, M]_R$. We shall make use of those notations in the following. Let $b_{\alpha_i\alpha_{i-1}}$ be in $[M_{\alpha_{i-1}}, M_{\alpha_i}]_R$ for $i=1, 2\cdots, n$. If $\alpha_1<\alpha_2\cdots<\alpha_n$, we denote briefly $b_{\alpha_n\alpha_{n-1}}b_{\alpha_{n-1}\alpha_{n-2}}\cdots b_{\alpha_2\alpha_1}$ by $b(\alpha_n, \alpha_{n-1}, \cdots\alpha_2, \alpha_1)$.

Lemma 3. *Let $\{M_\alpha\}_I$, M and $\mathfrak{C}$ be as above with $|I|$ infinite and $f=(b_{\sigma\tau})$ in $\mathfrak{C}\cap[M, M]_R$. We assume $\{M_\alpha\}_I$ a semi-T-nilpotent system with respect to $\mathfrak{C}$. We put $F_\tau=\{b(\alpha_n, \alpha_{n-1}, \cdots, \alpha_1)\,|\,\alpha_1=\tau$ and n is any integer $\geq 2\}$. Let x_τ be an* *element in* M_τ, *then* $b(\alpha_n, \alpha_{n-1}, \cdots, \alpha_1)(x_\tau)=0$ *for almost all* $b(\alpha_n, \alpha_{n-1}, \cdots, \alpha_1)$ *in* F_τ.

Proof. Since $\mathfrak{C}$ is an ideal, $b_{\sigma\tau}$ is in $\mathfrak{C}\cap[M_\tau, M_\sigma]_R$. Now, $\{b_{\alpha_2\tau}\}_{\alpha_2}$ is summable and hence, there exists a finite set T_1 such that $b_{\alpha_2\tau}(x_\tau)=0$ if $\alpha_2\notin T_1$. Since $\{b_{\alpha_3\alpha_2}\}_{\alpha_3}$ is summable for $\alpha_2\in T_1$, there exists a finite set T_2 such that $b(\alpha_3, \alpha_2, \tau)(x_\tau)=0$ for $\alpha_3\in T_2$, $\alpha_2\in T_1$. Repeating this argument, we obtain a family of finite set T_i such that $b(\alpha_t, \alpha_{t-1}, \cdots, \tau)(x_\tau)=0$ if $\alpha_k\notin T_k$ for some k. Hence, we obtain the lemma from Koning Graph Theorem and the assumption.

From Lemma 3, we know that $\sum_{\alpha_i} b(\sigma, \alpha_{n-1}, \cdots, \alpha_2, \tau)$ is in $[M_\tau, M_\sigma]_R$.

Lemma 4. *Let* M, $\{M_\alpha\}_I$ *and* $\mathfrak{C}$ *be as above and we assume* $\{M_\alpha\}_I$ *is a semi-T-nilpotent system with respect to* $\mathfrak{C}$. *Let* $(b_{\sigma\tau})$ *be in* $S_M\cap\mathfrak{C}$ *so that* $b_{\sigma\tau}=0$ *if* $\sigma\geq\tau$ *(resp.* $\sigma\leq\tau$*), then* $(b_{\sigma\tau})$ *is quasi-regular in* S_M.

Proof. It is clear from the proof of [5], Lemma 10.

Lemma 5. *Let* $\{M_\alpha\}_I$, M *and* $\mathfrak{C}$ *be as above. We assume the following.* 1) $\mathfrak{C}\cap S_\alpha\subseteq J(S_\alpha)$ *for every* $\alpha\in I$. 2) *if* $\{a_i\}_i$ *is a summable set in* $\mathfrak{C}\cap[M_\sigma, M_\tau]_R$, *then* $\sum_i a_i$ *is in* $\mathfrak{C}\cap[M_\sigma, M_\tau]_R$, *where* $S_\alpha=S_{M_\alpha}=[M_\alpha, M_\alpha]_R$, 3) $\{M_\alpha\}_I$ *is a semi-T-nilpotent system with respect to* $\mathfrak{C}$. *Then* $\mathfrak{C}\cap S_M\subseteq J(S_M)$.

Proof. Let $A'=(a'_{\sigma\tau})$ be in $\mathfrak{C}\cap S_M$ and put $A=E-A'=(a_{\sigma\tau})$, where E is the unit matrix. We shall show by the fundamental transformation of A that A is regular in S_M. Since $\mathfrak{C}$ is an ideal and $\mathfrak{C}\cap S_\alpha\subseteq J(S_\alpha)$, $a_{\sigma\sigma}=1-a'_{\sigma\sigma}$ is unit in S_σ. We put $b_{\sigma1}=-a_{\sigma1}a_{11}^{-1}$for $\sigma<1$, then $\{b_{\sigma1}\}_\sigma$ is summable and $b_{\sigma1}$ is in $\mathfrak{C}\cap[M_1, M_\sigma]_R$. We shall define $b_{\sigma\tau}$ for $\sigma<\tau$, satisfying the following conditions, by the transfinite induction on τ

1) $\{b_{\sigma\tau}\}_\sigma$ is summable and $b_{\sigma\tau}$ is in $\mathfrak{C}\cap[M_\tau, M_\sigma]_R$.
2) $b_{\sigma\tau}=-y_{\sigma\tau}y_{\tau\tau}^{-1}$, where for $\sigma\geq\tau$

$$y_{\sigma\tau}=a_{\sigma\tau}+\sum_{\tau>\alpha_t} b(\sigma, \alpha_t, \alpha_{t-1}, \cdots, \alpha_1)a_{\alpha_1\tau}\cdots(*).$$

We note that $\sum b(\sigma, \alpha_t, \cdots, \alpha_2, \alpha_1)a_{\alpha_1\tau}$ is defined and in $\mathfrak{C}\cap[M_\tau, M_\sigma]_R$ by 1), 2), the assumption and Lemma 3, and hence $y_{\tau\tau}$ is unit in S_τ, (note that $\{a_{i\tau}\}_i$ is summable). We assume $\{b_{\sigma\rho}\}$ is defined for all $\rho<\tau$, which satisfy the conditions 1) and 2). Then we can define $y_{\sigma\tau}$ for $\sigma\geq\tau$ from $(*)$ and define $b_{\sigma\tau}$ by 2). Since $\{y_{\sigma\tau}\}_\sigma$ is summable by Lemma 3, so is $\{b_{\sigma\tau}\}_\sigma$. Next, we put $c_{\sigma\tau}=\sum b(\sigma, \alpha_t, \cdots, \alpha_2, \tau)\in\mathfrak{C}\cap[M_\tau, M_\sigma]_R$ and $c_{\sigma\tau}=0$ if $\sigma<\tau$. Then $C=(c_{\sigma\tau})$ is in S_M by Lemma 3. We calculate the (σ, τ)-component $d_{\sigma\tau}$ is CA. For $\sigma>\tau>1$ we have $d_{\sigma\tau}=\sum_\rho c_{\sigma\rho}a_{\rho\tau}=\sum_{\sigma\geq\rho} c_{\sigma\rho}a_{\rho\tau}=\sum b(\sigma, \alpha_t, \cdots, \alpha_1)a_{\alpha_1\tau}+a_{\sigma\tau}=a_{\sigma\tau}+\sum_{\tau>\alpha_t} b(\sigma, \alpha_t, \cdots, \alpha_1)a_{\alpha_1\tau}+b_{\sigma\tau}(\sum b(\tau, \alpha_t, \cdots, \alpha_1)a_{\alpha_1\tau}+a_{\tau\tau})+\sum_{\sigma>\alpha_t>\tau} b_{\sigma\alpha_t}(\sum b(\alpha_t, \cdots, \sigma_1)a_{\alpha_1\tau}+a_{\alpha_t\tau})$. Hence, we have

3) $d_{\sigma\tau}=y_{\sigma\tau}+b_{\sigma\tau}y_{\tau\tau}+\sum b_{\sigma\alpha_t}d_{\alpha_t\tau}$.
It is clear that $d_{21}=0$. Now, we assume $d_{\alpha\beta}=0$ for $\sigma>\alpha>\beta$, then we obtain from 2) and 3), $d_{\sigma\tau}=0$ for $\sigma>\tau$. Thus, we have proved $d_{\sigma\tau}=0$ for all $\sigma>\tau$. Furthermore, $d_{\sigma\sigma}=\sum b(\sigma, \alpha_t, \cdots, \alpha_1)a_{\alpha_1\sigma}+a_{\sigma\sigma}$ is unit in S_σ from the assumptions. Finally, we put $C_1=\sum e_{\sigma\sigma}d_{\sigma\sigma}^{-1}$, where $\{e_{\sigma\tau}\}$ is a family of matrix units in S_M. Then $D=E-C_1CA=\sum e_{\sigma\tau}x_{\sigma\tau}$ and $x_{\sigma\tau}$ is in $\mathfrak{C}\cap[M_\tau, M_\sigma]_R$, since $b_{\sigma\tau}$ (resp. $a_{\sigma\tau}$) is in $\mathfrak{C}\cap[M_\tau, M_\sigma]_R$ if $\sigma>\tau$ (resp. $\sigma<\tau$). Hence, C_1CA is regular in S_M by Lemma 4. We know similarly that C is regular in S_M. Therefore, A is regular in S_M, which implies that $\mathfrak{C}\cap S_M\subseteq J(S_M)$.

Theorem 3. *Let $\{P_\alpha\}$ be a family of projective modules and $P=\sum_I\oplus P_\alpha$. Then $J(P)$ is small in P if and only if $J(P_\alpha)$ is small in P_α for every $\alpha\in I$ and $\{P_\alpha\}_I$ is a semi-T-nilpotent system if I is infinite.*

Proof. We assume $J(P)$ is small in P. Then $J(P_\alpha)$ is small in P_α. Let $\{P_{\alpha_i}\}_{i=1}^{\infty}$ be a sub-family of $\{P_\alpha\}$ and $f_i\in[P_{\alpha_i}, P_{\alpha_{i+1}}]_R\cap\mathfrak{J}$, where $\alpha_i\neq\alpha_j$ if $i\neq j$. Put $P'_i=\{p_i+f_i(p_i)\mid p_i\in P_{\alpha_i}\}$. Then $f_i(p_{\alpha_i})$ is in $J(P_{\alpha_{i+1}})$ by the definition and $P=\sum_{i=1}^{\infty}P'_{\alpha_i}+\sum_{\beta\neq\alpha_i}P_\beta+J(P)$. Hence, $P=\sum\oplus P'_{\alpha_i}\oplus\sum_{\beta\neq\alpha_i}\oplus P_\beta$. Therefore, $\{P_\alpha\}$ is a semi-T-nilpotent system, (see [5], Lemma 9). Conversely, if I is finite, the theorem is trivial. Hence, we assume that I is infinite. If $J(P_\alpha)$ is small in P_α, then $J(S_\alpha)=[P_\alpha, J(P_\alpha)]_R$ from Proposition 1. Now, we define an ideal $\mathfrak{C}$ in $\mathfrak{A}$ induced from $\{P_\alpha\}$ as follows: $\mathfrak{C}\cap[P_\alpha, P_\beta]_R=[P_\alpha, J(P_\beta)]_R$. Then $\mathfrak{C}$ satisfies the conditions in Lemma 5 by Corollary to Lemma 1 and hence, $\mathfrak{C}\cap S_P=[P, J(P)]_R\subseteq J(S_P)$. Therefore, $J(P)$ is small in P by Proposition 1.

Corollary 1 ([6], Theorems 6 and 7). *Let P and $\{P_\alpha\}_I$ be as above with I infinite. Then P is perfect (resp. semi-perfect) if and only if P_α is semi-perfect and $\{P_\alpha\}_I$ is a T-nilpotent (resp. semi-T-nilpotent) system.*

Proof. It is clear from Theorem 3 and [9], Theorem 5.1.

Corollary 2. *Let P be a projective module in which $J(P)$ is small. Then $J(F)$ is small in F for any directsum F of any copies of P if and only if $\{P\}$[3] is a T- nilpotent system with respect to $J(S_P)$.*

Proof. It is an immediate consequence of Theorem 3.

Corollary 3. *Let $\{P_\alpha\}$ be a family of perfect modules. Then $P=\sum_I\oplus P_\alpha$ is perfect if and only if $J(P)$ is small in P.*

Proof. "only if" part is clear. We may assume that $J(P)$ is small in P and P_α is completely indecomposable. If $|I|<\infty$, P is perfect. If $|I|=\infty$,

3) $\{P\}$ means $\{P_i\}$; $P_i\approx P$ for all i.

$\{P_\alpha\}$ is a semi-T-nilpotent system by Corollary 2. Since P_α is perfect, P is a T-nilpotent system. Therefore, P is perfect from Corollary 1.

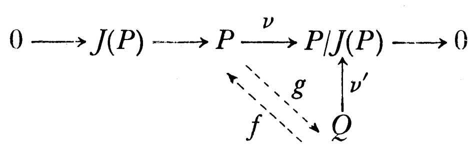
3. Quasi-perfect modules

We know from Corollary 1 to Theorem 3 that the perfect modules are special ones in projective modules with properties in Corollary 2. Thus, we call such a projective modules P *quasi-perfect*; namely $J(P)$ is small in P and $\{P\}$ is a T-nilpotent system with respect to $J(S_P)$, or equivalentely $\{P\}$ is a T-nilpotent system with respect to $[P, J(P)]_R$ by Proposition 1.

If $J(R)$ is right T-nilpotent, then for every projective module P, $J(P)$ is small in P and P is quasi-perfect by Theorem 3 and vice versa. If $R/J(R)$ is not artinian, then R is quasi-perfect, but not perfect. It is clear that a directsum of any copies (or direct summand) of a quasi-perfect module is also quasi-perfect. Hence, if a projective generator in $\mathfrak{M}_R$ is quasi-perfect, then so is every projective modules.

Lemma 6. *Let P be a projective module. We assume that $J(P)$ is small in P and $P/J(P)=\sum_I \oplus \bar{P}'_\alpha$ as $R/J(R)$-modules. If there exist projective R-modules Q_α so that $Q_\alpha/J(Q_\alpha)\approx\bar{P}_\alpha'$ for each $\alpha\in I$, then we have a direct decomposition $P=\sum_I \oplus P_\alpha$, which induces the above decomposition, and hence $J(Q_\alpha)$ is small in Q_α,* (cf. [9], Theorem 4.3).

Proof. Put $Q=\sum \oplus Q_\alpha$, then we have a diagram



where ν and ν' are natural epimorphisms from the assumption. Since Q is projective and $J(P)$ is small, P is a direct summand of Q via g; $Q=P\oplus Q'$. Hence, $Q=P+J(Q)=P\oplus J(Q')$. Therefore, $Q'=0$. It is clear that $J(Q_\alpha)$ is small in Q_α.

Theorem 4. *Let P be a quasi-perfect module. Then every direct decomposition of $P/J(P)$ is lifted to one of P.*

Proof. We assume that $P/J(P)=\bar{P}'_1\oplus\bar{P}'_2$ as $R/J(R)$-modules, and show that there exist P_i so that $P=P_1\oplus P_2$ induces the above decomposition. It is clear that $[P/J(P), P/J(P)]_{R/J(R)}=S/\mathfrak{J}$, where $S=S_P$ and $\mathfrak{J}=J(S_P)$. Let $a^2\equiv a$ (mod $\mathfrak{J}$) for $a\in S$. We shall show that there exists an idempotent e in S such that $e\equiv a$ (mod $\mathfrak{J}$). We use the same argument in [2], p. 546. We can find the following identities for each n from $1=(x-(1-x))^{2n}=\sum\binom{2n}{i}x^i(1-x)^{2n-i}$

4) $f_n(x)=f_{n-1}(x)+g_n(x)(x^2-x)^{n-1}$

5) $f_n(x)^2=f_n(x)+h_n(x)(x^2-x)^n$,

where $f_n(x)$, $g_n(x)$ and $h_n(x)$ are polynominals with coefficients of integers. From 4) we have $f_n(x)=x+g_0(x)(x^2-x)+\cdots+g_n(x)(x^2-x)^{n-1}$. Put $b=a^2-a\in\mathfrak{J}$ and $g_i(a)=c_i\in S$. Let p be an element in P, then $b^{n(p)}(p)=0$ for some integer $n(p)$ by the assumption. Put $A=a+\sum_{i=0}^{\infty}c_ib^{i+1}$. Since $\{c_ib^{i+1}\}_i$ is summable as above, A is in S. Furthermore, $(A^2-A)(p)=AA_{n(p)}(p)-A_{n(p)}(p)$, where $A_{n(p)}=a+\sum_{i=0}^{n(p)-1}c_ib^{i+1}$. Now, let $A_{n(p)}(p)=q$, and put $m=\max(n(p), n(q))$, then $AA_{n(p)}(p)=A_mA_{n(p)}(p)=A_mA_m(p)$. Hence, $(A^2-A)(p)=A_m^2(p)-A_m(p)$. We have similarly from 5) that $(A_{n'}^2-A_{n'})(p)=0$ for any $n'\geq$some n. Therefore, $A^2=A$. On the other hand, $A-a=\sum_i c_ib^{i+1}$ and $(\sum_i c_ib^{i+1})(p)\in J(P)$. Hence, $\sum_i c_ib^{i+1}\in[P, J(P)]_R=\mathfrak{J}$ by Corollary to Proposition 1. Therefore, we have porved the theorem by Lemma 6.

Corollary 1. *We assume that $R/J(R)$ is artinian. Then every quasi-perfect module is perfect.*

Proof. Since $P/J(P)$ is semi-simple, P is perfect from Theorem 4, Corollary to Theorem 3 and [9], Theorem 5.1.

Corollary 2. *We assume $J(R)$ is right T-nilpotent, then for a projective R-module P, a direct decomposition of $P/J(P)$ is lifted to one of P, and every idempotent in $R_I/J(R_I)$ is lifted to one in R_I for any set I. Furthermore, if $R/J(R)$ is a regular ring, then $\mathfrak{A}'_f/\mathfrak{J}$ is a spectral abelian category, where $\mathfrak{A}'_f$ is the full sub-category of finitely generated projective R-modules.*

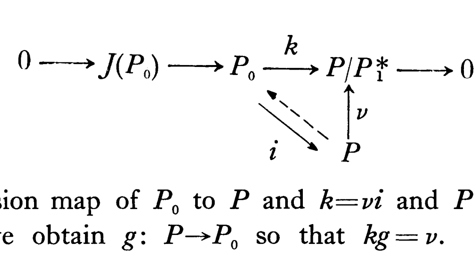
If P is perfect, then $P/J(P)$ is semi-simple and hence, $S_P/J(S_P)=\Pi\Delta^{\alpha}_{I_\alpha}$, where Δ^{α} are division rings. It is clear that $P'/J(P')$ is not semi-simple even though $S_{P'}/J(S_{P'})=\Pi\Delta^{\alpha}$ for a projective module P' We consider this situation.

Proposition 4. *Let P be a quasi-perfect module so that $S_P/J(S_P)=\prod_T\Delta^{\alpha}_{I_\alpha}$, then P contains a perfect module P_0 such that $S_{P_0}/J(S_{P_0})=\prod_T\Delta^{\alpha}_{I'_\alpha}$ and P is perfect if and only if P_0 is a direct summand of P, where $|I_\alpha|\geqslant|I'_\alpha|$ and $|I'_\alpha|\geqslant\aleph_0$ if $|I_\alpha|\geqslant\aleph_0$.*

Proof. Let $\bar{S}=S_P/J(S_P)$, $\bar{P}=P/J(P)$, and $\bar{e}_\alpha$ a projection of $\bar{S}$ to $\Delta^{\alpha}_{I_\alpha}$. Then there exists P_α in P which is a direct summand of P and $S_{P_\alpha}/J(S_{P_\alpha})=\bar{e}_\alpha\bar{S}_\alpha\bar{e}_\alpha\approx\Delta^{\alpha}_{I_\alpha}$. Let $\mathfrak{S}$ be the socle of $\Delta^{\alpha}_{I_\alpha}=\bar{S}_\alpha$, and $\mathfrak{S}\bar{P}(=\bar{P}_0)\subseteq\bar{P}$. Then the restriction φ of $\bar{S}_\alpha$ to $\bar{P}_0$ gives elements of $S_{P_0}=[\bar{P}_0, \bar{P}_0]_{R/J(R)}$. We first show that φ is a ring isomorphism. If $\operatorname{Ker}\varphi=\mathfrak{A}\neq 0$, then $\mathfrak{A}\supseteq\mathfrak{S}$. Since $\mathfrak{S}=\mathfrak{S}^2$, $\mathfrak{A}\mathfrak{S}\bar{P}=\bar{P}_0\neq 0$. Hence, $\operatorname{Ker}\varphi=0$. Since $\bar{P}_0=\sum e_{ii}\bar{P}$, where $\{e_{ij}\}$ is a family of matrix units of $\bar{S}_\alpha$, $\varphi(\mathfrak{S})$ is equal to the socle $\mathfrak{S}'$ of $S_{\bar{P}_0}$. Furthermore, $\bar{S}_\alpha=[\mathfrak{S},\mathfrak{S}]_{S_\alpha}$, and $S_{\bar{P}_0}=[\mathfrak{S}',\mathfrak{S}']_{S\bar{P}_0}$ as right modules. We may regard $\bar{S}_\alpha$ as a sub-ring of $S_{\bar{P}_0}$ by φ. Then $S_{\bar{P}_0}=[\mathfrak{S}',\mathfrak{S}']_{S\bar{P}_0}\subseteq[\mathfrak{S},\mathfrak{S}]_{\bar{S}_\alpha}=\bar{S}_\alpha$. Hence, φ is isomorphic. Now, since $\bar{P}_0=\sum\oplus e_{ii}\bar{P}$, P_α contains a direct summand $P_{\alpha J}$ for every finite set $J\subseteq I$ so that $\bar{P}_{\alpha J}=\sum_{i\in J}\oplus e_{ii}\bar{P}$. Let S be a family of projective submodules Q of P_α so that $Q=\sum_{i\in K}\oplus Q_i$, $\bar{Q}_i\approx e_{ii}\bar{P}$, for all i in K, and Q_J is a direct summan of P for any finite subset J of K. We can find a maximal element Q_α in S by defining a natural relation in S. We assume that Q_α is a direct summand of P and $\bar{Q}_\alpha\neq\bar{P}_0$. Since $\bar{Q}_\alpha$ is a direct summand of $\bar{P}_\alpha$ we can obtain a submodule U of P_α such that $P_\alpha=Q_\alpha\oplus U\oplus P'_\alpha$, which contradicts to the maximality of Q_α. Hence, $\bar{P}_0=\bar{Q}_\alpha$ in this case. On the other hand, since φ in the above is isomorphic, $\bar{P}_0=\bar{Q}_\alpha=\bar{P}_\alpha$. Finally, we put $P^*=\sum_{\alpha\in T}\oplus Q_\alpha=\sum_\alpha\sum_{i\in K_\alpha}\oplus Q_{i\alpha}$, and define a natural homomorphism f; $P^*\to P$. For any finite set J of $\cup K_\alpha$, $\bar{f}\,|\,P^*_J$ splits as $R/J(R)$-module. Hence, $f\,|\,P^*_J$ splits as an R-module, since $J(P^*_J)$ is small in P^*_J. Hence, f is monomorphic. Since $Q_{i\alpha}$ is projective and completely indecomposable, $Q_{i\alpha}$ is perfect from Corollary 2 to Theorem 3. Therefore, P^* is perfect by Corollary 1 to Theorem 3. If P^* is a direct summand of P, then Q_α is a direct summand of P_α, and hence, $Q_\alpha=P_\alpha$ from the first part. Let $P=P^*\oplus P_1$ and $\bar{g}$ a projection of $\bar{P}$ to $\bar{P}_1$. If $\bar{g}=\Pi f_\alpha(f_\alpha\in e_\alpha\bar{S}_P e_\alpha)$ is not zero, then $f_\alpha\neq 0$ for some α. However, φ is isomorphic, and hence $f_\alpha=0$. Therefore, $P^*=P$. Conversely, if P is perfect, P^* is a direct summand of P from Proposition 5 below.

Proposition 5. *Let P be a semi-perfect module and P_0 a projective R-module in P. Then P_0 is a direct summand of P if and only if $P_0\cap J(P)=J(P_0)$.*

Proof. We assume $J(P)\cap P_0=J(P_0)$. Then $P_0/J(P_0)$ is a $R/J(R)$-submodule of $P/J(P)$ and $P/J(P)=P_0/J(P_0)\oplus P_1/J(P_1)$ for some R-projective module P_P by [9], Theorem 4.3. Hence, $J(P_0)$ is small in P_0 by Lemma 6. Next, we have a diagram



where i is an inclusion map of P_0 to P and $k=\nu i$ and $P_1^*=P_1+J(P)$. Since P is projective, we obtain g: $P\to P_0$ so that $kg=\nu$. Let p_0 in P_0, then $(gi(p_0)-p_0)$ is in $J(S_{P_0})$. Therefore, gi is isomorphic, which means P_0 is a direct summand of P. The converse is clear.

Proposition 6. *There exists a semi-perfect module if and only if R contains a completely indecomposable and projective right ideal.*

Proof. If P is semi-perfect, then P contains a completely indecomposable semi-perfect module P_0 by [9], Corollary 5.3. Hence, $P_0/J(P_0)$ is a minimal $R/J(R)$-projective module. Since $J(P_0)$ is small, $P_0=pR$ for some $p\in P_0$. Hence, $P_0\approx eR$ for some idempotent e in R. The converse is clear from [6], Theorem 5.

4. Krull-Remak-Schmidt-Azumaya's theorem

In this section, we shall prove Kanbara's theorem in [7] as a corollary of Lemma 5. Let $\{M_\alpha\}_I$ be a family of completely indecomposable R-modules and $\mathfrak{A}$ the induced category from $\{M_\alpha\}$. We denote the ideal of $\mathfrak{A}$ defined in [5], §3 by $\mathfrak{J}'$. It is sufficient to prove that $J(S_M)=\mathfrak{J}'\cap S_M$ under the condition that $\{M_\alpha\}$ is a semi-T-nilpotent system with respect to $\mathfrak{J}'$, where $M=\sum_I\oplus M_\alpha$. However, if we use the argument in the proof of Lemma 5 in [5], we know that $\{M_\alpha\}$ satisfies the condition 2 in Lemma 5 if we take $\mathfrak{C}=\mathfrak{J}'$. It is clear that the conditions 1 and 3 are satisfied. Therefore, we obtain $J(S_M)=\mathfrak{J}'\cap S_M$ from Lemma 5.

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