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Author(s)	Miki, Kei
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FINITE DIMENSIONAL MODULES FOR THE q-TETRAHEDRON ALGEBRA

KEI MIKI

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Abstract

In [7] the q tetrahedron algebra $\boxtimes_q$ was introduced as a q analogue of the universal enveloping algebra of the three point loop algebra $sl_2 \otimes \mathbf{C}[t, t^{-1}, (t-1)^{-1}]$. In this paper the relation between finite dimensional $\boxtimes_q$ modules and finite dimensional modules for $U_q(L(sl_2))$, a q analogue of the loop algebra $L(sl_2)$, is studied. A connection between the $\boxtimes_q$ module structure and L-operators for $U_q(L(sl_2))$ is also discussed.

1. Introduction

In [1] a presentation of the three point loop algebra $sl_2 \otimes \mathbf{C}[t, t^{-1}, (t-1)^{-1}]$ in terms of generators and relations was obtained. The Lie algebra defined by the generators and the relations was named the tetrahedron algebra and denoted by $\boxtimes$, since the generators can be identified with the six edges of a tetrahedron. The relation between irreducible finite dimensional $\boxtimes$ modules and irreducible finite dimensional modules for the Onsager algebra were investigated in [2], using the notion of a tridiagonal pair. The tetrahedron algebra and its modules were further investigated in [3] and [4], and the universal central extension of this Lie algebra was studied in [5] and [6].

In [7] Ito and Terwilliger introduced the q-tetrahedron algebra $\boxtimes_q$, a q analogue of the universal enveloping algebra of the tetrahedron algebra. This algebra contains $U_q(L(sl_2))$ as a subalgebra and $U_q(L(sl_2))$ has A_q as a subalgebra. Here A_q is an algebra isomorphic to the subalgebra of $U_q(\widehat{sl}_2)$ generated by the Chevalley generators e_0 and e_1. Using the theory of a tridiagonal pair, Ito and Terwilliger studied the relation between A_q modules and $\boxtimes_q$ modules in [7] and the relation between $U_q(L(sl_2))$ modules and A_q modules in [8]. The relation between $\boxtimes_q$ modules and modules for the modified A_q was further studied in [9]. The combination of the results of [7] and [8] implies that there is a bijection between the isomorphism class of irreducible finite dimensional $\boxtimes_q$ modules of type 1 and the isomorphism class of irreducible finite dimensional $U_q(L(sl_2))$ modules V of type 1 whose associated Drinfeld polynomial $P_V(u)$ does not vanish at $u = 1$. (In [8] the condition for the Drinfeld polynomial was not $P_V(1) \neq 0$ but $P_V(q^{-1}(q - q^{-1})^{-2}) \neq 0$. The condition is modified to conform to the

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convention of the paper.) An extension of this correspondence to the case of reducible modules is the object of our study in this paper.

Now we explain our results. With a finite dimensional $U_q(L(sl_2))$ module V we can associate a polynomial $p_V(z)$ with constant term 1 so that for a polynomial $f(z)$ the following hold: $f(z)x^+(z) = 0$ on V if and only if $p_V(z)$ divides $f(z)$. Here $x^+(z) = \sum_{m\in\mathbf{Z}} x_m^+ z^{-m}$ with the x_m^+ the generators of $U_q(L(sl_2))$ in the Drinfeld realization. We call $p_V(z)$ the annihilating polynomial for V. Our main results are (i) answering the question which finite dimensional $U_q(L(sl_2))$ module V is extended to a $\boxtimes_q$ module in terms of $p_V(z)$ and (ii) the determination of the annihilating polynomial for a tensor product of evaluation modules and a finite dimensional highest weight module. These results are obtained without the use of the notion of a tridiagonal pair. More precisely, the first result is (a) the $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure if and only if $p_V(1) \neq 0$ and (b) in the case $p_V(1) \neq 0$ the extension is unique and V is irreducible as a $\boxtimes_q$ module if and only if it is irreducible as a $U_q(L(sl_2))$ module. This implies that there is a bijection between the isomorphism class of finite dimensional $\boxtimes_q$ modules and the isomorphism class of finite dimensional $U_q(L(sl_2))$ modules with $p_V(1) \neq 0$. Part of the second result is that $p_V(z) = z^{\deg P_V} P_V(z^{-1})$ (up to a nonzero multiplicative constant) for a finite dimensional highest weight $U_q(L(sl_2))$ module V with associated Drinfeld polynomial $P_V(u)$. Since $p_V(1) \neq 0$ if and only if $P_V(1) \neq 0$, our results reduce to those from [7] and [8] in the case of finite dimensional irreducible modules. In this paper we also discuss a tensor product module of finite dimensional $\boxtimes_q$ modules and a connection between the $\boxtimes_q$ module structure and L-operators for $U_q(L(sl_2))$.

This paper is organized as follows. In Section 2 the tetrahedron algebra is defined and the relation between finite dimensional $L(sl_2)$ modules and finite dimensional $\boxtimes$ modules is discussed. In Section 3, necessary results from [10] on finite dimensional highest weight $U_q(L(sl_2))$ modules are reviewed. In Section 4, the q-tetrahedron algebra $\boxtimes_q$ and its subalgebra isomorphic to $U_q(L(sl_2))$ are introduced. In Section 5, some basic results on finite dimensional $\boxtimes_q$ modules are derived. In Section 6, an annihilating polynomial $p_V(z)$ is introduced and the problem which finite dimensional $U_q(L(sl_2))$ module V is extended to a $\boxtimes_q$ module is studied. In Section 7, the annihilating polynomial is determined for a tensor product of evaluation modules and a finite dimensional highest weight module. In Section 8, it is shown that we can consider a tensor product module of finite dimensional $\boxtimes_q$ modules. In Section 9, a connection between the $\boxtimes_q$ module structure and L-operators for $U_q(L(sl_2))$ is studied.

2. The tetrahedron algebra $\boxtimes$

In this section we study the relation between finite dimensional $L(sl_2)$ modules and $\boxtimes$ modules. This result would be a help to understand a q analogue of it in Section 6.

2.1. The tetrahedron algebra $\boxtimes$. Let $\boxtimes$ denote the Lie algebra over $\mathbf{C}$ defined by generators

$$x_{ij} \quad (i \neq j \in \{0, 1, 2, 3\})$$

and the following relations:

$$x_{ij} + x_{ji} = 0, \tag{2.1}$$

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}, \tag{2.2}$$

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}] \tag{2.3}$$

where h, i, j, k are mutually distinct. Following [1], we call $\boxtimes$ the tetrahedron algebra.

Let us consider the following two $\mathbf{C}$-algebras:

$$\mathcal{A} = \mathbf{C}[t, t^{-1}, (t-1)^{-1}], \quad \mathcal{B} = \mathbf{C}[t, t^{-1}].$$

As usual, we endow the $\mathbf{C}$-vector space $sl_2 \otimes \mathcal{A}$ with a structure of Lie algebras over $\mathbf{C}$ by

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab \quad (u, v \in sl_2,\ a, b \in \mathcal{A}).$$

The Lie subalgebra $sl_2 \otimes \mathcal{B}$ of $sl_2 \otimes \mathcal{A}$ is the loop algebra of type sl_2 and will be denoted by $L(sl_2)$.

The tetrahedron algebra $\boxtimes$ is known to be isomorphic to $sl_2 \otimes \mathcal{A}$. (We use a slightly different convention from [1].)

Theorem 2.1 ([1]). (1) *There exists an isomorphism $\boxtimes \to sl_2 \otimes \mathcal{A}$ of Lie algebras over $\mathbf{C}$ determined by*

$$\begin{aligned}
&x_{12} \mapsto x \otimes 1, \quad x_{23} \mapsto y \otimes 1, \quad x_{31} \mapsto z \otimes 1,\\
&x_{03} \mapsto y \otimes t + z \otimes (t-1), \quad x_{01} \mapsto z \otimes t' + x \otimes (t'-1),\\
&x_{02} \mapsto x \otimes t'' + y \otimes (t''-1)
\end{aligned}$$

where $t' = 1 - t^{-1}$, $t'' = (1-t)^{-1}$ and

$$x = 2e - h, \quad y = -2f - h, \quad z = h$$

with e, f, h the standard generators of sl_2.
(2) *Under the isomorphism in* (1), *the subalgebra of $\boxtimes$ generated by x_{12}, x_{23}, x_{31}, x_{03} and x_{01} correspond to the subalgebra $L(sl_2)$ of $sl_2 \otimes \mathcal{A}$.*

In the following, we shall identify $sl_2 \otimes \mathcal{A}$ with $\boxtimes$ via the correspondence in the above theorem.

2.2. $L(sl_2)$ modules and $\boxtimes$ modules. By Theorem 2.1, $L(sl_2)$ is a subalgebra of $\boxtimes$. In this section we study the relation between finite dimensional $L(sl_2)$ modules and finite dimensional $\boxtimes$ modules. The main purpose of this paper is to investigate a q analogue of this relation.

For a Lie algebra L and an L module V, set

$$\mathrm{Ann}_L V = \{x \in L \mid xv = 0 \ (\forall v \in V)\}.$$

For annihilators of finite dimensional $L(sl_2)$ and $\boxtimes$ modules, the following hold.

Proposition 2.1. (1) *For a finite dimensional $L(sl_2)$ module V there exists a unique polynomial $p(t)$ with $p(0) = 1$ such that*

$$\mathrm{Ann}_{L(sl_2)} V = sl_2 \otimes p(t)\mathcal{B}.$$

(2) *For a finite dimensional $\boxtimes$ module V there exists a unique polynomial $p(t)$ with $p(0) = 1$ and $p(1) \neq 0$ such that*

$$\mathrm{Ann}_{\boxtimes} V = sl_2 \otimes p(t)\mathcal{A}.$$

This proposition follows from the following two simple lemmas.

Lemma 2.1 ([3, Theorems 5.2 and 5.3]). *For $A = \mathcal{A}, \mathcal{B}$, J is an ideal of the Lie algebra $sl_2 \otimes A$ if and only if there exists an ideal I of the* **C***-algebra A such that $J = sl_2 \otimes I$.*

Lemma 2.2. *For $A = \mathcal{A}, \mathcal{B}$ the following hold.*
(1) *For a nonzero ideal I of the* **C***-algebra A there exists a unique nonzero polynomial $p(t)$ (up to multiplication by a nonzero scalar) such that $I = p(t)A$.*
(2) *The polynomial in* (1) *satisfies $p(0), p(1) \neq 0$ in the case $A = \mathcal{A}$ and $p(0) \neq 0$ in the case $A = \mathcal{B}$.*

Proof. The polynomial $p(t)$ which generates the ideal $I \cap \mathbf{C}[t]$ of the polynomial algebra $\mathbf{C}[t]$ has the property in the lemma. $\square$

To state our result, we need the isomorphism φ_p in the following lemma.

Lemma 2.3. *Let $p(t)$ be a polynomial with $p(0), p(1) \neq 0$. Then the inclusion $L(sl_2) \to \boxtimes$ induces an isomorphism $\varphi_p\colon L(sl_2)/(sl_2 \otimes p(t)\mathcal{B}) \to \boxtimes/(sl_2 \otimes p(t)\mathcal{A})$ of Lie algebras over* **C**. *Let $g(t)$ be a polynomial such that $(t-1)g(t) \equiv 1 \bmod p(t)\mathbf{C}[t]$. Then the following holds*:

$$\varphi_p^{-1}(\overline{x_{02}}) = \overline{-(x+y) \otimes g(t) - y \otimes 1}.$$

Proof. Since $p(1) \neq 0$, we have $(p(t)\mathcal{A}) \cap \mathcal{B} = p(t)\mathcal{B}$. Hence the inclusion induces an injective homomorphism φ_p. The surjectivity of φ_p and the expression of $\varphi_p^{-1}(\overline{x_{02}})$ follow from the equality $g(t) \equiv (t-1)^{-1} \bmod p(t)\mathcal{A}$. □

The following theorem clarifies the relation between finite dimensional $L(sl_2)$ and $\boxtimes$ modules.

Theorem 2.2. *Let V be a finite dimensional $L(sl_2)$ module and $\rho\colon L(sl_2) \to gl(V)$ the homomorphism associated with V. Let further $p(t)$ be a polynomial with $p(0) = 1$ such that* $\mathrm{Ann}_{L(sl_2)} V = sl_2 \otimes p(t)\mathcal{B}$.
(1) *The $L(sl_2)$ module structure on V is extended to a $\boxtimes$ module structure if and only if $p(1) \neq 0$. In the case $p(1) \neq 0$, the action of $\boxtimes$ is uniquely given by the following composite map*:

$$\boxtimes \to \boxtimes/(sl_2 \otimes p(t)\mathcal{A}) \xrightarrow{\varphi_p^{-1}} L(sl_2)/(sl_2 \otimes p(t)\mathcal{B}) \to gl(V)$$

where the last map is the one induced by ρ. In particular, the action of x_{02} is given by the action of

$$-(x+y) \otimes g(t) - y \otimes 1$$

where $g(t)$ is a polynomial such that $(t-1)g(t) \equiv 1$ mod $p(t)\mathbf{C}[t]$ and x, y and z are those in Theorem 2.1.
(2) *In the case $p(1) \neq 0$, V is irreducible as a $\boxtimes$ module if and only if it is irreducible as an $L(sl_2)$ module.*

Proof. (1) Thanks to Lemma 2.3, it suffices to show that $p(1) \neq 0$ is necessary for the extension and that the extension is unique in the case $p(1) \neq 0$.

Suppose that the $L(sl_2)$ module structure on V is extended to a $\boxtimes$ module structure and let $r(t)$ be a polynomial with $r(0) = 1$ and $r(1) \neq 0$ such that $\mathrm{Ann}_{\boxtimes} V = sl_2 \otimes r(t)\mathcal{A}$. Then, since the original action of $L(sl_2)$ on V coincides with the action via the composite map $L(sl_2) \hookrightarrow \boxtimes \to gl(V)$ $(\star)$, we find that

$$\mathrm{Ann}_{L(sl_2)} V = (\mathrm{Ann}_{\boxtimes} V) \cap L(sl_2) \tag{2.4}$$

and that $(\star)$ induces the following homomorphism:

$$L(sl_2)/\mathrm{Ann}_{L(sl_2)} V \xrightarrow{\varphi} \boxtimes / \mathrm{Ann}_{\boxtimes} V \to gl(V). \tag{2.5}$$

By (2.4), $p(t)\mathcal{B} = (r(t)\mathcal{A}) \cap \mathcal{B} = r(t)\mathcal{B}$. This implies $r(t) = p(t)$. Hence $p(1) \neq 0$ and $\varphi = \varphi_p$. Since φ_p is an isomorphism by Lemma 2.3, we can see that the only possible $\boxtimes$ module structure on V is the one stated in the theorem in the case $p(1) \neq 0$.

(2) Follows from the fact that the action of x_{02} on V is a linear combination of the actions of the elements of $L(sl_2)$. □

3. $U_q(sl_2)$ and $U_q(L(sl_2))$

Leaving the study of a q analogue of the tetrahedron algebra ⊠ to later sections, we summarize necessary results on the quantum groups $U_q(sl_2)$ and $U_q(L(sl_2))$ [11], [12] and their finite dimensional modules in this section.

3.1. Notation. We fix a nonzero complex number q which is not a root of unity. For an integer n and a nonnegative integer m, we set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]! = [1][2]\cdots[m].$$

We further set

$$[a, b]_r = ab - rba$$

for a complex number r and elements a, b of a **C**-algebra.

3.2. $U_q(sl_2)$. We let $U_q(sl_2)$ be the **C**-algebra defined by generators $e,\ f,\ t,\ t^{-1}$ and relations

$$tt^{-1} = t^{-1}t = 1, \tag{3.1}$$

$$tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \tag{3.2}$$

$$[e,\ f] = \frac{t - t^{-1}}{q - q^{-1}}. \tag{3.3}$$

We consider a tensor product of $U_q(sl_2)$ modules via the following comultiplication Δ:

$$\Delta(t) = t \otimes t, \tag{3.4}$$

$$\Delta(e) = e \otimes 1 + t \otimes e, \tag{3.5}$$

$$\Delta(f) = f \otimes t^{-1} + 1 \otimes f. \tag{3.6}$$

3.3. $U_q(L(sl_2))$. We define $U_q(L(sl_2))$ to be the **C**-algebra generated by $e_i,\ f_i,\ t_i,\ t_i^{-1}$ $(i = 0, 1)$ subject to the relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \tag{3.7}$$

$$t_0 t_1 = t_1 t_0 = 1, \tag{3.8}$$

$$t_i e_j t_i^{-1} = q^{4\delta_{ij}-2} e_j, \quad t_i f_j t_i^{-1} = q^{2-4\delta_{ij}} f_j, \tag{3.9}$$

$$[e_i,\ f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \tag{3.10}$$

$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j), \tag{3.11}$$

$$f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j). \tag{3.12}$$

We consider a tensor product of $U_q(sl_2)$ modules via the following comultiplication Δ:

$$\Delta(t_i) = t_i \otimes t_i, \tag{3.13}$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \tag{3.14}$$

$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i. \tag{3.15}$$

It is well known [13], [14] that $U_q(L(sl_2))$ is also defined by generators

$$x_m^+,\ x_m^-,\ h_r,\ k,\ k^{-1} \quad (m \in \mathbf{Z},\ r \in \mathbf{Z} \setminus \{0\})$$

and relations

$$kk^{-1} = k^{-1}k = 1, \tag{3.16}$$

$$[k, h_r] = 0, \tag{3.17}$$

$$[h_r, h_s] = 0, \tag{3.18}$$

$$kx_m^{\pm}k^{-1} = q^{\pm 2}x_m^{\pm}, \tag{3.19}$$

$$[h_r, x_m^{\pm}] = \pm\frac{[2r]}{r}x_{r+m}^{\pm}, \tag{3.20}$$

$$[x_m^+, x_n^-] = \frac{1}{q-q^{-1}}(\Phi_{m+n}^{(+)} - \Phi_{m+n}^{(-)}), \tag{3.21}$$

$$[x_{m+1}^{\pm}, x_n^{\pm}]_{q^{\pm 2}} + [x_{n+1}^{\pm}, x_m^{\pm}]_{q^{\pm 2}} = 0 \tag{3.22}$$

where $\Phi_r^{(\pm)}$ $(r \in \mathbf{Z})$ is defined by $\Phi_{\pm r}^{(\pm)} = 0$ $(r < 0)$ and the generating series

$$\Phi^{(\pm)}(z) = \sum_{r\geq 0} \Phi_{\pm r}^{(\pm)} z^{\mp r} = k^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{r>0} h_{\pm r}z^{\mp r}\right).$$

The correspondence of the generators is given by

$$x_0^+ \leftrightarrow e_1, \quad x_0^- \leftrightarrow f_1, \quad k \leftrightarrow t_1, \tag{3.23}$$

$$x_1^- \leftrightarrow qt_1e_0, \quad x_{-1}^+ \leftrightarrow q^{-1}f_0t_1^{-1}, \tag{3.24}$$

$$h_1 \leftrightarrow q^{-1}[e_1, e_0]_{q^2}, \quad h_{-1} \leftrightarrow q[f_0, f_1]_{q^{-2}}. \tag{3.25}$$

Later we shall use the generating series $x^{\pm}(z) = \sum_{m\in\mathbf{Z}} x_m^{\pm}z^{-m}$ in addition to $\Phi^{(\pm)}(z)$.

We shall identify $U_q(sl_2)$ with the subalgebra of $U_q(L(sl_2))$ via the correspondence $e \mapsto e_1 = x_0^+$, $f \mapsto f_1 = x_0^-$ and $t \mapsto t_1 = k$.

3.4. Finite dimensional $U_q(sl_2)$ modules. For a nonnegative integer n and $\varepsilon \in \{+,-\}$, we let $V_{n,\varepsilon}$ denote the $n+1$ dimensional vector space with basis $v_0, v_1, \ldots, v_n$ on which the action of $U_q(sl_2)$ is defined by

$$ev_i = \varepsilon[n-i+1]v_{i-1}, \quad fv_i = [i+1]v_{i+1}, \quad tv_i = \varepsilon q^{n-2i}v_i \tag{3.26}$$

where $v_{-1} = v_{n+1} = 0$. Any nonzero scalar multiple of the vector v_0 is called a highest weight vector of $V_{n,\varepsilon}$.

The following is well known. See, for example, [15].

Proposition 3.1. (1) *The $U_q(sl_2)$ module $V_{n,\varepsilon}$ is irreducible.*
(2) *Any irreducible finite dimensional $U_q(sl_2)$ module is isomorphic to one of the $V_{n,\varepsilon}$.*
(3) *Any finite dimensional $U_q(sl_2)$ module is isomorphic to a direct sum of the $V_{n,\varepsilon}$.*

We shall call a $U_q(sl_2)$ module V of type 1 (resp. type -1) if V is isomorphic to a direct sum of the $V_{n,+}$ (resp. the $V_{n,-}$).

Later we need the following simple fact.

Lemma 3.1. *Let V be a finite dimensional $U_q(sl_2)$ module and X a linear map on V.*
(1) *If $tX = q^2 Xt$ and $[X, f] = 0$ on V, then $X = 0$.*
(2) *If $tX = q^{-2} Xt$ and $[e, X] = 0$ on V, then $X = 0$.*

Proof. We shall prove (1), the proof of (2) being similar. By Proposition 3.1 V admits a decomposition into a direct sum of irreducible finite dimensional $U_q(sl_2)$ submodules: $V = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(m)}$ with $V^{(i)} \simeq V_{n_i,\varepsilon_i}$. Let $v^{(i)}$ denote a highest weight vector of $V^{(i)}$. Since X commutes with f, it is sufficient to show that $Xv^{(i)} = 0$ for any i. Since $tX = q^2 Xt$, for any i there exist complex numbers c_j such that

$$Xv^{(i)} = \sum_j c_j f^{(n_j - n_i - 2)/2} v^{(j)}$$

where the sum is taken over j such that $n_j \equiv n_i \mod 2$, $n_j \geq n_i + 2$ and $\varepsilon_j = \varepsilon_i$. This implies

$$0 = Xf^{n_i+1} v^{(i)} = f^{n_i+1} Xv^{(i)} = \sum_j c_j f^{n_j - (n_j - n_i)/2} v^{(j)}.$$

Since $f^{n_j-(n_j-n_i)/2} v^{(j)} \neq 0$ for any j, we find that $c_j = 0$ for any j and hence that $Xv^{(i)} = 0$. This completes the proof. □

3.5. Finite dimensional $U_q(L(sl_2))$ modules.

3.5.1. Evaluation $U_q(L(sl_2))$ modules. For a nonzero complex number a we let $\mathrm{ev}_a \colon U_q(L(sl_2)) \to U_q(sl_2)$ be the homomorphism determined by

$$e_1 \mapsto e, \quad f_1 \mapsto f, \quad t_1 \mapsto t, \quad e_0 \mapsto af, \quad f_0 \mapsto a^{-1}e, \quad t_0 \mapsto t^{-1}. \tag{3.27}$$

Via the homomorphism ev_a we can endow a structure of $U_q(L(sl_2))$ modules on any $U_q(sl_2)$ module V. We shall denote the $U_q(L(sl_2))$ module thus obtained by $V(a)$ and call it an evaluation module.

3.5.2. Highest weight $U_q(L(sl_2))$ modules. Here we shall summarize necessary results on finite dimensional highest weight $U_q(L(sl_2))$ modules from [10]. We shall say that a $U_q(L(sl_2))$ module V is a highest weight module if V is generated by a nonzero vector v satisfying

$$x_m^+ v = 0 \ (m \in \mathbf{Z}), \quad \Phi_n^{(+)} v = d_n^+ v, \ \Phi_{-n}^{(-)} v = d_{-n}^- v \ (n \geq 0)$$

with complex numbers d_n^+ and d_{-n}^- satisfying $d_0^+ d_0^- = 1$. We call v a highest weight vector of V and $\mathbf{d} = (d_n^+, d_{-n}^-)_{n \in \mathbf{Z}_{\geq 0}}$ a highest weight of V.

Let us consider a highest weight $\mathbf{d}$ which is related to a polynomial $P(u)$ with constant term 1 in the following way:

$$\sum_{m \geq 0} d_m^+ u^m = q^{\deg P} \frac{P(q^{-2}u)}{P(u)}, \quad \sum_{m \geq 0} d_{-m}^- u^{-m} = q^{\deg P} \frac{P(q^{-2}u)}{P(u)}$$

where the r.h.s. of the first equality (resp. the second equality) should be understood as a Laurent expansion around 0 (resp. ∞). We shall say that a highest weight module with the above highest weight $\mathbf{d}$ is a highest weight module with Drinfeld polynomial $P(u)$. In particular we denote it by $V(P)$ if it is irreducible.

For $\varepsilon \in \{1, -1\}$, we shall say that a $U_q(L(sl_2))$ module V is of type ε if it is of type ε as a $U_q(sl_2)$ module.

The following were proven in [10].

Theorem 3.1. (1) *If a highest weight $U_q(L(sl_2))$ module of type 1 is finite dimensional, then it is a highest weight module with some $P(u)$ as an associated Drinfeld polynomial.*
(2) *A $U_q(L(sl_2))$ module V of type 1 is irreducible and finite dimensional if and only if it is isomorphic to one of the $V(P)$.*

Proposition 3.2. *The Drinfeld polynomial $P(u)$ associated with the evaluation $U_q(L(sl_2))$ module $V_{n,+}(a)$ is $P(u) = \prod_{1 \leq i \leq n}(1 - q^{n+1-2i} au)$.*

Proposition 3.3. *Let V and W be highest weight $U_q(L(sl_2))$ modules with Drinfeld polynomials $P_V(u)$ and $P_W(u)$, respectively. Then the submodule of $V \otimes W$ generated by the tensor product of the highest weight vectors is a highest weight module with Drinfeld polynomial $P_V(u)P_W(u)$.*

Theorem 3.2. (1) *$V_{n_1,+}(a_1) \otimes \cdots \otimes V_{n_N,+}(a_N)$ is irreducible if and only if $a_i/a_j \notin \{q^{n_i+n_j}, q^{n_i+n_j-2}, \ldots, q^{|n_i-n_j|+2}\}$ for any distinct i, j.*

(2) *Any irreducible finite dimensional* $U_q(L(sl_2))$ *module of type* 1 *is isomorphic to some* $V_{n_1,+}(a_1) \otimes \cdots \otimes V_{n_N,+}(a_N)$.

4. The q-tetrahedron algebra $\boxtimes_q$

4.1. The q-tetrahedron algebra $\boxtimes_q$.

4.1.1. In [7] the q-tetrahedron algebra $\boxtimes_q$, a q analogue of the universal enveloping algebra of $\boxtimes$, was introduced. Set

$$\{X, Y\}_r = \frac{rXY - r^{-1}YX}{r - r^{-1}}$$

for a complex number r different from 0 and ± 1 and elements X, Y of a $\mathbf{C}$-algebra. By definition, the q-tetrahedron algebra is the $\mathbf{C}$-algebra generated by X_{01}, X_{12}, X_{23}, X_{30}, X_{13}, X_{31}, X_{02}, X_{20} subject to the following relations:

$$X_{13}X_{31} = X_{31}X_{13} = 1, \tag{4.1}$$

$$X_{02}X_{20} = X_{20}X_{02} = 1, \tag{4.2}$$

$$\{X_{01}, X_{12}\}_q = 1, \quad \{X_{12}, X_{23}\}_q = 1, \quad \{X_{23}, X_{30}\}_q = 1, \quad \{X_{30}, X_{01}\}_q = 1, \tag{4.3}$$

$$\{X_{01}, X_{13}\}_q = 1, \quad \{X_{13}, X_{30}\}_q = 1, \quad \{X_{23}, X_{31}\}_q = 1, \quad \{X_{31}, X_{12}\}_q = 1, \tag{4.4}$$

$$\{X_{02}, X_{23}\}_q = 1, \quad \{X_{12}, X_{20}\}_q = 1, \quad \{X_{20}, X_{01}\}_q = 1, \quad \{X_{30}, X_{02}\}_q = 1, \tag{4.5}$$

$$X_{ij}^3 X_{kl} - [3]X_{ij}^2 X_{kl} X_{ij} + [3]X_{ij} X_{kl} X_{ij}^2 - X_{kl} X_{ij}^3 = 0 \quad ((i, j, k, l) \in J) \tag{4.6}$$

where $J = \{(0, 1, 2, 3), (2, 3, 0, 1), (1, 2, 3, 0), (3, 0, 1, 2)\}$.

This q analogue is related to $\boxtimes$ as follows. In terms of $y_{ij} = 2(1 - X_{ij})/(q - q^{-1})$, the relations $\{X_{ij}, X_{jk}\}_q = 1$ and (4.6) are rewritten as

$$q y_{ij} y_{jk} - q^{-1} y_{jk} y_{ij} = 2(y_{ij} + y_{jk})$$

and

$$[y_{ij}, [y_{ij}, [y_{ij}, y_{kl}]]] = 4X_{ij}[y_{ij}, y_{kl}]X_{ij},$$

respectively. Set $q = e^{\hbar}$. If $X_{ij} = 1 - \hbar x_{ij} + o(\hbar)$, then $y_{ij} \to x_{ij}$ in the limit $\hbar \to 0$ and the above two equalities reduce to (2.2) and (2.3) except for (2.3) with $(h, i, j, k) = (0, 2, 1, 3), (1, 3, 0, 2)$.

4.1.2. Let us consider the following three subalgebras of $\boxtimes_q$:

$$\boxtimes_q^- = \langle X_{01}, X_{23} \rangle, \quad \boxtimes_q^- = \langle X_{13}, X_{31}, X_{20}, X_{02} \rangle, \quad \boxtimes_q^+ = \langle X_{12}, X_{30} \rangle.$$

We define A_q to be the $\mathbf{C}$-algebra generated by X and Y subject to the following relations:

$$X^3Y - [3]X^2YX + [3]XYX^2 - YX^3 = 0, \tag{4.7}$$

$$Y^3X - [3]Y^2XY + [3]YXY^2 - XY^3 = 0. \tag{4.8}$$

In [7], it was noted that $\boxtimes_q = \boxtimes_q^- \boxtimes_q^0 \boxtimes_q^+$ and that both $\boxtimes_q^-$ and $\boxtimes_q^+$ are homomorphic images of A_q. More precisely the following proposition can be proven in a standard way (see, for example, [15, Chapter 4]). This result is due to J. Nakata and proves [7, Conjecture 19.6].

Proposition 4.1. (1) *The multiplication map $\boxtimes_q^- \otimes \boxtimes_q^0 \otimes \boxtimes_q^+ \to \boxtimes_q$ is an isomorphism of vector spaces.*
(2) *The subalgebras $\boxtimes_q^-$ and $\boxtimes_q^+$ are isomorphic to A_q via the correspondence $X_{01} \leftrightarrow X$, $X_{23} \leftrightarrow Y$ and the correspondence $X_{12} \leftrightarrow X$, $X_{30} \leftrightarrow Y$, respectively.*
(3) *The defining relations of $\boxtimes_q^0$ are*

$$X_{13}X_{31} = X_{31}X_{13} = 1, \quad X_{20}X_{02} = X_{02}X_{20} = 1.$$

4.2. The equitable presentation of $U_q(L(sl_2))$. By Theorem 2.1, $L(sl_2)$ is a subalgebra of $\boxtimes$. As a q analogue of this, we can prove that the q-tetrahedron algebra $\boxtimes_q$ contains $U_q(L(sl_2))$ as a subalgebra. For this it is convenient to introduce another presentation of $U_q(L(sl_2))$ called the equitable presentation [16], [17].

Let B_q be the $\mathbf{C}$-algebra generated by X_{01}, X_{12}, X_{23}, X_{30}, X_{13}, X_{31} subject to relations (4.1), (4.3), (4.4) and (4.6). This algebra is known to be isomorphic to $U_q(L(sl_2))$. (We use a slightly different convention from [7].)

Proposition 4.2 ([7]). (1) *There exists a $\mathbf{C}$-algebra isomorphism $B_q \to U_q(L(sl_2))$ determined by*

$$X_{12} \mapsto t_1 - (q-q^{-1})e_1, \quad X_{23} \mapsto t_1 + q(q-q^{-1})t_1 f_1, \quad X_{31} \mapsto t_1^{-1},$$
$$X_{30} \mapsto t_0 - (q-q^{-1})e_0, \quad X_{01} \mapsto t_0 + q(q-q^{-1})t_0 f_0, \quad X_{13} \mapsto t_0^{-1}.$$

(2) *If we identify B_q with $U_q(L(sl_2))$ via the correspondence in* (1), *then the comultiplication Δ of $U_q(L(sl_2))$ maps the elements $X_{ij} \in U_q(L(sl_2))$ as follows*:

$$\Delta(X_{13}) = X_{13} \otimes X_{13}, \tag{4.9}$$

$$\Delta(X_{12}) = (X_{12} - X_{13}) \otimes 1 + X_{13} \otimes X_{12}, \tag{4.10}$$

$$\Delta(X_{23}) = (X_{23} - X_{13}) \otimes 1 + X_{13} \otimes X_{23}, \tag{4.11}$$

$$\Delta(X_{30}) = (X_{30} - X_{31}) \otimes 1 + X_{31} \otimes X_{30}, \tag{4.12}$$

$$\Delta(X_{01}) = (X_{01} - X_{31}) \otimes 1 + X_{31} \otimes X_{01}. \tag{4.13}$$

As noted in [7], the subalgebra of $\boxtimes_q$ generated by X_{01}, X_{12}, X_{23}, X_{30}, X_{13} and X_{31} is a homomorphic image of B_q by the definition of B_q. Using Proposition 4.1, we can easily prove that this subalgebra is isomorphic to B_q.

Proposition 4.3. *The homomorphism $B_q \to \boxtimes_q$ determined by $X_{ij} \mapsto X_{ij}$ is injective.*

By Propositions 4.2 and 4.3, we can and do identify the subalgebra of $\boxtimes_q$ generated by X_{01}, X_{12}, X_{23}, X_{30}, X_{13} and X_{31} with $U_q(L(sl_2))$.

5. $\boxtimes_q$ modules

Now we start the study of finite dimensional $\boxtimes_q$ modules. First note that any $\boxtimes_q$ module can be regarded both as a $U_q(L(sl_2))$ module and as a $U_q(sl_2)$ module, since the q-tetrahedron algebra $\boxtimes_q$ contains $U_q(L(sl_2))$ as a subalgebra.

5.1. $\boxtimes_q$ modules.

Lemma 5.1. *Let V be a finite dimensional $U_q(sl_2)$ module. Then any linear map g on V admits the following unique decomposition*:

$$g = \sum_{\substack{m \in \mathbf{Z} \\ \varepsilon = +,-}} g_{m/2,\varepsilon} \ (\textit{finite sum}), \quad X_{13} g_{m/2,\varepsilon} X_{31} = \varepsilon q^m g_{m/2,\varepsilon}. \tag{5.1}$$

Proof. By Proposition 3.1 V is isomorphic to a direct sum of the $V_{n,\varepsilon}$. Hence $X_{13} = t_1$ is diagonalizable on V with eigenvalues εq^m ($\varepsilon \in \{+, -\}$, $m \in \mathbf{Z}$). The assertion follows from this since εq^m ($\varepsilon \in \{+, -\}$, $m \in \mathbf{Z}$) are distinct. □

For the actions of X_{02} and X_{20} on a finite dimensional $\boxtimes_q$ module, we can consider $(X_{02})_{m/2,\varepsilon}$ and $(X_{20})_{m/2,\varepsilon}$ as in the above lemma. We shall call them components of X_{02} and X_{20}. We can prove the following proposition for the components of X_{02} and X_{20}.

Proposition 5.1. *Let V be a finite dimensional $\boxtimes_q$ module. Then the following hold for the components $(X_{02})_{n,\varepsilon}$ and $(X_{20})_{n,\varepsilon}$ ($n \in \mathbf{Z}/2$, $\varepsilon = \pm$):*
(1) $(X_{02})_{n,\varepsilon} = 0$ *unless* $\varepsilon = +$, $n \in \mathbf{Z}$ *and* $n \le 1$.
(2) $(X_{20})_{n,\varepsilon} = 0$ *unless* $\varepsilon = +$, $n \in \mathbf{Z}$ *and* $n \ge -1$.

The proof of this proposition will be given in Section 5.2.

Now that we know $(X_{02})_{n,\varepsilon} = (X_{20})_{n,\varepsilon} = 0$ unless $\varepsilon = +$ and $n \in \mathbf{Z}$, we shall simply denote $(X_{02})_{n,+}$ and $(X_{20})_{n,+}$ by $(X_{02})_n$ and $(X_{20})_n$, respectively, and assume that the subscript n runs over integers. Then the following hold on a finite dimensional $\boxtimes_q$ module:

$$X_{02} = \sum_{n \in \mathbf{Z}_{\geq 0}} (X_{02})_{1-n}, \quad X_{13}(X_{02})_n X_{31} = q^{2n}(X_{02})_n, \tag{5.2}$$

$$X_{20} = \sum_{n \in \mathbf{Z}_{\geq 0}} (X_{20})_{n-1}, \quad X_{13}(X_{20})_n X_{31} = q^{2n}(X_{20})_n. \tag{5.3}$$

Proposition 5.2. *Let V be a finite dimensional $\boxtimes_q$ module. Then V admits a direct sum decomposition*

$$V = \bigoplus_{n \in \mathbf{Z}, \varepsilon = \pm} V(n, \varepsilon), \quad V(n, \varepsilon) = \{v \in V \mid X_{13} v = \varepsilon q^n v\}$$

and both $V(+) = \bigoplus_{n \in \mathbf{Z}} V(n, +)$ and $V(-) = \bigoplus_{n \in \mathbf{Z}} V(n, -)$ are submodules of the $\boxtimes_q$ module V.

Proof. The existence of the decomposition $V = \bigoplus_{n \in \mathbf{Z}, \varepsilon = \pm} V(n, \varepsilon)$ was already proved in the proof of Lemma 5.1. By (5.2) and (5.3), both X_{02} and X_{20} preserve each of $V(+)$ and $V(-)$. Since the algebra $\boxtimes_q$ is generated by the subalgebra $U_q(L(sl_2))$ and the elements X_{02} and X_{20}, $\boxtimes_q$ also preserves each of $V(+)$ and $V(-)$. $\square$

Following [7], we shall say that for $\varepsilon \in \{1, -1\}$ a finite dimensional $\boxtimes_q$ module V is of type ε if X_{31} is diagonalizable on V with eigenvalues in $\varepsilon q^{\mathbf{Z}}$. Any finite dimensional $\boxtimes_q$ module of type -1 is obtained from a finite dimensional $\boxtimes_q$ module of type 1 by the use of the automorphism of $\boxtimes_q$ such that $X_{ij} \mapsto -X_{ij}$. Hence by the above proposition we can consider only finite dimensional $\boxtimes_q$ modules of type 1 without loss of generality.

5.2. Proof of Proposition 5.1.

First we prove the following lemma.

Lemma 5.2. *On a finite dimensional $\boxtimes_q$ module, the following hold for $n \in \mathbf{Z}/2$ and $\varepsilon \in \{+, -\}$:*

(1) $\{(X_{02})_{n+1,\varepsilon}, X_{23} - X_{13}\}_q = \delta_{n,0}\delta_{\varepsilon,+} - \frac{q - \varepsilon q^{2n-1}}{q - q^{-1}} (X_{02})_{n,\varepsilon} X_{13}$,

(2) $\{X_{30} - X_{31}, (X_{02})_{n+1,\varepsilon}\}_q = \delta_{n,0}\delta_{\varepsilon,+} - \frac{q - \varepsilon q^{2n-1}}{q - q^{-1}} X_{31} (X_{02})_{n,\varepsilon}$,

(3) $\{X_{12} - X_{13}, (X_{20})_{n-1,\varepsilon}\}_q = \delta_{n,0}\delta_{\varepsilon,+} - \frac{\varepsilon q^{2n+1} - q^{-1}}{q - q^{-1}} (X_{20})_{n,\varepsilon} X_{13}$,

(4) $\{(X_{20})_{n-1,\varepsilon}, X_{01} - X_{31}\}_q = \delta_{n,0}\delta_{\varepsilon,+} - \frac{\varepsilon q^{2n+1} - q^{-1}}{q - q^{-1}} X_{31} (X_{20})_{n,\varepsilon}$.

Proof. By (4.4) and (4.5), $\{X_{23}, X_{31}\}_q = 1$ and $\{X_{02}, X_{23}\}_q = 1$. From these and the definition of the $(X_{02})_{n,\varepsilon}$, we find that

$$q(X_{23} - X_{13})X_{31} = q^{-1}X_{31}(X_{23} - X_{13})$$

and

$$\{X_{02}, X_{23} - X_{13}\}_q = 1 - \sum_{n,\varepsilon} \frac{q - \varepsilon q^{2n-1}}{q - q^{-1}}(X_{02})_{n,\varepsilon}X_{13}.$$

The former equality implies $X_{23} - X_{13} = (X_{23} - X_{13})_{-1,+}$. Noting this, we obtain from the latter that

$$\{(X_{02})_{n+1,\varepsilon}, X_{23} - X_{13}\}_q = \delta_{n,0}\delta_{\varepsilon,+} - \frac{q - \varepsilon q^{2n-1}}{q - q^{-1}}(X_{02})_{n,\varepsilon}X_{13}$$

for any $n \in \mathbf{Z}/2$ and $\varepsilon = \pm$. This proves (1). The proof of the rest of the assertions are similar. □

Now we can prove Proposition 5.1.

Proof of Proposition 5.1. We shall prove the assertions for the $(X_{02})_{n,\varepsilon}$, the proof of the case $(X_{20})_{n,\varepsilon}$ being similar. Suppose that for $\varepsilon \in \{+, -\}$ there exists an integer n such that $(X_{02})_{n,\varepsilon} \neq 0$ and let n_ε denote the largest one. Then, since $(X_{02})_{n_\varepsilon+1,\varepsilon} = 0$, we find from Lemma 5.2 (1) that

$$\frac{q - \varepsilon q^{2n_\varepsilon-1}}{q - q^{-1}}(X_{02})_{n_\varepsilon,\varepsilon}X_{13} = \delta_{n_\varepsilon,0}\delta_{\varepsilon,+}.$$

Since q is not a root of unity and X_{13} is invertible, the above equality is inconsistent in the case $\varepsilon = -$ and n_ε must be 0 or 1 in the case $\varepsilon = +$. This implies that $(X_{02})_{n,-} = 0$ for any integer n and that $(X_{02})_{n,+} = 0$ if n is an integer greater than or equal to 2. Similarly we can prove $(X_{02})_{n,\varepsilon} = 0$ for any $n \in \mathbf{Z} + 1/2$ and $\varepsilon = \pm$. This completes the proof. □

6. $U_q(L(sl_2))$ modules and $\boxtimes_q$ modules

Recall that $\boxtimes_q$ has $U_q(L(sl_2))$ as a subalgebra by Propositions 4.2 and 4.3. In this section, we shall study which finite dimensional $U_q(L(sl_2))$ module is extended to a $\boxtimes_q$ module. For this problem we can prove a q analogue of Theorem 2.2.

6.1. Main result 1. First we introduce an annihilating polynomial $p_V(z)$ for a finite dimensional $U_q(L(sl_2))$ module V, which is a q analogue of the polynomial $p(t)$ in Proposition 2.1 (1). The proof of the following proposition will be given in the next subsection.

Proposition 6.1. *For an integer m we set $\Psi_m = \Phi_m^{(+)} - \Phi_m^{(-)}$ and consider the generating series $\Psi(z) = \sum_{m\in\mathbf{Z}} \Psi_m z^{-m}$. Let V be a finite dimensional $U_q(L(sl_2))$ module and set*

$$I_V^\varepsilon = \{ f(z) \in \mathbf{C}[z] \mid f(z)x^\varepsilon(z) = 0 \text{ on } V\} \quad (\varepsilon = \pm),$$
$$I_V^0 = \{ f(z) \in \mathbf{C}[z] \mid f(z)\Psi(z) = 0 \text{ on } V\}.$$

Then $I_V^+ = I_V^- = I_V^0$ and this is a nonzero ideal of $\mathbf{C}[z]$ generated by a polynomial with nonzero constant term.

Let I_V signify the ideal $I_V^+ = I_V^- = I_V^0$ in the proposition. We shall denote the polynomial with constant term 1 generating the ideal I_V by $p_V(z)$ and call it the annihilating polynomial for V.

For a nilpotent linear map g on a vector space, we define its q-exponential $\exp_q g$ (see, for example, [18, Chapter IV]) by

$$\exp_q g = \sum_{l=0}^{\infty} q^{-l(l-1)/2} \frac{g^l}{[l]!}.$$

Using $p_V(z)$, we can state the following theorem, whose proof will be given in Subsection 6.3.

Theorem 6.1. *Let V be a finite dimensional $U_q(L(sl_2))$ module and $p_V(z)$ the annihilating polynomial for V.*
(1) *The $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure if and only if $p_V(1) \neq 0$.*
(2) *In the case $p_V(1) \neq 0$ the extension is unique. Let $g(z) = \sum_{j=0}^{M} d_j z^j$ be a polynomial such that $(z-1)g(z) \equiv 1 \mod p_V(z)\mathbf{C}[z]$. Then the action of X_{02} on V is given by*

$$X_{02} = (\exp_q x_0^-)\left(k^{-1} + (q - q^{-1}) \sum_{j=0}^{M} d_j x_j^+ \right)(\exp_q x_0^-)^{-1}.$$

(3) *In the case $p_V(1) \neq 0$, V is irreducible as a $\boxtimes_q$ module if and only if it is irreducible as a $U_q(L(sl_2))$ module.*

6.2. Proof of Proposition 6.1. For a polynomial $f(z) = \sum_{j=0}^{N} c_j z^j$ and a generating series $X(z) = \sum_{m\in\mathbf{Z}} X_m z^{-m}$, the following holds:

$$f(z)X(z) = \sum_{m\in\mathbf{Z}} \left(\sum_{j=0}^{N} c_j X_{j+m} \right) z^{-m}. \tag{6.1}$$

Noting this, we can prove Proposition 6.1.

Proof of Proposition 6.1. By Lemma 3.1 and the equality

$$\sum_{j=0}^{N} c_j \Psi_{j+m} = (q-q^{-1})\left[\sum_{j=0}^{N} c_j x^+_{j+m}, x^-_0\right] = (q-q^{-1})\left[x^+_0, \sum_{j=0}^{N} c_j x^-_{j+m}\right],$$

we can see that the following conditions (i)$_\varepsilon$ ($\varepsilon = \pm$) and (ii) for complex numbers $c_0, c_1, \ldots, c_N$ are equivalent:
(i)$_\varepsilon$ $\sum_{j=0}^{N} c_j x^\varepsilon_{j+m} = 0$ on V for any integer m,
(ii) $\sum_{j=0}^{N} c_j \Psi_{j+m} = 0$ on V for any integer m.
This and (6.1) imply that $I^+_V = I^-_V = I^0_V$. If $f(z)x^+(z) = 0$ on V for some polynomial $f(z)$, then $g(z)f(z)x^+(z) = 0$ and $z^m f(z)x^+(z) = 0$ on V for any polynomial $g(z)$ and any integer m. Hence I^+_V is an ideal of $\mathbf{C}[z]$ generated by a polynomial with nonzero constant term. □

6.3. Proof of Theorem 6.1. First we state one lemma and one proposition, the proof of which will be given in the next subsection.

Lemma 6.1. *On a finite dimensional $\boxtimes_q$ module the components $(X_{02})_n$ of X_{02} satisfy the following equalities*:
(1) $(X_{02})_{-n} = \delta_{n,0}k^{-1} - q^n[(X_{02})_{1-n}, x^-_0]_{q^{-2n}}/[n+1]$ $(n \geq 0)$,
(2) $[(X_{02})_1, x^-_{m+1} - x^-_m] = \Psi_m$ $(m \in \mathbf{Z})$.

Proposition 6.2. *Let V be a finite dimensional $U_q(L(sl_2))$ module and suppose that $y \in \mathrm{End}(V)$ satisfies*

$$kyk^{-1} = q^2 y, \tag{6.2}$$

$$[y, x^-_1 - x^-_0] = k - k^{-1}, \tag{6.3}$$

$$[x^+_0, y]_{q^2} = qy^2, \tag{6.4}$$

$$[x^+_{-1}, y]_{q^{-2}} = q^{-1}y^2. \tag{6.5}$$

Then the $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure by letting

$$X_{02} = (\exp_q x^-_0)(k^{-1} + y)(\exp_q x^-_0)^{-1}. \tag{6.6}$$

Assuming the above, we prove Theorem 6.1.

Proof of Theorem 6.1. If the $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure, then $(X_{02})_1$ satisfies (2) in Lemma 6.1. In terms of generating series, this is written as follows:

$$\Psi(z) = (z-1)[(X_{02})_1, x^-(z)]. \tag{6.7}$$

Suppose that $p_V(1) = 0$. Then $p_V(z)$ factorizes as $p_V(z) = (z-1)r(z)$ ($\star$) with some polynomial $r(z)$. Multiplying (6.7) by the polynomial $r(z)$, we find that

$$r(z)\Psi(z) = [(X_{02})_1,\ p_V(z)x^-(z)] = 0$$

on V and hence that $r(z) \in I_V$. This is inconsistent with ($\star$). Therefore $p_V(1) \neq 0$ is necessary for the extension.

Next, assuming that the extension is possible in the case $p_V(1) \neq 0$, we shall show the uniqueness. By Lemma 6.1 (1) $(X_{02})_n$ $(n \leq 0)$ is uniquely determined by $(X_{02})_1$ and X_{20} is the inverse of X_{02}. Hence it is sufficient to prove the uniqueness of $(X_{02})_1$. Suppose that the $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure in two ways and denote the difference of the two actions of X_{02} on V by δX_{02}. Then

$$(z-1)[(\delta X_{02})_1, x^-(z)] = 0$$

by (6.7). This implies

$$p_V(1)[(\delta X_{02})_1, x^-(z)] = (p_V(1) - p_V(z))[(\delta X_{02})_1, x^-(z)] = 0.$$

Since $p_V(1) \neq 0$, the coefficient of z^0 in the above equality gives $[(\delta X_{02})_1, x_0^-] = 0$. Hence we find that $(\delta X_{02})_1 = 0$ by Lemma 3.1. This completes the proof of the uniqueness.

Finally we shall prove that the extension is possible as in (2) in the case $p_V(1) \neq 0$. Let $g(z) = \sum_{j=0}^{M} d_j z^j$ be the the polynomial in part (2) of the theorem and set $y = (q - q^{-1}) \sum_{j=0}^{M} d_j x_j^+$. It is sufficient to show that y satisfies (6.2)–(6.5) in Proposition 6.2. Eq. (6.2) clearly holds. Since

$$[y, x^-(z)] = \sum_{m \in \mathbf{Z}} \left(\sum_{j=0}^{M} d_j \Psi_{j+m} \right) z^{-m} = g(z)\Psi(z),$$

we find that

$$(z-1)[y, x^-(z)] = (z-1)g(z)\Psi(z) = \Psi(z)$$

on V. The coefficient of z^0 in the above equality proves (6.3). The relation (3.22) for the x_m^+ is equivalent to

$$(z - q^2 w)x^+(z)x^+(w) + (w - q^2 z)x^+(w)x^+(z) = 0. \tag{6.8}$$

As

$$zg(z)x^+(z) = g(z)x^+(z) + (z-1)g(z)x^+(z) = g(z)x^+(z) + x^+(z)$$

on V, the coefficient of z^0 in $zg(z)x^+(z)$ is equal to $y/(q-q^{-1})+x_0^+$. Noting this, we find that the coefficient of z^0w^0 in $g(z)g(w)\times(6.8)$ yields (6.4). The proof of (6.5) is similar. This completes the proof of (1) and (2).

Part (3) follows from (2) since the actions of X_{02} and X_{20} on V are expressed in terms of the actions of the elements of $U_q(L(sl_2))$. □

6.4. Proof of Lemma 6.1 and Proposition 6.2. To prove Lemma 6.1 and Proposition 6.2, we need the following two lemmas. Recall that we are identifying $U_q(L(sl_2))$ with the subalgebra of $\boxtimes_q$ generated by X_{01}, X_{12}, X_{23}, X_{30}, X_{13} and X_{31} by Propositions 4.2 and 4.3.

Lemma 6.2. *Let V be a finite dimensional $U_q(L(sl_2))$ module.*
(1) *The $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure if and only if there exists an invertible linear map X_{02} on V which admits a decomposition* (5.2), *i.e.*,

$$X_{02} = \sum_{n\in\mathbf{Z}_{\geq 0}} (X_{02})_{1-n} \ (\textit{finite sum}), \quad k(X_{02})_n k^{-1} = q^{2n}(X_{02})_n \tag{6.9}$$

and satisfies

$$\{X_{02}, X_{23}\}_q = 1, \quad \{X_{30}, X_{02}\}_q = 1, \tag{6.10}$$

$$\{X_{02}, X_{12}\}_q = (X_{02})^2, \quad \{X_{01}, X_{02}\}_q = (X_{02})^2. \tag{6.11}$$

(2) *In terms of the components* $(X_{02})_n$ $(n \leq 1)$, (6.10) *and* (6.11) *are equivalent to the following equalities with* $n \geq 0$:

(i) $[(X_{02})_{1-n}, x_0^-]_{q^{-2n}} = \delta_{n,0}k^{-1} - q^{-n}[n+1](X_{02})_{-n}$,

(ii) $[(X_{02})_{1-n}, x_1^-]_{q^{2n}} = \delta_{n,0}k - q^{n}[n+1](X_{02})_{-n}$,

(iii) $[x_0^+, (X_{02})_{1-n}]_{q^2} = q(X_{02}^2)_{2-n} - q^{n-1}[n-1]k(X_{02})_{2-n}$,

(iv) $[x_{-1}^+, (X_{02})_{1-n}]_{q^{-2}} = q^{-1}(X_{02}^2)_{2-n} - q^{-(n-1)}[n-1]k^{-1}(X_{02})_{2-n}$

where $(X_{02}^2)_{2-n} = \sum_{\substack{l,m\geq 0\\ l+m=n}} (X_{02})_{1-l}(X_{02})_{1-m}$ *and* $(X_{02})_2 = 0$.

Proof. (1) The assertion follows from (4.2), (4.5) and Proposition 5.1.

(2) By Proposition 4.2, (3.23) and (3.24),

$$X_{23} = k + q(q-q^{-1})kx_0^-, \quad X_{30} = k^{-1} - q^{-1}(q-q^{-1})k^{-1}x_1^-,$$
$$X_{12} = k - (q-q^{-1})x_0^+, \qquad X_{01} = k^{-1} + (q-q^{-1})x_{-1}^+.$$

Using these, the assertion is easily proven. □

Lemma 6.3. *On a finite dimensional $\boxtimes_q$ module, the component $(X_{02})_1$ satisfies the following equalities*:

(1) $[h_1, (X_{02})_1]/[2] = (X_{02})_1 + (q - q^{-1})x_0^+$,
(2) $[h_{-1}, (X_{02})_1]/[2] = (X_{02})_1 - (q - q^{-1})x_{-1}^+$.

Proof. Set $y = (X_{02})_1$. The difference of (i) and (ii) of Lemma 6.2 (2) with $n = 0$ yields

$$[y, x_1^- - x_0^-] = k - k^{-1} \tag{6.12}$$

Using this, part (1) is derived by the following calculation:

$$\begin{aligned}
&[h_1, y]k - [2]yk \\
&= [h_1, y]k - q^{-2}[2]yk^{-1} - q^{-1}[y^2, x_1^- - x_0^-] && \text{(by (6.12))} \\
&= q^{-2}[[x_0^+, x_1^- - x_0^-], y]_{q^2} - q^{-2}[[x_0^+, y]_{q^2}, x_1^- - x_0^-] \\
& && \text{(by (3.21) and Lemma 6.2 (2) (iii) with } n = 0) \\
&= -q^{-2}[x_0^+, [y, x_1^- - x_0^-]]_{q^2} \\
&= -q^{-2}[x_0^+, k - k^{-1}]_{q^2} && \text{(by (6.12))} \\
&= (q^2 - q^{-2})x_0^+ k.
\end{aligned}$$

The proof of (2) is similar. □

Now we can prove Lemma 6.1 and Proposition 6.2.

Proof of Lemma 6.1. Part (1) immediately follows from (i) of Lemma 6.2 (2). Part (2) with $m = 0$ is nothing but (6.12). Since $[h_1, \Psi_m] = 0$ and $[h_1, x_n^-] = -[2]x_{n+1}^-$, the commutator of the case m of (2) and h_1 yields

$$[(X_{02})_1, x_{m+2}^- - x_{m+1}^-] = \frac{1}{[2]}[[h_1, (X_{02})_1], x_{m+1}^- - x_m^-].$$

Hence (2) with $m > 0$ is proven by induction on m, using Lemma 6.3 (1). Part (2) with $m < 0$ is similarly proven by the use of Lemma 6.3 (2). □

Proof of Proposition 6.2. We define Y_{-n} $(n \geq -1)$ inductively by $Y_1 = y$ and

$$Y_{-n} = \delta_{n,0}k^{-1} - \frac{q^n}{[n+1]}[Y_{1-n}, x_0^-]_{q^{-2n}} \quad (n \geq 0) \tag{6.13}$$

and set $Y = \sum_{n\geq 0} Y_{1-n}$. Let us denote (i)–(iv) of Lemma 6.2 (2) with $(X_{02})_n$ replaced by Y_n simply by (i)–(iv). By Lemma 6.2, we can see that letting $X_{02} = Y$ defines a $\boxtimes_q$ module structure on V if (a) the Y_n satisfy $kY_nk^{-1} = q^{2n}Y_n$ and (i)–(iv) and (b) Y is invertible.

First we shall show (a). By (6.2) and (6.13), the equalities $kY_nk^{-1} = q^{2n}Y_n$ hold. Part (i) is nothing but (6.13). Part (ii) is easily proven by induction on n, using (6.13), (6.3) and $[x_1^-, x_0^-]_{q^{-2}} = 0$. Using (6.13), we can show

$$[(Y^2)_{2-n}, x_0^-]_{q^{-2n}} = q^{-1}[2]Y_{1-n}k^{-1} - q^{-n}[n+1](Y^2)_{1-n} \quad (n \geq 0).$$

Utilizing (6.4), (6.13) and the above equality, we can show (iii) by induction on n. The proof of (iv) is similar.

Next we shall prove (b). By the definition of the Y_n,

$$Y_0 = k^{-1} + [x_0^-, Y_1], \quad Y_{-n} = \frac{q^{-n}}{[n+1]}[x_0^-, Y_{1-n}]_{q^{2n}} \quad (n \geq 1).$$

Hence we find that $Y = \sum_{n\geq 0} Z_{-n} + \sum_{n\geq -1} W_{-n}$ with $Z_0 = k^{-1}$, $W_1 = Y_1$ and

$$Z_{-n} = \frac{q^{-n(n+1)/2}}{[n+1]!}[x_0^-, \cdots [x_0^-, [x_0^-, k^{-1}]_{q^2}]_{q^4}, \cdots]_{q^{2n}} \quad (n \geq 1),$$
$$W_{-n} = \frac{q^{-n(n+1)/2}}{[n+1]!}[x_0^-, \cdots [x_0^-, [x_0^-, Y_1]_{q^0}]_{q^2}, \cdots]_{q^{2n}} \quad (n \geq 0).$$

Since Z_{-n} with $n \geq 1$ is rewritten as

$$Z_{-n} = \frac{q^{-n(n-1)/2}}{[n]!}[x_0^-, \cdots [x_0^-, [x_0^-, k^{-1}]_{q^0}]_{q^2}, \cdots]_{q^{2(n-1)}},$$

we find by Lemma 6.4 below that

$$Y = (\exp_q x_0^-)(k^{-1} + Y_1)(\exp_q x_0^-)^{-1}.$$

This expression proves (6.6) and the assertion (b). □

Lemma 6.4. *Let V be a vector space. For a nilpotent linear map g and a linear map X on V, the following hold*:
(1) $(\exp_q g)^{-1} = \sum_{l=0}^{\infty}(-1)^l q^{l(l-1)/2} g^l/[l]!$,
(2) $(\exp_q g)X(\exp_q g)^{-1} = X + \sum_{l\geq 1} q^{-l(l-1)/2}[g, \ldots, [g, [g, X]_{q^0}]_{q^2}, \ldots]_{q^{2(l-1)}}/[l]!$.

Proof. Part (1) is [18, Proposition IV.2.6]. Part (2) would be well known and is easily proven. □

7. Annihilating polynomial

By Theorem 6.1 the problem of telling whether a finite dimensional $U_q(L(sl_2))$ module V is extended to a $\boxtimes_q$ module is reduced to determining the annihilating polynomial $p_V(z)$. In this section, we shall determine $p_V(z)$ for a tensor product of evaluation $U_q(L(sl_2))$ modules and a finite dimensional highest weight $U_q(L(sl_2))$ module.

In this section we shall consider only finite dimensional $U_q(sl_2)$ and $U_q(L(sl_2))$ modules of type 1 and denote a $n+1$ dimensional irreducible $U_q(sl_2)$ module $V_{n,+}$ simply by V_n.

7.1. Main result 2. For a nonzero polynomial $P(u)$, we set

$$\hat{P}(z) = \frac{z^{\deg P} P(z^{-1})}{(z^{\deg P} P(z^{-1}))|_{z=0}}.$$

For annihilating polynomials the following two theorems hold, whose proof will be given in the next subsection.

Theorem 7.1. *For an evaluation $U_q(L(sl_2))$ module $V_n(a)$ and a tensor product of them, the following hold*:
(1) $p_{V_n(a)}(z) = \prod_{1 \le i \le n}(1 - q^{n+1-2i} a^{-1} z)$.
(2) $p_V(z) = \prod_{1 \le j \le N} p_{V_{n_j}(a_j)}(z)$ *for* $V = V_{n_1}(a_1) \otimes \cdots \otimes V_{n_N}(a_N)$.

Theorem 7.2. *Let V be a finite dimensional highest weight $U_q(L(sl_2))$ module of type* 1 *and $P(u)$ the associated Drinfeld polynomial. Then $p_V(z) = \hat{P}(z)$.*

7.2. Proof of Theorems 7.1 and 7.2. In this subsection we shall prove Theorems 7.1 and 7.2, assuming the following proposition. The proof of the proposition will be given in the next subsection.

Proposition 7.1. *Let V be a finite dimensional highest weight $U_q(L(sl_2))$ module of type* 1 *and $P(u)$ the associated Drinfeld polynomial. Then $p_V(z)$ divides $\hat{P}(z)$.*

First we prove Theorem 7.2, assuming Theorem 7.1.

Proof of Theorem 7.2. By Theorem 3.2 (2) $V(P)$ is isomorphic to some $V_{n_1}(a_1) \otimes \cdots \otimes V_{n_N}(a_N)$. By Propositions 3.2 and 3.3 and Theorem 7.1 (1), the n_j and the a_j satisfy $\hat{P}(z) = \prod_{1 \le j \le N} p_{V_{n_j}(a_j)}(z)$. Hence $p_{V(P)}(z) = \hat{P}(z)$ by Theorem 7.1 (2). Let W be a maximal submodule of V. Then V/W is isomorphic to $V(P)$. Therefore $\hat{P}(z)$ $(= p_{V(P)}(z))$ divides $p_V(z)$. Since $p_V(z)$ divides $\hat{P}(z)$ by Proposition 7.1, we obtain the assertion. □

In the rest of this subsection, we shall prove Theorem 7.1. First we prove three lemmas needed later. For an algebra A and a generating series $X(z) = \sum_{m \in \mathbf{Z}} X_m z^{-m}$ $(X_m \in A)$ we shall say that $X(z) \sim 0$ on a A module V if the action of X_m on V is zero for $m > 0$. We shall use the generating series $y^-(z) = \sum_{m \ge 1} x_m^- z^{-m}$.

Lemma 7.1. *Let V be a finite dimensional $U_q(L(sl_2))$ module. For a polynomial $f(z)$ the following conditions are equivalent*:
(1) $f(z)$ *is in* I_V.
(2) $f(z)y^-(z) \sim 0$ *on* V.
(3) $f(z)\Phi^{(+)}(z) \sim 0$ *on* V.

Proof. Let $f(z) = \sum_{0\le j\le N} c_j z^j$. Then the coefficient of z^{-m} in $f(z)y^-(z)$ is equal to $\sum_{0\le j\le N} c_j x^-_{j+m}$ for a positive integer m. Hence (1) $\Rightarrow$ (2) follows from the definition of I_V and (1) $\Leftarrow$ (2) follows from (3.20). The equivalence of (2) and (3) follows from Lemma 3.1. □

The following is a slight refinement of [10, Proposition 4.4]. Note that we are using the opposite of the comultiplication in [10].

Lemma 7.2. *For an integer l set*

$$U_l = \{u \in U_q(L(sl_2)) \mid kuk^{-1} = q^{2l}u\}.$$

Then the following hold:

$$\Delta(y^-(z)) = 1 \otimes y^-(z) \otimes 1 + y^-(z) \otimes \Phi^{(+)}(z) \mod \sum_{r\ge 1}(U_{-(r+1)} \otimes U_r)[[z^{-1}]].$$

Lemma 7.3. *On $V_n(a)$ the following hold*:
(1) $x^-(z)v_i = [i+1]\delta(z/aq^{n-2i-1})v_{i+1}$,
(2) $\Phi^{(+)}(z)v_i = q^{n-2i}\frac{(1-q^{n+1}az^{-1})(1-q^{-(n+1)}az^{-1})}{(1-q^{n-2i+1}az^{-1})(1-q^{n-2i-1}az^{-1})}v_i$.
Here $\delta(z) = \sum_{m\in\mathbf{Z}} z^m$ and the rational function of z on the right hand side of (2) *should be understood as a Laurent expansion around ∞.*

Proof. Part (1) is [10, Proposition 4.2]. (We are using a different correspondence of the generators (3.24), (3.25) and a different ev_a.) Part (2) easily follows from this, $x_0^+ = e$ and (3.26). □

Now we can prove Theorem 7.1.

Proof of Theorem 7.1. (1) By Lemma 7.3 (1), a polynomial $f(z)$ is in $I_{V_n(a)}$ if and only if $f(z)\delta(z/aq^{n-2i-1}) = f(aq^{n-2i-1})\delta(z/aq^{n-2i-1}) = 0$ for $0 \le i \le n-1$. This proves the assertion.

(2) We set $V_j = V_{n_j}(a_j)$, $p_j(z) = p_{V_j}(z)$, $W_N = V_1 \otimes \cdots \otimes V_N$ and $f_N(z) = p_1(z)\cdots p_N(z)$. We shall prove $p_{W_N}(z) = f_N(z)$ for any N.

First we shall show by induction on N that $f_N(z)$ divides $p_{W_N}(z)$. In the case $N=1$ there is nothing to prove. We assume the case $N-1$. By Lemma 7.2 we have

$$p_{W_N}(z)\Delta(y^-(z)) = 1\otimes p_{W_N}(z)y^-(z) + p_{W_N}(z)(y^-(z)\otimes \Phi^{(+)}(z)) \mod \sum_{r\ge 1}(U_{-(r+1)}\otimes U_r)((z^{-1})).$$

By the definition of $p_{W_N}(z)$ and Lemma 7.1, $p_{W_N}(z)y^-(z)\sim 0$ on $W_N = W_{N-1}\otimes V_N$. Hence the above equality implies the following:

(i) $p_{W_N}(z)y^-(z)\sim 0$ on V_N,

(ii) $p_{W_N}(z)(y^-(z)\otimes\Phi^{(+)}(z))\sim 0$ on $W_{N-1}\otimes V_N$.

By (i) and Lemma 7.1, $p_{W_N}(z)\in I_{V_N}$ and hence $p_N(z)$ divides $p_{W_N}(z)$. By this we can write $p_{W_N}(z) = g(z)p_N(z)$ with some polynomial $g(z)$. Then (ii) is rewritten as

$$g(z)y^-(z)\otimes p_N(z)\Phi^{(+)}(z)\sim 0 \quad \text{on} \quad W_{N-1}\otimes V_N. \tag{7.1}$$

By Lemma 7.3 (2) we find that

$$p_N(z)\Phi^{(+)}(z)v_i = \phi_i(z)v_i \quad (0\le i\le n_N) \tag{7.2}$$

on V_N, where $\phi_i(z) = \phi_{n_N,a_N,i}(z)$ with

$$\begin{aligned}\phi_{n,a,i}(z) &= p_{V_n(a)}(z)\times q^{n-2i}\frac{(1-q^{n+1}az^{-1})(1-q^{-(n+1)}az^{-1})}{(1-q^{n-2i+1}az^{-1})(1-q^{n-2i-1}az^{-1})}\\ &= q^{-(n-2i)}\prod_{\substack{0\le j\le n+1\\ j\ne i,i+1}}(1-q^{-(n-2j+1)}a^{-1}z).\end{aligned}$$

Eqs. (7.1) and (7.2) imply that $\phi_i(z)g(z)y^-(z)\sim 0$ on W_{N-1} for any i and hence that $p_{W_{N-1}}(z)$ divides $\phi_i(z)g(z)$ for any i. Since the g.c.d. of the $\phi_i(z)$ is 1, it follows that $p_{W_{N-1}}(z)$ divides $g(z)$. This, $p_{W_N}(z) = g(z)p_N(z)$ and the induction assumption prove that $f_N(z)$ divides $p_{W_N}(z)$.

Next, to complete the proof, we shall show that $p_{W_N}(z)$ divides $f_N(z)$. By Lemma 7.1, this is equivalent to ($\star$) $f_N(z)y^-(z)\sim 0$ on W_N. Suppose that W_N is irreducible. Then, by Propositions 3.2 and 3.3 and part (1) of this theorem, W_N is isomorphic to $V(P)$ such that $\hat{P}(z) = f_N(z)$. Hence we find that $p_{W_N}(z)$ divides $f_N(z)$ if W_N is irreducible by Proposition 7.1. This result proves that ($\star$) holds for any W_N, since W_N is irreducible for generic parameters a_i by Theorem 3.2 (1) and since the matrix elements of the action of x_m^- on W_N (relative to a certain basis) are Laurent polynomials in the a_j. □

7.3. Proof of Proposition 7.1. In the rest of this section, we shall prove Proposition 7.1, to complete the proof of Theorems 7.1 and 7.2. First we shall show that the proof is reduced to showing Lemma 7.4 below.

Following [10], define $\mathcal{P}_r \in U_q(L(sl_2))$ $(r \geq 0)$ inductively by $\mathcal{P}_0 = 1$ and

$$\mathcal{P}_r = \frac{\sum_{j=1}^{r} k^{-1}\Phi_j^{(+)}\mathcal{P}_{r-j}}{q^{-2r}-1} \quad (r \geq 1)$$

and set $\mathcal{P}(u) = \sum_{r\geq 0} \mathcal{P}_r u^r$, so that

$$\frac{k\mathcal{P}(q^{-2}u)}{\mathcal{P}(u)} = \sum_{r\geq 0} \Phi_r^{(+)} u^r. \tag{7.3}$$

Let V be a finite dimensional highest weight $U_q(L(sl_2))$ module of type 1 and $P(u) = \sum_{r=0}^{l} \lambda_r u^r$ $(\lambda_0 = 1,\ \lambda_l \neq 0)$ the associated Drinfeld polynomial. We denote a highest weight vector of V by v. Then

$$x_m^+ v = 0 \ (m \in \mathbf{Z}), \quad kv = q^l v, \quad \mathcal{P}_r v = \lambda_r v, \quad (x_1^-)^m v = 0 \ (m > l) \tag{7.4}$$

and V is spanned by the vectors $x_{r_1}^- x_{r_2}^- \cdots x_{r_n}^- v$ $(0 \leq n \leq l,\ r_1, r_2, \ldots, r_n \in \mathbf{Z})$. By Proposition 6.1, x_m^- with $m \leq 0$ is expressed as a linear combination of the x_m^- $(m \geq 1)$ on V. Hence we can see that

$$V = \sum_{0\leq n\leq l} \sum_{r_1, r_2, \ldots, r_n \geq 1} \mathbf{C} x_{r_1}^- x_{r_2}^- \cdots x_{r_n}^- v.$$

By the above and Lemma 7.1, we can see that, to prove Proposition 7.1, it suffices to show the following lemma.

Lemma 7.4. *Let v, l and the λ_r be as above. Then the following hold for $s > l$:*

$$\left(\sum_{r=0}^{l} \lambda_r x_{s-r}^-\right) x_{r_1}^- x_{r_2}^- \cdots x_{r_n}^- v = 0 \quad (r_1, r_2, \ldots, r_n \geq 1,\ 0 \leq n \leq l).$$

In the following, we shall show the above lemma. Let

$$X(u) = \sum_{m=1}^{\infty} x_m^- u^m, \quad \lambda(u) = \sum_{r=0}^{l} \lambda_r u^r, \quad Y(u) = \lambda(u)X(u).$$

Then

$$Y(u) = \sum_{m=1}^{\infty} y_m^- u^m, \quad y_m^- = \sum_{r=0}^{\min\{l, m-1\}} \lambda_r x_{m-r}^-.$$

For a generating series $A(u)$ in u, we denote the coefficient of u^s in $A(u)$ by $(A(u))_s$. As usual, we set $(x_m^{\pm})^{(n)} = (x_m^{\pm})^n/[n]!$ and $(X(u))^{(n)} = (X(u))^n/[n]!$ for a nonnegative integer n.

The following lemma plays an important role in the proof of Lemma 7.4. Part (1) is [19, Lemma 4.4] and part (2) follows from part (1) and (7.4).

Lemma 7.5. (1) *For* $s \ge r \ge 1$,

$$(x_0^+)^{(r)}(x_1^-)^{(s)} = (-1)^r q^{-rs}(X(u)^{(s-r)}\mathcal{P}(u))_s k^r \mod \sum_{l\in\mathbf{Z}} U_q(L(sl_2))x_l^+.$$

(2) *Let* v *and* l *be as above. Then the following hold for* $s > l$ *and* $0 \le r \le s-2$:

$$(Y(u)X(u)^r)_s v = 0.$$

To prove Lemma 7.4, we further need the following two lemmas.

Lemma 7.6. (1) $[x_m^-, x_n^-]_{q^2} = (1-q^2)\sum_{l=n}^m x_{n+m-l}^- x_l^-$ *for* $m \ge n$.
(2) $[y_m^-, x_1^-]_{q^2} = (1-q^2)(Y(u)X(u))_{m+1}$ *for* $m \ge 1$.
(3) $[y_{m+1}^-, x_n^-]_{q^{-2}} + [x_{n+1}^-, y_m^-]_{q^{-2}} = 0$ *for* $m > l$.
(4) *For* $r_1, r_2, \ldots, r_n \ge 1$, $x_{r_1}^- x_{r_2}^- \cdots x_{r_n}^-$ *is expressed as a linear combination of the following elements*:

$$x_{s_1}^- x_{s_2}^- \cdots x_{s_n}^- \quad (s_1 \ge s_2 \ge \cdots \ge s_n \ge 1,\ \textstyle\sum_j s_j = \sum_j r_j).$$

Proof. Part (1) follows from (3.22). Part (2) is a consequence of (1) and the definition of y_m^-. Part (3) follows from (3.22) and the definition of y_m^-. Part (4) is easily shown, using (1). □

Lemma 7.7. *For* $m, n \ge 1$,

$$y_m^-(x_1^-)^n - q^{2n}(x_1^-)^n y_m^- = \sum_{r=0}^{n-1} a_{n,r}(x_1^-)^r(Y(u)X(u)^{n-r})_{m+n-r}$$

where the $a_{n,r}$ *are some complex numbers.*

Assuming Lemma 7.7 for a while, we shall prove Lemma 7.4.

Proof of Lemma 7.4. We shall prove the assertion by induction on n. The case $n = 0$ follows from Lemma 7.5 (2) with $r = 0$. If $l = 0$, then we are done. Suppose that $l \ge 1$ and let $1 \le p \le l$. Assuming the case $n = p-1$, we show the case $n = p$ by induction on $N := \sum_{j=1}^p (r_j - 1)$. By Lemma 7.5 (2) and Lemma 7.7,

$$y_m^-(x_1^-)^p v = 0 \quad (m > l).$$

The case $N = 0$ follows from this. If $N > 0$, then $x_{r_1}^- x_{r_2}^- \cdots x_{r_p}^- v$ is rewritten as a linear combination of the vectors $x_{s_1}^- x_{s_2}^- \cdots x_{s_p}^- v$ with $s_1 > 1$, $s_j \ge 1$ $(2 \le j \le p)$ and $\sum_j s_j = \sum_j r_j$ by Lemma 7.6 (4). By the induction assumptions,

$$y_m^- x_{s_2}^- x_{s_3}^- \cdots x_{s_p}^- v = 0, \quad y_m^- x_{s_1-1}^- x_{s_2}^- \cdots x_{s_p}^- v = 0$$

for $m > l$. Hence we find

$$y_m^- x_{s_1}^- x_{s_2}^- \cdots x_{s_p}^- v = 0 \quad (m > l)$$

by Lemma 7.6 (3). This completes the proof. □

Next, to complete the proof, we shall show Lemma 7.7. For this we need the following.

Lemma 7.8. *For $n \geq 1$ and $m \in \mathbf{Z}$,*

$$(Y(u)X(u)^{n-1})_m x_1^- - q^{2n} x_1^- (Y(u)X(u)^{n-1})_m = (1-q^{2n})(Y(u)X(u)^n)_{m+1}.$$

Proof. The case $n = 1$ follows from Lemma 7.6 (2). Assuming the assertion for n, we find that

$$\begin{aligned} &(Y(u)X(u)^n)_m x_1^- - (1-q^{2n})(Y(u)X(u)^{n+1})_{m+1} \\ (\star) \qquad &= \sum_{r\geq 1} x_r^-((Y(u)X(u)^{n-1})_{m-r} x_1^- - (1-q^{2n})(Y(u)X(u)^n)_{m+1-r}) \\ &= q^{2n} \sum_{r \geq 1} x_r^- x_1^- (Y(u)X(u)^{n-1})_{m-r}. \end{aligned}$$

By Lemma 7.6 (1), $x_r^- x_1^- = q^2 x_1^- x_r^- + (1-q^2) \sum_{s=1}^r x_{r+1-s}^- x_s^-$ for $r \geq 1$. This implies

$$\sum_{r\geq 1} x_r^- x_1^- (Y(u)X(u)^{n-1})_{m-r} = q^2 x_1^- (Y(u)X(u)^n)_m + (1-q^2)(Y(u)X(u)^{n+1})_{m+1}.$$

By substituting this into $(\star)$, we obtain the assertion for $n+1$. □

Now we can prove Lemma 7.7.

Proof of Lemma 7.7. The case $n = 1$ follows from Lemma 7.6 (2). By the same claim,

$$\begin{aligned} &y_m^-(x_1^-)^{n+1} - q^{2(n+1)}(x_1^-)^{n+1} y_m^- \\ &= (y_m^-(x_1^-)^n - q^{2n}(x_1^-)^n y_m^-) x_1^- + (1-q^2) q^{2n} (x_1^-)^n (Y(u)X(u))_{m+1}. \end{aligned}$$

Substituting the assertion for n into the above and using Lemma 7.8, we obtain the assertion for $n+1$. □

8. Tensor product of $\boxtimes_q$ modules

In this section, we show that we can consider a tensor product module of finite dimensional $\boxtimes_q$ modules.

Theorem 8.1. *Let V, W and U be finite dimensional $\boxtimes_q$ modules.*
(1) *The $U_q(L(sl_2))$ module structure on $V \otimes W$ via the comultiplication* (4.9)–(4.13) *can be uniquely extended to a $\boxtimes_q$ module structure by the following actions of X_{20} and X_{02}:*

$$X_{02}(v \otimes w) = \sum_{n=0}^{\infty} (X_{02})_{1-n} v \otimes (X_{02})^n w, \quad X_{20}(v \otimes w) = \sum_{n=0}^{\infty} (X_{20})_{n-1} v \otimes (X_{20})^n w$$

where $(X_{02})_n$ *and* $(X_{20})_n$ *denote the components of* X_{02} *and* X_{20} *defined in* (5.2) *and* (5.3), *respectively.*
(2) *If we consider tensor product modules of* $\boxtimes_q$ *as in* (1), *then the associativity holds*:

$$V \otimes (W \otimes U) \simeq (V \otimes W) \otimes U.$$

Proof. (1) The extension is unique if possible by Theorem 6.1. Therefore it is sufficient to check the relations (4.2) and (4.5). Here we shall verify $X_{02}X_{20} = 1$ and $\{X_{02}, X_{23}\}_q = 1$ as examples, the proof of the remaining relations being similar.

Formally we write

$$\Delta(X_{02}) = \sum_{n=0}^{\infty}(X_{02})_{1-n} \otimes (X_{02})^n, \quad \Delta(X_{20}) = \sum_{n=0}^{\infty}(X_{20})_{n-1} \otimes (X_{20})^n. \tag{8.1}$$

Then

$$\Delta(X_{02})\Delta(X_{20}) = \sum_{m,n=0}^{\infty} (X_{02})_{1-m}(X_{20})_{n-1} \otimes (X_{02})^{m-n}. \tag{8.2}$$

By (5.2) and (5.3)

$$X_{02}X_{20} = \sum_{m,n\geq 0} (X_{02})_{1-m}(X_{20})_{n-1} = 1.$$

From this we find

$$\sum_{\substack{m,n\geq 0\\ n-m=r}} (X_{02})_{1-m}(X_{20})_{n-1} = \delta_{r,0}$$

for any integer r. Substituting this into (8.2), we obtain $\Delta(X_{02})\Delta(X_{20}) = 1$.

Next we consider $\{\Delta(X_{02}), \Delta(X_{23})\}_q = 1$. By (8.1) and (4.11)

$$\{\Delta(X_{02}), \Delta(X_{23})\}_q = \sum_{n=0}^{\infty}(\mathrm{I}_n + \mathrm{II}_n) \tag{8.3}$$

with

$$\mathrm{I}_n = \{(X_{02})_{1-n}, X_{23} - X_{13}\}_q \otimes (X_{02})^n, \quad \mathrm{II}_n = \{(X_{02})_{1-n} \otimes (X_{02})^n, X_{13} \otimes X_{23}\}_q.$$

By Lemma 5.2 (1), I_n is rewritten as follows:

$$\mathrm{I}_n = \delta_{n,0} - q^{-n}[n+1](X_{02})_{-n}X_{13} \otimes (X_{02})^n.$$

On the other hand, we find by (5.2) and $\{X_{02}, X_{23}\}_q = 1$ that

$$\begin{aligned}\mathrm{II}_n &= q^{1-n}[n](X_{02})_{1-n}X_{13} \otimes \{(X_{02})^n, X_{23}\}_{q^n}\\ &= q^{1-n}[n](X_{02})_{1-n}X_{13} \otimes (X_{02})^{n-1}.\end{aligned}$$

Substituting the above into (8.3), we find that $\{\Delta(X_{02}), \Delta(X_{23})\}_q = 1$, since $(X_{02})_n = 0$ for $n \ll 0$.

(2) By Theorem 6.1, the extended $\boxtimes_q$ module structure on the $U_q(L(sl_2))$ module $V \otimes W \otimes U$ is unique. This proves the assertion. It is also easy to prove the associativity directly, using the action in (1). $\square$

9. *L*-operators

At the end of the paper, we discuss an interesting connection between L-operators for $U_q(L(sl_2))$ and $\boxtimes_q$ module structures. The L-operators originated from the theory of exactly solvable models [20]. In this section we shall denote a $n+1$ dimensional irreducible $U_q(sl_2)$ module $V_{n,+}$ simply by V_n as before.

9.1. *L*-operators and $\boxtimes_q$ modules. Let V be a finite dimensional $U_q(L(sl_2))$ module. We shall call a linear map $L\colon V \otimes V_1(1) \to V \otimes V_1(1)$ satisfying

$$L\Delta(x) = \Delta^{\mathrm{op}}(x)L \quad (x \in U_q(L(sl_2))) \tag{9.1}$$

an L-operator for V. From an L-operator L for V, we define linear maps L_{ij} $(1 \leq i, j \leq 2)$ on V by

$$L = \sum_{i,j=1,2} L_{ij} \otimes E_{ij}$$

where the E_{ij} are the matrix units.

Proposition 9.1. *Let V be a finite dimensional $U_q(L(sl_2))$ module. Suppose that an L-operator L exists for V and that both L_{11} and L_{22} are invertible. Then the $U_q(L(sl_2))$ module structure on V is extended to a $\boxtimes_q$ module structure by the following action of X_{02} and X_{20}:*

$$X_{02} = (L_{22} + L_{21})(L_{11} + L_{12})^{-1}, \quad X_{20} = (L_{11} + L_{12})(L_{22} + L_{21})^{-1}. \tag{9.2}$$

Proof. It suffices to check (4.5). Here we prove $\{X_{30}, X_{02}\}_q = 1$ as an example. Among the relations that follow from (9.1), we need the following:

$$\begin{gathered}
t_0 L_{ii} = L_{ii} t_0, \quad t_0 L_{12} = q^2 L_{12} t_0, \quad t_0 L_{21} = q^{-2} L_{21} t_0, \\
e_0 L_{22} = q^{-1}(L_{22} e_0 - L_{12}), \quad e_0 L_{21} = q^{-1}(L_{21} e_0 - L_{11} + L_{22} t_0), \\
L_{11} e_0 = q^{-1} e_0 L_{11} - q^{-2} t_0 L_{12}, \quad L_{12} e_0 = q^{-1} e_0 L_{12}.
\end{gathered}$$

Using the above and $X_{30} = t_0 - (q - q^{-1})e_0$ from Proposition 4.2, it is easy to show that both $L_{11} + L_{12}$ and $L_{22} + L_{21}$ are invertible and that the following hold:

$$\begin{gathered}
qX_{30}(L_{22} + L_{21}) = (L_{22} + L_{21})Z + (q - q^{-1})(L_{11} + L_{12}), \\
(L_{11} + L_{12})Z = q^{-1} X_{30}(L_{11} + L_{12})
\end{gathered}$$

where $Z = q^{-1}t_0 - (q - q^{-1})e_0$. Eliminating Z from the above two equations, we obtain $\{X_{30}, X_{02}\}_q = 1$. $\square$

REMARK 9.1. In terms of a linear map $\mathcal{L} = \sum_{i,j=1,2} E_{ij} \otimes \mathcal{L}_{ij} : V_1(1) \otimes V \to V_1(1) \otimes V$ satisfying

$$\mathcal{L}\Delta(x) = \Delta^{\mathrm{op}}(x)\mathcal{L} \quad (x \in U_q(L(sl_2))),$$

we can show that

$$X_{02} = (\mathcal{L}_{22} - \mathcal{L}_{12})^{-1}(\mathcal{L}_{11} - \mathcal{L}_{21}), \quad X_{20} = (\mathcal{L}_{11} - \mathcal{L}_{21})^{-1}(\mathcal{L}_{22} - \mathcal{L}_{12}).$$

defines a $\boxtimes_q$ module structure on a $U_q(L(sl_2))$ module V if $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$ are invertible.

9.2. *L*-operators for a tensor product of evaluation modules. Recall that on a finite dimensional $U_q(sl_2)$ module V of type 1 the element t is diagonalizable with eigenvalues q^n ($n \in \mathbf{Z}$). Let us denote the eigenspace of t corresponding to q^n by $V[n]$. Fixing a square root $q^{1/2}$ of q, we define a linear map $t^{1/2}$ on V so that $t^{1/2}|_{V[n]} = q^{n/2}\mathrm{id}_{V[n]}$ for any n. Then this satisfies $(t^{1/2})^2 = t$, $t^{1/2}e = qet^{1/2}$ and $t^{1/2}f = q^{-1}ft^{1/2}$.

The following well known two propositions give an L-operator for a tensor product of evaluation $U_q(L(sl_2))$ modules of type 1. The L-operators in the first proposition first appeared in [21, Appendix]. See also [22, Proposition 2].

Proposition 9.2. *Let V be a finite dimensional $U_q(sl_2)$ module V of type 1 and consider the evaluation $U_q(L(sl_2))$ module $V(a)$. The linear map $L(a) = \sum_{i,j=1,2} L_{ij}(a) \otimes E_{ij}$ with*

$$L_{11}(a) = aq^{1/2}t^{1/2} - q^{-1/2}t^{-1/2}, \quad L_{12}(a) = (q - q^{-1})q^{-1/2}aft^{1/2},$$
$$L_{21}(a) = (q - q^{-1})q^{1/2}t^{-1/2}e, \quad L_{22}(a) = aq^{1/2}t^{-1/2} - q^{-1/2}t^{1/2}$$

is an L-operator for $V(a)$.

Proposition 9.3. *Let V and W be finite dimensional $U_q(L(sl_2))$ modules. Let L and L' be L-operators for V and W, respectively. Then*

$$\sum_{i,j=1,2} \left(\sum_{l=1,2} L_{il} \otimes L'_{lj} \right) \otimes E_{ij}$$

is an L-operator for $V \otimes W$.

9.3. $\boxtimes_q$ module structure on a tensor product of evaluation modules. Let us denote the L-operator associated with the evaluation module $V_n(a)$ as in Proposition 9.2

by $L(n,a) = \sum_{i,j=1,2} L(n,a)_{ij} \otimes E_{ij}$. Set $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_N)$. Then, by Proposition 9.3, $L(\mathbf{n}, \mathbf{a}) = \sum_{i,j=1,2} L(\mathbf{n}, \mathbf{a})_{ij} \otimes E_{ij}$ with

$$L(\mathbf{n}, \mathbf{a})_{ij} = \sum_{l_1, l_2, \ldots, l_{N-1}=1,2} L(n_1, a_1)_{i,l_1} \otimes L(n_2, a_2)_{l_1,l_2} \otimes \cdots \otimes L(n_N, a_N)_{l_{N-1},j}$$

is an L-operator for the tensor product $V_{\mathbf{n}}(\mathbf{a}) := V_{n_1}(a_1) \otimes V_{n_2}(a_2) \otimes \cdots \otimes V_{n_N}(a_N)$. Define $\mathrm{End}(V_{n_1} \otimes \cdots \otimes V_{n_N})$ valued rational functions $X_{02}(\mathbf{n}, \mathbf{a})$ and $X_{20}(\mathbf{n}, \mathbf{a})$ of $\mathbf{a}$ from the $L(\mathbf{n}, \mathbf{a})_{ij}$ as in (9.2).

Theorem 9.1. *Set $I_0 = \emptyset$ and $I_n = \{q^{n-1}, q^{n-3}, \ldots, q^{1-n}\}$ for a positive integer n.*
(1) The $U_q(L(sl_2))$ module structure on $V_{\mathbf{n}}(\mathbf{a})$ can be extended to a $\boxtimes_q$ module structure if and only if $a_i \notin I_{n_i}$ for any i.
(2) If $a_i \notin I_{n_i}$ for any i, then the assignment $X_{02} \mapsto X_{02}(\mathbf{n}, \mathbf{a})$, $X_{20} \mapsto X_{20}(\mathbf{n}, \mathbf{a})$ extends the $U_q(L(sl_2))$ module structure on $V_{\mathbf{n}}(\mathbf{a})$ to a $\boxtimes_q$ module structure.

Proof. (1) By Theorem 7.1, $a_i \notin I_{n_i}$ for any i if and only if $p_{V_{\mathbf{n}}(\mathbf{a})}(1) \neq 0$. Hence the assertion follows from Theorem 6.1.

(2) Let $J_n = I_n \cup \{q^{-(n+1)}\}$. We can easily show that

$$\det L_{jj}(n, a) = q^{(n+1)/2} \prod_{b \in J_n} (a - b), \quad \det L_{jj}(\mathbf{n}, \mathbf{a}) = \prod_{1 \le i \le n} (\det L_{jj}(n_i, a_i))^{m_i}$$

with $m_i = \prod_{l \neq i} (n_l + 1)$ for $j = 1, 2$. Hence, by Proposition 9.1, we can see that the assignment in the assertion defines a $\boxtimes_q$ module structure on $V_{\mathbf{n}}(\mathbf{a})$ if $a_i \notin J_{n_i}$ for any i.

We shall prove that the assignment works also in the case $a_i \notin I_{n_i}$ for any i. For this it is sufficient to show that the (matrix valued) rational functions $X_{02}(\mathbf{n}, \mathbf{a})$ and $X_{20}(\mathbf{n}, \mathbf{a})$ are not singular on $I'_{\mathbf{n}} := \prod_{1 \le i \le N} (\mathbf{C}^\times \setminus I_{n_i})$. This is done by induction on N. The case $N = 1$ is proven by checking that $X_{02}(n, a)$ and $X_{20}(n, a)$ do not have a pole at $a = q^{-(n+1)}$. Set $\mathbf{n}' = (n_1, n_2, \ldots, n_{N-1})$, $\mathbf{a}' = (a_1, a_2, \ldots, a_{N-1})$ and $V' = V_{n_1}(a_1) \otimes V_{n_2}(a_2) \otimes \cdots \otimes V_{n_{N-1}}(a_{N-1})$. Then if $a_i \notin J_{n_i}$ for any i, we have

$$X_{02}(\mathbf{n}, \mathbf{a})(v \otimes w) = \sum_{l \ge 0} (X_{02}(\mathbf{n}', \mathbf{a}'))_{1-l} v \otimes (X_{02}(n_N, a_N))^l w \quad (v \in V', \ w \in V_{n_N}(a_N))$$

by Theorems 8.1 and 6.1 (2). Assume the case $N - 1$. Then the r.h.s. of the above equality is not singular on $I'_{\mathbf{n}}$. This proves the case N for $X_{02}(\mathbf{n}, \mathbf{a})$. The proof of the case $X_{20}(\mathbf{n}, \mathbf{a})$ is similar. □

References

[1] B. Hartwig and P. Terwilliger: *The tetrahedron algebra, the Onsager algebra, and the $\mathfrak{sl}_2$ loop algebra*, J. Algebra **308** (2007), 840–863.

[2] B. Hartwig: *The tetrahedron algebra and its finite-dimensional irreducible modules*, Linear Algebra Appl. **422** (2007), 219–235.

[3] A. Elduque: *The S_4-action on tetrahedron algebra*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 1227–1248.

[4] T. Ito and P. Terwilliger: *Finite-dimensional irreducible modules for the three-point $\mathfrak{sl}_2$ loop algebra*, Comm. Algebra **36** (2008), 4557–4598.

[5] M. Bremner: *Generalized affine Kac-Moody Lie algebras over localizations of the polynomial ring in one variable*, Canad. Math. Bull. **37** (1994), 21–28.

[6] G. Benkart and P. Terwilliger: *The universal central extension of the three-point $\mathfrak{sl}_2$ loop algebra*, Proc. Amer. Math. Soc. **135** (2007), 1659–1668.

[7] T. Ito and P. Terwilliger: *The q-tetrahedron algebra and its finite dimensional irreducible modules*, Comm. Algebra **35** (2007), 3415–3439.

[8] T. Ito and P. Terwilliger: *Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations*, J. Algebra Appl. **6** (2007), 477–503.

[9] D. Funk-Neubauer: *Tridiagonal pairs and the q-tetrahedron algebra*, Linear Algebra Appl. **431** (2009), 903–925.

[10] V. Chari and A. Pressley: *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), 261–283.

[11] V.G. Drinfel'd: *Hopf algebras and the Yang-Baxter equation*, Sov. Math. Doklady **32** (1985), 254–258.

[12] M. Jimbo: *A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.

[13] V.G. Drinfel'd: *A new realization of Yangians and of quantum affine algebras*, Soviet Math. Dokl. **36** (1988), 212–216.

[14] J. Beck: *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994), 555–568.

[15] J.C. Jantzen: Lectures on Quantum Groups, Amer. Math. Soc., Providence, RI, 1996.

[16] T. Ito, P. Terwilliger and C. Weng: *The quantum algebra $U_q(\mathfrak{sl}_2)$ and its equitable presentation*, J. Algebra **298** (2006), 284–301.

[17] P. Terwilliger: *The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$*, J. Algebra **298** (2006), 302–319.

[18] C. Kassel: Quantum Groups, Graduate Texts in Mathematics **155**, Springer, New York, 1995.

[19] V. Chari and A. Pressley: *Weyl modules for classical and quantum affine algebras*, Represent. Theory **5** (2001), 191–223.

[20] P.P. Kulish and E.K. Sklyanin: *Solutions of the Yang-Baxter Equations*, J. Sov. Math. **19** (1982), 1596–1620.

[21] P.P. Kulish and N.Yu. Reshetikhin: *Quantum linear problem for the Sine-Gordon equation and higher representations*, J. Sov. Math. **23** (1983), 2435–2441.

[22] M. Jimbo: *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang–Baxter equation*, Lett. Math. Phys. **11** (1986), 247–252.

Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka 560-0043
Japan