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## SEIBERG-WITTEN INVARIANTS ON NON-SYMPLECTIC 4-MANIFOLDS

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Let  $X$  be an oriented, closed Riemannian 4-manifold. There is an integral cohomology class which reduces mod (2) to the second Stiefel-Whitney class  $w_2(X)$ . This integral cohomology class induces a  $Spin^c$ -structure on  $X$ . Seiberg and Witten in [10] introduced a new invariant on  $X$  which is a differential-topological invariant. Taubes in [9] proved that every closed symplectic 4-manifold has a non-trivial Seiberg-Witten invariant. The Seiberg-Witten invariants of connected sums of 4-manifolds with  $b_2^+ > 0$  identically vanish. Kotschick, Morgan and Taubes in [8] showed that there are closed oriented 4-manifolds with nontrivial Seiberg-Witten invariants which do not admit symplectic structures. They considered the case which is the first Betti number  $b_1(N)=0$ . We would like to generalize their theorem by giving a certain condition instead of  $b_1(N)=0$ , of course our case will cover their case. We introduce their theorem:

**Theorem ([8]).** *Let  $X$  be a manifold with a nontrivial Seiberg-Witten invariant with  $b_2^+(X) > 1$ , and let  $N$  be a manifold with  $b_1(N)=b_2^+(N)=0$  whose fundamental group has a nontrivial finite quotient. Then  $M=X\#N$  has a non-trivial Seiberg-Witten invariant but does not admit any symplectic structure.*

Let  $M$  be a closed symplectic 4-manifold and let  $M=X\#N$  be a smooth connected sum decomposition. By the vanishing theorem of Seiberg-Witten invariants and non-trivial Seiberg-Witten invariants for symplectic manifolds, one of the summands, say it  $N$ , has a negative definite intersection form. By Donaldson's Theorem [5] there is a basis  $\{e_1, \dots, e_n\}$  of the free part of  $H^2(N, \mathbf{Z})$  such that in this basis the intersection form of  $N$  is diagonal, where  $n$  is the rank of  $H^2(N, \mathbf{Z})$ . An element  $\alpha \in H^2(N, \mathbf{Z})$  is said to be characteristic if the intersection number  $\alpha \cdot x = x \cdot x \pmod{2}$  for any  $x \in H^2(N, \mathbf{Z})$ . If  $\alpha$  is characteristic, then  $\alpha \equiv w_2(N)$  modulo 2.

**Lemma 1.** *Let  $N$  be a closed oriented Riemannian 4-manifold with  $b_2^+(N)=0$  and let  $\{e_1, \dots, e_n\}$  is a basis for the free part of  $H^2(N, \mathbf{Z})$  such that  $e_i \cdot e_j = -\delta_{ij}$ .*

Then

1.  $e = e_1 + \dots + e_n$  is characteristic.
2.  $\alpha = (1 + \lambda_1)e_1 + \dots + (1 + \lambda_n)e_n$  is characteristic if and only if the  $\lambda_i$  are even.

Proof. It is sufficient to consider the free elements in the proof because the intersection numbers with torsion elements are zero. Let  $x = x_1e_1 + \dots + x_n e_n \in H^2(N, \mathbb{Z})$  where the  $x_i$  are integers  $i = 1, \dots, n$ .

Then

$$\begin{aligned} \alpha \cdot x &= -(1 + \lambda_1)x_1 - \dots - (1 + \lambda_n)x_n \quad \text{and} \\ x \cdot x &= -x_1^2 - \dots - x_n^2. \end{aligned}$$

$\alpha \cdot x = x \cdot x \pmod{2}$  for all  $x \in H^2(N, \mathbb{Z})$ .

$$\begin{aligned} &\Leftrightarrow -(1 + \lambda_1)x_1 - \dots - (1 + \lambda_n)x_n = -x_1^2 - \dots - x_n^2 \pmod{2} \text{ for all } x_1, \dots, x_n. \\ &\Leftrightarrow \lambda_1 x_1 + \dots + \lambda_n x_n = 0 \pmod{2} \text{ for all } x_1, \dots, x_n. \\ &\Leftrightarrow \lambda_1, \dots, \lambda_n \text{ are even.} \end{aligned}$$

If the fundamental group  $\pi_1(N)$  of  $N$  has a non-trivial finite quotient, then there is a connected covering of  $N$  with the cardinality of fiber  $> 1$  and so is a connected sum with  $N$ .

**Lemma 2** ([8]). *Let  $M = X \# N$  be a closed symplectic 4-manifold which decomposes as a connected sum. If  $N$  has a negative definite intersection form then its fundamental group does not admit nontrivial finite quotient.*

We recall briefly the Seiberg-Witten invariants for a compact, oriented Riemannian 4-manifold  $X$  with  $b_2^+(X) > 1$ .

Let  $e \in H^2(X, \mathbb{Z})$ , with  $e \equiv w_2(X) \pmod{2}$ .

The cohomology class  $e$  defines a  $Spin^c$ -structure on  $X$ . Let  $W^+(W^-) \rightarrow X$  be the positive (negative respectively) spinor bundle on  $X$  and  $L = \det(W^+)$  the determinant line bundle of  $W^+$ . Let  $\tau: \text{End}(W^+) \rightarrow \Lambda^+(T^*X) \otimes \mathbb{C}$  be the adjoint of Clifford multiplication. A connection  $A$  on  $L$  with the Levi-Civita connection on  $T^*X$  defines a covariant derivative  $\nabla_A: \Gamma(W^+) \rightarrow \Gamma(W^+ \otimes T^*X)$ . The composition of  $\nabla_A$  and Clifford multiplication define a Dirac operator

$$D_A: \Gamma(W^+) \rightarrow \Gamma(W^-).$$

For each connection on  $L$   $A \in \mathcal{A}(L)$  and  $\phi \in \Gamma(W^+)$ , the equations

$$\begin{cases} D_A \phi = 0 \\ F_A^+ = \frac{1}{4} \tau(\phi \otimes \phi^*) \end{cases}$$

are called the Seiberg-Witten monopole equations. The gauge group  $C^\infty(X, U(1))$  of the complex line bundle  $L$  acts on the space of solutions of the monopole equations. The moduli space  $\mathfrak{M}(X, e)$  is the quotient of the space of solutions by the gauge group. Then the moduli space is generically a compact smooth manifold with its dimension  $-(1/4)(2\chi(X) + 3\sigma(X)) + (1/4)c_1(L)^2$  and defines canonically an invariant which is so called the Seiberg-Witten invariants. For details see [5].

Let  $X$  and  $N$  be compact oriented 4-manifolds. Let  $\alpha \in H^2(X, \mathbb{Z})$  and  $\beta \in H^2(N, \mathbb{Z})$  such that  $\alpha \equiv w_2(X) \pmod{2}$ ,  $\beta \equiv w_2(N) \pmod{2}$ . Let  $M = X \# N$ , then  $\alpha + \beta \equiv w_2(M) \pmod{2}$ . Let the complex line bundles  $L_\alpha \rightarrow X$ ,  $L_\beta \rightarrow N$ ,  $L_{\alpha+\beta} \rightarrow M$  with their Chern classes  $c_1(L_\alpha) = \alpha$ ,  $c_1(L_\beta) = \beta$  and  $c_1(L_{\alpha+\beta}) = \alpha + \beta$  respectively. We can easily calculate the virtual dimensions of the moduli spaces.

**Lemma 3.**  $\dim \mathfrak{M}(M, \alpha + \beta) = \dim \mathfrak{M}(X, \alpha) + \dim \mathfrak{M}(N, \beta) + 1.$

*Proof.* The Euler characteristic is  $\chi(M) = \chi(X) + \chi(N) - 2$ . The signature is  $\sigma(M) = \sigma(X) + \sigma(N)$ . The first Chern classes are  $c_1(L_{\alpha+\beta}) = c_1(L_\alpha) + c_1(L_\beta)$  and  $\alpha \cdot \beta = 0$ . Thus

$$\begin{aligned} \dim \mathfrak{M}(M, \alpha + \beta) &= -\frac{1}{4}(2\chi(M) + 3\sigma(M)) + \frac{1}{4}c_1(L_{\alpha+\beta})^2 \\ &= \left[ -\frac{1}{4}(2\chi(X) + 3\sigma(X)) + \frac{1}{4}c_1(L_\alpha)^2 \right] \\ &\quad + \left[ -\frac{1}{4}(2\chi(N) + 3\sigma(N)) + \frac{1}{4}c_1(L_\beta)^2 \right] + 1 \\ &= \dim \mathfrak{M}(X, \alpha) + \dim \mathfrak{M}(N, \beta) + 1. \end{aligned}$$

Let  $N$  have a negative definite intersection form.

As in Lemma 1, let  $\{e_1, \dots, e_n\}$  be a basis of the free part of  $H^2(N, \mathbb{Z})$ . If  $\alpha = (1 + \lambda_1)e_1 + \dots + (1 + \lambda_n)e_n$  and the  $\lambda_i$  are even, the  $\alpha$  is characteristic.

**Lemma 4.** *If  $4b_1(N) = 2\lambda_1 + \dots + 2\lambda_n + \lambda_1^2 + \dots + \lambda_n^2$ , then  $\dim \mathfrak{M}(N, \alpha) = -1$ .*

**Corollary 5.** *If  $X$  is a symplectic manifold and  $K$  is the canonical line bundle on  $X$ , and  $M = X \# N$ , then  $\dim \mathfrak{M}(M, c_1(K) + \alpha) = \dim \mathfrak{M}(X, c_1(K)) = 0$ .*

*Proof.* For the proof use  $c_1(K)^2 = 2\chi + 3\sigma$  and Lemma 3, 4.

*Proof of Lemma 4.* The virtual dimension of the moduli space is

$$\dim \mathfrak{M}(N, \alpha) = -\frac{1}{4}(2\chi(N) + 3\sigma(N)) + \frac{1}{4}\alpha^2$$

$$\begin{aligned}
 &= -\frac{1}{4}\{2(2-2b_1(N)+b_2(N))+3(-b_2(N))\} \\
 &\quad +\frac{1}{4}[(1+\lambda_1)e_1+\dots+(1+\lambda_n)e_n]^2 \\
 &= -\frac{1}{4}[4-4b_1(N)-b_2(N)]+\frac{1}{4}[-(1+\lambda_1)^2-\dots-(1+\lambda_n)^2] \\
 &= -\frac{1}{4}[4-4b_1(N)+2\lambda_1+\lambda_1^2+\dots+2\lambda_n+\lambda_n^2] \\
 &= -1, \quad \text{since } 4b_1(N)=2\lambda_1+\dots+2\lambda_n+\lambda_1^2+\dots+\lambda_n^2.
 \end{aligned}$$

REMARK 1. For the equation  $4b_1(N)=2\lambda_1+\dots+2\lambda_n+\lambda_1^2+\dots+\lambda_n^2$ ,

1. If  $\lambda_2=\dots=\lambda_n=0$ ,  $b_1(N)=6$  and  $\lambda_1=4$  or  $-6$ , then the equation holds.
2. If  $\lambda_1=\dots=\lambda_n=0=b_1(N)$ , then the equation also holds.

**Theorem 6.** *Let  $X$  have a nontrivial Seiberg-Witten invariant and let  $N$  have a negative definite intersection form. If there are even integers  $\lambda_i$ ,  $i=1\cdots n$  such that  $4b_1(N)=2\lambda_1+\dots+2\lambda_n+\lambda_1^2+\dots+\lambda_n^2$ , then the connected sum  $M=X\#N$  has a nontrivial Seiberg-Witten invariant.*

Proof. Suppose  $N$  has a negative definite intersection form. As in Lemma 4, choose  $\alpha=(1+\lambda_1)e_1+\dots+(1+\lambda_n)e_n$  such that  $4b_1(N)=2\lambda_1+\dots+2\lambda_n+\lambda_1^2+\dots+\lambda_n^2$  and the  $\lambda_i$  are even. Then  $\alpha$  is characteristic by Lemma 1 and there is a  $Spin^c$ -structure on  $N$  with first Chern class  $\alpha$ . The Seiberg-Witten monopole equation is

$$\begin{cases} D_A\psi=0 \\ F_A^+ =\frac{1}{4}\tau(\psi\otimes\psi^*). \end{cases}$$

For a generic metric on  $N$  there is no non-abelian solution of the equations since  $\dim\mathfrak{M}(N,\alpha)=-1$ . We have a unique abelian solution  $(A_\alpha,0)$  given by the zero section of the positive spinor bundle and a connection  $A_\alpha$  whose curvature is the harmonic form representing  $\alpha=(i/2\pi)F_{A_\alpha}\in H^2(N,\mathbf{R})$ . The given  $Spin^c$ -structure  $e\in H^2(X,\mathbf{Z})$  on  $X$  and  $\alpha$  induce a  $Spin^c$ -structure on  $M$ . By choosing generic metrics on  $[X\setminus D^4]\cup[0,\infty)\times S^3$  and  $[N\setminus D^4]\cup[0,\infty)\times S^3$ , and product metric on the cylinder  $S^3\times\mathbf{R}$  and connecting them, we have a Riemannian metric on  $M=X\#N$ . The solutions of the Seiberg-Witten equations in  $\mathfrak{M}(M,e+\alpha)$  are given by gluing the solutions in  $\mathfrak{M}(X,e)$  on  $X$  to the unique solution  $(A_\alpha,0)$  in  $\mathfrak{M}(N,\alpha)$  on  $N$ .

In particular,  $\dim \mathfrak{M}(M, e + \alpha) = \dim \mathfrak{M}(X, e)$ .

By combining Lemma 1 to Theorem 6 we have the following Theorem.

**Theorem 7.** *Let  $X$  be a manifold with a nontrivial Seiberg-Witten invariant defined by  $e \in H^2(X, \mathbf{Z})(b_2^+(X) > 1)$ , and let  $N$  be a manifold with negative definite intersection form. If there are even integers  $\lambda_i, i = 1 \cdots n$  such that  $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$  and that the fundamental group of  $N$  has a nontrivial finite quotient, then the connected sum  $X \# N$  has a nontrivial Seiberg-Witten invariant but does not admit any symplectic structure.*

According to the Remark 1, Theorem 7 covers the Theorem [8] and there are many examples which are not included in Theorem [8].

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