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ON THE COHOMOLOGY OF FINITE GROUPS AND THE APPLICATIONS TO MODULAR REPRESENTATIONS

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1. Introduction

Let G be a finite group and k be a field of prime characteristic p. All modules considered here are assumed to be finite dimensional over k. In [2], Carslon introduced a certain condition on the cohomology ring of G to study the structure of periodic modules by homological techniques. Let us denote it by $C(n)$, where n is a positive integer. If G satisfies $C(n)$, then there are homogeneous elements of degree n having an interesting property releted to his notion of rank variety (see Section 3 for details). For a p-group P, he showed that there exists an integer $n(P)$ such that P satisfies $C(2n(P))$. And using this, he showed that the period of a periodic kP-module divides $2n(P)$.

The purpose of this paper is to extend Carlson's results to an arbitrary finite group G. In doing so, we shall give a stronger version of the condition $C(n)$, with a couple of equivalent conditions to it. Concerning Carlson's number $n(G)$ which can as well be defined for an arbitrary G, we shall prove that there exist cohomology elements of degree $2n(G)$ satisfying our new condition, so that G satisfies $C(2n(G))$. As an application of this result, we shall show that the period of a periodic kG-module divides $2n(G)$. As another application, we also give a homoligical criterion for a kG-module to be projective. A similar criterion has been given by Donovan [6], in response to a problem of Schultz [9].

2. Preliminaries

In this section we mention some preliminary facts needed in later arguments. For a kG-module M, set $\mathrm{Ext}^*_{kG}(M, M)=\sum_{n\geq 0}\mathrm{Ext}^n_{kG}(M, M)$. If H is a subgroup of G, then M_H denotes the restriction of M to a kH-module. First of all, we prove the following general fact.

Proposition 2.1. *Let* $0\to N_1\to M\xrightarrow{f} L\to 0$ *and* $0\to N_2\to M\xrightarrow{g} L\to 0$ *be exact* *sequences of kG-modules. If f-g factors through a projective kG-module Q, then* $N_1 \cong N_2$.

Proof. Suppose that $\alpha\colon M \to Q$ and $\beta\colon Q \to L$ give a factoriaztion of f-g through the projective module Q. Then we have the following two pull-back diagrams:

$$\begin{array}{ccccccccc} 0 & \to & N_1 & \to & S & \to & Q & \to & 0 \\ & & \| & & \downarrow & & \downarrow \beta & & \\ 0 & \to & N_1 & \to & M & \xrightarrow[f]{} & L & \to & 0\,, \end{array}$$

where $S = \{(x, y) \in M \oplus Q \mid f(x) = \beta(y)\}$,

$$\begin{array}{ccccccccc} 0 & \to & N_2 & \to & T & \to & Q & \to & 0 \\ & & \| & & \downarrow & & \downarrow \beta & & \\ 0 & \to & N_2 & \to & M & \xrightarrow[g]{} & L & \to & 0\,, \end{array}$$

where $T = \{(x, y) \in M \oplus Q \mid g(x) = \beta(y)\}$. Since f-$g = \beta \cdot \alpha$, we can define u: $S \to T$ by u: $(x, y) \to (x, y\text{-}\alpha(x))$. Then it is easy to see that u is a kG-isomorphism. Hence from $S \cong N_1 \oplus Q$ and $T \cong N_2 \oplus Q$, we have that $N_1 \cong N_2$.

It is well-known that there is a natural isomorphism between $\mathrm{Ext}^n_{kG}(k, k)$ and $\mathrm{Hom}_{kG}(\Omega^n(k), k)$. So, for an element ψ in $\mathrm{Ext}^n_{kG}(k, k)$, we denote by $\hat{\psi}$ the corresponding kG-homomorphism of $\Omega^n(k)$ into k.

Let $\langle a \rangle$ be a cyclic p-group. Define the $k\langle a \rangle$-homomorphisms $\hat{\xi}$: $\Omega^2(k) = k \to k$ by $\hat{\xi}$: $1 \mapsto 1$, and $\hat{\zeta}$: $\Omega(k) = \mathrm{Rad}\, k\langle a \rangle \to k$ by $\hat{\zeta}$: $(a-1) \mapsto 1$. Then we have the following (see, e.g., [5]):

$$\mathrm{Ext}^*_{k\langle a \rangle}(k, k) = k[\xi] \otimes \Lambda(\zeta)\,, \tag{2.2}$$

where $k[\xi]$ is the polynomial ring and $\Lambda(\zeta) = k + k\zeta$. If $|\langle a \rangle| > 2$, then $\zeta^2 = 0$. On the other hand, if $|\langle a \rangle| = 2$, then $\zeta^2 = \xi$ and so $1 \otimes \zeta^2 = \xi \otimes 1$.

Let A be an abelian p-group and $A = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ be a direct product of cyclic subgroups. It is well-known that Θ: $\mathrm{Ext}^*_{k\langle a_1 \rangle}(k, k) \otimes \cdots \otimes \mathrm{Ext}^*_{k\langle a_n \rangle}(k, k) \cong \mathrm{Ext}^*_{kA}(k, k)$ as k-algebras (see [4]). Let $\xi_i = \Theta(I \otimes \cdots \oplus \xi'_i \otimes \cdots \otimes I)$ and $\zeta_i = \Theta(I \otimes \cdots \otimes \zeta'_i \otimes \cdots \otimes I)$, where ξ'_i and ζ'_i are generators of $\mathrm{Ext}^*_{k\langle a_i \rangle}(k, k)$ as in (2.2). Then we have the following:

$$\mathrm{Ext}^*_{kA}(k, k) = k[\xi_1, \cdots, \xi_n] \otimes \Lambda(\zeta_1, \cdots, \zeta_n)\,.$$

Let E be the unique maximal elementary abelian subgroup of A, According to the decomposition of A, we decompose E into the form $E = \langle x_1, \cdots, x_n \rangle$ with $x_i \in \langle a_i \rangle$. From this decomposition, we obtain

$$\mathrm{Ext}^*_{kE}(k, k) = k[\rho_1, \cdots, \rho_n] \otimes \Lambda(\eta_1, \cdots, \eta_n)\,,$$

wjere ρ_i and η_i are defined similarly as ξ_i and ζ_i in the above. Then since $\Omega^2(k)=k$, we see that $\mathrm{res}_{\langle a_i\rangle,\langle x_i\rangle}(\xi_i)=\rho_i$. Regarding $\mathrm{Ext}^1_{k\langle a_i\rangle}(k, k)$ as $\mathrm{Hom}(\langle a_i\rangle, k)$, we see that if $|\langle a_i\rangle|>p$, then $\mathrm{res}_{\langle a_i\rangle,\langle x_i\rangle}(\zeta_i)=0$. Hence, using the argument of the tensor product of complexes (see [4]), we have the following.

Lemma 2.3. *With the above notations, we have that* $\mathrm{res}_{A,E}(\xi_i)=\rho_i$ *for* $i=1, \cdots, n$, $\mathrm{res}_{A,E}(\zeta_i)=0$ *for* $|\langle a_i\rangle|>p$ *and that* $\mathrm{res}_{A,E}(\zeta_i)=\eta_i$ *for* $|\langle a_i\rangle|=p$.

Here we recall the notion of a Bockstein element (see [8]). Suppose that H is a normal subgroup of a finite group G of index p. A Bokcstein element corresponding to H is an element β in $\mathrm{Ext}^2_{kG}(k, k)$ with $\mathrm{inf}_{G/H,G}(\mathrm{Ext}^2_{k(G/H)}(k, k))=k\cdot\beta$. Note that β is unique up to scalar multiples.

REMARK 2.4. Let $0\to k\to k_H{}^G\to k_H{}^G\to k\to 0$ be the part of the minimal projective $k(G/H)$-resolution of the trivial $k(G/H)$-module to the second syzygy. Then the above 2-extension represents a canonical generator in $\mathrm{Ext}^2_{k(G/H)}(k, k)$. Thus the Bockstein element β can be represented by $0\to k\to k_H{}^G\to k_H{}^G\to k\to 0$ as a sequence of kG-modules. Furthermore, it is known that β can be defined as the image under the Bockstein homomorphism of an element in $\mathrm{Ext}^1_{kG}(k, k)$ which vanishes under the restriction map $\mathrm{res}_{G,H}$. If G is an elementary abelian p-subgroup and $G=\langle x\rangle\times H$, then β can also be seen as a generator of the polynomial subring of $\mathrm{Ext}^*_{kG}(k, k)$ which coiresponds to $\langle x\rangle$ in the decomposition $G=\langle x\rangle\times H$.

Lemma 2.5. *Let A be an abelain p-group and E be the unique maximal elementary ablian subgroup of A. Let H be a maximal subgroup of E and τ be a Bockstein element corresponding to H. Then there exists an element σ in* $\mathrm{Ext}^2_{kA}(k, k)$ *such that* $\mathrm{res}_{A,E}(\sigma)=\tau$.

Proof. From Lemma 3.8 in [4], we have that, with the notation of Lemma 2.3, τ belongs to $k[\rho_1, \cdots, \rho_n]$. So the result is clear by Lemma 2.3.

3. Carlson's condition

Let G be a finite group and k be a field of characteristic $p>0$. Let ψ be an element in $\mathrm{Ext}^n_{kG}(k, k)\cong\mathrm{Hom}_{kG}(\Omega^n(k), k)$. Following Carlson, we let L_ψ be the kernel of $\hat{\psi}\colon \Omega^n(k)\to k$ for $\psi\neq 0$. If $\psi=0$, let $L_\psi=\Omega^n(k)\oplus\Omega(k)$. Carlson's condition is the following (which was originally defined in the case of p-groups):

Carlson's condition: Let n be a positive integer. We say that G satisfies *condition* $C(n)$, provided that for any maximal elementary abelian p-subgroup $E=\langle x_1, \cdots, x_r\rangle$ of G and for any element $u_\alpha=1+\sum_{j=1}^{r}\alpha_j(x_j-1)$ $(\alpha=(\alpha_j)\neq 0\in k^r)$, there exists an element ψ in $\mathrm{Ext}^n_{kG}(k, k)$ whose kernel L_ψ is free as a $k\langle u_\alpha\rangle$-module.

Remark 3.1. (1) The kernel L_ψ of ψ is free as a $k\langle u_\alpha\rangle$-module if and only if $\mathrm{res}_{G,\langle u_\alpha\rangle}(\psi)\neq 0$ (Lemma 3.9 in [4]).

(2) We may assume that k is an algebraically closed field. For, let K be an algebraic closure of k. If G satisfies condition $C(n)$ over K, then for any element u_α as above, there exists ψ in $\mathrm{Ext}^n_{KG}(K, K)$ with $\mathrm{res}_{KG,K\langle u_\alpha\rangle}(\psi)\neq 0$. Since $\mathrm{Ext}^n_{kG}(k, k)\otimes K\cong\mathrm{Ext}^n_{KG}(K, K)$, we may write $\psi=\sum_i \psi_i\otimes x_i$ with $\psi_i\in\mathrm{Ext}^n_{kG}(k, k)$ and $x_i\in K$. Then since $\mathrm{res}_{KG,K\langle u_\alpha\rangle}(\psi)=\sum_i \mathrm{res}_{kG,k\langle u_\alpha\rangle}(\psi_i)\otimes x_i$, there exists ψ_i such that $\mathrm{res}_{kG,k\langle u_\alpha\rangle}(\psi_i)\neq 0$. That is, G satisfies condition $C(n)$ over k.

(3) The condition $C(n)$ does not depend on the choices of generators of E (cf. Section 6 in [3]).

Now, we consider the following stronger condition than Carlson's one. Let $\psi_1, \cdots, \psi_t$ be elements in $\mathrm{Ext}^n_{kG}(k, k)$. We say that G satisfies condition $C(n)$ with $\psi_1, \cdots, \psi_t$, provided that for any element u_α as above, there exists ψ_i in $\{\psi_1, \cdots, \psi_n\}$ whose kernel L_{ψ_i} is free as a $k\langle u_\alpha\rangle$-module.

Before proceeding further, we put down here necessary results for the "cohomology variety". For a comprehensive treatment, we refer to [1] and [4].

Let K be an algebraically closed field of characteristic $p>0$. Let $H^*(G, K)=\sum_{i\geqq 0}\mathrm{Ext}^i_{KG}(K, K)$ if $p=2$ and $H^*(G, K)=\sum_{i\geqq 0}\mathrm{Ext}^{2i}_{KG}(K, K)$ if $p>2$. Then $H^*(G, K)$ has an associated affine variety $V_G(K)=\mathrm{Max}(H^*(G, K))$, which is the set of all maximal ideals of $H^*(G, K)$. Let M be a KG-module and $J_G(M)$ be the annihilator in $H^*(G, K)$ of $\mathrm{Ext}^*_{KG}(M, M)$. The variety $V_G(M)$ of M is defined as the subvariety of $V_G(K)$ associated to $J_G(M)$.

(3.2) Let $E=\langle x_1, \cdots, x_r\rangle$ be an elementary abelian p-group and $u_\alpha=1+\sum_{j=1}^{r}\alpha_j(x_j-1)$, $\alpha=(\alpha_j)\in K^r$. For a KE-module M, let $V_r(M)=\{0\}\cup\{\alpha\in K^r \mid M_{\langle u_\alpha\rangle}$ is not free as a $K\langle u_\alpha\rangle$-module$\}$. Then $V_r(M)$ is a subvariety of K^r, and via $V_E(K)\cong K^r$, we have that $V_E(M)\cong V_r(M)$.

Lemma 3.3. *Let M and N be KG-modules.*

(a) $V_G(M)=\{0\}$ *if and only if M is projective.*

(b) $V_G(M\otimes N)=V_G(M)\cap V_G(N)$.

(c) *For $\psi\in H^i(G, K)$, $V_G(L_\psi)=V(\psi)$, where $V(\psi)$ is the variety of the ideal $H^*(G, K)\cdot\psi$. That is,* $\sqrt{J_G(L_\psi)}=\sqrt{H^*(G, K)\cdot\psi}$.

Proposition 3.4. *Let n be a positive integer and $\psi_1, \cdots, \psi_t$ be elements in* $\mathrm{Ext}^{2n}_{KG}(K, K)$. *Then the following are equivalent.*

(1) *G satisfies condition $C(2n)$ with $\psi_1, \cdots, \psi_t$.*

(2) *$L_{\psi_1}\otimes\cdots\otimes L_{\psi_t}$ is projective.*

(3) $\sqrt{(\psi_1, \cdots, \psi_t)}=\sum_{i>0}\mathrm{Ext}^i_{KG}(K, K)$, *where* $\sqrt{(\psi_1, \cdots, \psi_t)}=\{\psi\in\mathrm{Ext}^*_{KG}(K, K) \mid \psi^c\in\sum_{i=1}^{t}\mathrm{Ext}^*_{KG}(K, K)\,\psi_i$ *for some* $c>0\}$.

Proof. By Chouinard's theorem ([4]), (2) holds if and only if $(L_{\psi_1}\otimes\cdots\otimes L_{\psi_t})_E$ is projective for every maximal elementary abelian p-subgroup E of G. Noting Lemma 3.3 and (3.2), we see that $(L_{\psi_1}\otimes\cdots\otimes L_{\psi_t})_E$ is projective for every E if and only if (1) holds.

By Lemma 3.3, (2) holds if and only if $V(\psi_1)\cap\cdots\cap V(\psi_t)=\{0\}$. We recall the fact that the point 0 in $V_G(K)$ is the maximal ideal $\sum_{i>0}\mathrm{Ext}^i_{KG}(G,K)$ ($\sum_{i>0}\mathrm{Ext}^{2i}_{KG}(K,K)$ for $p>2$) and that if $p>2$, the elements of odd degree in $\mathrm{Ext}^*_{KG}(K,K)$ are nilpotent. Then we see that $V(\psi_1)\cap\cdots\cap V(\psi_t)=\{0\}$ if and only if (3) holds.

REMARK 3.5. If $p=2$, then for any positive integer n, the above proposition is also true for elements $\psi_1,\cdots,\psi_t$ of degree n.

4. The main theorem

As before, G is a finite group and k is a field of characteristic $p>0$. The following definition is due to Carlson [2]:

DEFINITION 4.1. Let E be a maximal elementary abelian p-subgroup of G. Let A_E be an abelian p-subgroup of G which contains E and which has maximal order among such subgroups. Define $n(E)=|G:A_E|$ and $n(G)=L.C.M._{E\in\Gamma}\{n(E)\}$, where Γ is the set of all maximal elementary abelian p-subgroups of G.

The next theorem is the main result of this paper.

Theorem 4.2. *Let G be a finite group and k be a field of characteristic $p>0$. Then there exist $\psi_1,\cdots,\psi_t$ in $\mathrm{Ext}^{2n(G)}_{kG}(k,k)$ such that $L_{\psi_1}\otimes\cdots\otimes L_{\psi_t}$ is a projective kG-module.*

We shall prove the theorem with a series of lemmas. The first one is an analogue of a result of Quillen (see, e.g., Lemma 2.26.5 in [1]).

For $g\in G$ and a subgroup H of G, we write ${}^gH=gHg^{-1}$ and let ${}^g\gamma$ be the conjugation $\mathrm{con}_{H,{}^gH}(\gamma)\in\mathrm{Ext}^n_{k({}^gH)}(k,k)$ for $\gamma\in\mathrm{Ext}^n_{kH}(k,k)$. Let A be an abelian p-subgroup of G, E be the unique maximal elementary abelian subgroup of A and F be a elementary abelian subgroup of A.

Let g be an element in $G-N_G(F)$. We consider the following diagram:

$$\begin{tikzcd}
 & & {}^gE & \\
 & {}^gF({}^gA\cap F) \arrow[ur, dash] \arrow[dr, dash] & & W \arrow[ul, dash] \arrow[dl, dash] \\
{}^gF \arrow[ur, dash] \arrow[dr, dash] & & X \arrow[dl, dash] \arrow[dr, dash] & \\
 & {}^gF\cap X & & {}^gA\cap F,
\end{tikzcd}$$

where X is a maximal subgroup of ${}^gF({}^gA\cap F)$ which contains ${}^gA\cap F$, and W/X is a complement to ${}^gF({}^gA\cap F)/X$ in ${}^gE/X$. Let $M={}^{g^{-1}}W$ and $L={}^{g^{-1}}({}^gF\cap X)$. Then M and L are maximal subgroups of E and F. Now let $\tau\in \mathrm{Ext}^2_{kE}(k,k)$ be a Bockstein element corresponding to M. Then we know that $\nu=\mathrm{res}_{E,F}(\tau)\in \mathrm{Ext}^2_{kF}(k,k)$ is a Bockstein element corresponding to L.

Lemma 4.3. *Let A be an abelian p-subgroup of G and F be an elementary abelian subgroup of A. Then there exists an element ψ in $\mathrm{Ext}^{2|G:A|}_{kG}(k,k)$ such that $\mathrm{res}_{G,F}(\psi)$ is a product of Bockstein elements.*

Proof. We write $N=N_G(F)$ and let $\{g_1,g_2,\cdots,g_n\}$ be a set of representatives for the right cosets of N in G, with $g_1=1$.

As before, let E be the unique maximal elementary abelian subgroup of A. Let L_1 be a maximal subgroup of F and M_1/L_1 be the complement to F/L_1 in E/L_1. For the maximal subgroup M_1 of E, let $\tau_1\in\mathrm{Ext}^2_{kE}(k,k)$ be a Bockstein element corresponding to M_1. Then we see that $\nu_1=\mathrm{res}_{E,F}(\tau_1)\in\mathrm{Ext}^2_{kF}(k,k)$ is a Bockstein element corresponding to L_1. For each $g_i(i>1)$, we denote by $\tau_i\in \mathrm{Ext}^2_{kE}(k,k)$ and $\nu_i\in\mathrm{Ext}^2_{kF}(k,k)$ the Bockstein elements corresponding to M_i and L_i respectively. Then by Lemma 2.5, there exists an element σ_i in $\mathrm{Ext}^2_{kA}(k,k)$ such that $\mathrm{res}_{A,E}(\sigma_i)=\tau_i$ for $i=1,2,\cdots,n$. Now, define $\sigma\in\mathrm{Ext}^{2|G:N|}_{kA}(k,k)$ by $\sigma=\sigma_1\sigma_2\cdots\sigma_n$. Then, for g_i $(i>1)$, we have that

$$\begin{aligned}
\mathrm{res}_{{}^{g_i}A,{}^{g_i}A\cap F}({}^{g_i}\sigma_i) &= \mathrm{res}_{{}^{g_i}E,{}^{g_i}A\cap F}\,\mathrm{res}_{{}^{g_i}A,{}^{g_i}E}({}^{g_i}\sigma_i)\\
&= \mathrm{res}_{{}^{g_i}E,{}^{g_i}A\cap F}({}^{g_i}\tau_i)\\
&= \mathrm{res}_{{}^{g_i}M_i,{}^{g_i}A\cap F}\cdot\mathrm{res}_{{}^{g_i}E,{}^{g_i}M_i}({}^{g_i}\tau_i)\\
&= \mathrm{res}_{{}^{g_i}M_i,{}^{g_i}A\cap F}({}^{g_i}\mathrm{res}_{E,M_i}(\tau_i))\\
&=0\,.
\end{aligned}$$

Let x be an element in $G-N$. So $x=ug_i$ $(u\in N,\ i>1)$ and we have

$$\begin{aligned}
\mathrm{res}_{{}^xA,{}^xA\cap F}({}^x\sigma) &= \mathrm{res}_{{}^{ug_i}A,{}^{ug_i}A\cap F}({}^{ug_i}\sigma)\\
&= {}^u(\mathrm{res}_{{}^{g_i}A,{}^{g_i}A\cap F}({}^{g_i}\sigma))\\
&=0\,.
\end{aligned}$$

Therefore by the Mackey decomposition theorem for the norm map (Proposition 2 in [7]), we have

$$\begin{aligned}
\mathrm{res}_{G,F}\,\mathrm{norm}_{A,G}(1+\sigma) &= \prod_{x\in F\backslash G/A}\mathrm{norm}_{{}^xA\cap F,F}\,\mathrm{res}_{{}^xA,{}^xA\cap F}(1+{}^x\sigma)\\
&= \prod_{x\in N/A}\mathrm{res}_{{}^xA,F}(1+{}^x\sigma)\,.
\end{aligned}$$

So, if ψ denotes the homogeneous part of highest degree of $\mathrm{norm}_{A,G}(1+\sigma)$, we have

$$\mathrm{res}_{G,F}(\psi)=\prod_{x\in N/A}\mathrm{res}_{{}^xA,F}({}^x\sigma)=\prod_{x\in N/A}\prod_{i=1}^{n}\mathrm{res}_{{}^xE,F}({}^x\tau_i)$$
$$=\prod_{x\in N/F}\prod_{i=1}^{n}{}^x\nu_i\,.$$

Here ${}^x\nu_i$ is a Bockstein element corresponding to ${}^xL_i$, and ψ belongs to $\mathrm{Ext}_{kG}^{2|G:A|}(k,k)$. This completes the proof of the lemma.

The next result is Lemma 4.2 in Okuyama-Sasaki [8]. For the convenience of the reader, we give here a proof to it.

Lemma 4.4. *Let H be a normal subgroup of G of index p and β be a non zero Bockstein element corresponding to H. If a kG-module M is projective as a kH-module, then $L_\beta\otimes M$ is a projective kG-module.*

Proof. By Remark 3.6, we see that β can be represented by $0\to k\to k_H{}^G\to k_H{}^G\to k\to 0$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_\beta & \to & \mathrm{Ker}\,\lambda_1 & \to & \mathrm{Ker}\,\lambda_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^2(k) & \to & P_1 & \to & P_0 & \to & k \to 0 \\
 & & \downarrow\beta & & \downarrow\lambda_1 & & \downarrow\lambda_0 & & \| \\
0 & \to & k & \to & k_H{}^G & \to & k_H{}^G & \to & k \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0\ , & &
\end{array}$$

where P_0 and P_1 are the projective covers of k and $\Omega(k)$. By tensoring this diagram with M, we find readily that $L_\beta\otimes M$ is projective, since $k_H{}^G\otimes M\simeq M_H{}^G$ is projective.

Lemma 4.5. *Let M be a kG-module. For $\gamma_1\in\mathrm{Ext}_{kG}^n(k,k)$ and $\gamma_2\in\mathrm{Ext}_{kG}^m(k,k)$, suppose that $L_{\gamma_1}\otimes M$ and $L_{\gamma_2}\otimes M$ are projective kG-modules. Then $L_{\gamma_1\gamma_2}\otimes M$ is a projective kG-module.*

Proof. If γ_1 and γ_2 are non-zero, then, as is given in the proof of Theorem 8.5 in [4], there exists an exact sequence:

$$0\to\Omega^n(L_{\gamma_2})\to L_{\gamma_1\gamma_2}\oplus(\text{projective } kG\text{-module})\to L_{\gamma_1}\to 0\,.$$

Tensoring this sequence with M, we see that $L_{\gamma_1\gamma_2}\otimes M$ is projective. If $\gamma_1=0$ or $\gamma_2=0$, then the assertion is immediate from the definition of L_{γ_i}.

Lemma 4.6. *Let $\psi\in\mathrm{Ext}_{kG}^n(k,k)$ and H be a subgroup of G. Then $(L_\psi)_H\simeq L_{\mathrm{res}_{G,H}(\psi)}\oplus(\text{projective } kH\text{-module})$.*

Proof. We have $\Omega^n(k)_H = \Omega^n(k_H) \oplus Q$ with a projective kH-module Q. If $\gamma = \mathrm{res}_{G,H}(\psi) \neq 0$, then let $h\colon \Omega^n(k)_H = \Omega^n(k_H) \oplus Q \to k_H$ be the kH-homomorphism defined by $(w, w') \mapsto \hat{\gamma}(w)$ for $w \in \Omega^n(k_H)$, $w' \in Q$. By the definition of the restriction map, $\hat{\psi}_{\Omega^n(k_H)} = \hat{\gamma}$. Thus $\hat{\psi} - h$ is a projective kH-map and so by Proposition 2.1, $(L_\psi)_H \cong L_{\mathrm{res}_{G,H}(\psi)} \oplus Q$. If $\mathrm{res}_{G,H}(\psi) = 0$, it follows from Lemma 8.1 in [4] that $(L_\psi)_H \cong \Omega^n(k_H) \oplus \Omega(k_H) \oplus$(projective kH-module). This completes the proof.

Proof of Theorem 4.2. It suffices to show that given an elementary abelian p-subgroup F of G, there exist $\psi_1, \cdots, \psi_r$ in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_r})_F$ is a projective kF-module. For, if this is shown, then consider all those $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ taken over the elementary abelian p-subgroups of G. Then by Chouinard's theorem, we have that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is a projective kG-module.

We now prove the above assertion by induction on $|F|$. If F is cyclic, then our assertion has been proved in Lemmas 4.3 and 4.6. So we may assume that F is non-cyclic and that there exist elements $\psi_2, \cdots, \psi_r$ in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $(L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_L$ is projective for every maximal subgroup L of F. Now, Lemma 4.3 implies that there exists an element ψ_1 in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $\mathrm{res}_{G,F}(\psi_1)$ is a product of Bockstein elements. Then by our assumptions and Lemmas 4.4 and 4.5, we see that $L_{\mathrm{res}_{G,F}(\psi_1)} \otimes (L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_F$ is a projective kF-module. So from Lemma 4.6, we see that $(L_{\psi_1} \otimes L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_F$ is projective.

5. Applications

Let G be a finite group, k be a field of characteristic $p > 0$ and K be an algebraic closure of k. Let $n(G)$ be the integer given in Definition 4.1. Then Proposition 3.4 and Theorem 4.2 yield:

Corollary 5.1 (Periodicity of periodic modules). *The period of a periodic kG-module divides $2n(G)$.*

Proof. By Theorem 4.2, there exist $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is projective, so that $V_G(L_{\psi_1}{}^K \otimes \cdots \otimes L_{\psi_t}{}^K) = V_G(L_{\psi_1 \otimes I} \otimes \cdots \otimes L_{\psi_t \otimes I}) = \{0\}$. Then the assertion is followed by the same argument as in the proof of Theorem 8.7 in [4].

Corollary 4.7 (Criterion for a module to be projective). *A kG-module M is projective if and only if* $\mathrm{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$.

Proof. If $\mathrm{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$, then $\mathrm{Ext}_{KG}^{2n(G)}(M^K, M^K) = \{0\}$. Taking $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ as in Theorem 4.2, we have from the assumption that $\psi_1 \otimes I, \cdots, \psi_t \otimes I \in \mathrm{Ext}_{KG}^{2n(G)}(K, K)$ annihilate $\mathrm{Ext}_{KG}^*(M^K, M^K)$, so that $\sqrt{J_G(M^K)} \supset \sqrt{(\psi_1 \otimes I, \cdots, \psi_t \otimes I)}$. Then from Proposition 3.4, we see that $\sqrt{J_G(M^K)} =$

$\sum_{i>0} H^i(G, K)$, that is, $V_G(M^K)=\{0\}$. Therefore by Lemma 2.3, M^K is projective and so M is projective.

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