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ON THE COHOMOLOGY OF FINITE GROUPS AND THE APPLICATIONS TO MODULAR REPRESENTATIONS

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1. Introduction

Let G be a finite group and k be a field of prime characteristic p . All modules considered here are assumed to be finite dimensional over k . In [2], Carlson introduced a certain condition on the cohomology ring of G to study the structure of periodic modules by homological techniques. Let us denote it by $C(n)$, where n is a positive integer. If G satisfies $C(n)$, then there are homogeneous elements of degree n having an interesting property related to his notion of rank variety (see Section 3 for details). For a p -group P , he showed that there exists an integer $n(P)$ such that P satisfies $C(2n(P))$. And using this, he showed that the period of a periodic kP -module divides $2n(P)$.

The purpose of this paper is to extend Carlson's results to an arbitrary finite group G . In doing so, we shall give a stronger version of the condition $C(n)$, with a couple of equivalent conditions to it. Concerning Carlson's number $n(G)$ which can as well be defined for an arbitrary G , we shall prove that there exist cohomology elements of degree $2n(G)$ satisfying our new condition, so that G satisfies $C(2n(G))$. As an application of this result, we shall show that the period of a periodic kG -module divides $2n(G)$. As another application, we also give a homological criterion for a kG -module to be projective. A similar criterion has been given by Donovan [6], in response to a problem of Schultz [9].

2. Preliminaries

In this section we mention some preliminary facts needed in later arguments. For a kG -module M , set $\text{Ext}_{kG}^*(M, M) = \sum_{n \geq 0} \text{Ext}_{kG}^n(M, M)$. If H is a subgroup of G , then M_H denotes the restriction of M to a kH -module. First of all, we prove the following general fact.

Proposition 2.1. *Let $0 \rightarrow N_1 \rightarrow M \xrightarrow{f} L \rightarrow 0$ and $0 \rightarrow N_2 \rightarrow M \xrightarrow{g} L \rightarrow 0$ be exact*

sequences of kG -modules. If $f \circ g$ factors through a projective kG -module Q , then $N_1 \cong N_2$.

Proof. Suppose that $\alpha: M \rightarrow Q$ and $\beta: Q \rightarrow L$ give a factorization of $f \circ g$ through the projective module Q . Then we have the following two pull-back diagrams:

$$\begin{array}{ccccccc} 0 & \rightarrow & N_1 & \rightarrow & S & \rightarrow & Q \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \beta \\ 0 & \rightarrow & N_1 & \rightarrow & M & \xrightarrow{f} & L \rightarrow 0, \end{array}$$

where $S = \{(x, y) \in M \oplus Q \mid f(x) = \beta(y)\}$,

$$\begin{array}{ccccccc} 0 & \rightarrow & N_2 & \rightarrow & T & \rightarrow & Q \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \beta \\ 0 & \rightarrow & N_2 & \rightarrow & M & \xrightarrow{g} & L \rightarrow 0, \end{array}$$

where $T = \{(x, y) \in M \oplus Q \mid g(x) = \beta(y)\}$. Since $f \circ g = \beta \circ \alpha$, we can define $u: S \rightarrow T$ by $u: (x, y) \rightarrow (x, y - \alpha(x))$. Then it is easy to see that u is a kG -isomorphism. Hence from $S \cong N_1 \oplus Q$ and $T \cong N_2 \oplus Q$, we have that $N_1 \cong N_2$.

It is well-known that there is a natural isomorphism between $\text{Ext}_{kG}^n(k, k)$ and $\text{Hom}_{kG}(\Omega^n(k), k)$. So, for an element ψ in $\text{Ext}_{kG}^n(k, k)$, we denote by $\hat{\psi}$ the corresponding kG -homomorphism of $\Omega^n(k)$ into k .

Let $\langle a \rangle$ be a cyclic p -group. Define the $k\langle a \rangle$ -homomorphisms $\hat{\xi}: \Omega^2(k) \rightarrow k \rightarrow k$ by $\hat{\xi}: 1 \mapsto 1$, and $\hat{\zeta}: \Omega(k) = \text{Rad } k\langle a \rangle \rightarrow k$ by $\hat{\zeta}: (a-1) \mapsto 1$. Then we have the following (see, e.g., [5]):

$$(2.2) \quad \text{Ext}_{k\langle a \rangle}^*(k, k) = k[\hat{\xi}] \otimes \Lambda(\hat{\zeta}),$$

where $k[\hat{\xi}]$ is the polynomial ring and $\Lambda(\hat{\zeta}) = k + k\hat{\zeta}$. If $|\langle a \rangle| > 2$, then $\hat{\zeta}^2 = 0$. On the other hand, if $|\langle a \rangle| = 2$, then $\hat{\zeta}^2 = \hat{\xi}$ and so $1 \otimes \hat{\zeta}^2 = \hat{\xi} \otimes 1$.

Let A be an abelian p -group and $A = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ be a direct product of cyclic subgroups. It is well-known that $\Theta: \text{Ext}_{k\langle a_1 \rangle}^*(k, k) \otimes \cdots \otimes \text{Ext}_{k\langle a_n \rangle}^*(k, k) \cong \text{Ext}_{kA}^*(k, k)$ as k -algebras (see [4]). Let $\xi_i = \Theta(I \otimes \cdots \otimes \xi'_i \otimes \cdots \otimes I)$ and $\zeta_i = \Theta(I \otimes \cdots \otimes \zeta'_i \otimes \cdots \otimes I)$, where ξ'_i and ζ'_i are generators of $\text{Ext}_{k\langle a_i \rangle}^*(k, k)$ as in (2.2). Then we have the following:

$$\text{Ext}_{kA}^*(k, k) = k[\xi_1, \dots, \xi_n] \otimes \Lambda(\zeta_1, \dots, \zeta_n).$$

Let E be the unique maximal elementary abelian subgroup of A . According to the decomposition of A , we decompose E into the form $E = \langle x_1, \dots, x_n \rangle$ with $x_i \in \langle a_i \rangle$. From this decomposition, we obtain

$$\text{Ext}_{kE}^*(k, k) = k[\rho_1, \dots, \rho_n] \otimes \Lambda(\eta_1, \dots, \eta_n),$$

wjere ρ_i and η_i are defined similarly as ξ_i and ζ_i in the above. Then since $\Omega^2(k)=k$, we see that $\text{res}_{\langle a_i \rangle, \langle x_i \rangle}(\xi_i)=\rho_i$. Regarding $\text{Ext}_{k\langle a_i \rangle}^1(k, k)$ as $\text{Hom}(\langle a_i \rangle, k)$, we see that if $|\langle a_i \rangle| > p$, then $\text{res}_{\langle a_i \rangle, \langle x_i \rangle}(\zeta_i)=0$. Hence, using the argument of the tensor product of complexes (see [4]), we have the following.

Lemma 2.3. *With the above notations, we have that $\text{res}_{A,E}(\xi_i)=\rho_i$ for $i=1, \dots, n$, $\text{res}_{A,E}(\zeta_i)=0$ for $|\langle a_i \rangle| > p$ and that $\text{res}_{A,E}(\zeta_i)=\eta_i$ for $|\langle a_i \rangle|=p$.*

Here we recall the notion of a Bockstein element (see [8]). Suppose that H is a normal subgroup of a finite group G of index p . A Bockstein element corresponding to H is an element β in $\text{Ext}_{kG}^2(k, k)$ with $\text{inf}_{G/H, G}(\text{Ext}_{k(G/H)}^2(k, k))=k \cdot \beta$. Note that β is unique up to scalar multiples.

REMARK 2.4. Let $0 \rightarrow k \rightarrow k_H^G \rightarrow k_H^G \rightarrow k \rightarrow 0$ be the part of the minimal projective $k(G/H)$ -resolution of the trivial $k(G/H)$ -module to the second syzygy. Then the above 2-extension represents a canonical generator in $\text{Ext}_{k(G/H)}^2(k, k)$. Thus the Bockstein element β can be represented by $0 \rightarrow k \rightarrow k_H^G \rightarrow k_H^G \rightarrow k \rightarrow 0$ as a sequence of kG -modules. Furthermore, it is known that β can be defined as the image under the Bockstein homomorphism of an element in $\text{Ext}_{kG}^1(k, k)$ which vanishes under the restriction map $\text{res}_{G, H}$. If G is an elementary abelian p -subgroup and $G = \langle x \rangle \times H$, then β can also be seen as a generator of the polynomial subring of $\text{Ext}_{kG}^*(k, k)$ which corresponds to $\langle x \rangle$ in the decomposition $G = \langle x \rangle \times H$.

Lemma 2.5. *Let A be an abelian p -group and E be the unique maximal elementary abelian subgroup of A . Let H be a maximal subgroup of E and τ be a Bockstein element corresponding to H . Then there exists an element σ in $\text{Ext}_{kA}^2(k, k)$ such that $\text{res}_{A,E}(\sigma)=\tau$.*

Proof. From Lemma 3.8 in [4], we have that, with the notation of Lemma 2.3, τ belongs to $k[\rho_1, \dots, \rho_n]$. So the result is clear by Lemma 2.3.

3. Carlson's condition

Let G be a finite group and k be a field of characteristic $p > 0$. Let ψ be an element in $\text{Ext}_{kG}^n(k, k) \cong \text{Hom}_{kG}(\Omega^n(k), k)$. Following Carlson, we let L_ψ be the kernel of $\hat{\psi}: \Omega^n(k) \rightarrow k$ for $\psi \neq 0$. If $\psi = 0$, let $L_\psi = \Omega^n(k) \oplus \Omega(k)$. Carlson's condition is the following (which was originally defined in the case of p -groups):

Carlson's condition: Let n be a positive integer. We say that G satisfies *condition C(n)*, provided that for any maximal elementary abelian p -subgroup $E = \langle x_1, \dots, x_r \rangle$ of G and for any element $u_\alpha = 1 + \sum_{j=1}^r \alpha_j(x_j - 1)$ ($\alpha = (\alpha_j) \neq 0 \in k^r$), there exists an element ψ in $\text{Ext}_{kG}^n(k, k)$ whose kernel L_ψ is free as a $k\langle u_\alpha \rangle$ -module.

REMARK 3.1. (1) The kernel L_ψ of ψ is free as a $k\langle u_\alpha \rangle$ -module if and only if $\text{res}_{G, \langle u_\alpha \rangle}(\psi) \neq 0$ (Lemma 3.9 in [4]).

(2) We may assume that k is an algebraically closed field. For, let K be an algebraic closure of k . If G satisfies condition $C(n)$ over K , then for any element u_α as above, there exists ψ in $\text{Ext}_{KG}^n(K, K)$ with $\text{res}_{KG, K\langle u_\alpha \rangle}(\psi) \neq 0$. Since $\text{Ext}_{kG}^n(k, k) \otimes K \cong \text{Ext}_{KG}^n(K, K)$, we may write $\psi = \sum_i \psi_i \otimes x_i$ with $\psi_i \in \text{Ext}_{kG}^n(k, k)$ and $x_i \in K$. Then since $\text{res}_{KG, K\langle u_\alpha \rangle}(\psi) = \sum_i \text{res}_{kG, k\langle u_\alpha \rangle}(\psi_i) \otimes x_i$, there exists ψ_i such that $\text{res}_{kG, k\langle u_\alpha \rangle}(\psi_i) \neq 0$. That is, G satisfies condition $C(n)$ over k .

(3) The condition $C(n)$ does not depend on the choices of generators of E (cf. Section 6 in [3]).

Now, we consider the following stronger condition than Carlson's one. Let ψ_1, \dots, ψ_t be elements in $\text{Ext}_{kG}^n(k, k)$. We say that G satisfies condition $C(n)$ with ψ_1, \dots, ψ_t , provided that for any element u_α as above, there exists ψ_i in $\{\psi_1, \dots, \psi_t\}$ whose kernel L_{ψ_i} is free as a $k\langle u_\alpha \rangle$ -module.

Before proceeding further, we put down here necessary results for the "cohomology variety". For a comprehensive treatment, we refer to [1] and [4].

Let K be an algebraically closed field of characteristic $p > 0$. Let $H^*(G, K) = \sum_{i \geq 0} \text{Ext}_{kG}^i(K, K)$ if $p = 2$ and $H^*(G, K) = \sum_{i \geq 0} \text{Ext}_{kG}^{2i}(K, K)$ if $p > 2$. Then $H^*(G, K)$ has an associated affine variety $V_G(K) = \text{Max}(H^*(G, K))$, which is the set of all maximal ideals of $H^*(G, K)$. Let M be a KG -module and $J_G(M)$ be the annihilator in $H^*(G, K)$ of $\text{Ext}_{kG}^*(M, M)$. The variety $V_G(M)$ of M is defined as the subvariety of $V_G(K)$ associated to $J_G(M)$.

(3.2) Let $E = \langle x_1, \dots, x_r \rangle$ be an elementary abelian p -group and $u_\alpha = 1 + \sum_{j=1}^r \alpha_j(x_j - 1)$, $\alpha = (\alpha_j) \in K^r$. For a KE -module M , let $V_r(M) = \{0\} \cup \{\alpha \in K^r \mid M_{\langle u_\alpha \rangle} \text{ is not free as a } K\langle u_\alpha \rangle\text{-module}\}$. Then $V_r(M)$ is a subvariety of K^r , and via $V_E(K) \cong K^r$, we have that $V_E(M) \cong V_r(M)$.

Lemma 3.3. *Let M and N be KG -modules.*

(a) $V_G(M) = \{0\}$ if and only if M is projective.

(b) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.

(c) For $\psi \in H^i(G, K)$, $V_G(L_\psi) = V(\psi)$, where $V(\psi)$ is the variety of the ideal $H^*(G, K) \cdot \psi$. That is, $\sqrt{J_G(L_\psi)} = \sqrt{H^*(G, K) \cdot \psi}$.

Proposition 3.4. *Let n be a positive integer and ψ_1, \dots, ψ_t be elements in $\text{Ext}_{kG}^{2n}(K, K)$. Then the following are equivalent.*

(1) G satisfies condition $C(2n)$ with ψ_1, \dots, ψ_t .

(2) $L_{\psi_1} \otimes \dots \otimes L_{\psi_t}$ is projective.

(3) $\sqrt{(\psi_1, \dots, \psi_t)} = \sum_{i>0} \text{Ext}_{kG}^i(K, K)$, where $\sqrt{(\psi_1, \dots, \psi_t)} = \{\psi \in \text{Ext}_{kG}^*(K, K) \mid \psi^c \in \sum_{i=1}^t \text{Ext}_{kG}^i(K, K) \psi_i \text{ for some } c > 0\}$.

Proof. By Chouinard's theorem ([4]), (2) holds if and only if $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_t})_E$ is projective for every maximal elementary abelian p -subgroup E of G . Noting Lemma 3.3 and (3.2), we see that $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_t})_E$ is projective for every E if and only if (1) holds.

By Lemma 3.3, (2) holds if and only if $V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}$. We recall the fact that the point 0 in $V_G(K)$ is the maximal ideal $\sum_{i>0} \text{Ext}_{kG}^i(G, K)$ ($\sum_{i>0} \text{Ext}_{kG}^{2i}(K, K)$ for $p>2$) and that if $p>2$, the elements of odd degree in $\text{Ext}_{kG}^*(K, K)$ are nilpotent. Then we see that $V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}$ if and only if (3) holds.

REMARK 3.5. If $p=2$, then for any positive integer n , the above proposition is also true for elements ψ_1, \dots, ψ_t of degree n .

4. The main theorem

As before, G is a finite group and k is a field of characteristic $p>0$. The following definition is due to Carlson [2]:

DEFINITION 4.1. Let E be a maximal elementary abelian p -subgroup of G . Let A_E be an abelian p -subgroup of G which contains E and which has maximal order among such subgroups. Define $n(E) = |G : A_E|$ and $n(G) = L.C.M._{E \in \Gamma} \{n(E)\}$, where Γ is the set of all maximal elementary abelian p -subgroups of G .

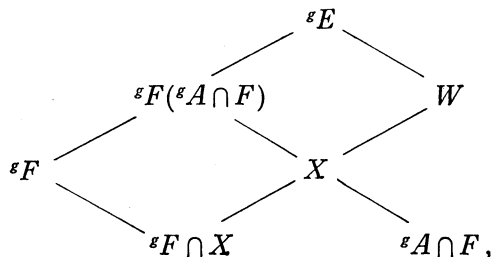
The next theorem is the main result of this paper.

Theorem 4.2. *Let G be a finite group and k be a field of characteristic $p>0$. Then there exist ψ_1, \dots, ψ_t in $\text{Ext}_{kG}^{2n(G)}(k, k)$ such that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is a projective kG -module.*

We shall prove the theorem with a series of lemmas. The first one is an analogue of a result of Quillen (see, e.g., Lemma 2.26.5 in [1]).

For $g \in G$ and a subgroup H of G , we write ${}^gH = gHg^{-1}$ and let ${}^g\gamma$ be the conjugation $\text{con}_H, {}^g\gamma \in \text{Ext}_{k({}^gH)}^n(k, k)$ for $\gamma \in \text{Ext}_{kH}^n(k, k)$. Let A be an abelian p -subgroup of G , E be the unique maximal elementary abelian subgroup of A and F be a elementary abelian subgroup of A .

Let g be an element in $G - N_G(F)$. We consider the following diagram:



where X is a maximal subgroup of ${}^g F({}^g A \cap F)$ which contains ${}^g A \cap F$, and W/X is a complement to ${}^g F({}^g A \cap F)/X$ in ${}^g E/X$. Let $M = {}^g^{-1}W$ and $L = {}^g^{-1}({}^g F \cap X)$. Then M and L are maximal subgroups of E and F . Now let $\tau \in \text{Ext}_{kE}^2(k, k)$ be a Bockstein element corresponding to M . Then we know that $\nu = \text{res}_{E,F}(\tau) \in \text{Ext}_{kF}^2(k, k)$ is a Bockstein element corresponding to L .

Lemma 4.3. *Let A be an abelian p -subgroup of G and F be an elementary abelian subgroup of A . Then there exists an element ψ in $\text{Ext}_{kG}^{2|G:A|}(k, k)$ such that $\text{res}_{G,F}(\psi)$ is a product of Bockstein elements.*

Proof. We write $N = N_G(F)$ and let $\{g_1, g_2, \dots, g_n\}$ be a set of representatives for the right cosets of N in G , with $g_1 = 1$.

As before, let E be the unique maximal elementary abelian subgroup of A . Let L_1 be a maximal subgroup of F and M_1/L_1 be the complement to F/L_1 in E/L_1 . For the maximal subgroup M_1 of E , let $\tau_1 \in \text{Ext}_{kE}^2(k, k)$ be a Bockstein element corresponding to M_1 . Then we see that $\nu_1 = \text{res}_{E,F}(\tau_1) \in \text{Ext}_{kF}^2(k, k)$ is a Bockstein element corresponding to L_1 . For each $g_i (i > 1)$, we denote by $\tau_i \in \text{Ext}_{kE}^2(k, k)$ and $\nu_i \in \text{Ext}_{kF}^2(k, k)$ the Bockstein elements corresponding to M_i and L_i respectively. Then by Lemma 2.5, there exists an element σ_i in $\text{Ext}_{kA}^2(k, k)$ such that $\text{res}_{A,E}(\sigma_i) = \tau_i$ for $i = 1, 2, \dots, n$. Now, define $\sigma \in \text{Ext}_{kA}^{2|G:A|}(k, k)$ by $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$. Then, for $g_i (i > 1)$, we have that

$$\begin{aligned} \text{res}_{g_i A, g_i A \cap F}({}^{g_i} \sigma_i) &= \text{res}_{g_i B, g_i A \cap F} \text{res}_{g_i A, g_i B}({}^{g_i} \sigma_i) \\ &= \text{res}_{g_i B, g_i A \cap F}({}^{g_i} \tau_i) \\ &= \text{res}_{g_i M_i, g_i A \cap F} \cdot \text{res}_{g_i B, g_i M_i}({}^{g_i} \tau_i) \\ &= \text{res}_{g_i M_i, g_i A \cap F}({}^{g_i} \text{res}_{E, M_i}(\tau_i)) \\ &= 0. \end{aligned}$$

Let x be an element in $G - N$. So $x = u g_i (u \in N, i > 1)$ and we have

$$\begin{aligned} \text{res}_{x A, x A \cap F}(x \sigma) &= \text{res}_{u g_i A, u g_i A \cap F}({}^{u g_i} \sigma) \\ &= {}^u (\text{res}_{g_i A, g_i A \cap F}({}^{g_i} \sigma)) \\ &= 0. \end{aligned}$$

Therefore by the Mackey decomposition theorem for the norm map (Proposition 2 in [7]), we have

$$\begin{aligned} \text{res}_{G,F} \text{norm}_{A,G}(1 + \sigma) &= \prod_{x \in F \backslash G/A} \text{norm}_{x A \cap F, F} \text{res}_{x A, x A \cap F}(1 + x \sigma) \\ &= \prod_{x \in N/A} \text{res}_{x A, F}(1 + x \sigma). \end{aligned}$$

So, if ψ denotes the homogeneous part of highest degree of $\text{norm}_{A,G}(1 + \sigma)$, we have

$$\begin{aligned} \text{res}_{G,F}(\psi) &= \prod_{x \in \mathcal{N}/\mathcal{A}} \text{res}_{x_{A,F}}({}^x\sigma) = \prod_{x \in \mathcal{N}/\mathcal{A}} \prod_{i=1}^n \text{res}_{x_{E,F}}({}^x\tau_i) \\ &= \prod_{x \in \mathcal{N}/F} \prod_{i=1}^n {}^x\nu_i. \end{aligned}$$

Here ${}^x\nu_i$ is a Bockstein element corresponding to xL_i , and ψ belongs to $\text{Ext}_{kG}^{2|G|}(k, k)$. This completes the proof of the lemma.

The next result is Lemma 4.2 in Okuyama-Sasaki [8]. For the convenience of the reader, we give here a proof to it.

Lemma 4.4. *Let H be a normal subgroup of G of index p and β be a non zero Bockstein element corresponding to H . If a kG -module M is projective as a kH -module, then $L_\beta \otimes M$ is a projective kG -module.*

Proof. By Remark 3.6, we see that β can be represented by $0 \rightarrow k \rightarrow k_H^G \rightarrow k_H^G \rightarrow k \rightarrow 0$. Then we have a commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & L_\beta & \rightarrow & \text{Ker } \lambda_1 & \rightarrow & \text{Ker } \lambda_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \Omega^2(k) & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow k \rightarrow 0 \\ & & \downarrow \beta & & \downarrow \lambda_1 & & \downarrow \lambda_0 \quad \parallel \\ 0 & \rightarrow & k & \rightarrow & k_H^G & \rightarrow & k_H^G \rightarrow k \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array},$$

where P_0 and P_1 are the projective covers of k and $\Omega(k)$. By tensoring this diagram with M , we find readily that $L_\beta \otimes M$ is projective, since $k_H^G \otimes M \cong M_H^G$ is projective.

Lemma 4.5. *Let M be a kG -module. For $\gamma_1 \in \text{Ext}_{kG}^n(k, k)$ and $\gamma_2 \in \text{Ext}_{kG}^m(k, k)$, suppose that $L_{\gamma_1} \otimes M$ and $L_{\gamma_2} \otimes M$ are projective kG -modules. Then $L_{\gamma_1\gamma_2} \otimes M$ is a projective kG -module.*

Proof. If γ_1 and γ_2 are non-zero, then, as is given in the proof of Theorem 8.5 in [4], there exists an exact sequence:

$$0 \rightarrow \Omega^n(L_{\gamma_2}) \rightarrow L_{\gamma_1\gamma_2} \oplus (\text{projective } kG\text{-module}) \rightarrow L_{\gamma_1} \rightarrow 0.$$

Tensoring this sequence with M , we see that $L_{\gamma_1\gamma_2} \otimes M$ is projective. If $\gamma_1=0$ or $\gamma_2=0$, then the assertion is immediate from the definition of L_{γ_i} .

Lemma 4.6. *Let $\psi \in \text{Ext}_{kG}^n(k, k)$ and H be a subgroup of G . Then $(L_\psi)_H \cong L_{\text{res}_{G,H}(\psi)} \oplus (\text{projective } kH\text{-module})$.*

Proof. We have $\Omega^n(k)_H = \Omega^n(k_H) \oplus Q$ with a projective kH -module Q . If $r = \text{res}_{G,H}(\psi) \neq 0$, then let $h: \Omega^n(k)_H = \Omega^n(k_H) \oplus Q \rightarrow k_H$ be the kH -homomorphism defined by $(w, w') \mapsto \hat{\gamma}(w)$ for $w \in \Omega^n(k_H), w' \in Q$. By the definition of the restriction map, $\hat{\psi}_{\Omega^n(k_H)} = \hat{\gamma}$. Thus $\hat{\psi} - h$ is a projective kH -map and so by Proposition 2.1, $(L_\psi)_H \cong L_{\text{res}_{G,H}(\psi)} \oplus Q$. If $\text{res}_{G,H}(\psi) = 0$, it follows from Lemma 8.1 in [4] that $(L_\psi)_H \cong \Omega^n(k_H) \oplus \Omega(k_H) \oplus (\text{projective } kH\text{-module})$. This completes the proof.

Proof of Theorem 4.2. It suffices to show that given an elementary abelian p -subgroup F of G , there exist ψ_1, \dots, ψ_r in $\text{Ext}_k^{2n(G)}(k, k)$ such that $(L_{\psi_1} \otimes \dots \otimes L_{\psi_r})_F$ is a projective kF -module. For, if this is shown, then consider all those $\psi_1, \dots, \psi_i \in \text{Ext}_k^{2n(G)}(k, k)$ taken over the elementary abelian p -subgroups of G . Then by Chouinard's theorem, we have that $L_{\psi_1} \otimes \dots \otimes L_{\psi_i}$ is a projective kG -module.

We now prove the above assertion by induction on $|F|$. If F is cyclic, then our assertion has been proved in Lemmas 4.3 and 4.6. So we may assume that F is non-cyclic and that there exist elements ψ_2, \dots, ψ_r in $\text{Ext}_k^{2n(G)}(k, k)$ such that $(L_{\psi_2} \otimes \dots \otimes L_{\psi_r})_L$ is projective for every maximal subgroup L of F . Now, Lemma 4.3 implies that there exists an element ψ_1 in $\text{Ext}_k^{2n(G)}(k, k)$ such that $\text{res}_{G,F}(\psi_1)$ is a product of Bockstein elements. Then by our assumptions and Lemmas 4.4 and 4.5, we see that $L_{\text{res}_{G,F}(\psi_1)} \otimes (L_{\psi_2} \otimes \dots \otimes L_{\psi_r})_F$ is a projective kF -module. So from Lemma 4.6, we see that $(L_{\psi_1} \otimes L_{\psi_2} \otimes \dots \otimes L_{\psi_r})_F$ is projective.

5. Applications

Let G be a finite group, k be a field of characteristic $p > 0$ and K be an algebraic closure of k . Let $n(G)$ be the integer given in Definition 4.1. Then Proposition 3.4 and Theorem 4.2 yield:

Corollary 5.1 (Periodicity of periodic modules). *The period of a periodic kG -module divides $2n(G)$.*

Proof. By Theorem 4.2, there exist $\psi_1, \dots, \psi_t \in \text{Ext}_k^{2n(G)}(k, k)$ such that $L_{\psi_1} \otimes \dots \otimes L_{\psi_t}$ is projective, so that $V_G(L_{\psi_1}^K \otimes \dots \otimes L_{\psi_t}^K) = V_G(L_{\psi_1 \otimes I} \otimes \dots \otimes L_{\psi_t \otimes I}) = \{0\}$. Then the assertion is followed by the same argument as in the proof of Theorem 8.7 in [4].

Corollary 4.7 (Criterion for a module to be projective). *A kG -module M is projective if and only if $\text{Ext}_k^{2n(G)}(M, M) = \{0\}$.*

Proof. If $\text{Ext}_k^{2n(G)}(M, M) = \{0\}$, then $\text{Ext}_K^{2n(G)}(M^K, M^K) = \{0\}$. Taking $\psi_1, \dots, \psi_t \in \text{Ext}_k^{2n(G)}(k, k)$ as in Theorem 4.2, we have from the assumption that $\psi_1 \otimes I, \dots, \psi_t \otimes I \in \text{Ext}_K^{2n(G)}(K, K)$ annihilate $\text{Ext}_K^*(M^K, M^K)$, so that $\sqrt{J_G(M^K)} \supseteq \sqrt{(\psi_1 \otimes I, \dots, \psi_t \otimes I)}$. Then from Proposition 3.4, we see that $\sqrt{J_G(M^K)} =$

$\sum_{i>0} H^i(G, K)$, that is, $V_G(M^K) = \{0\}$. Therefore by Lemma 2.3, M^K is projective and so M is projective.

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