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ON THE COHOMOLOGY OF FINITE GROUPS AND THE APPLICATIONS TO MODULAR REPRESENTATIONS

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1. Introduction

Let $G$ be a finite group and $k$ be a field of prime characteristic $p$. All modules considered here are assumed to be finite dimensional over $k$. In [2], Carlslon introduced a certain condition on the cohomology ring of $G$ to study the structure of periodic modules by homological techniques. Let us denote it by $C(n)$, where $n$ is a positive integer. If $G$ satisfies $C(n)$, then there are homogeneous elements of degree $n$ having an interesting property related to his notion of rank variety (see Section 3 for details). For a $p$-group $P$, he showed that there exists an integer $n(P)$ such that $P$ satisfies $C(2n(P))$. And using this, he showed that the period of a periodic $kP$-module divides $2n(P)$.

The purpose of this paper is to extend Carlson's results to an arbitrary finite group $G$. In doing so, we shall give a stronger version of the condition $C(n)$, with a couple of equivalent conditions to it. Concerning Carlson's number $n(G)$ which can as well be defined for an arbitrary $G$, we shall prove that there exist cohomology elements of degree $2n(G)$ satisfying our new condition, so that $G$ satisfies $C(2n(G))$. As an application of this result, we shall show that the period of a periodic $kG$-module divides $2n(G)$. As another application, we also give a homological criterion for a $kG$-module to be projective. A similar criterion has been given by Donovan [6], in response to a problem of Schultz [9].

2. Preliminaries

In this section we mention some preliminary facts needed in later arguments. For a $kG$-module $M$, set $\text{Ext}^*_G(M, M) = \sum_{n\geq 0} \text{Ext}^*_G(M, M)$. If $H$ is a subgroup of $G$, then $M_H$ denotes the restriction of $M$ to a $kH$-module. First of all, we prove the following general fact.

**Proposition 2.1.** Let $0 \rightarrow N_1 \rightarrow M \rightarrow L \rightarrow 0$ and $0 \rightarrow N_2 \rightarrow M \overset{f}{\rightarrow} L \rightarrow 0$ be exact
sequences of \( kG \)-modules. If \( f-g \) factors through a projective \( kG \)-module \( Q \), then \( N_1 \cong N_2 \).

Proof. Suppose that \( \alpha: M \rightarrow Q \) and \( \beta: Q \rightarrow L \) give a factorization of \( f-g \) through the projective module \( Q \). Then we have the following two pull-back diagrams:

\[
\begin{array}{c}
0 \rightarrow N_1 \rightarrow S \rightarrow Q \rightarrow 0 \\
\downarrow \quad \downarrow \beta \\
0 \rightarrow N_1 \rightarrow M \rightarrow L \rightarrow 0 ,
\end{array}
\]

where \( S = \{(x, y) \in M \oplus Q \mid f(x) = \beta(y)\} \),

\[
\begin{array}{c}
0 \rightarrow N_2 \rightarrow T \rightarrow Q \rightarrow 0 \\
\downarrow \quad \downarrow \beta \\
0 \rightarrow N_2 \rightarrow M \rightarrow L \rightarrow 0 ,
\end{array}
\]

where \( T = \{(x, y) \in M \oplus Q \mid g(x) = \beta(y)\} \). Since \( f-g = \beta \cdot \alpha \), we can define \( u: S \rightarrow T \) by \( u: (x, y) \mapsto (x, y - \alpha(x)) \). Then it is easy to see that \( u \) is a \( kG \)-isomorphism. Hence from \( S \cong N_1 \oplus Q \) and \( T \cong N_2 \oplus Q \), we have that \( N_1 \cong N_2 \).

It is well-known that there is a natural isomorphism between \( \text{Ext}_{kG}^1(k, k) \) and \( \text{Hom}_{kG}(Q^*(k), k) \). So, for an element \( \psi \) in \( \text{Ext}_{kG}^1(k, k) \), we denote by \( \tilde{\psi} \) the corresponding \( kG \)-homomorphism of \( Q^*(k) \) into \( k \).

Let \( \langle a \rangle \) be a cyclic \( p \)-group. Define the \( k \langle a \rangle \)-homomorphisms \( \xi: \Omega^2(k) = k \rightarrow k \) by \( \xi: 1 \mapsto 1 \), and \( \tilde{\xi}: \Omega(k) = \text{Rad } k \langle a \rangle \rightarrow k \) by \( \tilde{\xi}: (a-1) \mapsto 1 \). Then we have the following (see, e.g., [5]):

\[
(2.2) \quad \text{Ext}_G^{\langle a \rangle}(k, k) = k[\xi] \otimes \Lambda(\xi),
\]

where \( k[\xi] \) is the polynomial ring and \( \Lambda(\xi) = k + k\xi \). If \( |\langle a \rangle| > 2 \), then \( \xi^2 = 0 \). On the other hand, if \( |\langle a \rangle| = 2 \), then \( \xi^2 = \xi \) and so \( 1 \otimes \xi^2 = \xi \otimes 1 \).

Let \( A \) be an abelian \( p \)-group and \( A = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle \) be a direct product of cyclic subgroups. It is well-known that \( \Theta: \text{Ext}_{kG}^1(k, k) \otimes \cdots \otimes \text{Ext}_{kG}^1(k, k) \cong \text{Ext}_{kG}^n(k, k) \) as \( k \)-algebras (see [4]). Let \( \xi_i = \Theta(1 \otimes \cdots \otimes \xi_i \otimes \cdots \otimes I) \) and \( \epsilon_i = \Theta(1 \otimes \cdots \otimes \xi_i \otimes \cdots \otimes I) \), where \( \xi_i \) and \( \epsilon_i \) are generators of \( \text{Ext}_{kG}^i(k, k) \) as in (2.2). Then we have the following:

\[
\text{Ext}_{kG}^1(k, k) = k[\xi_1, \ldots, \xi_n] \otimes \Lambda(\xi_1, \ldots, \xi_n).
\]

Let \( E \) be the unique maximal elementary abelian subgroup of \( A \). According to the decomposition of \( A \), we decompose \( E \) into the form \( E = \langle x_1, \ldots, x_n \rangle \) with \( x_i \in \langle a_i \rangle \). From this decomposition, we obtain

\[
\text{Ext}_{kG}^1(k, k) = k[\rho_1, \ldots, \rho_n] \otimes \Lambda(\eta_1, \ldots, \eta_n),
\]
where \( \rho_i \) and \( \eta_i \) are defined similarly as \( \xi_i \) and \( \zeta_i \) in the above. Then since \( \Omega^2(k) = k \), we see that \( \operatorname{res}_{\langle \alpha_i \rangle \langle \xi_i \rangle}^{\langle \alpha_i \rangle \langle \xi_i \rangle} = \rho_i \). Regarding \( \operatorname{Ext}^{1}_{\langle \alpha_i \rangle \langle \xi_i \rangle}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (k, k) \) as \( \operatorname{Hom}(\langle \alpha_i \rangle, k) \), we see that if \( |\langle \alpha_i \rangle| > p \), then \( \operatorname{res}_{\langle \alpha_i \rangle \langle \xi_i \rangle}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (\xi_i) = 0 \). Hence, using the argument of the tensor product of complexes (see [4]), we have the following.

**Lemma 2.3.** With the above notations, we have that \( \operatorname{res}_{A, E}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (\xi_i) = \rho_i \) for \( i = 1, \ldots, n \), \( \operatorname{res}_{A, E}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (\xi_i) = 0 \) for \( |\langle \alpha_i \rangle| > p \) and that \( \operatorname{res}_{A, E}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (\xi_i) = \eta_i \) for \( |\langle \alpha_i \rangle| = p \).

Here we recall the notion of a Bockstein element (see [8]). Suppose that \( H \) is a normal subgroup of a finite group \( G \) of index \( p^m \). A Bockstein element corresponding to \( H \) is an element \( \beta \) in \( \operatorname{Ext}^{1}_{G/H} (\mathbb{Z}, k) \) with \( \inf_{G/H} (\operatorname{Ext}^{1}_{G/H} (\mathbb{Z}, k)) = \mathbb{Z} \cdot \beta \). Note that \( \beta \) is unique up to scalar multiples.

**Remark 2.4.** Let \( 0 \to k \to k \to k \to 0 \) be the part of the minimal projective \( k(G/H) \)-resolution of the trivial \( k(G/H) \)-module to the second syzygy. Then the above 2-extension represents a canonical generator in \( \operatorname{Ext}^{1}_{kG} (k, k) \). Thus the Bockstein element \( \beta \) can be represented by \( 0 \to k \to k \to k \to 0 \) as a sequence of \( kG \)-modules. Furthermore, it is known that \( \beta \) can be defined as the image under the Bockstein homomorphism of an element in \( \operatorname{Ext}^{1}_{kG} (k, k) \) which vanishes under the restriction map \( \operatorname{res}_{G/H} \). If \( G \) is an elementary abelian \( p \)-subgroup and \( G = \langle x \rangle \times H \), then \( \beta \) can also be seen as a generator of the polynomial subring of \( \operatorname{Ext}^{1}_{kG} (k, k) \) which corresponds to \( \langle x \rangle \) in the decomposition \( G = \langle x \rangle \times H \).

**Lemma 2.5.** Let \( A \) be an abelian \( p \)-group and \( E \) be the unique maximal elementary abelian \( p \)-subgroup of \( A \). Let \( H \) be a maximal subgroup of \( E \) and \( \tau \) be a Bockstein element corresponding to \( H \). Then there exists an element \( \sigma \) in \( \operatorname{Ext}^{1}_{kG} (k, k) \) such that \( \operatorname{res}_{A, E}^{\langle \alpha_i \rangle \langle \xi_i \rangle} (\sigma) = \tau \).

**Proof.** From Lemma 3.8 in [4], we have that, with the notation of Lemma 2.3, \( \tau \) belongs to \( k[p_1, \ldots, p_n] \). So the result is clear by Lemma 2.3.

### 3. Carlson's condition

Let \( G \) be a finite group and \( k \) be a field of characteristic \( p > 0 \). Let \( \psi \) be an element in \( \operatorname{Ext}^{1}_{kG} (k, k) = \operatorname{Hom}_{kG} (\Omega^1(k), k) \). Following Carlson, we let \( L_\psi \) be the kernel of \( \psi : \Omega^1(k) \to k \) for \( \psi \neq 0 \). If \( \psi = 0 \), let \( L_\psi = \Omega^1(k) \oplus \Omega^1(k) \). Carlson's condition is the following (which was originally defined in the case of \( p \)-groups):

**Carlson's condition:** Let \( n \) be a positive integer. We say that \( G \) satisfies condition \( C(n) \), provided that for any maximal elementary abelian \( p \)-subgroup \( E = \langle x_1, \ldots, x_n \rangle \) of \( G \) and for any element \( u_\alpha = 1 + \sum_{j=1}^{n} \alpha_j (x_{j} - 1) (\alpha = (\alpha_j) \in k^n) \), there exists an element \( \psi_\alpha \) in \( \operatorname{Ext}_G^1 (k, k) \) whose kernel \( L_\psi_\alpha \) is free as a \( k \langle u_\alpha \rangle \)-module.
REMARK 3.1. (1) The kernel \( L_\psi \) of \( \psi \) is free as a \( k\langle u_a \rangle \)-module if and only if \( \text{res}_{G, K^{(a)}}(\psi) \neq 0 \) (Lemma 3.9 in [4]).

(2) We may assume that \( k \) is an algebraically closed field. For, let \( K \) be an algebraic closure of \( k \). If \( G \) satisfies condition \( C(n) \) over \( K \), then for any element \( u_a \) as above, there exists \( \psi \in \text{Ext}_G(K, K) \) with \( \text{res}_{G, K^{(a)}}(\psi) \neq 0 \). Since \( \text{Ext}_G^*(k, k) \otimes K \cong \text{Ext}_G^*(K, K) \), we may write \( \psi = \sum_t \psi_t \otimes x_t \) with \( \psi_t \in \text{Ext}_G^*(k, k) \) and \( x_t \in K \). Then since \( \text{res}_{G, K^{(a)}}(\psi) = \sum_t \text{res}_{G, K^{(a)}}(\psi_t) \otimes x_t \), there exists \( \psi_t \) such that \( \text{res}_{G, K^{(a)}}(\psi_t) \neq 0 \). That is, \( G \) satisfies condition \( C(n) \) over \( k \).

(3) The condition \( C(n) \) does not depend on the choices of generators of \( E \) (cf. Section 6 in [3]).

Now, we consider the following stronger condition than Carlson's one.

Let \( \psi_1, \ldots, \psi_t \) be elements in \( \text{Ext}_G^*(k, k) \). We say that \( G \) satisfies condition \( C(n) \) with \( \psi_1, \ldots, \psi_t \), provided that for any element \( u_a \) as above, there exists \( \psi_i \) in \( \{ \psi_1, \ldots, \psi_t \} \) whose kernel \( L_{\psi_i} \) is free as a \( k\langle u_a \rangle \)-module.

Before proceeding further, we put down here necessary results for the "cohomology variety". For a comprehensive treatment, we refer to [1] and [4].

Let \( K \) be an algebraically closed field of characteristic \( p > 0 \). Let \( H^*(G, K) = \bigoplus_{m \geq 0} \text{Ext}_{KG}^*(K, K) \) if \( p = 2 \) and \( H^*(G, K) = \bigoplus_{m \geq 0} \text{Ext}_{KG}^*(K, K) \) if \( p > 2 \). Then \( H^*(G, K) \) has an associated affine variety \( V_G(K) = \text{Max}(H^*(G, K)) \), which is the set of all maximal ideals of \( H^*(G, K) \). Let \( M \) be a \( KG \)-module and \( J_G(M) \) be the annihilator in \( H^*(G, K) \) of \( \text{Ext}_{KG}^*(M, M) \). The variety \( V_G(M) \) of \( M \) is defined as the subvariety of \( V_G(K) \) associated to \( J_G(M) \).

Lemma 3.3. Let \( M \) and \( N \) be \( KG \)-modules.

(a) \( V_G(M) = \{0\} \) if and only if \( M \) is protective.

(b) \( V_G(M \otimes N) = V_G(M) \cap V_G(N) \).

(c) For \( \psi \in H^*(G, K) \), \( V_G(L_\psi) = V(\psi) \), where \( V(\psi) \) is the variety of the ideal \( H^*(G, K) \cdot \psi \). That is, \( \sqrt{J_G(L_\psi)} = \sqrt{H^*(G, K) \cdot \psi} \).

Proposition 3.4. Let \( n \) be a positive integer and \( \psi_1, \ldots, \psi_t \) be elements in \( \text{Ext}_{KG}^*(K, K) \). Then the following are equivalent.

(1) \( G \) satisfies condition \( C(2n) \) with \( \psi_1, \ldots, \psi_t \).

(2) \( L_{\psi_1} \otimes \cdots \otimes L_{\psi_t} \) is projective.

(3) \( \sqrt{(\psi_1, \ldots, \psi_t)} = \sum_{i=1}^t \text{Ext}_{KG}^*(K, K) \psi_i \) for some \( c > 0 \).
Proof. By Chouinard's theorem ([4]), (2) holds if and only if \((L\psi_1 \otimes \cdots \otimes L\psi_t)_E\) is projective for every maximal elementary abelian \(p\)-subgroup \(E\) of \(G\). Noting Lemma 3.3 and (3.2), we see that \((L\psi_1 \otimes \cdots \otimes L\psi_t)_E\) is projective for every \(E\) if and only if (1) holds.

By Lemma 3.3, (2) holds if and only if \(V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}\). We recall the fact that the point 0 in \(V_G(K)\) is the maximal ideal \(\sum_{i>0} \text{Ext}^{i}_G(G, K)(\sum_{i>0} \text{Ext}^{2i}_G(K, K))\) for \(p>2\) and that if \(p>2\), the elements of odd degree in \(\text{Ext}^{i}_G(K, K)\) are nilpotent. Then we see that \(V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}\) if and only if (3) holds.

Remark 3.5. If \(p=2\), then for any positive integer \(n\), the above proposition is also true for elements \(\psi_1, \ldots, \psi_t\) of degree \(n\).

4. The main theorem

As before, \(G\) is a finite group and \(k\) is a field of characteristic \(p>0\). The following definition is due to Carlson [2]:

**Definition 4.1.** Let \(E\) be a maximal elementary abelian \(p\)-subgroup of \(G\). Let \(A_E\) be an abelian \(p\)-subgroup of \(G\) which contains \(E\) and which has maximal order among such subgroups. Define \(n(E)=|G:A_E|\) and \(n(G)=\text{L.C.M.}_{\Gamma} \{n(E)\}\), where \(\Gamma\) is the set of all maximal elementary abelian \(p\)-subgroups of \(G\).

The next theorem is the main result of this paper.

**Theorem 4.2.** Let \(G\) be a finite group and \(k\) be a field of characteristic \(p>0\). Then there exist \(\psi_1, \ldots, \psi_t\) in \(\text{Ext}^{i}_{\text{G}}(k, k)\) such that \(L\psi_1 \otimes \cdots \otimes L\psi_t\) is a projective \(kG\)-module.

We shall prove the theorem with a series of lemmas. The first one is an analogue of a result of Quillen (see, e.g., Lemma 2.26.5 in [1]).

For \(g \in G\) and a subgroup \(H\) of \(G\), we write \(\varepsilon H = gHg^{-1}\) and let \(\varepsilon \gamma\) be the conjugation \(\text{con}_{H, \gamma} \in \text{Ext}^{1}_{\varepsilon H}(k, k)\) for \(\gamma \in \text{Ext}^{1}_{\varepsilon H}(k, k)\). Let \(A\) be an abelian \(p\)-subgroup of \(G\), \(E\) be the unique maximal elementary abelian subgroup of \(A\) and \(F\) be a elementary abelian subgroup of \(A\).

Let \(g\) be an element in \(G - N_G(F)\). We consider the following diagram:

\[\begin{array}{ccc}
\varepsilon E & \rightarrow & W \\
\downarrow \varepsilon F(\varepsilon A \cap F) & & \\
\varepsilon F \cap X & \rightarrow & \varepsilon A \cap F,
\end{array}\]
where $X$ is a maximal subgroup of $\mathfrak{s}F(\mathfrak{s}A \cap F)$ which contains $\mathfrak{s}A \cap F$, and $W/X$ is a complement to $\mathfrak{s}F(\mathfrak{s}A \cap F)/X$ in $\mathfrak{s}E/X$. Let $M=\mathfrak{s}^{-1}W$ and $L=\mathfrak{s}^{-1}(\mathfrak{s}F \cap X)$. Then $M$ and $L$ are maximal subgroups of $E$ and $F$. Now let $\tau \in \text{Ext}^2_{E}(k, k)$ be a Bockstein element corresponding to $M$. Then we know that $\nu=\text{res}_{E,F}(\tau) \in \text{Ext}^2_{F}(k, k)$ is a Bockstein element corresponding to $L$.

**Lemma 4.3.** Let $A$ be an abelian $p$-subgroup of $G$ and $F$ be an elementary abelian subgroup of $A$. Then there exists an element $\psi$ in $\text{Ext}^2_{A}(k, k)$ such that $\text{res}_{A,F}(\psi)$ is a product of Bockstein elements.

**Proof.** We write $N=N_G(F)$ and let $\{g_1, g_2, \ldots, g_n\}$ be a set of representatives for the right cosets of $N$ in $G$, with $g_1=1$.

As before, let $E$ be the unique maximal elementary abelian subgroup of $A$. Let $L_1$ be a maximal subgroup of $F$ and $M/L_1$ be the complement to $F/L_1$ in $E/L_1$. For the maximal subgroup $M_1$ of $E$, let $\tau_i \in \text{Ext}^2_{E}(k, k)$ be a Bockstein element corresponding to $M_1$. Then we see that $\nu_i=\text{res}_{E,F}(\tau_i) \in \text{Ext}^2_{F}(k, k)$ is a Bockstein element corresponding to $L_i$. For each $g_i(i>1)$, we denote by $\tau_i \in \text{Ext}^2_{E}(k, k)$ and $\nu_i \in \text{Ext}^2_{F}(k, k)$ the Bockstein elements corresponding to $M_i$ and $L_i$ respectively. Then by Lemma 2.5, there exists an element $\sigma_i$ in $\text{Ext}^2_{A}(k, k)$ such that $\text{res}_{A,E}(\sigma_i)=\tau_i$ for $i=1, 2, \ldots, n$. Now, define $\sigma \in \text{Ext}^2_{A}(k, k)$ by $\sigma=\sigma_1 \sigma_2 \ldots \sigma_n$. Then, for $g_i(i>1)$, we have that

$$\text{res}_{A,F}(\varepsilon_i \sigma_i) = \text{res}_{E,F}(\varepsilon_i \sigma_i) \cdot \text{res}_{E,F}(\varepsilon_i \sigma_i) \cdot \text{res}_{E,F}(\varepsilon_i \sigma_i) = \text{res}_{E,F}(\varepsilon_i \sigma_i) \cdot \text{res}_{E,F}(\varepsilon_i \sigma_i) \cdot \text{res}_{E,F}(\varepsilon_i \sigma_i) = 0.$$

Let $x$ be an element in $G-N$. So $x=ug_i$ ($u \in N$, $i>1$) and we have

$$\text{res}_{A,F}(\varepsilon_i \sigma) = \text{res}_{E,F}(\varepsilon_i \sigma) \cdot \text{res}_{E,F}(\varepsilon_i \sigma) = \text{res}_{E,F}(\varepsilon_i \sigma) \cdot \text{res}_{E,F}(\varepsilon_i \sigma) = 0.$$

Therefore by the Mackey decomposition theorem for the norm map (Proposition 2 in [7]), we have

$$\text{res}_{G,F} \text{norm}_{A,O}(1+\sigma) = \prod_{x \in \text{norm}_{A,F,F} \text{res}_{A,F}(1+\varepsilon \sigma)} \text{res}_{A,F}(1+\varepsilon \sigma).$$

So, if $\psi$ denotes the homogeneous part of highest degree of norm$_{A,O}(1+\sigma)$, we have
\[ \text{res}_{G,F}(\psi) = \prod_{x \in H/A} \text{res}_{x,A,F}(\tau) = \prod_{x \in H/A} \prod_{i=1}^{t} \text{res}_{x,F}(\tau_i) = \prod_{x \in H/A} \prod_{i=1}^{t} \nu_i. \]

Here \( \nu_i \) is a Bockstein element corresponding to \( \nu^L \), and \( \psi \) belongs to \( \text{Ext}_{kG}^{2}[\Lambda^1](k,k) \). This completes the proof of the lemma.

The next result is Lemma 4.2 in Okuyama-Sasaki [8]. For the convenience of the reader, we give here a proof to it.

**Lemma 4.4.** Let \( H \) be a normal subgroup of \( G \) of index \( p \) and \( \beta \) be a non-zero Bockstein element corresponding to \( H \). If a \( kG \)-module \( M \) is projective as a \( kH \)-module, then \( L_\beta \otimes M \) is a projective \( kG \)-module.

**Proof.** By Remark 3.6, we see that \( \beta \) can be represented by \( 0 \to k \to k^H \to k^H \to 0 \). Then we have a commutative diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to L_\beta & \to \text{Ker } \lambda_1 & \to \text{Ker } \lambda_0 & \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to \Omega^2(k) & \to P_1 & \to P_0 & \to k & \to 0 \\
\downarrow \beta & \downarrow \lambda_1 & \downarrow \lambda_0 & | \\
0 \to k & \to k^H & \to k^H & \to k & \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0,
\end{array}
\]

where \( P_0 \) and \( P_1 \) are the projective covers of \( k \) and \( \Omega(k) \). By tensoring this diagram with \( M \), we find readily that \( L_\beta \otimes M \) is projective, since \( k^H \otimes M \cong M^H \) is projective.

**Lemma 4.5.** Let \( M \) be a \( kG \)-module. For \( \gamma_1 \in \text{Ext}_{kG}^1(k,k) \) and \( \gamma_2 \in \text{Ext}_{kG}^1(k,k) \), suppose that \( L_{\gamma_1} \otimes M \) and \( L_{\gamma_2} \otimes M \) are projective \( kG \)-modules. Then \( L_{\gamma_1} \gamma_2 \otimes M \) is a projective \( kG \)-module.

**Proof.** If \( \gamma_1 \) and \( \gamma_2 \) are non-zero, then, as is given in the proof of Theorem 8.5 in [4], there exists an exact sequence:

\[ 0 \to \Omega^*(L_{\gamma_2}) \to L_{\gamma_1} \gamma_2 \otimes (\text{projective } kG\text{-module}) \to L_{\gamma_1} \to 0. \]

Tensoring this sequence with \( M \), we see that \( L_{\gamma_1} \gamma_2 \otimes M \) is projective. If \( \gamma_1 = 0 \) or \( \gamma_2 = 0 \), then the assertion is immediate from the definition of \( L_{\gamma_1} \).

**Lemma 4.6.** Let \( \psi \in \text{Ext}_{kG}^1(k,k) \) and \( H \) be a subgroup of \( G \). Then \( (L_{\psi})_H \cong L_{\text{res}_{G,H}(\psi)} \otimes (\text{projective } kH\text{-module}) \).
Proof. We have $$\Omega^n(k_H) = \Omega^n(k_H) \oplus \Omega^n(k_H)$$ with a projective $kH$-module $Q$. If $r = \text{res}_{G,H}(\psi) \neq 0$, then let $h: \Omega^n(k_H) = \Omega^n(k_H) \oplus \Omega^n(k_H) \rightarrow k_H$ be the $kH$-homomorphism defined by $(w, w') \mapsto \gamma(w)$ for $w \in \Omega^n(k_H)$, $w' \in Q$. By the definition of the restriction map, $\gamma'_{\text{res}_{G,H}}(\psi) = \gamma$. Thus $\gamma - h$ is a projective $kH$-map and so by Proposition 2.1, $(L\psi)_{\text{res}_{G,H}} = \text{res}_{G,H}(\psi) \oplus \Omega^n(k_H)$ (projective $kH$-module). This completes the proof.

Proof of Theorem 4.2. It suffices to show that given an elementary abelian $p$-subgroup $F$ of $G$, there exist $\psi_1, \ldots, \psi_r \in \text{Ext}_{kG}^i(k, k)$ such that $(L\psi_1 \otimes \cdots \otimes L\psi_r)_F$ is a projective $kF$-module. For, if this is shown, then consider all those $\psi_1, \ldots, \psi_1 \in \text{Ext}_{kG}^i(k, k)$ taken over the elementary abelian $p$-subgroups of $G$. Then by Chouinard’s theorem, we have that $L\psi_1 \otimes \cdots \otimes L\psi_r$ is a projective $kG$-module.

We now prove the above assertion by induction on $|F|$. If $F$ is cyclic, then our assertion has been proved in Lemmas 4.3 and 4.6. So we may assume that $F$ is non-cyclic and that there exist elements $\psi_1, \ldots, \psi_r \in \text{Ext}_{kG}^i(k, k)$ such that $(L\psi_1 \otimes \cdots \otimes L\psi_r)_L$ is projective for every maximal subgroup $L$ of $F$. Now, Lemma 4.3 implies that there exists an element $\psi_1 \in \text{Ext}_{kG}^i(k, k)$ such that $\text{res}_{G,F}(\psi_1)$ is a product of Bockstein elements. Then by our assumptions and Lemmas 4.4 and 4.5, we see that $(L\psi_1 \otimes \cdots \otimes L\psi_r)_F$ is a projective $kF$-module. So from Lemma 4.6, we see that $(L\psi_1 \otimes \cdots \otimes L\psi_r)_F$ is projective.

5. Applications

Let $G$ be a finite group, $k$ be a field of characteristic $p > 0$ and $K$ be an algebraic closure of $k$. Let $n(G)$ be the integer given in Definition 4.1. Then Proposition 3.4 and Theorem 4.2 yield:

**Corollary 5.1** (Periodicity of periodic modules). The period of a periodic $kG$-module divides $2n(G)$.

Proof. By Theorem 4.2, there exist $\psi_1, \ldots, \psi_r \in \text{Ext}_{kG}^{2n(G)}(k, k)$ such that $L\psi_1 \otimes \cdots \otimes L\psi_r$ is projective, so that $V^G_{\text{res}}(L\psi_1 \otimes \cdots \otimes L\psi_r) = V^G_{\text{res}}(L\psi_1 \otimes \cdots \otimes L\psi_r) = \{0\}$. Then the assertion is followed by the same argument as in the proof of Theorem 8.7 in [4].

**Corollary 4.7** (Criterion for a module to be projective). A $kG$-module $M$ is projective if and only if $\text{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$.

Proof. If $\text{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$, then $\text{Ext}_{kG}^{2n(G)}(M^k, M^k) = \{0\}$. Taking $\psi_1, \ldots, \psi_r \in \text{Ext}_{kG}^{2n(G)}(k, k)$ as in Theorem 4.2, we have from the assumption that $V^G_{\text{res}}(I \otimes I, \ldots, I \otimes I) \in \text{Ext}_{kG}^{2n(G)}(K, K)$ annihilate $\text{Ext}_{kG}^{2n(G)}(M^k, M^k)$, so that $\sqrt{f_G(M^k)} = \sqrt{(\psi_1 \otimes I, \ldots, \psi_r \otimes I)}$. Then from Proposition 3.4, we see that $\sqrt{f_G(M^k)} = \ldots$
\[ \sum_{i>0} H^i(G,K), \text{ that is, } V_o(M^k) = \{0\}. \] Therefore by Lemma 2.3, \( M^k \) is projective and so \( M \) is projective.

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References


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