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ON THE COHOMOLOGY OF FINITE GROUPS AND THE APPLICATIONS TO MODULAR REPRESENTATIONS

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1. Introduction

Let G be a finite group and k be a field of prime characteristic p. All modules considered here are assumed to be finite dimensional over k. In [2], Carslon introduced a certain condition on the cohomology ring of G to study the structure of periodic modules by homological techniques. Let us denote it by $C(n)$, where n is a positive integer. If G satisfies $C(n)$, then there are homogeneous elements of degree n having an interesting property releted to his notion of rank variety (see Section 3 for details). For a p-group P, he showed that there exists an integer $n(P)$ such that P satisfies $C(2n(P))$. And using this, he showed that the period of a periodic kP-module divides $2n(P)$.

The purpose of this paper is to extend Carlson's results to an arbitrary finite group G. In doing so, we shall give a stronger version of the condition $C(n)$, with a couple of equivalent conditions to it. Concerning Carlson's number $n(G)$ which can as well be defined for an arbitrary G, we shall prove that there exist cohomology elements of degree $2n(G)$ satisfying our new condition, so that G satisfies $C(2n(G))$. As an application of this result, we shall show that the period of a periodic kG-module divides $2n(G)$. As another application, we also give a homoligical criterion for a kG-module to be projective. A similar criterion has been given by Donovan [6], in response to a problem of Schultz [9].

2. Preliminaries

In this section we mention some preliminary facts needed in later arguments. For a kG-module M, set $\mathrm{Ext}^*_{kG}(M, M) = \sum_{n\geq 0} \mathrm{Ext}^n_{kG}(M, M)$. If H is a subgroup of G, then M_H denotes the restriction of M to a kH-module. First of all, we prove the following general fact.

Proposition 2.1. *Let* $0 \to N_1 \to M \overset{f}{\to} L \to 0$ *and* $0 \to N_2 \to M \overset{g}{\to} L \to 0$ *be exact*

sequences of kG-modules. If f-g factors through a projective kG-module Q, then $N_1 \cong N_2$.

Proof. Suppose that $\alpha: M \to Q$ and $\beta: Q \to L$ give a factoriaztion of f-g through the projective module Q. Then we have the following two pull-back diagrams:

$$0 \to N_1 \to S \to Q \to 0$$
$$\| \qquad \downarrow \qquad \downarrow \beta$$
$$0 \to N_1 \to M \underset{f}{\to} L \to 0,$$

where $S = \{(x, y) \in M \oplus Q \,|\, f(x) = \beta(y)\}$,

$$0 \to N_2 \to T \to Q \to 0$$
$$\| \qquad \downarrow \qquad \downarrow \beta$$
$$0 \to N_2 \to M \underset{g}{\to} L \to 0,$$

where $T = \{(x, y) \in M \oplus Q \,|\, g(x) = \beta(y)\}$. Since f-$g = \beta \cdot \alpha$, we can define u: $S \to T$ by u: $(x, y) \to (x, y - \alpha(x))$. Then it is easy to see that u is a kG-isomorphism. Hence from $S \cong N_1 \oplus Q$ and $T \cong N_2 \oplus Q$, we have that $N_1 \cong N_2$.

It is well-known that there is a natural isomorphism between $\text{Ext}_{kG}^n(k, k)$ and $\text{Hom}_{kG}(\Omega^n(k), k)$. So, for an element ψ in $\text{Ext}_{kG}^n(k, k)$, we denote by $\hat\psi$ the corresponding kG-homomorphism of $\Omega^n(k)$ into k.

Let $\langle a \rangle$ be a cyclic p-group. Define the $k\langle a \rangle$-homomorphisms $\hat\xi$: $\Omega^2(k) = k \to k$ by $\hat\xi$: $1 \mapsto 1$, and $\hat\zeta$: $\Omega(k) = \text{Rad } k\langle a \rangle \to k$ by $\hat\zeta$: $(a-1) \mapsto 1$. Then we have the following (see, e.g., [5]):

(2.2) $\text{Ext}_{k\langle a \rangle}^*(k, k) = k[\xi] \otimes \Lambda(\zeta)$,

where $k[\xi]$ is the polynomial ring and $\Lambda(\zeta) = k + k\zeta$. If $|\langle a \rangle| > 2$, then $\zeta^2 = 0$. On the other hand, if $|\langle a \rangle| = 2$, then $\zeta^2 = \xi$ and so $1 \otimes \zeta^2 = \xi \otimes 1$.

Let A be an abelian p-group and $A = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ be a direct product of cyclic subgroups. It is well-known that Θ: $\text{Ext}_{k\langle a_1 \rangle}^*(k, k) \otimes \cdots \otimes \text{Ext}_{k\langle a_n \rangle}^*(k, k) \cong \text{Ext}_{kA}^*(k, k)$ as k-algebras (see [4]). Let $\xi_i = \Theta(I \otimes \cdots \oplus \xi_i' \otimes \cdots \otimes I)$ and $\zeta_i = \Theta(I \otimes \cdots \otimes \zeta_i' \otimes \cdots \otimes I)$, where ξ_i' and ζ_i' are generators of $\text{Ext}_{k\langle a_i \rangle}^*(k, k)$ as in (2.2). Then we have the following:

$$\text{Ext}_{kA}^*(k, k) = k[\xi_1, \cdots, \xi_n] \otimes \Lambda(\zeta_1, \cdots, \zeta_n).$$

Let E be the unique maximal elementary abelian subgroup of A, According to the decomposition of A, we decompose E into the form $E = \langle x_1, \cdots, x_n \rangle$ with $x_i \in \langle a_i \rangle$. From this decomposition, we obtain

$$\text{Ext}_{kE}^*(k, k) = k[\rho_1, \cdots, \rho_n] \otimes \Lambda(\eta_1, \cdots, \eta_n),$$

wjere ρ_i and η_i are defined similarly as ξ_i and ζ_i in the above. Then since $\Omega^2(k)=k$, we see that $\mathrm{res}_{\langle a_i\rangle,\langle x_i\rangle}(\xi_i)=\rho_i$. Regarding $\mathrm{Ext}^1_{k\langle a_i\rangle}(k, k)$ as $\mathrm{Hom}(\langle a_i\rangle, k)$, we see that if $|\langle a_i\rangle|>p$, then $\mathrm{res}_{\langle a_i\rangle,\langle x_i\rangle}(\zeta_i)=0$. Hence, using the argument of the tensor product of complexes (see [4]), we have the following.

Lemma 2.3. *With the above notations, we have that* $\mathrm{res}_{A,E}(\xi_i)=\rho_i$ *for* $i=1$, $\cdots, n$, $\mathrm{res}_{A,E}(\zeta_i)=0$ *for* $|\langle a_i\rangle|>p$ *and that* $\mathrm{res}_{A,E}(\zeta_i)=\eta_i$ *for* $|\langle a_i\rangle|=p$.

Here we recall the notion of a Bockstein element (see [8]). Suppose that H is a normal subgroup of a finite group G of index p. A Bokcstein element corresponding to H is an element β in $\mathrm{Ext}^2_{kG}(k, k)$ with $\mathrm{inf}_{G/H,G}(\mathrm{Ext}^2_{k(G/H)}(k, k))= k\cdot\beta$. Note that β is unique up to scalar multiples.

REMARK 2.4. Let $0\to k\to k_H{}^G\to k_H{}^G\to k\to0$ be the part of the minimal projective $k(G/H)$-resolution of the trivial $k(G/H)$-module to the second syzygy. Then the above 2-extension represents a canonical generator in $\mathrm{Ext}^2_{k(G/H)}(k, k)$. Thus the Bockstein element β can be represented by $0\to k\to k_H{}^G\to k_H{}^G\to k\to0$ as a sequence of kG-modules. Furthermore, it is known that β can be defined as the image under the Bockstein homomorphism of an element in $\mathrm{Ext}^1_{kG}(k, k)$ which vanishes under the restriction map $\mathrm{res}_{G,H}$. If G is an elementary abelian p-subgroup and $G=\langle x\rangle\times H$, then β can also be seen as a generator of the polynomial subring of $\mathrm{Ext}^*_{kG}(k, k)$ which coiresponds to $\langle x\rangle$ in the decomposition $G=\langle x\rangle\times H$.

Lemma 2.5. *Let A be an abelain p-group and E be the unique maximal elementary ablian subgroup of A. Let H be a maximal subgroup of E and τ be a Bockstein element corresponding to H. Then there exists an element σ in $\mathrm{Ext}^2_{kA}(k, k)$ such that* $\mathrm{res}_{A,E}(\sigma)=\tau$.

Proof. From Lemma 3.8 in [4], we have that, with the notation of Lemma 2.3, τ belongs to $k[\rho_1, \cdots, \rho_n]$. So the result is clear by Lemma 2.3.

3. Carlson's condition

Let G be a finite group and k be a field of characteristic $p>0$. Let ψ be an element in $\mathrm{Ext}^n_{kG}(k, k)\cong\mathrm{Hom}_{kG}(\Omega^n(k), k)$. Following Carlson, we let L_ψ be the kernel of $\hat\psi\colon \Omega^n(k)\to k$ for $\psi\neq0$. If $\psi=0$, let $L_\psi=\Omega^n(k)\oplus\Omega(k)$. Carlson's condition is the following (which was originally defined in the case of p-groups):

Carlson's condition: Let n be a positive integer. We say that G satisfies *condition $C(n)$*, provided that for any maximal elementary abelian p-subgroup $E=\langle x_1, \cdots, x_r\rangle$ of G and for any element $u_\alpha=1+\sum_{j=1}^{r} \alpha_j(x_j-1)$ $(\alpha=(\alpha_j)\neq 0\in k^r)$, there exists an element ψ in $\mathrm{Ext}^n_{kG}(k, k)$ whose kernel L_ψ is free as a $k\langle u_\alpha\rangle$-module.

REMARK 3.1. (1) The kernel L_ψ of ψ is free as a $k\langle u_\alpha\rangle$-module if and only if $\mathrm{res}_{G,\langle u_\alpha\rangle}(\psi)\neq 0$ (Lemma 3.9 in [4]).

(2) We may assume that k is an algebraically closed field. For, let K be an algebraic closure of k. If G satisfies condition $C(n)$ over K, then for any element u_α as above, there exists ψ in $\mathrm{Ext}^n_{KG}(K, K)$ with $\mathrm{res}_{KG,K\langle u_\alpha\rangle}(\psi)\neq 0$. Since $\mathrm{Ext}^n_{kG}(k, k)\otimes K\cong \mathrm{Ext}^n_{KG}(K, K)$, we may write $\psi=\sum_i \psi_i\otimes x_i$ with $\psi_i\in$ $\mathrm{Ext}^n_{kG}(k, k)$ and $x_i\in K$. Then since $\mathrm{res}_{KG,K\langle u_\alpha\rangle}(\psi)=\sum_i \mathrm{res}_{kG,k\langle u_\alpha\rangle}(\psi_i)\otimes x_i$, there exists ψ_i such that $\mathrm{res}_{kG,k\langle u_\alpha\rangle}(\psi_i)\neq 0$. That is, G satisfies condition $C(n)$ over k.

(3) The condition $C(n)$ does not depend on the choices of generators of E (cf. Section 6 in [3]).

Now, we consider the following stronger condition than Carlson's one. Let $\psi_1, \cdots, \psi_t$ be elements in $\mathrm{Ext}^n_{kG}(k, k)$. We say that G satisfies condition $C(n)$ with $\psi_1, \cdots, \psi_t$, provided that for any element u_α as above, there exists ψ_i in $\{\psi_1, \cdots, \psi_n\}$ whose kernel L_{ψ_i} is free as a $k\langle u_\alpha\rangle$-module.

Before proceeding further, we put down here necessary results for the "cohomology variety". For a comprehensive treatment, we refer to [1] and [4].

Let K be an algebraically closed field of characteristic $p>0$. Let $H^*(G, K)=$ $\sum_{i\geq 0} \mathrm{Ext}^i_{KG}(K, K)$ if $p=2$ and $H^*(G, K)=\sum_{i\geq 0} \mathrm{Ext}^{2i}_{KG}(K, K)$ if $p>2$. Then $H^*(G, K)$ has an associated affine variety $V_G(K)=\mathrm{Max}(H^*(G, K))$, which is the set of all maximal ideals of $H^*(G, K)$. Let M be a KG-module and $J_G(M)$ be the annihilator in $H^*(G, K)$ of $\mathrm{Ext}^*_{KG}(M, M)$. The variety $V_G(M)$ of M is defined as the subvariety of $V_G(K)$ associated to $J_G(M)$.

(3.2)   Let $E=\langle x_1, \cdots, x_r\rangle$ be an elementary abelian p-group and $u_\alpha=1+\sum_{j=1}^r \alpha_j(x_j-1)$, $\alpha=(\alpha_j)\in K^r$. For a KE-module M, let $V_r(M)=\{0\}\cup\{\alpha\in K^r\mid M_{\langle u_\alpha\rangle}$ is not free as a $K\langle u_\alpha\rangle$-module$\}$. Then $V_r(M)$ is a subvariety of K^r, and via $V_E(K)\cong K^r$, we have that $V_E(M)\cong V_r(M)$.

Lemma 3.3. *Let M and N be KG-modules.*
(a) $V_G(M)=\{0\}$ *if and only if M is projective.*
(b) $V_G(M\otimes N)=V_G(M)\cap V_G(N)$.
(c) *For $\psi\in H^i(G, K)$, $V_G(L_\psi)=V(\psi)$, where $V(\psi)$ is the variety of the ideal $H^*(G, K)\cdot\psi$. That is, $\sqrt{J_G(L_\psi)}=\sqrt{H^*(G, K)\cdot\psi}$.*

Proposition 3.4. *Let n be a positive integer and $\psi_1, \cdots, \psi_t$ be elements in $\mathrm{Ext}^{2n}_{KG}(K, K)$. Then the following are equivalent.*
(1) *G satisfies condition $C(2n)$ with $\psi_1, \cdots, \psi_t$.*
(2) *$L_{\psi_1}\otimes\cdots\otimes L_{\psi_t}$ is projective.*
(3) *$\sqrt{(\psi_1, \cdots, \psi_t)}=\sum_{i>0}\mathrm{Ext}^i_{KG}(K, K)$, where $\sqrt{(\psi_1, \cdots, \psi_t)}=\{\psi\in\mathrm{Ext}^*_{KG}$ $(K, K)\mid\psi^c\in\sum_{i=1}^t\mathrm{Ext}^*_{KG}(K, K)\psi_i$ for some $c>0\}$.*

Proof. By Chouinard's theorem ([4]), (2) holds if and only if $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_t})_E$ is projective for every maximal elementary abelian p-subgroup E of G. Noting Lemma 3.3 and (3.2), we see that $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_t})_E$ is projective for every E if and only if (1) holds.

By Lemma 3.3, (2) holds if and only if $V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}$. We recall the fact that the point 0 in $V_G(K)$ is the maximal ideal $\sum_{i>0} \mathrm{Ext}^i_{KG}(G, K)$ $(\sum_{i>0} \mathrm{Ext}^{2i}_{KG}(K, K)$ for $p>2)$ and that if $p>2$, the elements of odd degree in $\mathrm{Ext}^*_{KG}(K, K)$ are nilpotent. Then we see that $V(\psi_1) \cap \cdots \cap V(\psi_t) = \{0\}$ if and only if (3) holds.

REMARK 3.5. If $p=2$, then for any positive integer n, the above proposition is also true for elements $\psi_1, \cdots, \psi_t$ of degree n.

4. The main theorem

As before, G is a finite group and k is a field of characteristic $p>0$. The following definition is due to Carlson [2]:

DEFINITION 4.1. Let E be a maximal elementary abelian p-subgroup of G. Let A_E be an abelian p-subgroup of G which contains E and which has maximal order among such subgroups. Define $n(E) = |G : A_E|$ and $n(G) = L.C.M._{E \in \Gamma} \{n(E)\}$, where Γ is the set of all maximal elementary abelian p-subgroups of G.
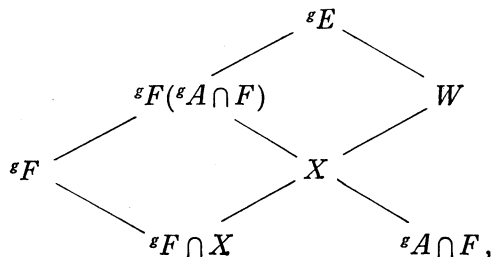
The next theorem is the main result of this paper.

Theorem 4.2. *Let G be a finite group and k be a field of characteristic $p>0$. Then there exist $\psi_1, \cdots, \psi_t$ in $\mathrm{Ext}^{2n(G)}_{kG}(k, k)$ such that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is a projective kG-module.*

We shall prove the theorem with a series of lemmas. The first one is an analogue of a result of Quillen (see, e.g., Lemma 2.26.5 in [1]).

For $g \in G$ and a subgroup H of G, we write ${}^gH = gHg^{-1}$ and let ${}^g\gamma$ be the conjugation con_H, ${}_H(\gamma) \in \mathrm{Ext}^n_{k({}^gH)}(k, k)$ for $\gamma \in \mathrm{Ext}^n_{kH}(k, k)$. Let A be an abelian p-subgroup of G, E be the unique maximal elementary abelian subgroup of A and F be a elementary abelian subgroup of A.

Let g be an element in $G - N_G(F)$. We consider the following diagram:

$$
\begin{array}{ccccc}
 & & {}^gE & & \\
 & \diagup & & \diagdown & \\
{}^gF({}^gA \cap F) & & & & W \\
\diagup & & \diagdown & \diagup & \\
{}^gF & & & X & \\
 & \diagdown & & \diagup & \diagdown \\
 & & {}^gF \cap X & & {}^gA \cap F,
\end{array}
$$

where X is a maximal subgroup of ${}^g F({}^g A \cap F)$ which contains ${}^g A \cap F$, and W/X is a complement to ${}^g F({}^g A \cap F)/X$ in ${}^g E/X$. Let $M = g^{-1} W$ and $L = g^{-1}({}^g F \cap X)$. Then M and L are maximal subgroups of E and F. Now let $\tau \in \mathrm{Ext}_{kE}^2(k, k)$ be a Bockstein element corresponding to M. Then we know that $\nu = \mathrm{res}_{E,F}(\tau) \in \mathrm{Ext}_{kF}^2(k, k)$ is a Bockstein element corresponding to L.

Lemma 4.3. *Let A be an abelian p-subgroup of G and F be an elementary abelian subgroup of A. Then there exists an element ψ in $\mathrm{Ext}_{kG}^{2|G:A|}(k, k)$ such that $\mathrm{res}_{G,F}(\psi)$ is a product of Bockstein elements.*

Proof. We write $N = N_G(F)$ and let $\{g_1, g_2, \cdots, g_n\}$ be a set of representatives for the right cosets of N in G, with $g_1 = 1$.

As before, let E be the unique maximal elementary abelian subgroup of A. Let L_1 be a maximal subgroup of F and M_1/L_1 be the complement to F/L_1 in E/L_1. For the maximal subgroup M_1 of E, let $\tau_1 \in \mathrm{Ext}_{kE}^2(k, k)$ be a Bockstein element corresponding to M_1. Then we see that $\nu_1 = \mathrm{res}_{E,F}(\tau_1) \in \mathrm{Ext}_{kF}^2(k, k)$ is a Bockstein element corresponding to L_1. For each $g_i (i > 1)$, we denote by $\tau_i \in \mathrm{Ext}_{kE}^2(k, k)$ and $\nu_i \in \mathrm{Ext}_{kF}^2(k, k)$ the Bockstein elements corresponding to M_i and L_i respectively. Then by Lemma 2.5, there exists an element σ_i in $\mathrm{Ext}_{kA}^2(k, k)$ such that $\mathrm{res}_{A,E}(\sigma_i) = \tau_i$ for $i = 1, 2, \cdots, n$. Now, define $\sigma \in \mathrm{Ext}_{kA}^{2|G:N|}(k, k)$ by $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$. Then, for $g_i (i > 1)$, we have that

$$
\begin{aligned}
\mathrm{res}_{{}^{g_i}A, {}^{g_i}A \cap F}({}^{g_i}\sigma_i) &= \mathrm{res}_{{}^{g_i}E, {}^{g_i}A \cap F} \, \mathrm{res}_{{}^{g_i}A, {}^{g_i}E}({}^{g_i}\sigma_i) \\
&= \mathrm{res}_{{}^{g_i}E, {}^{g_i}A \cap F}({}^{g_i}\tau_i) \\
&= \mathrm{res}_{{}^{g_i}M_i, {}^{g_i}A \cap F} \cdot \mathrm{res}_{{}^{g_i}E, {}^{g_i}M_i}({}^{g_i}\tau_i) \\
&= \mathrm{res}_{{}^{g_i}M_i, {}^{g_i}A \cap F}({}^{g_i}\mathrm{res}_{E,M_i}(\tau_i)) \\
&= 0.
\end{aligned}
$$

Let x be an element in $G - N$. So $x = u g_i$ ($u \in N$, $i > 1$) and we have

$$
\begin{aligned}
\mathrm{res}_{{}^x A, {}^x A \cap F}({}^x \sigma) &= \mathrm{res}_{{}^{u g_i}A, {}^{u g_i}A \cap F}({}^{u g_i}\sigma) \\
&= {}^u(\mathrm{res}_{{}^{g_i}A, {}^{g_i}A \cap F}({}^{g_i}\sigma)) \\
&= 0.
\end{aligned}
$$

Therefore by the Mackey decomposition theorem for the norm map (Proposition 2 in [7]), we have

$$
\begin{aligned}
\mathrm{res}_{G,F} \, \mathrm{norm}_{A,G}(1 + \sigma) &= \prod_{x \in F \backslash G / A} \mathrm{norm}_{{}^x A \cap F, F} \, \mathrm{res}_{{}^x A, {}^x A \cap F}(1 + {}^x \sigma) \\
&= \prod_{x \in N/A} \mathrm{res}_{{}^x A, F}(1 + {}^x \sigma).
\end{aligned}
$$

So, if ψ denotes the homogeneous part of highest degree of $\mathrm{norm}_{A,G}(1 + \sigma)$, we have

$$\mathrm{res}_{G,F}(\psi) = \prod_{x \in N/A} \mathrm{res}_{x_{A,F}}(^x\sigma) = \prod_{x \in N/A} \prod_{i=1}^{n} \mathrm{res}_{x_{E,F}}(^x\tau_i)$$

$$= \prod_{x \in N/F} \prod_{i=1}^{n} {}^x v_i \ .$$

Here ${}^x v_i$ is a Bockstein element corresponding to ${}^x L_i$, and ψ belongs to $\mathrm{Ext}^{2|G\,:\,A|}_{kG}$ (k, k). This completes the proof of the lemma.

The next result is Lemma 4.2 in Okuyama-Sasaki [8]. For the convenience of the reader, we give here a proof to it.

Lemma 4.4. *Let H be a normal subgroup of G of index p and β be a non zero Bockstein element corresponding to H. If a kG-module M is projective as a kH-module, then $L_\beta \otimes M$ is a projective kG-module.*

Proof. By Remark 3.6, we see that β can be represented by $0 \to k \to k_H{}^G \to k_H{}^G \to k \to 0$. Then we have a commutative diagram:

$$
\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
 & \downarrow & \downarrow & \downarrow & \\
0 \to & L_\beta & \to \mathrm{Ker}\,\lambda_1 \to \mathrm{Ker}\,\lambda_0 \to 0 & & \\
 & \downarrow & \downarrow & \downarrow & \\
0 \to & \Omega^2(k) \to & P_1 & \to P_0 & \to k \to 0 \\
 & \downarrow \beta & \downarrow \lambda_1 & \downarrow \lambda_0 & \| \\
0 \to & k & \to k_H{}^G & \to k_H{}^G & \to k \to 0 \\
 & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 & ,
\end{array}
$$

where P_0 and P_1 are the projective covers of k and $\Omega(k)$. By tensoring this diagram with M, we find readily that $L_\beta \otimes M$ is projective, since $k_H{}^G \otimes M \cong M_H{}^G$ is projective.

Lemma 4.5. *Let M be a kG-module. For $\gamma_1 \in \mathrm{Ext}^n_{kG}(k, k)$ and $\gamma_2 \in \mathrm{Ext}^m_{kG}$ (k, k), suppose that $L_{\gamma_1} \otimes M$ and $L_{\gamma_2} \otimes M$ are projective kG-modules. Then $L_{\gamma_1\gamma_2} \otimes M$ is a projective kG-module.*

Proof. If γ_1 and γ_2 are non-zero, then, as is given in the proof of Theorem 8.5 in [4], there exists an exact sequence:

$$0 \to \Omega^n(L_{\gamma_2}) \to L_{\gamma_1\gamma_2} \oplus (\text{projective } kG\text{-module}) \to L_{\gamma_1} \to 0 \ .$$

Tensoring this sequence with M, we see that $L_{\gamma_1\gamma_2} \otimes M$ is projective. If $\gamma_1 = 0$ or $\gamma_2 = 0$, then the assertion is immediate from the definition of L_{γ_i}.

Lemma 4.6. *Let $\psi \in \mathrm{Ext}^n_{kG}(k, k)$ and H be a subgroup of G. Then $(L_\psi)_H \cong L_{\mathrm{res}_{G,H}(\psi)} \oplus (\text{projective } kH\text{-module})$.*

Proof.   We have $\Omega^n(k)_H = \Omega^n(k_H) \oplus Q$ with a projective kH-module Q.   If $r = \mathrm{res}_{G,H}(\psi) \neq 0$, then let $h: \Omega^n(k)_H = \Omega^n(k_H) \oplus Q \to k_H$ be the kH-homomorphism defined by $(w, w') \mapsto \hat{\gamma}(w)$ for $w \in \Omega^n(k_H)$, $w' \in Q$.   By the definition of the restriction map, $\psi_{\Omega^n(k_H)} = \hat{\gamma}$.   Thus $\psi - h$ is a projective kH-map and so by Proposition 2.1, $(L_\psi)_H \cong L_{\mathrm{res}_{G,H}(\psi)} \oplus Q$.   If $\mathrm{res}_{G,H}(\psi) = 0$, it follows from Lemma 8.1 in [4] that $(L_\psi)_H \cong \Omega^n(k_H) \oplus \Omega(k_H) \oplus (\text{projective } kH\text{-module})$.   This completes the proof.

Proof of Theorem 4.2.   It suffices to show that given an elementary abelian p-subgroup F of G, there exist $\psi_1, \cdots, \psi_r$ in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $(L_{\psi_1} \otimes \cdots \otimes L_{\psi_r})_F$ is a projective kF-module.   For, if this is shown, then consider all those $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ taken over the elementary abelian p-subgroups of G. Then by Chouinard's theorem, we have that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is a projective kG-module.

We now prove the above assertion by induction on $|F|$.   If F is cyclic, then our assertion has been proved in Lemmas 4.3 and 4.6.   So we may assume that F is non-cyclic and that there exist elements $\psi_2, \cdots, \psi_r$ in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $(L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_L$ is projective for every maximal subgroup L of F.   Now, Lemma 4.3 implies that there exists an element ψ_1 in $\mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $\mathrm{res}_{G,F}(\psi_1)$ is a product of Bockstein elements.   Then by our assumptions and Lemmas 4.4 and 4.5, we see that $L_{\mathrm{res}_{G,F}(\psi_1)} \otimes (L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_F$ is a projective kF-module.   So from Lemma 4.6, we see that $(L_{\psi_1} \otimes L_{\psi_2} \otimes \cdots \otimes L_{\psi_r})_F$ is projective.

5.   Applications

Let G be a finite group, k be a field of characteristic $p > 0$ and K be an algebraic closure of k.   Let $n(G)$ be the integer given in Definition 4.1.   Then Proposition 3.4 and Theorem 4.2 yield:

Corollary 5.1 (Periodicity of periodic modules).   *The period of a periodic kG-module divides $2n(G)$.*

Proof.   By Theorem 4.2, there exist $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ such that $L_{\psi_1} \otimes \cdots \otimes L_{\psi_t}$ is projective, so that $V_G(L_{\psi_1}{}^K \otimes \cdots \otimes L_{\psi_t}{}^K) = V_G(L_{\psi_1 \otimes I} \otimes \cdots \otimes L_{\psi_t \otimes I}) = \{0\}$.   Then the assertion is followed by the same argument as in the proof of Theorem 8.7 in [4].

Corollary 4.7 (Criterion for a module to be projective).   *A kG-module M is projective if and only if $\mathrm{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$.*

Proof.   If $\mathrm{Ext}_{kG}^{2n(G)}(M, M) = \{0\}$, then $\mathrm{Ext}_{KG}^{2n(G)}(M^K, M^K) = \{0\}$.   Taking $\psi_1, \cdots, \psi_t \in \mathrm{Ext}_{kG}^{2n(G)}(k, k)$ as in Theorem 4.2, we have from the assumption that $\psi_1 \otimes I, \cdots, \psi_t \otimes I \in \mathrm{Ext}_{KG}^{2n(G)}(K, K)$ annihilate $\mathrm{Ext}_{KG}^*(M^K, M^K)$, so that $\sqrt{J_G(M^K)} \supset \sqrt{(\psi_1 \otimes I, \cdots, \psi_t \otimes I)}$.   Then from Proposition 3.4, we see that $\sqrt{J_G(M^K)} =$

$\sum_{i>0} H^i(G, K)$, that is, $V_G(M^K) = \{0\}$.   Therefore by Lemma 2.3, M^K is projective and so M is projective.

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