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Author(s)	Ku, Hsü Tung; Ku, Mei Chin; Mann, L. N.
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Ku, H.-T., Ku M.-C. and Mann, L. N.  
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## NEWMAN'S THEOREM FOR PSEUDO-SUBMERSIONS

Hsu-TUNG KU, MEI-CHIN KU AND L.N. MANN

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**1. Introduction.** In 1931 M.H.A. Newman [N] proved the following result.

**Theorem** (Newman). *If  $M$  is a connected topological manifold with metric  $d$ , there exists a number  $\varepsilon=\varepsilon(M, d)>0$ , depending only upon  $M$  and  $d$ , such that every finite group  $G$  acting effectively on  $M$  has at least one orbit of diameter at least  $\varepsilon$ .*

P.A. Smith [S] in 1941 proved a version of Newman's Theorem in terms of coverings of  $M$  and Dress [D] in 1969 gave a simplified proof of Newman's Theorem based on Newman's original approach and using a modern version of local degree.

In another direction Cernavskii [C] in 1964 generalized Newman's Theorem to the setting of finite-to-one open mappings on manifolds. His techniques were based upon those of Smith. Recently McAuley and Robinson [M-R] and Deane Montgomery [MO] have expanded upon Cernavskii's results. In fact McAuley and Robinson, using the techniques of Dress, have obtained the following version of Cernavskii's result. [M-R, Theorem 3].

**Theorem** (Cernavskii-McAuley-Robinson). *If  $M$  is a compact connected topological manifold with metric  $d$ , there exists a number  $\varepsilon=\varepsilon(M, d)>0$  such that if  $Y$  is a metric space and  $f: M \rightarrow Y$  a continuous finite-to-one proper open surjective mapping which is not a homeomorphism, then there is at least one  $y \in Y$  such that  $\text{diam } f^{-1}(y) \geq \varepsilon$ .*

In [H-M] we gave estimates of the  $\varepsilon$  in Newman's Theorem for *Riemannian* manifolds  $M$  in terms of convexity and curvature invariants of  $M$ . In this note we apply the techniques of [H-M] to obtain estimates of  $\varepsilon$  for the Cernavskii-McAuley-Robinson result for the case where  $M$  is a Riemannian manifold. In particular, if  $S^n$  is the standard unit sphere with standard metric, we show  $\varepsilon > \pi/2$ , i.e. if  $f: S^n \rightarrow Y$  is as above, there exists  $y \in Y$  with  $\text{diam } f^{-1}(y) > \pi/2$ . We also obtain a cohomology version of Newman's Theorem for compact orientable Riemannian manifolds which generalizes Theorem 3 of

[H-M].

We wish to thank McAuley and Robinson for sending us [M-R] prior to its publication.

## 2. Generalized Newman's theorem for Riemannian manifolds.

We shall call an open finite-to-one proper surjective map  $f: M \rightarrow Y$ ,  $Y$  a metric space, which is not a homeomorphism, a *pseudo-submersion*, and  $f^{-1}(f(x))$  an *orbit of  $f$  at  $x$*  and denoted by  $O_f(x)$ .

Now let  $M$  be a connected Riemannian manifold with a metric induced from the Riemannian metric of  $M$ . Assume that there exists at least one pseudo-submersion  $f: M \rightarrow Y$ . Define the *Newman's diameter*  $d^T(M)$  of  $M$  by

$$d^T(M) = \sup \left\{ \varepsilon \left| \begin{array}{l} \text{for every pseudo-submersion } f: M \rightarrow Y, \\ \text{there exists } x \in M \text{ such that } \text{diam } O_f(x) \geq \varepsilon \end{array} \right. \right\}$$

Define the *cardinality* of  $f$  by  $\text{Card } f = \max \{ \text{card } O_f(x) : x \in M \}$ . Suppose there exists at least one pseudo-submersion  $f: M \rightarrow Y$  with  $\text{Card } f = p > 1$ ; we define the *mod  $p$  Newman's diameter*  $d_p^T(M)$  as the supremum of the numbers  $\varepsilon > 0$  such that for every pseudo-submersion  $g: M \rightarrow Y$  with  $\text{Card } g = p$ , there exists an orbit of diameter at least  $\varepsilon$ .

We call a subset  $S$  of a Riemannian manifold  $M$  *convex* if for every pair of points in  $S$  there exists a unique distance measuring geodesic in  $S$  joining them. For  $x \in M$ , the *radius of convexity of  $M$  at  $x$* , which we denote by  $r_x$ , is defined as the supremum of the radii of all convex embedded open balls centered at  $x$ .

The following result is based upon Lemma 3 in [D] and appears as Theorem 2 in [M-R].

**Proposition 2.1** (Dress-McAuley-Robinson). *Let  $U$  be an open, connected, relatively compact subset of  $R^n$  and  $f: \bar{U} \rightarrow Y$  a pseudo-submersion. Then*

$$\begin{aligned} D &= \text{Max} \{ \text{Min} \{ \|x - y\| : y \in \partial \bar{U} \} : x \in U \} \\ &\leq C = \text{Max} \{ \text{diam } O_f(x) : x \in \partial \bar{U} \}. \end{aligned}$$

Here  $\|x - y\|$  is the euclidean norm.

It is well-known that the exponential map locally stretches distances for manifolds of nonpositive curvature. In [H-M] the following analogous result was obtained for manifolds of bounded curvature.

**Proposition 2.2.** *Suppose  $K \leq b^2$ ,  $b > 0$ , (respectively  $K \leq 0$ ) on a Riemannian manifold  $M$  with distance function  $d$ . Let  $B_r(z) = \{y : d(y, z) < r\}$  be a convex embedded ball centered at  $z$  in  $M$ . Suppose further that  $r < \pi b^{-1}/2$  (respectively  $0 < r < \infty$  when  $K \leq 0$ ). For any  $x, y \in B_r(z)$ , if  $\hat{x} = \exp_z^{-1}x$  and  $\hat{y} = \exp_z^{-1}y$ , then*

$d(x, y) \geq (2/\pi) \|\hat{x} - \hat{y}\|$  (respectively  $d(x, y) \geq \|\hat{x} - \hat{y}\|$  when  $K \leq 0$ ). Here  $\|\hat{x} - \hat{y}\|$  is the euclidean norm in the tangent space  $M_z$ .

Using Propositions 2.1 and 2.2 and the techniques of [H-M] we are able to prove the main result of this section.

**Theorem 2.3.** *Let*

$$\bar{r} = \sup_{x \in M} r_x.$$

(1) *If  $K \leq 0$ ,  $d^T(M) \geq \bar{r}/2$ . In particular if  $\bar{r} = +\infty$ , there exist point inverses of arbitrarily large diameters.*

(2) *If  $K \leq b^2$ , and  $a = \min\{\pi/2b, \bar{r}\}$ ,  $d^T(M) \geq 2a/(\pi+2)$ .*

Proof. Fix any  $z \in M$  and let  $r_z$  = the radius of convexity at  $z$ . For any  $r > 0$  satisfying

$$r < \begin{cases} r_z & \text{if } K \leq 0 \\ \min\{r_z, \pi b^{-1}/2\} & \text{if } K \leq b^2, \end{cases}$$

and any  $\alpha$ ,  $\frac{1}{2} \leq \alpha < 1$ , suppose that

$$(H) \quad \text{diam } O_f(x) < (1-\alpha)r, \text{ all } x \in M.$$

Define  $U = f^{-1}[f(B_{\alpha r}(z))]$ . Clearly  $U$  is open. We claim  $U$  is connected. Let  $V$  be a component of  $U$ . Now it is known [C], [MO] that  $V$  maps onto  $f(U) = f(B_{\alpha r}(z))$ . Hence,  $V$  intersects  $O_f(z)$ . But since

$$\begin{aligned} \text{diam } O_f(z) &< (1-\alpha)r \leq \alpha r, \\ O_f(z) &\subset B_{\alpha r}(z). \quad \text{Furthermore by (H),} \\ B_{\alpha r}(z) &\subset U \subset B_r(z). \end{aligned}$$

Let  $U_\wedge = \exp_z^{-1}U$ . Then  $U_\wedge$  is an open and connected subset of  $R^n = M_z$ . It can be verified that

$$\bar{U}_\wedge = \exp_z^{-1} \circ f^{-1}[f(\bar{B}_{\alpha r}(z))].$$

Consequently we can apply Proposition 2.1 to  $f_\wedge = f \circ \exp_z$ :  $\bar{U}_\wedge \rightarrow Y$ . Now

$$\begin{aligned} \{\hat{x} \in M_z \mid \|\hat{x}\| \leq \alpha r\} &= \exp_z^{-1} \bar{B}_{\alpha r}(z) \subset \bar{U}_\wedge \\ &\subset \exp_z^{-1} \bar{B}_r(z) = \{\hat{x} \in M_z \mid \|\hat{x}\| \leq r\} \end{aligned}$$

The left-hand inclusion implies

$$D = \max \{ \min \{ \|\hat{x} - \hat{y}\| \mid \hat{y} \in \partial \bar{U}_\wedge \mid \hat{x} \in U_\wedge \} \geq \alpha r \} \quad (\text{Simply let } \hat{x} = 0)$$

Since  $\bar{B}_r(z)$  is a convex, embedded ball with  $r < \pi b^{-1}/2$  when  $K \leq b^2$  ( $r < \infty$  when  $K \leq 0$ ), we may apply Proposition 2.2. So

$$\begin{aligned}
C &= \operatorname{Max} \{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}_\lambda\} \\
&\leq \begin{cases} \operatorname{Max} \{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq 0 \\ \pi/2 \operatorname{Max} \{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq b^2 \end{cases} \\
&< \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases}
\end{aligned}$$

by (H).

By Proposition 2.1,  $D \leq C$ . Consequently

$$\alpha r < \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases}$$

or

$$\alpha < \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}$$

Consequently, (H) is *false* for

$$a = \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}$$

So there exists an  $x \in M$  with  $\operatorname{diam} O_f(x) \geq r/2$  if  $K \leq 0$ ;  $2r/(\pi+2)$  if  $K \leq b^2$ .

It is possible to obtain a version of Theorem 2.3 in terms of *injectivity radius*. For a complete connected Riemannian manifold  $M$  define the *injectivity radius*  $i(M)$  by

$$i(M) = \sup \{d(x, C(x)) : x \in M\}$$

where  $C(x)$  denotes the cut locus of  $x$ .

#### Theorem 2.4.

- (1) If  $K \leq 0$ ,  $d^T(M) \geq i(M)/2$ .
- (2) If  $K \leq b^2$ ,  $M$  is compact and  $a = \operatorname{Min} \{\pi/2b, i(M)/2\}$ ,  $d^T(M) \geq 2a/\pi$ .

**3. Estimate of Newman's diameter  $d^T(S^n)$  and related topics.** We use the notion of *degree of a map* defined by Dress [D].

Let  $f: M^n \rightarrow Y$  be a pseudo-submersion. The *branch set*  $B_f$  of  $f$  is defined as  $B_f = \{x \in M : f \text{ is not a local homeomorphism at } x\}$ . By [C] or [M-R],  $M - f^{-1}(f(B_f))$  is a dense open subset of  $M^n$ .

**Lemma 3.1: Newman's Lemma** (Dress [D], McAuley-Robinson [M-R]). *Let  $f: M \rightarrow Y$  be a pseudo-submersion,  $X$  a locally compact metric space,  $g: M \rightarrow X$  and  $j: Y \rightarrow X$  be a proper map such that  $g = j \circ f$ . Let  $x \in X$  be such that*

$$g^{-1}(x) \cap f^{-1}(f(B_f)) = \emptyset,$$

and  $y \in j^{-1}(x)$ . If  $\text{Card } f^{-1}(y) = p$ , then  $g$  is inessential at  $x$  for  $Z_p$ ; that is, the degree of  $g$  at  $x$ ,  $d(g, x)$ , is zero (with  $Z_p$  as coefficients).

**Theorem 3.2.** *Let  $M$  be a compact connected oriented topological  $n$ -manifold and  $f: M^n \rightarrow Y$  be a pseudo-submersion with  $\text{Card } O_f(x_0) = p > 1$  for some  $x_0 \in M - f^{-1}(f(B_f))$ . Suppose  $\varphi: M \rightarrow S^n$  is a map such that  $\deg \varphi \neq 0 \pmod{p}$ . If we denote  $\varphi(z)$  by  $\bar{z}$ , then there exists  $x \in M$  such that the following holds:*

$$(1) \quad \sum_{z \in O_f(x)} \bar{z} = c\bar{x} \text{ in } R^{n+1} \text{ for some } c \leq 0.$$

$$(2) \quad \begin{cases} = \pi & \text{if } \text{Card } O_f(x) = 2 \\ \geq 2\pi/3, \text{ and } \angle \bar{x}o\bar{z} = \angle \bar{x}o\bar{y}, \text{ if } \text{Card } O_f(x) = 3 \text{ and} \\ & O_f(x) = \{x, y, z\} \\ \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2 & \text{if } \text{Card } O_f(x) \geq 4 \end{cases}$$

for some  $z \in O_f(x)$ , where  $\angle \bar{x}o\bar{z}$  denotes the angle between  $o\bar{x}$  and  $o\bar{z}$ ,  $o \in R^{n+1}$  the origin, and  $S^n$  the standard unit sphere in  $R^{n+1}$ .

Proof. (1) Suppose on the contrary, then  $\sum_{z \in O_f(x)} \bar{z} \neq 0$  for all  $x$  in  $M$ . Define a map  $g: M^n \rightarrow S^n$  by

$$g(x) = \sum_{z \in O_f(x)} \bar{z} / \left| \sum_{z \in O_f(x)} \bar{z} \right|.$$

Then for any  $z \in O_f(x)$ ,  $g(z) = g(x)$ . Hence  $g$  induces a map  $j: Y \rightarrow S^n$  such that  $g = j \circ f$ . It follows from Lemma 3.1 that  $g$  is inessential at  $g(x)$  for  $Z_p$ .

On the other hand, by hypothesis there is a well defined homotopy  $H: M \times [0, 1] \rightarrow S^n$  between  $\varphi$  and  $g$  defined by

$$H(x, t) = \{t\varphi(x) + (1-t)g(x)\} / |t\varphi(x) + (1-t)g(x)|.$$

Hence,  $\deg \varphi = \deg g = d(g, g(x)) = 0 \pmod{p}$ . This is a contradiction.

(2) For any  $y, z \in O_f(x)$ , set  $\theta_{yz} = \angle \bar{y}o\bar{z}$ . Let  $\langle \cdot, \cdot \rangle$  be the standard inner product in  $R^{n+1}$ . From (1) there exists an element  $x$  in  $M$  such that

$$\langle \bar{x}, \bar{x} \rangle + \sum_{z \neq x, z \in O_f(x)} \langle \bar{x}, \bar{z} \rangle = c \langle \bar{x}, \bar{x} \rangle$$

for some  $c \leq 0$ ; that is,

$$(**) \quad \sum_{x \neq z, z \in O_f(x)} \cos \theta_{xz} = c - 1 \leq -1$$

If  $\text{Card } f = 2$ , it is easy to see from (\*\*) that  $c = 0$ , and  $\theta_{xz} = \pi$ .

If  $\text{Card } f = 3$ , then  $\cos \theta_{xy} + \cos \theta_{xz} = c - 1$ . From (1) we have

$$|(1-c)\bar{x} + \bar{z}|^2 = |-\bar{y}|^2.$$

Hence  $\cos \theta_{xy} = \cos \theta_{xz} = (c-1)/2$ . That is,  $\theta_{xy} = \theta_{xz} \geq 2\pi/3$ . If  $\text{Card } f = p \geq 4$ , there exists at least one  $z \in O_f(x)$  such that  $\cos \theta_{xz} \leq -1/(p-1)$ ; that is,  $\theta_{xz} \geq \pi$

$$-\cos^{-1}(1/(p-1)) > \pi/2.$$

Theorem 3.2 implies the following:

**Corollary 3.3.** (1)  $d_2^T(S^n) = \pi$ , i.e., for any pseudo-submersion  $f: S^n \rightarrow Y$  with  $\text{Card } f = 2$ , there exists  $x \in S^n$  such that  $f^{-1}(f(x)) = \{x, -x\}$ .  
 (2)  $d_3^T(S^n) = 2\pi/3$ .  
 (3)  $(p-1)\pi/p \geq d_p^T(S^n) \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2$  if  $p \geq 4$ .  
 (4)  $2\pi/3 \geq d^T(S^n) > \pi/2$ .

Proof. In [K], the *equivariant diameter*  $D(M)$  and *modulo p equivariant diameter*  $D_p(M)$  have been defined. They are precisely defined by the pseudo-submersions  $\pi: M \rightarrow M/G$  which are orbit maps of isometric actions of compact Lie groups  $G$  or  $G = Z_p$  on  $M$  respectively. Hence  $D(M) \geq d^T(M)$  and  $D_p(M) \geq d_p^T(M)$  for some  $p$ . But  $D(S^n) = 2\pi/3$  and  $D_p(S^n) = (p-1)\pi/p$  if  $p \geq 3$  by [K]. Hence, the result follows from Theorem 3.2 by applying it to the identity map  $S^n \rightarrow S^n$ .

**REMARKS.** (i) The statement (1) extends the following well known result: *For any non-trivial involution g of  $S^n$ , there exists  $x \in S^n$  such that  $gx = -x$ .*  
 (ii) By using the arguments of Milnor in [MI] we can also show the following: Let  $f: M^n \rightarrow Y$  and  $\tilde{f}: \tilde{M} \rightarrow \tilde{Y}$  be pseudo-submersions with  $\text{Card } f = \text{Card } \tilde{f} = 2$ ,  $B_f = B_{\tilde{f}} = \phi$ , where  $M$  is a compact connected oriented  $n$ -manifold and  $\tilde{M}$  a mod 2 homology  $n$ -sphere. Suppose there exists a map  $\varphi: M \rightarrow \tilde{M}$  of odd degree. Then there exists  $x$  in  $M$  such that  $\varphi O_f(x) = O_{\tilde{f}}(\varphi x)$ .

**Theorem 3.4.** Let  $M$  be a compact connected  $n$ -dimensional submanifold of  $R^{n+1}$ ,  $n \geq 2$ , and let  $y \in R^{n+1} - M$  be in a bounded component. Suppose  $f: M \rightarrow Y$  is a pseudo-submersion. Then there exists  $x \in M$  such that

- (1) If  $\text{Card } f = 2$ ,  $\{O_f(x), y\}$  lies on a line in  $R^{n+1}$ .
- (2) If  $\text{Card } f = 3$ , and  $O_f(x) = \{x, u, v\}$ , then

$$\angle xyu = \angle uyu = \angle vyu = 2\pi/3.$$

In particular  $\{O_f(x), y\}$  lies in a 2-plane in  $R^{n+1}$ .

- (3) If  $\text{Card } f = p \geq 4$ , then  $\angle uyu \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2$  for some  $u, v \in O_f(x)$ , and  $\{O_f(x), y\} \subset R^{p-1} \cap M$ , for some  $(p-1)$ -plane  $R^{p-1}$  of  $R^{n+1}$  (if  $n \geq p-2$ ) passing through the origin.

Proof. Apply Theorem 3.2 to the map  $\varphi: M \rightarrow S^n$  defined by  $\varphi(x) = (y-x)/\|y-x\|$  because  $\deg \varphi = \pm 1$ . The equality in (2) follows from Corollary 3.3 (2).

#### 4. Cohomology version of Newman's theorem for pseudo-submersions

Let  $f: M \rightarrow Y$  be a pseudo-submersion. A subset  $A$  of  $M$  is called *satur-*

ated if  $A=O_f(A)$ , where  $O_f(A)=\cup\{O_f(x): x\in A\}$ , or equivalently  $A=f^{-1}(f(A))$ . Let  $x\in M-f^{-1}(f(B_f))$ . Then there exists an open neighborhood  $V$  of  $x$  which is homeomorphic to  $R^n$  and  $f|V: V\rightarrow f(V)$  is a homeomorphism. Hence by excision we have

$$H_n(Y, Y-f(x); Z_p) \approx H_n(f(V), f(V)-f(x); Z_p) \approx Z_p,$$

where  $p=\text{Card } O_f(x)$ .

We shall say a pseudo-submersion  $f: M\rightarrow Y$  satisfies the (LOA) (*local orientable condition* for  $A$ ) if  $A$  is a closed saturated subset of  $M$ ,  $B=f(A)$  is closed in  $Y$  and such that the inclusion  $i_B: (Y, B)\rightarrow(Y, Y-x)$  induces an isomorphism

$$i_{B*}: H_n(Y, B; Z_p) \rightarrow H_n(Y, Y-f(x); Z_p)$$

for some  $x\in M-f^{-1}(f(B_f))$ ,  $\text{Card } O_f(x)=p$ .

The following result extends the cohomology version of Newman's Theorem for group actions [B], [S] due to Smith.

**Theorem 4.1.** *Let  $A$  be a closed subspace of a compact oriented  $n$ -manifold  $M$  such that  $H_n(M, A; Z_p)\approx Z_p$ . Let  $\mathcal{U}$  be any open covering of  $M$  such that*

$$H^*(K(\mathcal{U}), K(\mathcal{U}|A); Z_p) \rightarrow H^*(M, A; Z_p)$$

*is surjective, where  $K(\mathcal{U})$  denotes the nerve of the covering  $\mathcal{U}$ . Then there does not exist a pseudo-submersion  $f: M\rightarrow Y$  satisfying (LOA) and such that each orbit of  $f$  is contained in some open set in  $\mathcal{U}$ .*

**Proof.** Suppose the conclusion is false. Then there exists a pseudo-submersion  $f: M\rightarrow Y$  satisfying (LOA) and each orbit  $O_f(x)$  is contained in a saturated open set  $V_x$  which is contained in some member of  $\mathcal{U}$ . Let  $\mathcal{V}=\{f(V_x): x\in V\}$ . Then  $f^{-1}\mathcal{V}$  is a refinement of  $\mathcal{U}$ . By [B, p. 154],  $f^*: H^*(Y, B; Z_p) \rightarrow H^*(M, A; Z_p)$  is an epimorphism. But the Kronecker product induces a canonical epimorphism [G, p. 132]

$$\alpha: H^*(M, A; Z_p) \rightarrow H_n(M, A; Z_p)^* = \text{Hom}(H_n(M, A; Z_p); Z_p);$$

hence we have an isomorphism  $f_*: H_n(M, A; Z_p) \rightarrow H_n(Y, B; Z_p)$ .

Let  $K=O_f(x)$ , and  $O_K\in H_n(M, M-K; Z_p)$  be the fundamental class which is the element such that for any  $z\in K$ , the inclusion  $i_z: (M, M-K)\rightarrow(M, M-z)$  satisfies  $i_z^*(O_K)=1_z$ , the identity element of  $H_n(M, M-z; Z_p)\approx Z_p$  (cf. [D]). We have the following commutative diagram

$$\begin{array}{ccccc} Z_p = H_n(M; Z_p) & \xrightarrow{\quad} & H_n(M, A; Z_p) & \xrightarrow{\quad} & \\ \downarrow & \searrow & \nearrow i_* & \downarrow & \\ H_n(M, M-K; Z_p) & \xleftarrow{\quad} & H_n(M, M-z; Z_p) = Z_p & \xleftarrow{\quad} & \\ & \swarrow k_z^* & \nearrow i_z^* & & \end{array}$$

where all homomorphisms are induced by inclusions. Since  $k_{z*}$  is an isomorphism for all  $z$  in  $K$ , there exists an element  $a$  in  $H_n(M, A; Z_p)$  such that  $i_*(a) = O_K$ . Now we consider the following commutative diagram

$$\begin{array}{ccc} H_n(M, A; Z_p) & \xrightarrow[f_*]{\approx} & H_n(Y, B; Z_p) \\ i_* \downarrow & & \approx \downarrow i_{B*} \\ H_n(M, M-K; Z_p) & \xrightarrow[f_*]{\approx} & H_n(Y, Y-f(x); Z_p) \end{array}$$

By definition,  $d(f, f(x)) = f_*(O_K)$  (cf. [D]). It follows that

$$d(f, f(x)) = f_*i_*(a) = i_{B*}f_*(a) \neq 0.$$

On the other hand, we can apply Lemma 3.1 to the map  $f$ , with  $f = j \circ f$ , to obtain  $d(f, f(x)) = 0$ , where  $j$  is the identity map. This is an obvious contradiction and the proof of the theorem is complete.

**Corollary 4.2.** *Let  $M$  be a compact connected oriented  $n$ -manifold, and  $\mathcal{U}$  an open covering of  $M$  such that*

$$(*) \quad H^q(|\sigma|; Z_p) = 0 \text{ for any } \sigma \in K(\mathcal{U}) \text{ and any } q \geq 1.$$

*Then there does not exist a pseudo-submersion  $f: M \rightarrow Y$  such that*

- (1)  $i_{x*}: H_n(Y; Z_p) \xrightarrow{\approx} H_n(Y, Y-x; Z_p)$ , where  $i_x: Y \rightarrow (Y, Y-x)$  is inclusion,  $x \in M - f^{-1}(f(B_f))$ ,  $\text{Card } O_f(x) = p$ , and
- (2) Each orbit of  $f$  is contained in some member of  $\mathcal{U}$ .

Proof. The hypothesis  $(*)$  implies that

$$H^q(K(\mathcal{U}); Z_p) \xrightarrow{\approx} H^q(M; Z_p)$$

for all  $q \geq 0$  by Leray's Theorem [G-R, p. 189].

*As an example, if  $M$  is a compact connected oriented Riemannian manifold, and  $\mathcal{U}$  consists of all open convex proper subsets of  $M$ , then the condition  $(*)$  of Corollary 4.2 is satisfied.*

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Department of Mathematics  
 University of Massachusetts  
 Amherst, MA 01003  
 U.S.A.

