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NEWMAN'S THEOREM FOR PSEUDO-SUBMERSIONS

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1. Introduction. In 1931 M.H.A. Newman [N] proved the following result.

Theorem (Newman). *If M is a connected topological manifold with metric d, there exists a number $\varepsilon=\varepsilon(M, d)>0$, depending only upon M and d, such that every finite group G acting effectively on M has at least one orbit of diameter at least ε.*

P.A. Smith [S] in 1941 proved a version of Newman's Theorem in terms of coverings of M and Dress [D] in 1969 gave a simplified proof of Newman's Theorem based on Newman's original approach and using a modern version of local degree.

In another direction Cernavskii [C] in 1964 generalized Newman's Theorem to the setting of finite-to-one open mappings on manifolds. His techniques were based upon those of Smith. Recently McAuley and Robinson [M-R] and Deane Montgomery [MO] have expanded upon Cernavskii's results. In fact McAuley and Robinson, using the techniques of Dress, have obtained the following version of Cernavskii's result. [M-R, Theorem 3].

Theorem (Cernavskii-McAuley-Robinson). *If M is a compact connected topological manifold with metric d, there exists a number $\varepsilon=\varepsilon(M, d)>0$ such that if Y is a metric space and f: $M\to Y$ a continuous finite-to-one proper open surjective mapping which is not a homeomorphism, then there is at least one $y\in Y$ such that* $\operatorname{diam} f^{-1}(y)\geq\varepsilon$.

In [H-M] we gave estimates of the ε in Newman's Theorem for *Riemannian* manifolds M in terms of convexity and curvature invariants of M. In this note we apply the techniques of [H-M] to obtain estimates of ε for the Cernavskii-McAuley-Robinson result for the case where M is a Riemannian manifold. In particular, if S^n is the standard unit sphere with standard metric, we show $\varepsilon>\pi/2$, i.e. if f: $S^n\to Y$ is as above, there exists $y\in Y$ with diam $f^{-1}(y)>\pi/2$. We also obtain a cohomology version of Newman's Theorem for compact orientable Riemannian manifolds which generalizes Theorem 3 of

[H-M].

We wish to thank McAuley and Robinson for sending us [M-R] prior to its publication.

2. Generalized Newman's theorem for Riemannian manifolds. We shall call an open finite-to-one proper surjective map $f\colon M\to Y$, Y a metric space, which is not a homeomorphism, a *pseudo-submersion*, and $f^{-1}(f(x))$ an *orbit of f at x* and denoted by $O_f(x)$.

Now let M be a connected Riemannian manifold with a metric induced from the Riemannian metric of M. Assume that there exists at least one pseudo-submersion $f\colon M\to Y$. Define the *Newman's diameter* $d^T(M)$ of M by

$$d^T(M)=\sup\left\{\varepsilon\ \middle|\ \begin{array}{l}\text{for every pseudo-submersion } f\colon M\to Y.\\ \text{there exists } x\in M \text{ such that diam } O_f(x)\geq\varepsilon\end{array}\right\}$$

Define the *cardinality* of f by Card $f=\max\{\text{card } O_f(x)\colon x\in M\}$. Suppose there exists at least one pseudo-submersion $f\colon M\to Y$ with Card $f=p>1$; we define the *mod p Newman's diameter* $d_p^T(M)$ as the supremum of the numbers $\varepsilon>0$ such that for every pseudo-submersion $g\colon M\to Y$ with Card $g=p$, there exists an orbit of diameter at least ε.

We call a subset S of a Riemannian manifold M *convex* if for every pair of points in S there exists a unique distance measuring geodesic in S joining them. For $x\in M$, the *radius of convexity of M at x*, which we denote by r_x, is defined as the supremum of the radii of all convex embedded open balls centered at x.

The following result is based upon Lemma 3 in [D] and appears as Theorem 2 in [M-R].

Proposition 2.1 (Dress-McAuley-Robinson). *Let U be an open, connected, relatively compact subset of R^n and $f\colon \bar{U}\to Y$ a pseudo-submersion. Then*

$$\begin{aligned}D&=\operatorname{Max}\{\operatorname{Min}\{\|x-y\|\colon y\in\partial\bar{U}\}\colon x\in U\}\\ &\leq C=\operatorname{Max}\{\text{diam } O_f(x)\colon x\in\partial\bar{U}\}\,.\end{aligned}$$

Here $\|x-y\|$ is the euclidean norm.

It is well-known that the exponential map locally stretches distances for manifolds of nonpositive curvature. In [H-M] the following analogous result was obtained for manifolds of bounded curvature.

Proposition 2.2. *Suppose $K\leq b^2$, $b>0$, (respectively $K\leq 0$) on a Riemannian manifold M with distance function d. Let $B_r(z)=\{y\colon d(y,z)<r\}$ be a convex embedded ball centered at z in M. Suppose further that $r<\pi b^{-1}/2$ (respectively $0<r<\infty$ when $K\leq 0$). For any $x,y\in B_r(z)$, if $\hat{x}=\exp_z^{-1}x$ and $\hat{y}=\exp_z^{-1}y$, then*

$d(x, y) \geq (2/\pi)\|\hat{x}-\hat{y}\|$ (respectively $d(x, y) \geq \|\hat{x}-\hat{y}\|$ when $K \leq 0$). Here $\|\hat{x}-\hat{y}\|$ is the euclidean norm in the tangent space M_z.

Using Propositions 2.1 and 2.2 and the techniques of [H-M] we are able to prove the main result of this section.

Theorem 2.3. *Let*

$$\bar{r} = \sup_{x \in M} r_x .$$

(1) *If $K \leq 0$, $d^T(M) \geq \bar{r}/2$. In particular if $\bar{r} = +\infty$, there exist point inverses of arbitrarily large diameters.*

(2) *If $K \leq b^2$, and $a = \mathrm{Min}\{\pi/2b, \bar{r}\}$, $d^T(M) \geq 2a/(\pi+2)$.*

Proof. Fix any $z \in M$ and let r_z = the radius of convexity at z. For any $r > 0$ satisfying

$$r < \begin{cases} r_z & \text{if } K \leq 0 \\ \mathrm{Min}\{r_z, \pi b^{-1}/2\} & \text{if } K \leq b^2 , \end{cases}$$

and any α, $\frac{1}{2} \leq \alpha < 1$, suppose that

$$(H) \ \ \mathrm{diam}\ O_f(x) < (1-\alpha) r, \text{ all } x \in M .$$

Define $U = f^{-1}[f(B_{\alpha r}(z))]$. Clearly U is open. We claim U is connected. Let V be a component of U. Now it is known [C], [MO] that V maps *onto* $f(U) = f(B_{\alpha r}(z))$. Hence, V intersects $O_f(z)$. But since

$$\mathrm{diam}\ O_f(z) < (1-\alpha) r \leq \alpha r ,$$

$O_f(z) \subset B_{\alpha r}(z)$. Furthermore by (H),

$$B_{\alpha r}(z) \subset U \subset B_r(z) .$$

Let $U_\wedge = \exp_z^{-1} U$. Then $U_\wedge$ is an open and connected subset of $R^n = M_z$. It can be varified that

$$\bar{U}_\wedge = \exp_z^{-1} \circ f^{-1}[f(\bar{B}_{\alpha r}(z))] .$$

Consequently we can apply Proposition 2.1 to $f_\wedge = f \circ \exp_z$: $\bar{U}_\wedge \to Y$. Now

$$\{\hat{x} \in M_z \mid \|\hat{x}\| \leq \alpha r\} = \exp_z^{-1} \bar{B}_{\alpha r}(z) \subset \bar{U}_\wedge \subset \exp_z^{-1} \bar{B}_r(z) = \{\hat{x} \in M_z \mid \|\hat{x}\| \leq r\}$$

The left-hand inclusion implies

$$D = \mathrm{Max}\{\mathrm{Min}\{\|\hat{x}-\hat{y}\| \mid \hat{y} \in \partial \bar{U}_\wedge \mid \hat{x} \in U_\wedge\} \geq \alpha r \quad (\text{Simply let } \hat{x} = 0)$$

Since $\bar{B}_r(z)$ is a convex, embedded ball with $r < \pi b^{-1}/2$ when $K \leq b^2$ ($r < \infty$ when $K \leq 0$), we may apply Proposition 2.2. So

$$\begin{aligned} C &= \operatorname{Max}\{\operatorname{diam} O_f(\hat{x}) \mid \hat{x} \in \partial \bar{U}_\wedge\} \\ &\leq \begin{cases} \operatorname{Max}\{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq 0 \\ \pi/2 \operatorname{Max}\{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq b^2 \end{cases} \\ &< \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases} \end{aligned}$$

by (H).

By Proposition 2.1, $D \leq C$. Consequently

$$\alpha r < \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases}$$

or

$$\alpha < \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}$$

Consequently, (H) is *false* for

$$a = \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}$$

So there exists an $x \in M$ with diam $O_f(x) \geq r/2$ if $K \leq 0$; $2r/(\pi+2)$ if $K \leq b^2$.

It is possible to obtain a version of Theorem 2.3 in terms of *injectivity* radius. For a complete connected Riemannian manifold M define the *injectivity radius* $\mathring{i}(M)$ by

$$\mathring{i}(M) = \sup\{d(x, C(x)) \colon x \in M\}$$

where $C(x)$ denotes the cut locus of x.

Theorem 2.4.
(1) *If* $K \leq 0$, $d^T(M) \geq \mathring{i}(M)/2$.
(2) *If* $K \leq b^2$, *M is compact and* $a = \operatorname{Min}\{\pi/2b, \mathring{i}(M)/2\}$, $d^T(M) \geq 2a/\pi$.

3. Estimate of Newman's diameter $d^T(S^n)$ and related topics. We use the notion of *degree of a map* defined by Dress [D].

Let $f\colon M^n \to Y$ be a pseudo-submersion. The *branch set* B_f of f is defined as $B_f = \{x \in M\colon f$ is not a local homeomorphism at $x\}$. By [C] or [M-R], $M - f^{-1}(f(B_f))$ is a dense open subset of M^n.

Lemma 3.1: Newman's Lemma (Dress [D], McAuley-Robinson [M-R]). *Let* $f\colon M \to Y$ *be a pseudo-submersion,* X *a locally compact metric space,* $g\colon M \to X$ *and* $j\colon Y \to X$ *be a proper map such that* $g = j \circ f$. *Let* $x \in X$ *be such that*

$$g^{-1}(x) \cap f^{-1}(f(B_f)) = \phi\,,$$

and $y \in j^{-1}(x)$. If Card $f^{-1}(y)=p$, *then g is inessential at x for Z_p; that is, the degree of g at x, $d(g, x)$, is zero (with Z_p as coefficients).*

Theorem 3.2. *Let M be a compact connected oriented topological n-manifold and f: $M^n \to Y$ be a pseudo-submersion with* Card $O_f(x_0)=p>1$ *for some $x_0 \in M-f^{-1}(f(B_f))$. Suppose φ: $M \to S^n$ is a map such that the $\deg\varphi \neq 0$* mod p. *If we denote $\varphi(z)$ by $\bar{z}$, then there exists $x \in M$ such that the following holds:*

$$(1) \qquad \sum_{z \in O_f(x)} \bar{z} = c\bar{x} \text{ in } R^{n+1} \text{ for some } c \leq 0 .$$

$$(2) \qquad \angle \bar{x}o\bar{z} \begin{cases} = \pi \text{ if Card } O_f(x) = 2 \\ \geq 2\pi/3, \text{ and } \angle \bar{x}o\bar{z} = \angle \bar{x}o\bar{y}, \text{ if Card } O_f(x) = 3 \text{ and } \\ \qquad\qquad\qquad\qquad\qquad\qquad O_f(x) = \{x, y, z\} . \\ \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2 \text{ if Card } O_f(x) \geqq 4 \end{cases}$$

for some $z \in O_f(x)$, where $\angle \bar{x}o\bar{z}$ denotes the angle between $o\bar{x}$ and $o\bar{z}$, $o \in R^{n+1}$ the origin, and S^n the standard unit sphere in R^{n+1}.

Proof. (1) Suppose on the contrary, then $\sum_{z \in O_f(x)} \bar{z} \neq 0$ for all x in M. Define a map g: $M^n \to S^n$ by

$$g(x) = \sum_{z \in O_f(x)} \bar{z} \Big/ \Big| \sum_{z \in O_f(x)} \bar{z} \Big| .$$

Then for any $z \in O_f(x)$, $g(z)=g(x)$. Hence g induces a map j: $Y \to S^n$ such that $g=j \circ f$. It follows from Lemma 3.1 that g is inessential at $g(x)$ for Z_p.

On the other hand, by hypothesis there is a well defined homotopy H: $M \times [0, 1] \to S^n$ between φ and g defined by

$$H(x, t) = \{t\varphi(x)+(1-t)g(x)\} \,/\, |t\varphi(x)+(1-t)g(x)| .$$

Hence, $\deg \varphi = \deg g = d(g, g(x)) = 0$ mod p. This is a contradiction.

(2) For any $y, z \in O_f(x)$, set $\theta_{yz} = \angle \bar{y}o\bar{z}$. Let $\langle\ ,\ \rangle$ be the standard inner product in R^{n+1}. From (1) there exists an element x in M such that

$$\langle \bar{x}, \bar{x} \rangle + \sum_{z \neq x, z \in O_f(x)} \langle \bar{x}, \bar{z} \rangle = c \langle \bar{x}, \bar{x} \rangle$$

for some $c \leq 0$; that is,

$$(**) \qquad \sum_{x \neq z, z \in O_f(x)} \cos \theta_{xz} = c-1 \leq -1$$

If Card $f=2$, it is easy to see from $(**)$ that $c=0$, and $\theta_{xz}=\pi$.
If Card $f=3$, then $\cos \theta_{xy} + \cos \theta_{xz} = c-1$. From (1) we have

$$|(1-c)\bar{x}+\bar{z}|^2 = |-\bar{y}|^2 .$$

Hence $\cos\theta_{xy} = \cos\theta_{xz} = (c-1)/2$. That is, $\theta_{xy} = \theta_{xz} \geq 2\pi/3$. If Card $f=p \geq 4$, there exists at least one $z \in O_f(x)$ such that $\cos\theta_{xz} \leq -1/(p-1)$; that is, $\theta_{xz} \geq \pi$

$-\cos^{-1}(1/(p-1))>\pi/2$.

Theorem 3.2 implies the following:

Corollary 3.3. (1) $d_2^T(S^n)=\pi$, i.e., *for any pseudo-submersion* $f\colon S^n\to Y$ *with* Card $f=2$, *there exists* $x\in S^n$ *such that* $f^{-1}(f(x))=\{x, -x\}$.
(2) $d_3^T(S^n)=2\pi/3$.
(3) $(p-1)\pi/p\geq d_p^T(S^n)\geq\pi-\cos^{-1}(1/(p-1))>\pi/2$ *if* $p\geq 4$.
(4) $2\pi/3\geq d^T(S^n)>\pi/2$.

Proof. In [K], the *equivariant diameter* $D(M)$ and *modulo* p *equivariant diameter* $D_p(M)$ have been defined. They are precisely defined by the pseudo-submersions $\pi\colon M\to M/G$ which are orbit maps of isometric actions of compact Lie groups G or $G=Z_p$ on M respectively. Hence $D(M)\geq d^T(M)$ and $D_p(M)\geq d_p^T(M)$ for some p. But $D(S^n)=2\pi/3$ and $D_p(S^n)=(p-1)\pi/p$ if $p\geq 3$ by [K]. Hence, the result follows from Theorem 3.2 by applying it to the identity map $S^n\to S^n$.

REMARKS. (i) The statement (1) extends the following well known result: *For any non-trivial involution* g *of* S^n, *there exists* $x\in S^n$ *such that* $gx=-x$.
(ii) By using the arguments of Milnor in [MI] we can also show the following: Let $f\colon M^n\to Y$ and $\tilde{f}\colon\tilde{M}\to\tilde{Y}$ be pseudo-submersions with Card $f=$Card $\tilde{f}=2$, $B_f=B_{\tilde{f}}=\phi$, where M is a compact connected oriented n-manifold and $\tilde{M}$ a mod 2 homology n-sphere. Suppose there exists a map $\varphi\colon M\to\tilde{M}$ of odd degree. Then there exists x in M such that $\varphi O_f(x)=O_{\tilde{f}}(\varphi x)$.

Theorem 3.4. *Let* M *be a compact connected* n*-dimensional submanifold of* R^{n+1}, $n\geq 2$, *and let* $y\in R^{n+1}-M$ *be in a bounded component. Suppose* $f\colon M\to Y$ *is a pseudo-submersion. Then there exists* $x\in M$ *such that*
(1) *If* Card $f=2$, $\{O_f(x), y\}$ *lies on a line in* R^{n+1}.
(2) *If* Card $f=3$, *and* $O_f(x)=\{x, u, v\}$, *then*

$$\angle xyu=\angle uyv=\angle vyx=2\pi/3\,.$$

In particular $\{O_f(x), y\}$ *lies in a* 2*-plane in* R^{n+1}.
(3) *If* Card $f=p\geq 4$, *then* $\angle uyv\geq\pi-\cos^{-1}(1/(p-1))>\pi/2$ *for some* $u, v\in O_f(x)$, *and* $\{O_f(x), y\{\subset R^{p-1}\cap M$, *for some* $(p-1)$*-plane* R^{p-1} *of* R^{n+1} (*if* $n\geq p-2$) *passing through the origin.*

Proof. Apply Theorem 3.2 to the map $\varphi\colon M\to S^n$ defined by $\varphi(x)=(y-x)/||y-x||$ because $\deg\varphi=\pm 1$. The equality in (2) follows from Corollary 3.3 (2).

4. Cohomology version of Newman's theorem for pseudo-submersions

Let $f\colon M\to Y$ be a pseudo-submersion. A subset A of M is called *satur-*

ated if $A=O_f(A)$, where $O_f(A)=\cup\{O_f(x)\colon x\in A\}$, or equivalently $A=f^{-1}(f(A))$. Let $x\in M-f^{-1}(f(B_f))$. Then there exists an open neighborhood V of x which is homeomorphic to R^n and $f\,|\,V\colon V\to f(V)$ is a homeomorphism. Hence by excision we have

$$H_n(Y,\ Y-f(x);\ Z_p)\approx H_n(f(V), f(V)-f(x);\ Z_p)\approx Z_p\,,$$

where $p=\text{Card } O_f(x)$.

We shall say a pseudo-submersion $f\colon M\to Y$ satisfies the (LOA) (*local orientable condition* for A) if A is a closed saturated subset of M, $B=f(A)$ is closed in Y and such that the inclusion $i_B\colon (Y, B)\to(Y,\ Y-x)$ induces an isomorphism

$$i_{B*}\colon H_n(Y,\ B;\ Z_p)\to H_n(Y,\ Y-f(x);\ Z_p)$$

for some $x\in M-f^{-1}(f(B_f))$, Card $O_f(x)=p$.

The following result extends the cohomology version of Newman's Theorem for group actions [B], [S] due to Smith.

Theorem 4.1. *Let A be a closed subspace of a compact oriented n-manifold M such that $H_n(M,\ A;\ Z_p)\approx Z_p$. Let $\mathcal{U}$ be any open covering of M such that*

$$H^n(K(\mathcal{U}),\ K(\mathcal{U}\,|\,A);\ Z_p)\to H^n(M,\ A;\ Z_p)$$

is surjective, where $K(\mathcal{U})$ denotes the nerve of the covering $\mathcal{U}$. Then there does not exist a pseudo-submersion $f\colon M\to Y$ satisfying (LOA) *and such that each orbit of f is contained in some open set in $\mathcal{U}$.*

Proof. Suppose the conclusion is false. Then there exists a pseudo-submersion $f\colon M\to Y$ satisfying (LOA) and each orbit $O_f(x)$ is contained in a saturated open set V_x which is contained in some member of $\mathcal{U}$. Let $\mathcal{V}=\{f(V_x)\colon x\in V\}$. Then $f^{-1}\mathcal{V}$ is a refinement of $\mathcal{U}$. By [B, p. 154], $f^*\colon H^n(Y,\ B;\ Z_p)\to H^n(M,\ A;\ Z_p)$ is an epimorphism. But the Kronecker product induces a canonical epimorphism [G, p. 132]

$$\alpha\colon H^n(M,\ A;\ Z_p)\to H_n(M,\ A;\ Z_p)^*=\mathrm{Hom}(H_n(M,\ A;\ Z_p);\ Z_p);$$

hence we have an isomorphism $f_*\colon H_n(M,\ A;\ Z_p)\to H_n(Y,\ B;\ Z_p)$.

Let $K=O_f(x)$, and $O_K\in H_n(M,\ M-K;\ Z_p)$ be the fundamental class which is the element such that for any $z\in K$, the inclusion $i_z\colon (M,\ M-K)\to(M,\ M-z)$ satisfies $i_{z*}(O_K)=1_z$, the identity element of $H_n(M,\ M-z;\ Z_p)\approx Z_p$ (cf. [D]). We have the following commutative diagram

$$\begin{tikzcd}
Z_p=H_n(M;\ Z_p) \arrow[rr] \arrow[d] \arrow[drr, "k_{z*}"] & & H_n(M,\ A;\ Z_p) \arrow[d] \arrow[dll, "i_*"'] \\
H_n(M,\ M-K;\ Z_p) \arrow[rr, "i_{z*}"'] & & H_n(M,\ M-z;\ Z_p)=Z_p
\end{tikzcd}$$

where all homomorphisms are induced by inclusions. Since k_{z*} is an isomorphism for all z in K, there exists an element a in $H_n(M, A; Z_p)$ such that $i_*(a) = O_K$. Now we consider the following commutative diagram

$$\begin{tikzcd}
H_n(M, A; Z_p) \arrow[rr, "f_*", "\approx"'] \arrow[d, "i_*"'] & & H_n(Y, B; Z_p) \arrow[d, "\approx"', "i_{B*}"] \\
H_n(M, M-K; Z_p) \arrow[rr, "f_*"] & & H_n(Y, Y-f(x); Z_p)
\end{tikzcd}$$

By definition, $d(f, f(x)) = f_*(O_K)$ (cf. [D]). It follows that

$$d(f, f(x)) = f_* i_*(a) = i_{B*} f_*(a) \neq 0 .$$

On the other hand, we can apply Lemma 3.1 to the map f, with $f = j \circ f$, to obtain $d(f, f(x)) = 0$, where j is the identity map. This is an obvious contradiction and the proof of the theorem is complete.

Corollary 4.2. *Let M be a compact connected oriented n-manifold, and $\mathcal{U}$ an open covering of M such that*

$$(*) \quad H^q(|\sigma|; Z_p) = 0 \text{ for any } \sigma \in K(\mathcal{U}) \text{ and any } q \geq 1.$$

Then there does not exist a pseudo-submersion $f: M \to Y$ such that

(1) *$i_{x*}: H_n(Y; Z_p) \overset{\approx}{\to} H_n(Y, Y-x; Z_p)$, where $i_x: Y \to (Y, Y-x)$ is inclusion, $x \in M - f^{-1}(f(B_f))$, Card $O_f(x) = p$, and*

(2) *Each orbit of f is contained in some member of $\mathcal{U}$.*

Proof. The hypothesis $(*)$ implies that

$$H^q(K(\mathcal{U}); Z_p) \overset{\approx}{\to} H^q(M; Z_p)$$

for all $q \geq 0$ by Leray's Theorem [G-R, p. 189].

As an example, if M is a compact connected oriented Riemannian manifold, and $\mathcal{U}$ consists of all open convex proper subsets of M, then the condition $(*)$ of Corollary 4.2 is satisfied.

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