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# NEWMAN'S THEOREM FOR PSEUDO-SUBMERSIONS

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**1. Introduction.** In 1931 M.H.A. Newman [N] proved the following result.

**Theorem** (Newman). *If $M$ is a connected topological manifold with metric $d$, there exists a number $\varepsilon=\varepsilon(M, d)>0$, depending only upon $M$ and $d$, such that every finite group $G$ acting effectively on $M$ has at least one orbit of diameter at least $\varepsilon$.*

P.A. Smith [S] in 1941 proved a version of Newman's Theorem in terms of coverings of $M$ and Dress [D] in 1969 gave a simplified proof of Newman's Theorem based on Newman's original approach and using a modern version of local degree.

In another direction Cernavskii [C] in 1964 generalized Newman's Theorem to the setting of finite-to-one open mappings on manifolds. His techniques were based upon those of Smith. Recently McAuley and Robinson [M-R] and Deane Montgomery [MO] have expanded upon Cernavskii's results. In fact McAuley and Robinson, using the techniques of Dress, have obtained the following version of Cernavskii's result. [M-R, Theorem 3].

**Theorem** (Cernavskii-McAuley-Robinson). *If $M$ is a compact connected topological manifold with metric $d$, there exists a number $\varepsilon=\varepsilon(M, d)>0$ such that if $Y$ is a metric space and $f$: $M\rightarrow Y$ a continuous finite-to-one proper open surjective mapping which is not a homeomorphism, then there is at least one $y\in Y$ such that* $\operatorname{diam} f^{-1}(y)\geq\varepsilon$.

In [H-M] we gave estimates of the $\varepsilon$ in Newman's Theorem for *Riemannian* manifolds $M$ in terms of convexity and curvature invariants of $M$. In this note we apply the techniques of [H-M] to obtain estimates of $\varepsilon$ for the Cernavskii-McAuley-Robinson result for the case where $M$ is a Riemannian manifold. In particular, if $S^n$ is the standard unit sphere with standard metric, we show $\varepsilon>\pi/2$, i.e. if $f$: $S^n\rightarrow Y$ is as above, there exists $y\in Y$ with diam $f^{-1}(y)>\pi/2$. We also obtain a cohomology version of Newman's Theorem for compact orientable Riemannian manifolds which generalizes Theorem 3 of

[H-M].

We wish to thank McAuley and Robinson for sending us [M-R] prior to its publication.

**2. Generalized Newman's theorem for Riemannian manifolds.** We shall call an open finite-to-one proper surjective map $f: M \to Y$, $Y$ a metric space, which is not a homeomorphism, a *pseudo-submersion,* and $f^{-1}(f(x))$ an *orbit of $f$ at $x$* and denoted by $O_f(x)$.

Now let $M$ be a connected Riemannian manifold with a metric induced from the Riemannian metric of $M$. Assume that there exists at least one pseudo-submersion $f: M \to Y$. Define the *Newman's diameter* $d^T(M)$ of $M$ by

$$d^T(M) = \sup\left\{ \varepsilon \,\middle|\, \begin{array}{l} \text{for every pseudo-submersion } f\colon M \to Y. \\ \text{there exists } x \in M \text{ such that diam } O_f(x) \geq \varepsilon \end{array} \right\}$$

Define the *cardinality* of $f$ by Card $f = \max\{\text{card } O_f(x) \colon x \in M\}$. Suppose there exists at least one pseudo-submersion $f: M \to Y$ with Card $f = p > 1$; we define the *mod $p$ Newman's diameter* $d_p^T(M)$ as the supremum of the numbers $\varepsilon > 0$ such that for every pseudo-submersion $g: M \to Y$ with Card $g = p$, there exists an orbit of diameter at least $\varepsilon$.

We call a subset $S$ of a Riemannian manifold $M$ *convex* if for every pair of points in $S$ there exists a unique distance measuring geodesic in $S$ joining them. For $x \in M$, the *radius of convexity of $M$ at $x$*, which we denote by $r_x$, is defined as the supremum of the radii of all convex embedded open balls centered at $x$.

The following result is based upon Lemma 3 in [D] and appears as Theorem 2 in [M-R].

**Proposition 2.1** (Dress-McAuley-Robinson). *Let $U$ be an open, connected, relatively compact subset of $R^n$ and $f$: $\bar{U} \to Y$ a pseudo-submersion. Then*

$$\begin{aligned} D &= \operatorname{Max}\{\operatorname{Min}\{\|x-y\| \colon y \in \partial\bar{U}\} \colon x \in U\} \\ &\leq C = \operatorname{Max}\{\operatorname{diam} O_f(x) \colon x \in \partial\bar{U}\}\ . \end{aligned}$$

*Here $\|x-y\|$ is the euclidean norm.*

It is well-known that the exponential map locally stretches distances for manifolds of nonpositive curvature. In [H-M] the following analogous result was obtained for manifolds of bounded curvature.

**Proposition 2.2.** *Suppose $K \leq b^2$, $b > 0$, (respectively $K \leq 0$) on a Riemannian manifold $M$ with distance function $d$. Let $B_r(z) = \{y\colon d(y, z) < r\}$ be a convex embedded ball centered at $z$ in $M$. Suppose further that $r < \pi b^{-1}/2$ (respectively $0 < r < \infty$ when $K \leq 0$). For any $x, y \in B_r(z)$, if $\hat{x} = \exp_z^{-1} x$ and $\hat{y} = \exp_z^{-1} y$, then*

*$d(x, y) \geq (2/\pi)\|\hat{x}-\hat{y}\|$ (respectively $d(x, y) \geq \|\hat{x}-\hat{y}\|$ when $K \leq 0$). Here $\|\hat{x}-\hat{y}\|$ is the euclidean norm in the tangent space $M_z$.*

Using Propositions 2.1 and 2.2 and the techniques of [H-M] we are able to prove the main result of this section.

**Theorem 2.3.** *Let*

$$\bar{r} = \sup_{x \in M} r_x .$$

(1) *If $K \leq 0$, $d^T(M) \geq \bar{r}/2$. In particular if $\bar{r} = +\infty$, there exist point inverses of arbitrarily large diameters.*

(2) *If $K \leq b^2$, and $a = \mathrm{Min}\{\pi/2b, \bar{r}\}$, $d^T(M) \geq 2a/(\pi+2)$.*

Proof. Fix any $z \in M$ and let $r_z$=the radius of convexity at $z$. For any $r>0$ satisfying

$$r < \begin{cases} r_z & \text{if } K \leq 0 \\ \mathrm{Min}\{r_z, \pi b^{-1}/2\} & \text{if } K \leq b^2 , \end{cases}$$

and any $\alpha$, $\frac{1}{2} \leq \alpha < 1$, suppose that

$$(H) \ \ \mathrm{diam}\ O_f(x) < (1-\alpha)r, \text{ all } x \in M .$$

Define $U = f^{-1}[f(B_{\alpha r}(z))]$. Clearly $U$ is open. We claim $U$ is connected. Let $V$ be a component of $U$. Now it is known [C], [MO] that $V$ maps *onto* $f(U) = f(B_{\alpha r}(z))$. Hence, $V$ intersects $O_f(z)$. But since

$$\mathrm{diam}\ O_f(z) < (1-\alpha)r \leq \alpha r ,$$
$O_f(z) \subset B_{\alpha r}(z)$. Furthermore by $(H)$,
$$B_{\alpha r}(z) \subset U \subset B_r(z) .$$

Let $U_\wedge = \exp_z^{-1} U$. Then $U_\wedge$ is an open and connected subset of $R^n = M_z$. It can be varified that

$$\bar{U}_\wedge = \exp_z^{-1} \circ f^{-1}[f(\bar{B}_{\alpha r}(z))] .$$

Consequently we can apply Proposition 2.1 to $f_\wedge = f \circ \exp_z \colon \bar{U}_\wedge \to Y$. Now

$$\{\hat{x} \in M_z \mid \|\hat{x}\| \leq \alpha r\} = \exp_z^{-1} \bar{B}_{\alpha r}(z) \subset \bar{U}_\wedge$$
$$\subset \exp_z^{-1} \bar{B}_r(z) = \{\hat{x} \in M_z \mid \|\hat{x}\| \leq r\}$$

The left-hand inclusion implies

$$D = \mathrm{Max}\{\mathrm{Min}\{\|\hat{x}-\hat{y}\| \mid \hat{y} \in \partial\bar{U}_\wedge \mid \hat{x} \in U_\wedge\} \geq \alpha r \quad (\text{Simply let } \hat{x} = 0)$$

Since $\bar{B}_r(z)$ is a convex, embedded ball with $r < \pi b^{-1}/2$ when $K \leq b^2$ ($r < \infty$ when $K \leq 0$), we may apply Proposition 2.2. So

$$
\begin{aligned}
C &= \operatorname{Max}\{\operatorname{diam} O_f(\hat{x}) \mid \hat{x} \in \partial \bar{U}_\wedge\} \\
&\leq \begin{cases} \operatorname{Max}\{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq 0 \\ \pi/2 \operatorname{Max}\{\operatorname{diam} O_f(x) \mid x \in \partial \bar{U}\} & \text{if } K \leq b^2 \end{cases} \\
&< \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases}
\end{aligned}
$$

by $(H)$.

By Proposition 2.1, $D \leq C$. Consequently

$$
\alpha r < \begin{cases} (1-\alpha)r & \text{if } K \leq 0 \\ (1-\alpha)\pi r/2 & \text{if } K \leq b^2 \end{cases}
$$

or

$$
\alpha < \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}
$$

Consequently, $(H)$ is *false* for

$$
a = \begin{cases} 1/2 & \text{if } K \leq 0 \\ \pi/(\pi+2) & \text{if } K \leq b^2 \end{cases}
$$

So there exists an $x \in M$ with diam $O_f(x) \geq r/2$ if $K \leq 0$; $2r/(\pi+2)$ if $K \leq b^2$.

It is possible to obtain a version of Theorem 2.3 in terms of *injectivity* radius. For a complete connected Riemannian manifold $M$ define the *injectivity radius* $\mathring{i}(M)$ by

$$
\mathring{i}(M) = \sup\{d(x, C(x)) \colon x \in M\}
$$

where $C(x)$ denotes the cut locus of $x$.

**Theorem 2.4.**
(1) *If* $K \leq 0$, $d^T(M) \geq \mathring{i}(M)/2$.
(2) *If* $K \leq b^2$, *$M$ is compact and* $a = \operatorname{Min}\{\pi/2b, \mathring{i}(M)/2\}$, $d^T(M) \geq 2a/\pi$.

**3. Estimate of Newman's diameter $d^T(S^n)$ and related topics.** We use the notion of *degree of a map* defined by Dress [D].

Let $f\colon M^n \to Y$ be a pseudo-submersion. The *branch set* $B_f$ of $f$ is defined as $B_f = \{x \in M \colon f$ is not a local homeomorphism at $x\}$. By [C] or [M-R], $M - f^{-1}(f(B_f))$ is a dense open subset of $M^n$.

**Lemma 3.1: Newman's Lemma** (Dress [D], McAuley-Robinson [M-R]). *Let $f\colon M \to Y$ be a pseudo-submersion, $X$ a locally compact metric space, $g\colon M \to X$ and $j\colon Y \to X$ be a proper map such that $g = j \circ f$. Let $x \in X$ be such that*

$$
g^{-1}(x) \cap f^{-1}(f(B_f)) = \phi\,,
$$

*and* $y \in j^{-1}(x)$. *If* Card $f^{-1}(y)=p$, *then* $g$ *is inessential at* $x$ *for* $Z_p$; *that is, the degree of* $g$ *at* $x$, $d(g, x)$, *is zero* (*with* $Z_p$ *as coefficients*).

**Theorem 3.2.** *Let* $M$ *be a compact connected oriented topological n-manifold and* $f$: $M^n \to Y$ *be a pseudo-submersion with* Card $O_f(x_0)=p>1$ *for some* $x_0 \in M-f^{-1}(f(B_f))$. *Suppose* $\varphi$: $M \to S^n$ *is a map such that the* $\deg \varphi \neq 0 \bmod p$. *If we denote* $\varphi(z)$ *by* $\bar{z}$, *then there exists* $x \in M$ *such that the following holds*:

$$(1) \qquad \sum_{z \in O_f(x)} \bar{z} = c\bar{x} \text{ in } R^{n+1} \text{ for some } c \leq 0 .$$

$$(2) \qquad \angle \bar{x}o\bar{z} \begin{cases} = \pi \text{ if } \operatorname{Card} O_f(x) = 2 \\ \geq 2\pi/3, \text{ and } \angle \bar{x}o\bar{z} = \angle \bar{x}o\bar{y}, \text{ if } \operatorname{Card} O_f(x) = 3 \text{ and } \\ \qquad O_f(x) = \{x, y, z\} . \\ \geq \pi - \cos^{-1}(1/(p-1)) > \pi/2 \text{ if } \operatorname{Card} O_f(x) \geqq 4 \end{cases}$$

*for some* $z \in O_f(x)$, *where* $\angle \bar{x}o\bar{z}$ *denotes the angle between* $o\bar{x}$ *and* $o\bar{z}$, $o \in R^{n+1}$ *the origin, and* $S^n$ *the standard unit sphere in* $R^{n+1}$.

Proof. (1) Suppose on the contrary, then $\sum_{z \in O_f(x)} \bar{z} \neq 0$ for all $x$ in $M$. Define a map $g$: $M^n \to S^n$ by

$$g(x) = \sum_{z \in O_f(x)} \bar{z} / \Big| \sum_{z \in O_f(x)} \bar{z} \Big| .$$

Then for any $z \in O_f(x)$, $g(z)=g(x)$. Hence $g$ induces a map $j$: $Y \to S^n$ such that $g=j \circ f$. It follows from Lemma 3.1 that $g$ is inessential at $g(x)$ for $Z_p$.

On the other hand, by hypothesis there is a well defined homotopy $H$: $M \times [0, 1] \to S^n$ between $\varphi$ and $g$ defined by

$$H(x, t) = \{t\varphi(x)+(1-t)g(x)\} / |t\varphi(x)+(1-t)g(x)| .$$

Hence, $\deg \varphi = \deg g = d(g, g(x)) = 0 \bmod p$. This is a contradiction.

(2) For any $y, z \in O_f(x)$, set $\theta_{yz} = \angle \bar{y}o\bar{z}$. Let $\langle\ ,\ \rangle$ be the standard inner product in $R^{n+1}$. From (1) there exists an element $x$ in $M$ such that

$$\langle \bar{x}, \bar{x} \rangle + \sum_{z \neq x, z \in O_f(x)} \langle \bar{x}, \bar{z} \rangle = c \langle \bar{x}, \bar{x} \rangle$$

for some $c \leq 0$; that is,

$$(**) \qquad \sum_{x \neq z, z \in O_f(x)} \cos \theta_{xz} = c-1 \leq -1$$

If Card $f=2$, it is easy to see from (**) that $c=0$, and $\theta_{xz}=\pi$.
If Card $f=3$, then $\cos \theta_{xy} + \cos \theta_{xz} = c-1$. From (1) we have

$$|(1-c)\bar{x}+\bar{z}|^2 = |-\bar{y}|^2 .$$

Hence $\cos \theta_{xy} = \cos \theta_{xz} = (c-1)/2$. That is, $\theta_{xy} = \theta_{xz} \geq 2\pi/3$. If Card $f=p \geq 4$, there exists at least one $z \in O_f(x)$ such that $\cos \theta_{xz} \leq -1/(p-1)$; that is, $\theta_{xz} \geq \pi$

$-\cos^{-1}(1/(p-1))>\pi/2$.

Theorem 3.2 implies the following:

**Corollary 3.3.** (1) $d_2^T(S^n)=\pi$, i.e., *for any pseudo-submersion* $f\colon S^n\to Y$ *with* Card $f=2$, *there exists* $x\in S^n$ *such that* $f^{-1}(f(x))=\{x, -x\}$.
(2) $d_3^T(S^n)=2\pi/3$.
(3) $(p-1)\pi/p\geq d_p^T(S^n)\geq\pi-\cos^{-1}(1/(p-1))>\pi/2$ *if* $p\geq 4$.
(4) $2\pi/3\geq d^T(S^n)>\pi/2$.

Proof. In [K], the *equivariant diameter* $D(M)$ and *modulo* $p$ *equivariant diameter* $D_p(M)$ have been defined. They are precisely defined by the pseudo-submersions $\pi\colon M\to M/G$ which are orbit maps of isometric actions of compact Lie groups $G$ or $G=Z_p$ on $M$ respectively. Hence $D(M)\geq d^T(M)$ and $D_p(M)\geq d_p^T(M)$ for some $p$. But $D(S^n)=2\pi/3$ and $D_p(S^n)=(p-1)\pi/p$ if $p\geq 3$ by [K]. Hence, the result follows from Theorem 3.2 by applying it to the identity map $S^n\to S^n$.

REMARKS. (i) The statement (1) extends the following well known result: *For any non-trivial involution* $g$ *of* $S^n$, *there exists* $x\in S^n$ *such that* $gx=-x$.
(ii) By using the arguments of Milnor in [MI] we can also show the following: Let $f\colon M^n\to Y$ and $\tilde{f}\colon \tilde{M}\to\tilde{Y}$ be pseudo-submersions with Card $f=$Card $\tilde{f}=2$, $B_f=B_{\tilde{f}}=\phi$, where $M$ is a compact connected oriented $n$-manifold and $\tilde{M}$ a mod 2 homology $n$-sphere. Suppose there exists a map $\varphi\colon M\to\tilde{M}$ of odd degree. Then there exists $x$ in $M$ such that $\varphi O_f(x)=O_{\tilde{f}}(\varphi x)$.

**Theorem 3.4.** *Let* $M$ *be a compact connected* $n$*-dimensional submanifold of* $R^{n+1}$, $n\geq 2$, *and let* $y\in R^{n+1}-M$ *be in a bounded component. Suppose* $f\colon M\to Y$ *is a pseudo-submersion. Then there exists* $x\in M$ *such that*
(1) *If* Card $f=2$, $\{O_f(x), y\}$ *lies on a line in* $R^{n+1}$.
(2) *If* Card $f=3$, *and* $O_f(x)=\{x, u, v\}$, *then*

$$\angle xyu=\angle uyv=\angle vyx=2\pi/3\,.$$

*In particular* $\{O_f(x), y\}$ *lies in a* 2*-plane in* $R^{n+1}$.
(3) *If* Card $f=p\geq 4$, *then* $\angle uyv\geq\pi-\cos^{-1}(1/(p-1))>\pi/2$ *for some* $u, v\in O_f(x)$, *and* $\{O_f(x), y\{\subset R^{p-1}\cap M$, *for some* $(p-1)$*-plane* $R^{p-1}$ *of* $R^{n+1}$ *(if* $n\geq p-2$*) passing through the origin.*

Proof. Apply Theorem 3.2 to the map $\varphi\colon M\to S^n$ defined by $\varphi(x)=(y-x)/||y-x||$ because $\deg\varphi=\pm1$. The equality in (2) follows from Corollary 3.3 (2).

## 4. Cohomology version of Newman's theorem for pseudo-submersions

Let $f\colon M\to Y$ be a pseudo-submersion. A subset $A$ of $M$ is called *satur-*

*ated* if $A=O_f(A)$, where $O_f(A)=\cup\{O_f(x)\colon x\in A\}$, or equivalently $A=f^{-1}(f(A))$. Let $x\in M-f^{-1}(f(B_f))$. Then there exists an open neighborhood $V$ of $x$ which is homeomorphic to $R^n$ and $f\,|\,V\colon V\to f(V)$ is a homeomorphism. Hence by excision we have

$$H_n(Y,\ Y-f(x);\ Z_p)\approx H_n(f(V), f(V)-f(x);\ Z_p)\approx Z_p\,,$$

where $p=\text{Card } O_f(x)$.

We shall say a pseudo-submersion $f\colon M\to Y$ satisfies the (LOA) (*local orientable condition* for $A$) if $A$ is a closed saturated subset of $M$, $B=f(A)$ is closed in $Y$ and such that the inclusion $i_B\colon (Y, B)\to(Y,\ Y-x)$ induces an isomorphism

$$i_{B*}\colon H_n(Y,\ B;\ Z_p)\to H_n(Y,\ Y-f(x);\ Z_p)$$

for some $x\in M-f^{-1}(f(B_f))$, Card $O_f(x)=p$.

The following result extends the cohomology version of Newman's Theorem for group actions [B], [S] due to Smith.

**Theorem 4.1.** *Let $A$ be a closed subspace of a compact oriented n-manifold $M$ such that $H_n(M,\ A;\ Z_p)\approx Z_p$. Let $\mathcal{U}$ be any open covering of $M$ such that*

$$H^n(K(\mathcal{U}),\ K(\mathcal{U}\,|\,A);\ Z_p)\to H^n(M,\ A;\ Z_p)$$

*is surjective, where $K(\mathcal{U})$ denotes the nerve of the covering $\mathcal{U}$. Then there does not exist a pseudo-submersion $f\colon M\to Y$ satisfying* (LOA) *and such that each orbit of $f$ is contained in some open set in $\mathcal{U}$.*

Proof. Suppose the conclusion is false. Then there exists a pseudo-submersion $f\colon M\to Y$ satisfying (LOA) and each orbit $O_f(x)$ is contained in a saturated open set $V_x$ which is contained in some member of $\mathcal{U}$. Let $\mathcal{V}=\{f(V_x)\colon x\in V\}$. Then $f^{-1}\mathcal{V}$ is a refinement of $\mathcal{U}$. By [B, p. 154], $f^*\colon H^n(Y,\ B;\ Z_p)\to H^n(M,\ A;\ Z_p)$ is an epimorphism. But the Kronecker product induces a canonical epimorphism [G, p. 132]

$$\alpha\colon H^n(M,\ A;\ Z_p)\to H_n(M,\ A;\ Z_p)^*=\text{Hom}(H_n(M,\ A;\ Z_p);\ Z_p);$$

hence we have an isomorphism $f_*\colon H_n(M,\ A;\ Z_p)\to H_n(Y,\ B;\ Z_p)$.

Let $K=O_f(x)$, and $O_K\in H_n(M,\ M-K;\ Z_p)$ be the fundamental class which is the element such that for any $z\in K$, the inclusion $i_z\colon (M,\ M-K)\to(M,\ M-z)$ satisfies $i_{z*}(O_K)=1_z$, the identity element of $H_n(M,\ M-z;\ Z_p)\approx Z_p$ (cf. [D]). We have the following commutative diagram

$$\begin{tikzcd}
Z_p = H_n(M;\ Z_p) \arrow[rr] \arrow[d] \arrow[drr, "k_{z*}"'] & & H_n(M,\ A;\ Z_p) \arrow[d] \arrow[dll, "i_*"] \\
H_n(M,\ M-K;\ Z_p) \arrow[rr, "i_{z*}"'] & & H_n(M,\ M-z;\ Z_p) = Z_p
\end{tikzcd}$$

where all homomorphisms are induced by inclusions. Since $k_{z*}$ is an isomorphism for all $z$ in $K$, there exists an element $a$ in $H_n(M, A; Z_p)$ such that $i_*(a) = O_K$. Now we consider the following commutative diagram

$$\begin{tikzcd}
H_n(M, A; Z_p) \arrow[r, "f_*", "\approx"'] \arrow[d, "i_*"'] & H_n(Y, B; Z_p) \arrow[d, "\approx"', "i_{B*}"] \\
H_n(M, M-K; Z_p) \arrow[r, "f_*"] & H_n(Y, Y-f(x); Z_p)
\end{tikzcd}$$

By definition, $d(f, f(x)) = f_*(O_K)$ (cf. [D]). It follows that

$$d(f, f(x)) = f_* i_*(a) = i_{B*} f_*(a) \neq 0 .$$

On the other hand, we can apply Lemma 3.1 to the map $f$, with $f = j \circ f$, to obtain $d(f, f(x)) = 0$, where $j$ is the identity map. This is an obvious contradiction and the proof of the theorem is complete.

**Corollary 4.2.** *Let $M$ be a compact connected oriented $n$-manifold, and $\mathcal{U}$ an open covering of $M$ such that*

$$(*) \quad H^q(|\sigma|; Z_p) = 0 \textit{ for any } \sigma \in K(\mathcal{U}) \textit{ and any } q \geq 1.$$

*Then there does not exist a pseudo-submersion $f\colon M \to Y$ such that*

(1) *$i_{x*}\colon H_n(Y; Z_p) \xrightarrow{\approx} H_n(Y, Y-x; Z_p)$, where $i_x\colon Y \to (Y, Y-x)$ is inclusion, $x \in M - f^{-1}(f(B_f))$,* Card *$O_f(x) = p$, and*

(2) *Each orbit of $f$ is contained in some member of $\mathcal{U}$.*

Proof. The hypothesis $(*)$ implies that

$$H^q(K(\mathcal{U}); Z_p) \xrightarrow{\approx} H^q(M; Z_p)$$

for all $q \geq 0$ by Leray's Theorem [G-R, p. 189].

*As an example, if $M$ is a compact connected oriented Riemannian manifold, and $\mathcal{U}$ consists of all open convex proper subsets of $M$, then the condition $(*)$ of Corollary 4.2 is satisfied.*

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