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GEOMETRY OF PLANE CURVES VIA TSCHIRNHAUSEN RESOLUTION TOWER

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1. Introduction.

The weight vectors of a resolution tower of toric modifications for an irreducible germ of a plane curve C carry enough information to read off invariants such as the Puiseux pairs, multiplicities, etc [29]. However, each step of the inductive construction of a tower of toric modifications depends on a choice of the modification local coordinates. This ambiguity makes it difficult to study the equi-singularity problem of a family of germs of plane curves or to study a global curve. It is the purpose of this paper to make a canonical choice of the modification local coordinates (u_i,v_i) (Theorem 4.5), and to obtain a canonical sequence of germs of curves $\{C_i; i=1,\cdots,k\}$ $(C_k=C)$ such that the local knot of the curve C_i is a compound torus knot around the local knot of the curve C_{i-1}. We will show that the local equations $h_i(x,y)$ of the the germs $\{C_i; i=1,\cdots,k\}$ are the Tschirnhausen approximate polynomials of the local equation $f(x,y)$ for C, provided that $f(x,y)$ is a monic polynomial in y.

The importance of the Tschirnhausen approximate polynomials was first observed by Abhyankar-Moh [3,4], and our work is very much influenced by them. However, our result gives not only a geometric interpretation of [3,4] but also a new method to study the equi-singularity problem, see [35], for a given family of germs of irreducible plane curves $f(x,y,t)=0$ whose Tschirnhausen approximate polynomials $h_i(x,y)$, $i=1,\cdots,k-1$ do not depend on t.

In section 6, we show that a family of germs of plane curves $\{f_t(x,y)=0\}$ with Tschirnhausen approximate polynomials $h_i(x,y)$, $i=1,\cdots,k-1$ not depending upon t and satisfying an additional intersection condition is equi-singular (Theorem 6.2). In section 8, we will give a new proof and a generalization of the Abhyankar-Moh-Suzuki theorem from the viewpoint of the equi-singularity at infinity (Theorems 8.2, 8.3, 8.7).

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2. Tschirnhausen approximate polynomials of a monic polynomial.

Let $f(y)=y^n+\Sigma_{i=1}^n c_i y^{n-i}$ be a monic polynomial in y of degree n with coefficients in an integral domain R which contains the field of rational numbers $\boldsymbol{Q}$, and let a be a positive integer such that a divides n. The *n/a-th Tschirnhausen approximate polynomial* (*or the n/a-th Tschirnhausen approximate root*) of $f(y)$ is the monic polynomial $h(y)\in R[y]$ of degree a such that degree $(f(y)-h(y)^{n/a})<n-a$. The coefficients of $h(y)=y^a+\Sigma_{i=1}^a \alpha_i y^{a-i}$ are inductively determined by: $\alpha_0=1$ and $c_i(x)=\Sigma_{j_1+\cdots+j_l=i}\alpha_{j_1}(x)\cdots\alpha_{j_l}(x)$ for $i=1,\cdots,a$. The coefficient α_j is a weighted homogeneous polynomial of degree j in the variables $c_1,\cdots,c_a$ with weight$(c_j)=j$, $1\le j\le a$. In our application R will be the ring $\boldsymbol{C}\{x\}$ or $\boldsymbol{C}[x]$. For further detail, we refer to [3,4,32]. From the Euclidean division algorithm, it follows that

Proposition 2.1. *Let $h(y)\in R[y]$ be monic of degree a in y, and let $P(y)\in R[y]$ such that $sa\le \deg_y P(y)<(s+1)a$. Then there exits a unique expansion, called the Euclidian expansion, $P(y)=\Sigma_{i=0}^s \alpha_i(y)h(y)^{s-i}$, where $\alpha_i(y)\in R[y]$, $i=0,\cdots,s$, satisfy $\deg_y\alpha_i(y)<a$. In particular, we can expand $f(y)$ with respect to its n/a-th Tschirnhausen approximate polynomial, as $f(y)=h(y)^{n/a}+\Sigma_{i=2}^{n/a}c_i(y)h(y)^{n/a-i}$, $\deg_y c_i(y)<a$. If $f(y)=f(x,y)\in \boldsymbol{C}[x][y]$, the coefficients $c_i(y)=c_i(x,y)$ are also polynomials in x and y.*

The second assertion is immediate from the Euclidian expansion of $f-h^{n/a}$. We call the above expansion *the n/a-th Tschirnhausen expansion* of $f(x,y)$. The expansion of $P(x,y)$ with respect to $h(x,y)$ will also be called *the Tschirnhausen expansion* if $h(x,y)$ is a Tschirnhausen approximate polynomial. Tschirnhausen approximate polynomials behave hereditarily in the following sense.

Proposition 2.2. *Assume that $a,b\ge 2$ are integers such that $ab\mid n$. Let h and h' be respectively the n/a-th and n/ab-th Tschirnhausen approximate polynomials of f and let $h'=h^b+\Sigma_{i=1}^b c_i h^{b-i}$, $\deg_y c_i<a$, be the Tschirnhausen expansion of h' with respect to h. The first coefficient c_1 is zero and h is the ab/a-th Tschirnhausen approximate polynomial of h'.*

Proof. With $m:=n/ab$, we have $\deg_y(f-h^{mb})<n-a$ and $\deg_y(f-h'^m)<n-ab$. Using the expansion of h' with respect to $h: h'=h^b+\Sigma_{i=1}^b c_i h^{b-i}$, $\deg_y c_i<a$, we get

$$h'^m=\left((h^b+c_1h^{b-1})+\sum_{i=2}^{b}c_ih^{b-i}\right)^m=h^{mb}+mc_1h^{mb-1}+R_1+R_2,$$

where $R_1:=\Sigma_{i=2}^m\binom{m}{i}c_1^ih^{mb-i}$ and $R_2:=\Sigma_{i=1}^m\binom{m}{i}(h^b+c_1h^{b-1})^{m-i}\,(\Sigma_{i=2}^b c_ih^{b-i})^i$. If $c_1\ne 0$, we would first conclude $\deg_y R_1<n-a+\deg_y c_1$, $\deg_y R_2<n-a$, and then

$$n-a>\deg_y(f-h^{mb})=\deg_y(f-h'^m+mc_1h^{mb-1}+R_1+R_2)$$
$$=\deg_y(c_1h^{mb-1})\geq(mb-1)a=n-a.$$

So $c_1=0$ and it follows that $\deg_y(h'-h^b)=\deg_y(\Sigma_{i=2}^b c_ih^{b-i})<ab-a$. By the uniqueness of the Tschirnhausen approximate polynomial, the above inequality implies that h is the ab/a-th Tschirnhausen approximate polynomial of h'. Q.E.D.

The generalized binomial formula: $(1+z)^r=\Sigma_{j=0}^{\infty}\binom{r}{j}z^j$ for $r>0$, with coefficients $\binom{r}{j}:=r(r-1)\cdots(r-j+1)/j!$, converges for $|z|<1$. When r is a rational number p/q, the identity: $((1+z)^{p/q})^q=(1+z)^p$ gives a recurrent computation of the coefficients of $(1+z)^{p/q}$. In particular, with $\mathrm{Trunc}^{(\ell)}(1+z)^{p/q}:=\Sigma_{j=0}^{\ell}\binom{p/q}{j}z^j$, it follows that

$$\text{(2.2.1)} \qquad \mathrm{val}_z((1+z)^p-(\mathrm{Trunc}^{(\ell)}(1+z)^{p/q})^q)>\ell$$

For a real number $x\in\boldsymbol{R}$, denote by $[x]$ the largest integer n such that $n\leq x$.

Lemma 2.3. *Assume that a, b, c, d are positive integers such that* $\gcd(a,b)=1$ *and that d divides ac. Let $F(y,z)=(y^a+z^b)^c$ and $H(y,z)=y^{ac/d}\mathrm{Trunc}^{([c/d])}(1+z^b/y^a)^{c/d}$. Then H is the d-th Tschirnhausen approximate polynomial of $F(y,z)$ as a polynomial of y.*

Proof. The polynomials $F(y,z)$ and $H(y,z)$ are weighted homogeneous of degree abc and abc/d respectively with respect to the weight vector $P={}^t(b,a)$. In particular, the monomials in $F(y,z)$ and $H(y,z)^d$ have the form $y^{ai}z^{bj}$ with $i+j=c$. Note also that $\deg_yF(y,z)=ac$, $\deg_yH=ac/d$ and $\deg_y(F-H^a)<ac-a[c/d]$ by (2.2.1). As $ac-ac/d>ac-a[c/d]-a$, this implies the inequality: $\deg_y(F(y,z)-H(y,z)^d)<ac-ac/d$. Q.E.D.

3. Toric modifications and strict transforms

3.1. Basic properties of toric modifications (see [26,29,30,33]). Let (x,y) be a fixed system of local (or global) coordinates of $\boldsymbol{C}^2$ at the origin. Let N be the lattice of integral weights for the monomials in (x,y). The weights $E_1(x^ay^b)=a$ and $E_2(x^ay^b)=b$ span the lattice N, and a weight $\alpha_iE_1+\beta_iE_2$ will be denoted by the integral column vector ${}^t(\alpha_i,\beta_i)$. Let N^+ be the space of positive weight vectors of N, and similarly let $N_{\boldsymbol{R}}^+$ be the positive cone in $N_{\boldsymbol{R}}:=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. A simplicial cone subdivision Σ^* of $N_{\boldsymbol{R}}^+$ is a sequence $(T_1,\cdots,T_m)$ of primitive weights in N^+, called the vertices, such that $T_0=E_1$, $T_{m+1}=E_2$ and $\det(T_i,T_{i+1})=\det_{\{E_1,E_2\}}(T_i,T_{i+1})\geq 1$ holds. The $m+1$ cones $\mathrm{Cone}(T_i,T_{i+1}):=\{tT_i+sT_{i+1}\,;t,s\geq 0\}$, $i=0,\cdots,m$, cover without overlap the cone $N_{\boldsymbol{R}}^+$. The subdivision Σ^* is called *regular* if $\det(T_i,T_{i+1})=1$ for each $i=0,\cdots,m$. Let σ_i be the integral matrix mapping E_1 to T_i and E_2 to T_{i+1}.

Using a birational mapping $\phi_M:\boldsymbol{C}^2\to\boldsymbol{C}^2$, $\phi_M(x,y)=(x^ay^b,x^cy^d)$ for an integral unimodular matrix $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the toric modification $p:X\to \boldsymbol{C}^2$ associated with a regular simplicial cone subdivision Σ^* is defined as follows. The non-singular complex manifold X is covered with $m+1$ so-called *toric coordinate charts* $\{\boldsymbol{C}^2_{\sigma_i}, (x_{\sigma_i}, y_{\sigma_i})\}$, $i=0,\cdots,m$, where points $(x_{\sigma_i}, y_{\sigma_i})\in \boldsymbol{C}^2_{\sigma_i}$ and $(x_{\sigma_j}, y_{\sigma_j})\in \boldsymbol{C}^2_{\sigma_j}$ are identified if and only if the birational map $\phi_{\sigma_j^{-1}\sigma_i}$ is defined at the point $(x_{\sigma_i}, y_{\sigma_i})\in \boldsymbol{C}^2_{\sigma_i}$ and $\phi_{\sigma_j^{-1}\sigma_i}(x_{\sigma_i}, y_{\sigma_i})=(x_{\sigma_j}, y_{\sigma_j})$. The morphism $\pi_{\sigma_i}:\boldsymbol{C}^2_{\sigma_i}\to \boldsymbol{C}^2$ defined by $\pi_{\sigma_i}(x_{\sigma_i}, y_{\sigma_i}) =\phi_{\sigma_i}(x_{\sigma_i}, y_{\sigma_i})$ are compatible with the identifications and define a proper birational analytic map $p:X\to \boldsymbol{C}^2$. A toric modification is a composition of finite blowing-ups (see [18]). The exceptional divisor $p^{-1}(O)$ is the union of m rational curves $\{\hat{E}(T_i); i=1,\cdots,m\}$ and each one is covered by *its left chart* $\boldsymbol{C}^2_{\sigma_{i-1}}$ and *its right chart* $\boldsymbol{C}^2_{\sigma_i}$ and defined by the equations $\{x_{\sigma_i}=0\}$, $\{y_{\sigma_{i-1}}=0\}$. Thus only $\hat{E}(T_i)$ and $\hat{E}(T_{i+1})$ intersect transversely at the origin of the chart $\boldsymbol{C}^2_{\sigma_i}$. The non-compact divisors $\hat{E}(E_1):=\{x_{\sigma_0}=0\}$ and $\hat{E}(E_2):=\{y_{\sigma_m}=0\}$ map isomorphically onto the axis $x=0$ and $y=0$.

3.2. Admissible toric modifications. Let $f(x,y)=\Sigma a_{\alpha,\beta}x^\alpha y^\beta$ be the Taylor expansion of a germ of a holomorphic function f with $f(O)=0$. The Newton polygon $\Gamma_+(f;(x,y))$ of $f(x,y)$ is the convex hull in $N^*_{\boldsymbol{R}}$ of $\{(\alpha+s,\beta+t)\in \boldsymbol{R}^2; a_{\alpha,\beta}\neq 0, s\geq 0, t\geq 0\}$ and the Newton boundary $\Gamma(f;(x,y))$ is the union of the compact faces of $\Gamma_+(f;(x,y))$ (see [26,27,29] for instance). The Newton boundary $\Gamma(f;(x,y))$ contains only a finite number of faces of dimension one. Each positive weight vector $P={}^t(p,q)\in N^+$ defines a non-negative function on $\Gamma_+(f;(x,y))$, for which we denote by $d(P;f)$ its minimal value and by $\Delta(P;f)$ the face or the vertex where this minimal value is taken. We consider on N^+ the equivalence relation: $P\sim Q$ if and only if $\Delta(P;f)=\Delta(Q;f)$. *The dual Newton diagram* $\Gamma^*(f;(x,y))$ of $f(x,y)$ is the conical subdivision of N^+ given by the equivalence classes. Let $P_i={}^t(a_i,b_i)\in N^+$, $i=1,\cdots,m$ be the ordered list of primitive weight vectors such that $\Delta(P_i;f)$ is the list of the one-dimensional faces of $\Gamma^*(f;(x,y))$ and $\det(P_i, {}_{i+1})=a_ib_{i+1}-a_{i+1}b_i>0$, $i=1,\cdots,m-1$. The *face function* $f_{P_i}(x,y):=\Sigma_{(\alpha,\beta)\in\Delta(P_i;f)}a_{\alpha,\beta}x^\alpha y^\beta$ admits a product decomposition $f_{P_i}(x,y)=c_ix^{r_i}y^{s_i}\Pi_{j=1}^{k_i}(y^{a_i}-\gamma_{i,j}x^{b_i})^{\nu_{i,j}}$ with distinct non-zero complex numbers $\gamma_{i,1},\cdots,\gamma_{i,k_i}$. Recall that $f(x,y)$ is *non-degenerate* if and only if $\nu_{i,j}=1$ for any i, j. The partial sum $\mathcal{N}(f)(x,y)=\Sigma'' a_{\alpha,\beta}x^\alpha y^\beta$ over all $(\alpha,\beta)\in\Gamma(f;(x,y))$ is the *Newton principal part* $\mathcal{N}(f)(x,y)$.

A regular simplicial cone subdivision Σ^* with vertices $\{T_0=E_1, T_1,\cdots,T_l, T_{l+1} =E_2\}$ is called *admissible* for $f(x,y)$ if Σ^* is a refinement of the dual Newton diagram $\Gamma^*(f;(x,y))$, meaning $P_i={}^t(a_i,b_i)\in\{T_0, T_1,\cdots,T_\ell, T_{\ell+1}\}$, $i=1,\cdots,m$. Note that Σ^* is admissible for $f(x,y)$ if and only if $\Delta(T_j;f)\cap\Delta(T_{j+1};f)\neq\emptyset$, $j=0,\cdots,\ell$. The corresponding toric modification $p:X\to \boldsymbol{C}^2$ is called *admissible* for $f(x,y)$.

Basic properties of admissible toric modifications are:

(3.2.A) The divisor $\hat{E}(T_j)$ meets the proper transform $\tilde{C}$ if and only if T_j is a primitive weight P_i.

(3.2.B) The divisor $\hat{E}(P_i)$ intersects $\tilde{C}$ at k_i points. In the right toric chart $\{C_{\sigma_j},(x_{\sigma_j},y_{\sigma_j})\}$, $\sigma_j=\mathrm{Cone}(T_j,T_{j+1})$, $P_i=T_j$, the intersection $\tilde{C}\cap\hat{E}(P_i)$ is $\{(0,\gamma_{i,1}),\cdots,(0,\gamma_{i,k_i})\}$.

(3.2.C) The divisor of the pull back p^*f of the function f is given by

$$(p^*f)=\sum_{i=1}^{m}\sum_{\ell=1}^{k_i}\tilde{C}_{i,\ell}+\sum_{j=0}^{\ell+1}d(T_j;f)\hat{E}(T_j)$$

where $\tilde{C}_{i,\ell}$ is the union of components of $\tilde{C}$ which pass through $(0,\gamma_{i,\ell})$.

(3.2.D) If $f(x,y)$ is irreducible as a germ of a function at the origin, then $m=1$ and $k_1=1$.

(3.2.E) If f is non-degenerate, the curve $\tilde{C}_{i,j}$ is smooth and $\tilde{C}_{i,j}$ intersects transversely with $\hat{E}(P_i)$. Thus, if $f(x,y)$ is non-degenerate, the modification p is a good resolution of $f(x,y)$ (see [18]).

3.3. Intersection multiplicity with a reduced irreducible germ. Let $C=\{f(x,y)=0\}$ be a reduced irreducible germ of a curve. The defining function admits for a weight $P_1={}^t(a_1,b_1)$ an initial expansion $f(x,y)=(y^{a_1}+\xi_1x^{b_1})^{A_2}+$(higher terms) with $\xi_1\neq 0$ and $\gcd(a_1,b_1)=1$, where "higher terms" collects the monomials of P_1-degree strictly greater than $a_1b_1A_2$. Let C' be another (not necessarily irreducible) germ of a curve defined by $C'=\{(x,y)\in U;g(x,y)=0\}$. Let $p:X\to \boldsymbol{C}^2$ be a toric modification admissible both for C and C', and let Ξ_1 be the interesection point of $\tilde{C}$ and $E(P_1)$. Then

Proposition 3.3.1 (Lemma 7.12, [29]). *The intersection multiplicity of C and C' at the origin is $I(C,C';O)=d(P_1;g)A_2+I(\tilde{C},\tilde{C}';\Xi_1)$. The term $I(\tilde{C},\tilde{C}';\Xi_1)$ vanishes if and only if $g_{P_1}(x,y)$ is not divisible by $(y^{a_1}+\xi_1x^{b_1})$. If $g(x,y)$ has for a primitive weight vector $P_1'={}^t(a_1',b_1')$ the initial expansion $g(x,y)=(y^{a_1'}+\xi_1'x^{b_1'})^{A_2'}+$(higher terms), then $d(P_1;g)A_2=\min(a_1b_1',a_1'b_1)\times A_2A_2'$ and moreover $I(\tilde{C},\tilde{C}';\Xi_1)=0$ if and only if either $P_1\neq P_1'$ or $P_1=P_1'$ and $\xi_1\neq\xi_1'$.*

3.4. A resolution tower of toric modification for an irreducible germ. Let C be an irreducible germ of a plane curve and let

$$\mathscr{T}=\{X_k\xrightarrow{p_k}X_{k-1}\xrightarrow{p_{k-1}}\cdots\to X_1\xrightarrow{p_1}X_0\}$$

be a sequence of non-trivial toric modifications where each $p_{i+1}:X_{i+1}\to X_i$ is the toric modification associated with a regular simplicial cone subdivision Σ_i^* of the cone $N_{\boldsymbol{R}}^+$ in the space of weights for a local system of coordinates (u_i,v_i) of X_i, centered at the center $\Xi_i\in X_i$ of the modification p_{i+1}. Let $E_{i,1},\cdots,E_{i,s_i}$ be the exceptional divisors of $p_i:X_i\to X_{i-1}$. By abuse of the notation, we denote by the same $E_{i,j}$, the strict transform of $E_{i,j}$ to X_ℓ for any $\ell \geq i$. Thus the exceptional divisors of the modification $\Phi_k := p_1 \circ \cdots \circ p_k : X_k \to X_0$ are $\{E_{i,j}\}$, $1 \leq i \leq k$, $1 \leq j \leq s_i$. Denote by $\Xi_i \in E_{i,\beta_i} \cap C^{(i)}$ the preimage of the singularity in the strict transform $C^{(i)}$ of C to X_i. We call $\mathscr{T}$ *a resolution tower of admissible toric modifications* if the following conditions are satisfied ([29]).

(i) X_0 is an open neighborhood of the origin O of $\mathbf{C}^2$, $(u_0,v_0)=(x,y)$ and $\Xi_0 = O$.

(ii) The modification $p_{i+1} : X_{i+1} \to X_i$ is non-trivial and admissible for $\Phi_i^* f(u_i,v_i)$, $i>0$.

(iii) The coordinate u_i is simply the restriction $u_i = x_{\sigma_i} \mid W_i$ of the coordinate x_{σ_i} of the right toric chart of E_{i,β_i} to a neighborhood W_i of Ξ_i.

(iv) $p_i(\Xi_i) = \Xi_{i-1}$.

(v) The composition $\Phi_k : X_k \to X_0$ is a good resolution of C.

The weight vectors $P_i = {}^t(a_i,b_i)$ corresponding to the exceptional divisors E_{i,β_i} for $i=1,\cdots,k$ are *the weight vectors of the tower* ([29]). If the tower $\mathscr{T}$ is admissible for C, there exist for $i=0,\cdots,k-1$ non-zero complex numbers $\xi_i \in \mathbf{C}$ so that $C^{(i)} = \{(u_i,v_i) \in W_i ; (v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A_{i+2}} + (\text{higher terms}) = 0\}$, where $C^{(0)} = C$ and $A_j = a_j \cdots a_k$, $j \leq k$ and $A_{k+1} = 1$.

Let $D = \{(x,y) \in \mathbf{C}^2 ; g(x,y) = 0\}$ be an irreducible, not necessarily reduced, germ of a plane curve at the origin of $\mathbf{C}^2 = X_0$ and let $D^{(i)}$ be the strict transform of D to X_i. If D has the same toric tangential direction of depth θ with C with respect to $\mathscr{T}$, i.e. if $\Xi_i \in D^{(i)}$ for $i \leq \theta$ and $\Xi_{\theta+1} \notin D^{(\theta+1)}$, there exist a non-zero complex number $\xi'_{\theta+1}$, a positive integer $A_{\theta+2}{}'$ and a primitive weight vector $P'_{\theta+1} := {}^t(a'_{\theta+1}, b'_{\theta+1})$ such that

$$(3.4.1) \qquad D^{(i)} = \begin{cases} \{u_i,v_i) \in W_i ; (v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A'_{i+2}} + (\text{higher terms}) = 0\}, & i<\theta \\ \{u_\theta,v_\theta) \in W_\theta ; (v_\theta^{a'_{\theta+1}} + \xi'_{\theta+1} u_\theta^{b'_{\theta+1}})^{A'_{\theta+2}} + (\text{higher terms}) = 0\}, & i=\theta \end{cases}$$

where $A'_j = a_j \cdots a_\theta a'_{\theta+1} A'_{\theta+2}$, $j \leq \theta+1$. If $P'_{\theta+1} = {}^t(1,0)$, the transform $D^{(\theta)}$ is defined by $\{v_\theta^{A'_{\theta+2}} = 0\}$ since D is irreducible. The case $P'_{\theta+1} = {}^t(0,1)$ does not occur as $\{u_{\theta+1} = 0\}$ is nothing but $\hat{E}(P_\theta)$. Put

$$I(P_{\theta+1}, P'_{\theta+1}) := \begin{cases} \min(a_{\theta+1} b'_{\theta+1}, a'_{\theta+1} b_{\theta+1}), & \text{if } a_{\theta+1} b'_{\theta+1}, a'_{\theta+1} b_{\theta+1} > 0 \\ b_{\theta+1}, & \text{if } P'_{\theta+1} = {}^t(1,0) \end{cases}$$

By induction, using Proposition (3.3.1), we get

Lemma 3.4.2 ([29]). *Assume that D has the same toric tangential direction of depth θ with C with respect to $\mathscr{T}$. Under the assumption* (3.4.1) *on D, the local intersection multiplicity is*

$$I(C,D;O) = \sum_{i=1}^{\theta} a_i b_i A_{i+1} A'_{i+1} + I(P_{\theta+1}, P'_{\theta+1}) \times A_{\theta+2} A'_{\theta+2}$$

Let $D_1,\cdots,D_r$ be the irreducible components of a reducible plane curve germ D. We say that the reducible germ D has *the same toric tangetial direction of depth* θ *with* C *with respect to* $\mathscr{T}$ if $\Xi_i \in D_j^{(i)}$ for any $j=1,\cdots,r$ and $i \le \theta$ and $\Xi_{\theta+1} \notin D_{j_0}^{(\theta+1)}$ for some j_0.

4. A Tschirnhausen resolution tower for an irreducible germ

Lemma 4.1. *Let* $p: X \to C^2$ *be a toric modification with respect to a regular simplicial cone subdivision* Σ^* *of* N^+. *Let* $\sigma = \mathrm{Cone}(P,P')$ *be a cone in* Σ^* *and* $g(x,y) \in C\{x,y\}$, *such that* $\Delta(P;g)$ *is a vertex. At each point* $\Xi \in \hat{E}(P) - \bigcup_{Q \ne P} \hat{E}(Q)$ *the function* $p^*g / x_\sigma^{d(P;g)}$ *is a unit.*

Proof. Let $\{(\nu_1,\nu_2)\} = \Delta(P;g)$ and $c \ne 0$ be the coefficient of $x^{\nu_1}y^{\nu_2}$ in $g(x,y)$. Then the pullback p^*g factors $x_\sigma^{d(P;g)} y_\sigma^{d(P';g)} \{c y_\sigma^\alpha + x_\sigma g'(x_\sigma, y_\sigma)\}$ for some analytic function $g'(x_\sigma,y_\sigma)$ and $\alpha \ge 0$. Moreover, $\alpha = 0$ if and only if $\Delta(P';g) \supset \Delta(P;g)$. In conclusion, $p^*g / x_\sigma^{d(P;g)}$ is a unit at Ξ since y_σ is. Q.E.D.

In particular, if $P = {}^t(a,b)$ and $\Gamma(g;(x,y)) \subset \{(\nu_1,\nu_2); \nu_2 < a\}$ or if $g(x,y) \in C\{x\}[y]$ and $\deg_y g < a$, the face $\Delta(P;g)$ is a vertex, and the lemma applies.

A. Tschirnhausen resolution tower

4.2. Let $f(x,y) \in C\{x\}[y]$ be monic of degree n and irreducible with the initial expansion

$$f(x,y) = (y^{a_1} + \xi_1 x^{b_1})^{A_2} + (\text{higher terms}), \quad a_1 > 1 \tag{4.2.1}$$

for the primitive weight vector $P_1 = {}^t(a_1,b_1)$ with $n = a_1 A_2$. The n/a-th Tschirnhausen approximate polynomial $H_a(x,y)$ is a monic polynomial of degree a in y and defines at the origin the germ of the curve $D_a := \{H_a(x,y) = 0\}$.

4.3. First observation. Let $p_1: X_1 \to C^2$ be an admissible toric modification with respect to a regular simplicial cone subdivision Σ_0^* for $f(x,y)$. The strict transform $C^{(1)}$ of C to X_1 intersects only with $\hat{E}(P_1)$, say at the point Ξ_1. In the chart $C_{\sigma_1}^2$, where $P_1' = {}^t(a_1',b_1')$ and $\sigma_1 = (P_1,P_1')$ is the right cone of $\hat{E}(P_1)$, we have $\Xi_1 = (0,-\xi_1)$. Put $h_1(x,y) := H_{a_1}$, $C_1 := D_{a_1}$. For a multiple a of a_1 with $a \mid n$, the A_2-th (resp. n/a-th) Tschirnhausen approximate polynomial of $(y^{a_1} + \xi_1 x^{b_1})^{A_1}$ is the face function h_{1P_1} (resp. H_{aP_1}), hence h_1 and H_a can be written as:

$$\begin{cases} h_1(x,y) = y^{a_1} + \xi_1 x^{b_1} + (\text{higher terms}) \\ H_a(x,y) = (y^{a_1} + \xi_1 x^{b_1})^{a/a_1} + (\text{higher terms}), \quad \text{if } a_1 \mid a \end{cases} \tag{4.3.1}$$

In particular, $h_1(x,y)$ is non-degenerate. As $p_1^*(y^{a_1} + \xi_1 x^{b_1}) = x_{\sigma_1}^{a_1 b_1} y_{\sigma_1}^{a_1' b_1}(y_{\sigma_1} + \xi_1)$, we can write $p_1^*h_1(x_{\sigma_1},y_{\sigma_1})=x_{\sigma_1}^{a_1b_1}y_{\sigma_1}^{a_1'b_1}((y_{\sigma_1}+\xi_1)+x_{\sigma_1}R(x_{\sigma_1},y_{\sigma_1}))$, $R(x_{\sigma_1},y_{\sigma_1})\in C\{x_{\sigma_1},y_{\sigma_1}\}$. The functions

$$u_1=x_{\sigma_1},\qquad v_1=p_1^*h_1/x_{\sigma_1}^{a_1b_1}=y_{\sigma_1}^{a_1'b_1}((y_{\sigma_1}+\xi_1)+x_{\sigma_1}R(x_{\sigma_1},y_{\sigma_1}))$$

give a system of coordinates (u_1,v_1) in a neighbourhood W_1 of Ξ_1. The strict transform $C_1^{(1)}$ of C_1 to X_1 intersects only with $\hat{E}(P_1)$ and $p_1^*h_1=u_1^{a_1b_1}v_1$, $C_1^{(1)}=\{v_1=0\}$, so C_1 is irreducible and p_1 is a good resolution of C_1.
If $A_2=1$, we have $f=h_1$ and we have nothing to do further. If $A_2\geq 2$, the pull back $p_1^*f(u_1,v_1)$ has an initial expansion

$$p_1^*f(u_1,v_1)=u_1^{m_1(f)}(v_1^{a_2}+\xi_2u_1^{b_2})^{A_3}+(\text{higher terms}) \tag{4.3.2}$$

with primitive weight vector $P_2={}^t(a_2,b_2)$ where the multiplicity of Φ_1^*f on E_1 is $m_1(f)=a_1b_1A_2$ by (4.2.1). Note also $A_2=a_2A_3$ and $I(C_1,C;O)=a_1b_1A_2+b_2A_3$ by Lemma 3.4.2. The advantage of the "Tschirnhausen coordinates" is the inequality $a_2\geq 2$. In fact, in the Tschirnhausen expansion $f(x,y)=h_1(x,y)^{A_2}+\Sigma_{j=2}^{A_2}c_j(x,y)h_1(x,y)^{A_2-j}$ of $f(x,y)$ with respect to h_1 we have $c_j(x,y)\in C\{x\}[y]$ and $\deg_y c_j(x,y)<a_1$, $j=2,\cdots,A_2$, so the face $\Delta(P_1,c_j)$ is necessarily a vertex. Therefore by the definition of the coordinate (u_1,v_1) and Lemma 4.1 in a smaller neighbourhood W_1 of Ξ_1 the pull-backs are: $p_1^*h_1(u_1,v_1)=u_1^{m_1(h_1)}v_1$ with $m_1(h_1)=a_1b_1$ and $p_1^*c_j(u_1,v_1)=u_1^{m_j}U_j$, where $m_j=d(P_1,c_j)$ and U_j is a unit for $j\geq 2$ with $c_j\neq 0$. If $c_j=0$, we put $U_j=0$ for simplicity. Thus we have

$$p_1^*f(u_1,v_1)=(u_1^{m_1(h_1)}v_1)^{A_2}+\sum_{j=2}^{A_2}u_1^{m_j}U_j(u_1^{m_1(h_1)}v_1)^{A_2-j}, \tag{4.3.3}$$

hence, with $Q_0=(m_1(h_1)A_2,A_2)$ and $Q_j=(m_j+(A_2-j)m_1(h_1),A_2-j)$, the Newton polygon $\Gamma_+(p_1^*f;(u_1,v_1))$ is the convex hull of the sets $\{Q_0+\boldsymbol{R}_+^2\}$ and $\{Q_j+\boldsymbol{R}_+^2\}$, $2\leq j\leq A_2$, $c_j\neq 0$. The Newton principal part $\mathcal{N}(p_1^*f)(u_1,v_1)$ contains $(m_1(f)+b_2, A_2-a_2)$ by (4.3.2). It follows that $(m_1(f)+b_2,A_2-a_2)=Q_j$ for $j=0$ or for some $j\geq 2$, hence $a_2\geq 2$. Moreover, if $c_j\neq 0$, we have $d(P_2;p_1^*c_jh_1^{A_2-j})\geq d(P_2;p_1^*f)$, with equality if and only if $a_2\,|\,j$.

Let $a_1\,|\,a$ and $a\,|\,n$. The following Tschirnhausen expansions start at $j=2$ by Proposition 2.2:

$$\begin{cases} H_a=h_1^{a/a_1}+\Sigma_{j=2}^{a/a_1}d_jh_1^{a/a_1-j}\in C\{x\}[y][h_1], & \deg_y d_j<a_1\\ f\;=H_a^{n/a}+\Sigma_{j=2}^{n/a}c_jH_a^{n/a-j}\in C\{x\}[y][H_a], & \deg_y c_j<a \end{cases} \tag{4.3.4}$$

By (4.3.3), the principal part of $p_1^*f(u_1,v_1)$ with respect to the weight vector P_2 is

$$p_1^*f_{P_2}(u_1,v_1)=u_1^{m_1(f)}(v_1^{a_2}+\xi_2u_1^{b_2})^{A_3} \tag{4.3.5}$$

With $R_a:=\Sigma_{j=2}^{n/a}c_jH_a^{n/a-j}$, we have that $\deg_y R_a<n-a$ and therefore the Tschirnhausen

expansion of R_a with respect to h_1 can be written as $R_a=\Sigma_{\ell=0}^{A_2-a/a_1-1}\beta_\ell h_1^\ell$ for some $\beta_\ell\in C\{x\}[y]$ and $\deg_y\beta_\ell<a_1$. If $\beta_\ell\neq 0$, we can write $p_1^*(\beta_\ell h_1^\ell)=U_\ell u_1^{\gamma_\ell}v_1^\ell$ by Lemma 4.1 for a unit U_ℓ and a non-negative integer $\gamma_\ell\in N$. Thus we have for the Newton principal part

$$\deg_{v_1}\mathcal{N}(p_1^*R_a)(u_1,v_1)\leq A_2-a/a_1-1. \tag{4.3.6}$$

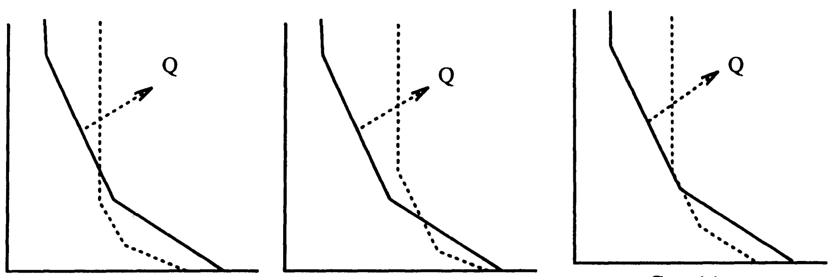
So, comparing the pull-back $p_1^*f(u_1,v_1)=p_1^*H_a^{n/a}+\Sigma_{j=2}^{n/a}U_\ell u_1^{\gamma_\ell}v_1^\ell$ of (4.3.4) and (4.3.5), we see that the monomials $u_1^{m_1(f)}\times\binom{A_3}{i}(v_1^{a_2})^i(\xi_2u_1^{b_2})^{A_3-i}$ of $u_1^{m_1(f)}(v_1^{a_2}+\xi_2u_1^{b_2})^{A_s}$ for $i>A_3-a/a_1a_2-1/a_2$ come from $p_1^*H_a^{n/a}$. The expansions (4.3.4) and (4.3.1) give with some analytic functions g_a, G_a

$$\begin{cases}p_1^*H_a(u_1,v_1)=u_1^{m_1(H_a)}(v_1^{a/a_1}+u_1g_a(u_1,v_1))\\ p_1^*H_a^{n/a}(u_1,v_1)=u_1^{m_1(f)}(v_1^{A_2}+u_1G_a(u_1,v_1))\end{cases} \tag{4.3.7}$$

Note that $m_1(H_a)n/a=m_1(f)$. Applying the above argument to $p_1^*H_a(u_1,v_1)$, we can conclude that the Newton boundary $\Gamma(p_1^*H_a;(u_1,v_1))$ is situated in the region $\{(v_1,v_2)\in \boldsymbol{R}^2;0\leq v_2\leq a/a_1\}$ and that $B_a:=(m_1(H_a),a/a_1)$ is the vertex of the left end of $\Gamma(p_1^*f;(u_1,v_1))$ by (4.3.7). Note also that $(n/a)B_a=(m_1(f),A_2)$ is the left end of $\Gamma(p_1^*f;(u_1,v_1))$ by (4.3.7). Let Δ_a be the first face of $\Gamma(p_1^*H_a)$ which contains B_a and let $Q={}^t(p_2,q_2)$ be the weight vector of Δ_a.

Assertion 4.3.8. *The inequality $q_2/p_2\geq b_2/a_2$ holds.*

Proof. Assuming by contradiction that $q_2/p_2<b_2/a_2$, we have $p_1^*f_Q(u_1,v_1)=u^{m_1(f)}v_1^{A_2}$, and we will prove the assertion by excluding the following three cases: (a) $d(Q;p_1^*H_a^{n/a})>d(Q;p_1^*R_a)$, (b) $d(Q;p_1^*H_a^{n/a})<d(Q;p_1^*R_a)$, (c) $d(Q;p_1^*H_a^{n/a})=d(Q;p_1^*R_a)$. Figure (4.3.A) indicates the respective situations. In case (a), $u^{m_1(f)}v_1^{A_2}=(p_1^*R_a)_Q(u_1,v_1)$ holds, which is impossible by (4.3.6). The case (b) is impossible as $(p_1^*H_a)_Q(u_1,v_1)^{n/a}\neq u^{m_1(f)}v_1^{A_2}$ by the assumption. If case (c) holds, from (4.3.6) it follows $(p_1^*H_a)_Q(u_1,v_1)^{n/a}+(p_1^*R_a)_Q(u_1,v_1)\neq 0$, and then $d(Q;p_1^*H_a^{n/a})=d(Q;p_1^*R_a)$



Case (a) Case (b) Case (c)

Figure (4.3.A)

$=d(Q;p_1^*f)$ and finally the equality $u^{m_1(f)}v_1^{A_2}=(p_1^*H_a)_Q(u_1,v_1)^{n/a}+(p_1^*R_a)_Q(u_1,v_1)$. But this equality is impossible. In fact, let us write $(p_1^*H_a)_Q(u_1,v_1)=u_1^{m_1(H_a)}v_1^{a/a_1}+\gamma u_1^{\alpha_1}v_1^{\beta_1}+S(u_1,v_1)$ where $\gamma\neq 0, 0\leq\beta_1<a/a_1$ and $\deg_{v_1}S(u_1,v_1)<\beta_1$ if $S\neq 0$. Then $(p_1^*H_a)_Q(u_1,v_1)^{n/a}=u_1^{m_1(f)}v_1^{A_2}+n/a\cdot\gamma u_1^{\alpha_1'}v_1^{\beta_1'}+S'(u_1,v_1)$ with $\deg_{v_1}S'<\beta_1'$, where $\alpha_1'=\alpha_1+(n/a-1)m_1(H_a)$ and $\beta_1'=A_2-a/a_1+\beta_1\geq A_2-a/a_1$. On the other hand, the second term of the right side of the equality has no monomial $u_1^{\nu_1}v_1^{\nu_2}$ with $\nu_2\geq A_2-a/a_1$. Q.E.D.

By Assertion 4.3.8, the face function $(p_1^*H_a)_{P_2}(u_1,v_1)$ for the weight vector P_2 is divisible by $u_1^{m_1(H_a)}$, hence $H_a'(u_1,v_1):=(p_1^*H_a)_{P_2}(u_1,v_1)/u_1^{m_1(H_a)}$ is a polynomial. By a similar discussion as above, we conclude: $d(P_2;p_1^*f(u_1,v_1))=d(P_2;p_1^*H_a(u_1,v_1)^{n/a})=d(P_2;p_1^*R_a(u_1,v_1)),(p_1^*f)_{P_2}=(p_1^*H_a)_{P_2}^{n/a}+(p_1^*R_a)_{P_2}$ and $\deg_{v_1}((v_1^{a_2}+\xi_2u_1^{b_2})^{A_3}-H_a'^{n/a}(u_1,v_1))\leq A_2-a/a_1-1$. In other words, $H_a'(u_1,v_1)$ is the n/a-th Tschirnhausen approximate polynomial of $(v_1^{a_2}+\xi_2u_1^{b_2})^{A_3}$. In particular, if a_1a_2 divides a, $A_3/(n/a)=a/a_1a_2$ is an integer and we can see easily that $H_a'(u_1,v_1)=(v_1^{a_2}+\xi_2u_1^{b_2})^{a/a_1a_2}$ if $a_1a_2|a$ and $p_1^*H_a(u_1,v_1)=u_1^{m_1(H_a)}(v_1^{a_2}+\xi_2u_1^{b_2})^{a/a_1a_2}+$(higher terms). Putting $h_2=H_{a_1a_2}$, $C_2=D_{a_1a_2}$ and $a=a_1a_2$, we observe that $p_1^*h_2(u_1,v_1)=u_1^{m_1(h_2)}(v_1^{a_2}+\xi_2u_1^{b_2})+$(higher terms) and therefore $p_1^*h_2(u_1,v_1)$ is clearly non-degenerate.

4.4. Inductive construction of a tower. Let $\mathscr{T}_j=\{X_j\xrightarrow{p_j}X_{j-1}\to\cdots\to X_1\xrightarrow{p_1}X_0=\boldsymbol{C}^2\}$ be a tower of toric modifications with the corresponding weight vectors $P_i={}^t(a_i,b_i)$ such that $a_1\cdots a_j|n$ and $a_i\geq 2$, $i=1,\cdots,j$. Put $A_{i+1}:=n/a_1\cdots a_i$, $i\leq j$ and for simplicity $h_i(x,y)=H_{a_1\cdots a_i}(x,y)$, $C_i=D_{a_1\cdots a_i}$ and $\Phi_i=p_1\circ\cdots\circ p_i:X_i\to X_0$. Let $D_a^{(l)}$ and $C_i^{(l)}$, $(i\geq l)$ be the strict transforms of D_a and C_i to X_l respectively. The map $p_i:X_i\to X_{i-1}$ is an admissible toric modification for Φ_{i-1}^*f associated with a regular simplicial cone subdivision Σ_{i-1}^*. Let $\Xi_i=C^{(i)}\cap X_i$ be the center of the modification p_{i+1} and let (u_i,v_i) be the chosen modification local coordinate system with the center Ξ_i so that $\{u_i=0\}$ is the defining equation of the exceptional divisor $E_i:=\hat{E}(P_i)$ for $i=1,\cdots,j$. We assume the following properties (1-j), (2-j) and (3-j) for the tower.

(1-j) (C_i,O) is a germ of an irreducible curve at the origin for $i=1,\cdots,j$ and the strict transform $C_i^{(i)}$ to X_i is smooth and is defined by $\{v_i=0\}$. The pull backs of f and h_i, $i\leq j$ equal:

$$(4.4.1)\qquad \Phi_i^*f(u_i,v_i)=\begin{cases}u_j^{m_j(f)}v_j, & i=j \text{ and } A_{j+1}=1\\ u_i^{m_i(f)}(v_i^{a_{i+1}}+\xi_{i+1}u_i^{b_{i+1}})^{A_{i+2}}+\text{(higher terms)}, & \text{otherwise}\end{cases}$$

$$(4.4.2)\qquad \Phi_i^*h_l(u_i,v_i)=\begin{cases}u_i^{m_i(h_l)}v_i, & i=l\\ u_i^{m_i(h_l)}(v_i^{a_{i+1}}+\xi_{i+1}u_i^{b_{i+1}})^{A_{i+2}/A_{l+1}}+\text{(higher terms)}, & i<l\end{cases}$$

The modification coordinates (u_i, v_i) are characterized by (4.4.2). We assume $a_{j+1} \geq 2$ in (4.4.1), if $A_{j+1} \geq 2$. More generally, for any positive integer a with $a|n$ and $a_1 \cdots a_{i+1} | a$, we have

(4.4.3) $$\Phi_i^* H_a(u_i, v_i) = u_i^{m_i(h_i)}(v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{a/a_1 \cdots a_{i+1}} + (\text{higher terms})$$

Here $m_i(h_i)$, $m_i(H_a)$ and $m_i(f)$ are the respective multiplicities of the pull backs $\Phi_i^* h_i$, $\Phi_i^* H_a$ and $\Phi_i^* f$ on the exceptional divisor E_i and they satisfy the equalities:

$$m_i(h_i) \times A_{i+1} = m_i(H_a) \times n/a = m_i(f)$$

$$m_1(f) = a_1 b_1 A_2, \quad m_i(f) = a_i m_{i-1}(f) + a_i b_i A_{i+1}$$

(2-j) The local intersection multiplicities at the origin are given by

$$I(C_i, C; O) = \Sigma_{s=1}^{i+1} a_s b_s A_{s+1}^2 / A_{i+1},$$

$$I(C_i, C_l; O) = \Sigma_{s=1}^{i+1} a_s b_s A_{s+1}^2 / (A_{i+1} A_{l+1}), \quad 1 \leq i < l \leq j$$

More generally, $I(D_a, C; O) = \Sigma_{s=1}^{j} a_s b_s A_{s+1}^2 / (n/a) + I(D_a^{(j)}, C^{(j)}; \Xi_j)$, if $a|n$ and $a_1 \cdots a_j | a$.

(3-j) For any non-zero polynomial $\alpha(x,y) \in C\{x\}[y]$ with $\deg_y \alpha(x,y) < a_1 \cdots a_j$, the pull back $\Phi_j^* \alpha$ can be written as $\Phi_j^* \alpha = U \times u_j^s$ in a small neighbourhood W_j of Ξ_j for some integer $s \geq 0$.

If $A_{j+1} = 1$, then $h_j = f$ and (4.4.1) says that $\Phi_j : X_j \to X_0$ is a good resolution of C. If $A_{j+1} \geq 2$, we will add to the tower a toric modification $p_{j+1} : X_{j+1} \to X_j$ keeping the above properties. Let $P_{j+1} = {}^t(a_{j+1}, b_{j+1})$ be the weight vector of the unique face of $\Gamma(\Phi_j^* f; (u_j, v_j))$ characterized by (4.4.1) and (4.4.2): $\Phi_j^* f(u_j, v_j) = u_j^{m_j(f)}(v_j^{a_{j+1}} + \xi_{j+1} u_j^{b_{j+1}})^{A_{j+2}} + (\text{higher terms})$. Choose a regular simplicial cone subdivision Σ_j^* of the $\Gamma^*(\Phi_j^* f; (u_j, v_j))$ and make the corresponding modification $p_{j+1} : X_{j+1} \to X_j$ with center $\Xi_j \in E_j$. Then $\Phi_j^* h_{j+1}(u_j, v_j)$ is non-degenerate by (4.4.2), so in the right toric chart $\sigma = (P_{j+1}, P'_{j+1})$ we can write $\Phi_{j+1}^* h_{j+1}(x_\sigma, y_\sigma) = x_\sigma^{m_{j+1}(h_{j+1})} y_\sigma^{m'_{j+1}(h_{j+1})}((y_\sigma + \xi_{j+1}) + x_\sigma G)$ where $m_{j+1}(h_{j+1})$ and $m'_{j+1}(h_{j+1})$ are multiplicities on $E_{j+1} = \hat{E}(P_{j+1})$ and $\hat{E}(P'_{j+1})$ respectively. The functions $u_{j+1} := x_\sigma$ and $v_{j+1} := y_\sigma^{m'_{j+1}(h_{j+1})}((y_\sigma + \xi_{j+1}) x_\sigma G)$ give a system of coordinates in a neighborhood W_{j+1} of the intersection point Ξ_{j+1} of $C_{j+1}^{(j+1)}$ and E_{j+1}. By the definition the strict transform $C_{j+1}^{(j+1)}$ is smooth and is defined by $\{v_{j+1} = 0\}$ in W_{j+1}. We show (3-(j+1)) first. For $\alpha(x,y) \in C\{x\}[y]$ with $\deg_y \alpha < a_1 \cdots a_{j+1}$, its Tschirnhausen expansion with respect to $h_j : \alpha(x,y) = \Sigma_{i=1}^{a_{j+1}} \alpha_i(x,y) h_j^{a_{j+1}-i}(x,y)$ has coefficients with $\deg_y \alpha_i < a_1 \cdots a_j$. Applying inductively if $\alpha_i \neq 0$ we get $\Phi_j^*(\alpha_i h_j^{a_{j+1}-i}) = U_i u_j^{v_i} v_j^{a_{j+1}-i}$ with $v_i \geq 0$ and a unit U_i. So by Lemma 4.1, $\Phi_{j+1}^* \alpha = p_{j+1}^*(\Phi_j^* \alpha) = U \times u_{j+1}^s$ for a unit U on W_{j+1} and $s \geq 0$.

If $a_{j+1} = A_{j+1}$ i.e., $A_{j+2} = 1$, the modification $\Phi_{j+1} : X_{j+1} \to X_0$ is a good resolution of C, so clearly we have (1-(j+1)) and (2-(j+1)). If $A_{j+2}\geq 2$, we write

(4.4.4) $$\Phi_{j+1}^{*}f(u_{j+1},v_{j+1})=u_{j+1}^{m_{j+1}(f)}(v_{j+1}^{a_{j+2}}+\xi_{j+2}u_{j+1}^{b_{j+2}})^{A_{j+3}}+(\text{higher terms})$$

Note that $m_{j+1}(f)=a_{j+1}m_j(f)+a_{j+1}b_{j+1}A_{j+2}$. Using the A_{j+2}-th Tschirnhausen expansions of $f: f(x,y)=h_{j+1}^{A_{j+2}}+\Sigma_{i=2}^{A_{j+2}}c_{j+1,i}h_{j+1}^{A_{j+2}-i}$, and repeating the argument in 4.3, we will prove $a_{j+2}\geq 2$. As above, if $c_{j+1,i}\neq 0$, write $\Phi_{j+1}^{*}(c_{j+1,i}h_{j+1}^{A_{j+1}-i})(u_{j+1}, v_{j+1})=U_{j+1,i}u_{j+1}^{m_i}v_{j+1}^{A_{j+1}-i}$ for some integer m_i, and a unit $U_{j+1,i}$. The Newton principal part $\mathcal{N}(\Phi_{j+1}^{*}f)(u_1,v_1)$ contains the exponent $(m_{j+1}(f)+b_{j+2},A_{j+2}-a_{j+2})$ by (4.4.4) and we conclude that $a_{j+2}\geq 2$ as in 4.3.

Now we show (1-(j+1)). For a with $a|n$ and $a_1\cdots a_{j+1}|a$, consider the Tschirnhausen expansions:

$$f(x,y)=H_a^{n/a}+\sum_{i=2}^{n/a}c_iH_a^{n/a-i},\qquad H_a=h_{j+1}^{\beta_{j+1}}+\sum_{i=2}^{\beta_{j+1}}d_ih_{j+1}^{\beta_{j+1}-i}$$

with $\deg_y c_i<a$ and $\deg_y d_i<a_1\cdots a_{j+1}$ where $\beta_{j+1}:=a/a_1\cdots a_{j+1}$. Applying the same argument to the h_{j+1}-expansion of $R:=f-H_a^{n/a}=\Sigma_{i=2}^{n/a}c_iH_a^{n/a-i}$, we see: $\deg_{v_{j+1}}\Phi_{j+1}^{*}(R)<A_{j+2}-\beta_{j+1}$. But from (4.4.3) with $g_a,G_a\in\mathbf{C}\{u_{j+1},v_{j+1}\}$ follows:

$$\begin{aligned}\Phi_{j+1}^{*}H_a(u_{j+1},v_{j+1})&=p_{j+1}^{*}(\Phi_j^{*}H_a)(u_{j+1},v_{j+1})\\&=u_{j+1}^{m_{j+1}(H_a)}((v_{j+1}^{\beta_{j+1}}+u_{j+1}g_a(u_{j+1},v_{j+1}))\\ \Phi_{j+1}^{*}H_a^{n/a}(u_{j+1},v_{j+1})&=u_{j+1}^{m_{j+1}(f)}((v_{j+1}^{A_2}+u_{j+1}G_a(u_{j+1},v_{j+1}))\end{aligned}$$

So $B_a:=(m_{j+1}(H_a),\beta_{j+1})$ is the left end vertex of $\Gamma(\Phi_{j+1}^{*}H_a)$, $n/a\times B_a$ is the left end vertex of $\Gamma(\Phi_{j+1}^{*}H_a^{n/a})$ and also of $\Gamma(\Phi_{j+1}^{*}f;(u_{j+1},v_{j+1}))$. By the arguments of 4.3 and (4.4.3), the first face Δ_a of $\Gamma(\Phi_{j+1}^{*}H_a;(u_{j+1},v_{j+1}))$, which contains B_a, has the weight vector $P_{j+2}={}^t(a_{j+2},b_{j+2})$, hence

(4.4.5) $$\begin{cases}d(P_{j+2};\Phi_{j+1}^{*}f)=d(P_{j+2};\Phi_{j+1}^{*}H_a^{n/a})=d(P_{j+2};\Phi_{j+1}^{*}H_a)\times n/a\\ \deg_{v_{j+1}}((\Phi_{j+1}^{*}f)_{P_{j+2}}-(\Phi_{j+1}^{*}H_a)_{P_{j+2}}^{n/a})(u_{j+1},v_{j+1})<A_{j+2}-\beta_{j+1}\end{cases}$$

Note: $\deg_{v_{j+1}}\Phi_{j+1}^{*}f = d(P_{j+1};\Phi_j^{*}f)$. The polynomial $H_a'(u_{j+1},v_{j+1}):=(\Phi_{j+1}^{*}H_a)_{P_{j+2}}/u_{j+1}^{m_{j+1}(H_a)}$ is monic in v_{j+1} of degree $\beta_{j+1}=a/a_1\cdots a_{j+1}$, implying with the inequality of (4.4.5) the

Assertion 4.4.6. *If $a_1\cdots a_{j+1}|a$, then $H_a'(u_{j+1},v_{j+1})$ is the n/a-th Tschirnhausen approximate polynomial of $(\Phi_{j+1}^{*}f)_{P_{j+2}}(u_{j+1},v_{j+1})/u^{m_{j+1}(f)}=(v_{j+1}^{a_{j+2}}+\xi_{j+2}u_{j+1}^{b_{j+2}})^{A_{j+2}}\in\mathbf{C}\{u_{j+1}\}[v_{j+1}]$. In particular, if $a_1\cdots a_{j+2}|a$, then $H_a'(u_{j+1},v_{j+1}) = (v_{j+1}^{a_{j+2}}+\xi_{j+2}u_{j+1}^{b_{j+2}})^{\beta_{j+2}}$, $\Phi_{j+1}^{*}H_a(u_{j+1},v_{j+1})=u_{j+1}^{m_{j+1}(H_a)}(v_{j+1}^{a_{j+2}}+\xi_{j+2}u_{j+1}^{b_{j+2}})^{\beta_{j+2}}+$(higher terms), with $\beta_{j+2}:=a/a_1\cdots a_{j+2}$.*

This proves (1-(j+1)). The assertion about the intersection multiplicities

(2-(j + 1)) follows immediately from Lemma 3.4.2.

As $a_1 \cdots a_i$ divides n and $a_i \geq 2$ for each $i = 1, \cdots, k$, the above inductive construction stops after a finite number of toric modifications. In fact, k (respectively $k-1$) is the number of Puiseux pairs if $b_1 > 1$ (resp. if $b_1 = 1$.) See [29] and [18]. Thus we have proved the following.

Theorem 4.5. *Let $f(x,y) \in C\{x\}[y]$ be monic of degree n with the initial expansion*

$$f(x,y) = (y^{a_1} + \xi_1 x^{b_1})^{A_2} + (\textit{heigher terms}), \qquad n = a_1 A_2, \qquad a_1 > 1$$

and defining in a neighbourhood W_0 an irreducible curve $C := \{(x,y) \in W_0; f(x,y) = 0\}$ at the origin. There exits a resolution tower $\mathscr{T}$, satisfying the following conditions (1) and (2), of toric modifications: $\mathscr{T} = \{X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 = C^2\}$ having the weight vectors $\{P_i = {}^t(a_i, b_i); i = 1, \cdots, k\}$ where $n = a_1 \cdots a_k$, $a_i \geq 2$, $i = 1, \cdots, k$. With $A_i = a_i a_{i+1} \cdots a_k$, let $h_i(x,y)$ be the A_{i+1}-th Tschirnhausen approximate polynomial of $f(x,y)$ and let $C_i = \{(x,y) \in C^2; h_i(x,y) = 0\}$, $i = 1, \cdots, k$. Note $h_k = f$ and $C_k = C$. Denote by $\Xi_i \in E_i := \hat{E}(P_i)$ the center of p_{i+1}, by (u_i, v_i) the modification local coordinate centered at Ξ_i so that $\{u_i = 0\}$ is the defining equation of the divisor E_i. Put $\Phi_i = p_1 \circ \cdots \circ p_i : X_i \to X_0$.

(1) For each $i = 1, \cdots, k$, C_i is an irreducible curve at the origin having the good resolution Φ_i, such that the strict transform $C_i^{(i)}$ in X_i is defined by $\{v_i = 0\}$. The pull backs are

$$\Phi_i^* h_\ell(u_i, v_i) = \begin{cases} u_i^{m_i(h_i)} v_i, & i = \ell \\ u_i^{m_i(h_l)} (v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A_{i+2}/A_{l+1}} + (\textit{higher terms}), & i < \ell \end{cases}$$

In particular, putting $\ell = k$,

$$\text{(4.5.1)} \qquad \Phi_i^* f(u_i, v_i) = \begin{cases} u_k^{m_k(f)} v_k, & i = k \\ u_i^{m_i(f)} (v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A_{i+2}} + (\textit{higher terms}), & i < k \end{cases}$$

where the multiplicities $m_i(h_\ell)$ and $m_i(f)$ of the pull backs $\Phi_i^* h_\ell$ and $\Phi_i^* f$ on E_i satisfy the equalities: $m_i(h_\ell) = m_i(f) / A_{\ell+1}$, $m_1(f) = a_1 b_1 A_2$ and $m_i(f) = a_i m_{i-1}(f) + a_i b_i A_{i+1}$ for $i = 1, \cdots, l$. More explicitly

$$\text{(4.5.2)} \qquad \begin{cases} m_i(f) = a_i b_i A_{i+1} + \cdots + a_i \cdots a_1 b_1 A_2 = (\Sigma_{\ell=1}^i a_\ell b_\ell A_{\ell+1}^2) / A_{i+1} \\ m_i(h_\ell) = (\Sigma_{j=1}^i a_j b_j A_{j+1}^2) / (A_{i+1} A_{\ell+1}), \qquad i \leq \ell \end{cases}$$

(2) The local intersection multiplicities are

$$I(C_\ell, C_j; O) = \Sigma_{i=1}^{\ell+1} a_i b_i A_{i+1}^2 / (A_{\ell+1} A_{j+1}), \ \ell < j \leq k.$$

The equality (4.5.2) follows from (4.5.1). The other assertions are etablished in the inductive argument.

DEFINITION 4.5.3. The toric tower of Theorem 4.5 is *a Tschirnhausen resolution tower of toric modifications* of C, the coordinates (u_i,v_i) of W_i are *Tschirnhausen coordinates centered at* Ξ_i, and the curve C_i is *the* A_{i+1}*-th Tschirnhausen approximate curve* of C.

The combinatorial choice of the admissible subdivisions Σ_i^*'s determines completely the Tschirnhausen resolution tower of toric modifications. In Theorem 4.7, we will show that the length of the tower k and the sequence of the weight vectors $\{P_1,\cdots,P_k\}$ are independent of the choice of a certain resolution tower of toric modifications.

REMARK 4.5.4. Let $\tilde{a}_1=\min(a_1,b_1)$ and $\tilde{b}_1=\max(a_1,b_1)$ and let $n_1=\tilde{a}_1$, $m_1=\tilde{b}_1$ and $n_i=a_i$, $m_i=b_i+b_{i-1}a_i+\cdots+b_2a_3\cdots a_i+\tilde{b}_1a_2\cdots a_i$ for $i\geq 2$. Then we have shown in Corollary 6.8 of [29] that the Puiseux pairs of C_j is given by $\{(n_i,m_i);i=1,\cdots,j\}$, $(b_1>1)$ or $\{(n_i,m_i);i=2,\cdots,j\}$, $(b_1=1)$. The isotopy class of the knot depends only on the set of Puiseux pairs. Thus the knot given by C_j at the origin can be considered as a compound torus knot along the knot given by C_{j-1} for $j=2,\cdots,k$. There exist tori in the Milnor sphere for an ireeducible plane curve singularity, which are transversal to the Milnor fibration of the singularity, such that the tori give a decomposition of the complement of the knot and of the monodromy diffeomorphism of the singularity. For instance, on each piece of this decomposition the monodromy can be realized by a monodromy vector field having all its orbits closed and a surface of genus 0 as orbit space. In particular, the monodromy is in this decomposition piecewise of finite order (see [1]). More precisely, using the Tschirnhausen resolution and the modification coordinates, this decomposition of [1] is given explicitly as follows. First, the modifications $\Phi_i: X_i\to \boldsymbol{C}^2$ are isomorphisms above the spheres S_r of radius $r>0$ around $0\in\boldsymbol{C}^2$. Let (u_i,v_i), $1\leq i\leq k$ be the modification coordinates of $\Xi_i\in X_i$ as in Theorem 4.5. The strict transforms $C_j^{(i)}$, $j=i,\cdots,k$ give germs of irreducible curves at Ξ_i and $C_i^{(i)}$ is given by $\{v_i=0\}$. The sphere S_r is isotopic to $|u_i|=r'$ for some $r'>0$ in a neighborhood of Ξ_i. For $\varepsilon>0$, let $T_{i,\varepsilon,r}=\{(u_i,v_i)\in S_r;|v_i|\leq\varepsilon\}$. For sufficiently small r and ε, $T_{i,\varepsilon,r}$ is diffeomorphic to the product $K_i\times D_\varepsilon$ where $K_i:=C_i^{(i)}\cap S_r$ and $D_\varepsilon:=\{\eta\in\boldsymbol{C};|\eta|\leq\varepsilon\}$ and $T_{i,\varepsilon,r}$ gives a canonical tubular neighbourhood of K_i. We can take positive numbers $r_i,\varepsilon_i>0$ for $i=1,\cdots,k$ so that $C_j^{(i)}$ intersect transversely with S_α for any $\alpha\leq r_i$ and $j=i,\cdots,k$ and $C_j^{(i)}\cap S_\alpha\subset T_{i,\varepsilon_i/2,\alpha}$. By the inductive argument, we can assume also that $T_{i,\varepsilon,\alpha}\cap\partial T_{i-1,\varepsilon_{i-1}/2,\alpha}=\emptyset$. Now taking $r_0=\min(r_1,\cdots,r_k)$, we get

Theorem 4.5.5. *Let* $f(x,y)$ *be as in Theorem* 4.5. *Then for every* $0<r\leq r_0$, *the following properties hold.*

(1) *$T_{i,\varepsilon_i/2,r} \supset T_{i+1,\varepsilon_{i+1},r}$ for $i=1,\cdots,k-1$ where,*
(2) *the boundary of $T_{i,\varepsilon_i,r}$ is a torus transversal to the Milnor fibration of the singularity of f,*
(3) *the restrictions of $f/|f|$ to $S_r - T_{i,\varepsilon_i,r}$ and $T_{i,\varepsilon_i,r} - T_{i+1,\varepsilon_{i+1},r}$, $i=1,\cdots,k-1$ are locally trivial fibrations over the circle and*
(4) *the monodromy diffeomorphism of the restriction to the differences $S_r - T_{1,\varepsilon_1,r}$ and $T_{i,\varepsilon_i,r} - T_{i+1,\varepsilon_{i+1},r}$, $i=1,\cdots,k-1$ can be chosen to be of finite order.*

B. Intersections of other Tschirnhausen approximate polynomials. Let, as before, $H_a(x,y)$ be the n/a-th Tschirnhausen apporoximate polynomial and $D_a = \{H_a(x,y)=0\}$.

Theorem 4.6. *If $a \mid n$, $a_1 \cdots a_s \mid a$, $a_1 \cdots a_{s+1} \nmid a$ and $a \neq a_1 \cdots a_s$, then D_a and C have the same toric tangential direction of depth s and $I(D_a, C_i; O) = \Sigma_{j=1}^{\alpha+1} a_j b_j A_{j+1}^2 / (A_{i+1} n/a)$ where $\alpha = \min(s,i)$.*

Proof. Recall that $\Phi_s^* f(u_s, v_s) = u_s^{m_s(f)}(v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}})^{A_{s+2}} + (\text{higher terms})$. We consider the face function of the pull-back $\Psi_s^* H_a$ and put $H'_a(u_s, v_s) := (\Psi_s^* H_a)_{P_{s+1}}(u_s, v_s) / u_s^{m_s(H_a)}$. We have seen in the inductive construction of the Tschirnhausen tower that D_a has the same toric tangential direction at least of depth s with C. We have shown in Assertion 4.4.6 that $H'_a(u_s, v_s)$ is the n/a-th Tschirnhausen approximate polynomial of $(v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}})^{A_{s+2}}$. Now the main step of the proof is the following.

Lemma 4.6.1. *The constant term of the polynomial $H'_a(u_s, v_s) \in \mathbf{C}\{u_s\}[v_s]$ is zero and $v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}}$ does not divide $H'_a(u_s, v_s)$.*

Proof. Put $\beta_j = a / a_1 \cdots a_j$. The point is that $\beta_{s+1} := A_{s+2} / (n/a)$ is not an integer. As $H'_a(u_s, v_s)$ is the n/a-th Tschirnhausen approximate polynomial of $(v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}})^{A_{s+2}}$, we have $H'_a(u_s, v_s) = v_s^{\beta_s} \mathrm{Trunc}^{([\beta_{s+1}])}(1 + \xi_{s+1} u_s^{b_{s+1}} v_s^{-a_{s+1}})^{\beta_{s+1}} = v_s^{\beta_s} \Sigma_{j=0}^{([\beta_{s+1}])} \binom{\beta^{s+1}}{j} (\xi_{s+1} u_s^{b_{s+1}} v_s^{-a_{s+1}})^j$ by Lemma 2.3. Thus $H'_a(u_s, v_s)$ does not have a constant term as a polynomial of v_s. If $v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}}$ divides $H'_a(u_s, v_s)$, we will get a contradiction. In fact, the polynomial

$$H''_a(u_s, v_s) := (v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}})^{-1} H'_a(u_s, v_s) \tag{4.6.4}$$

is the n/a-th Tschirnhausen approximate polynomial of $(v_s^{a_{s+1}} + \xi_{s+1} u_s^{b_{s+1}})^{A_{s+2} - n/a}$. By the generalized binomial formula again, we have

$$H''_a(u_s, v_s) = v_s^{\beta_s - a_{s+1}} \sum_{j=0}^{[\beta_{s+1}]-1} \binom{\beta_{s+1} - 1}{j} (\xi_{s+1} u_s^{b_{s+1}} v_s^{-a_{s+1}})^j.$$

Comparing the coefficients of $v_{s+1}^{\beta_s - [\beta_{s+1}] a_{s+1}}$ in (4.6.4), we get: $\binom{\beta_{s+1}}{[\beta_{s+1}]} = \binom{\beta_{s+1} - 1}{[\beta_{s+1}] - 1}$, which is a contradiction as $\beta_{s+1} \neq [\beta_{s+1}]$. Q.E.D.

Now by the Lemma the curve D_a has the same toric tangential direction of depth s but not of depth $s+1$ with C. In particular, $D_a^{(s+1)} \cap C^{(s+1)} = \emptyset$. The main problem in proving the assertion about the intersection multiplicity is that D_a may be neither irreducible nor reduced. See Example 4.9. Let $D_{a,1}, \cdots, D_{a,\ell}$ be the irreducible components. Let $k_a(u_s, v_s)$ and $k_{a,j}(u_s, v_s)$, $j=1,\cdots,\ell$, be the defining functions of the strict transforms $D_a^{(s)}$ and $D_{a,j}^{(s)}$ for $j=1,\cdots,\ell$. Then we can write $k_a(u_s, v_s) = v_s^{\beta_s} + \Sigma_{t=1}^{\beta_s} \gamma_t(u_s) v_s^{\beta_s - t}$, $\gamma_t(u_s) \in \mathbf{C}\{u_s\}$ and

$$(4.6.5) \qquad k_{a,j}(u_s, v_s) = \begin{cases} (v_s^{a_{s+1,j}} + \xi_{a,j} u_s^{b_{s+1,j}})^{A_{s+2,j}} + (\text{higher terms}), & b_{s+1,j} \neq 0 \\ v_s^{A_{s+2,j}} U_j, \qquad b_{s+1,j} = 0, & a_{s+1,j} = 1 \end{cases}$$

where $\gcd(a_{s+1,j}, b_{s+1,j}) = 1$ and U_j is a unit. They satisfy:

$$(4.6.6) \qquad \beta_s = \sum_{j=1}^{\ell} a_{s+1,j} A_{s+2,j}$$

Recall that the weight vector of the unique face of $\Gamma(k_{a,j}; (u_s, v_s))$ corresponds to the weight vector of a face of $\Gamma(k_a; (u_s, v_s))$. By Assertion 4.4.6, the Newton boundary $\Gamma(k_a; (u_s, v_s))$ starts with the face (possibly a vertex) of the weight vector P_{s+1} and any other face has a milder slope. Therefore we have $b_{s+1,j}/a_{s+1,j} \geq b_{s+1}/a_{s+1}$ if $b_{s+1,j} \neq 0$. Now we apply Lemma 3.4.2 to compute the intersection numbers. For $i \leq s$, we have $I(D_a, C_i) = \Sigma_{j=1}^{i+1} a_j b_j A_{j+1}^2 / (A_{i+1} n/a)$, $i \leq s$ and for $i \geq s$, with $P_{s+1,t} := {}^t(a_{s+1,t}, b_{s+1,t})$ we have

$$\begin{aligned} I(D_a, C_i) &= \sum_{j=1}^{s} a_j b_j A_{j+1}^2 / (A_{i+1} n/a) + \sum_{t=1}^{\ell} I(P_{s+1}, P_{t,s+1}) A_{s+2,t} A_{s+2} / A_{i+1} \\ &= \sum_{j=1}^{s} a_j b_j A_{j+1}^2 / (A_{i+1} n/a) + \sum_{t=1}^{\ell} b_{s+1} \beta_{s,t} A_{s+2,t} A_{s+2} / A_{i+1} \qquad \text{by (4.6.6)} \\ &= \sum_{j=1}^{s+1} a_j b_j A_{j+1}^2 / (A_{i+1} n/a), \qquad i > s \qquad \text{by (4.6.5).} \end{aligned}$$

where $I(P_{s+1}, P_{t,s+1})$ is defined as in Lemma 3.4.2.

C. Relations with other toric towers. Consider two toric resolution towers:

$$\mathscr{T} = \{X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 = \mathbf{C}^2\}$$

$$\mathscr{Q} = \{Y_s \xrightarrow{q_s} Y_{s-1} \to \cdots \to Y_1 \xrightarrow{q_1} Y_0 = \mathbf{C}^2\}$$

where $\mathscr{T}$ is a Tschirnhausen tower of resolution with the weight vectors $P_i = {}^t(a_i, b_i)$,

$i=1,\cdots,k$ and $n=a_1\cdots a_k$, $a_i\geq 2$, $i=1,\cdots,k$ as in Theorem 4.5. Let $A_i=a_ia_{i+1}\cdots a_s$ and let $h_i(x,y)$ be the A_{i+1}-th Tschirnhausen approximate polynomial of $f(x,y)$ and let C_i be the corresponding Tschirnhausen curve for $i=1,\cdots,k$ as before. Let $Q_i={}^t(\alpha_i,\beta_i)$, $i=1,\cdots,s$ be the corresponding weight vectors of $\mathcal{Q}$ with $n=\alpha_1\cdots\alpha_s$. We assume that $\alpha_i\geq 2$, $i=1,\cdots,s$ and $Q_1=P_1$. We call such a toric tower $\mathcal{Q}$ a *Tschirnhausen-good resolution tower.* A Tschirnhausen resolution tower is a Tschirnhausen-good resolution tower by Theorem 4.5. Now Theorem 4.5 can be generalized as follows.

Theorem 4.7. *Let $f(x,y)$ be as in Theorem 4.5. Let $\mathcal{T}$ and $\mathcal{Q}$ be as above. Assume that $q_{i+1}: Y_{i+1}\to Y_i$ is a toric modification centered at $\Theta_i\in E_i':=\hat{E}(Q_i)$ with the modification local coordinate system (w_i,z_i), so that $\{w_i=0\}$ defines the divisor E_i'. Put $\Psi_i=q_1\circ\cdots\circ q_i: Y_i\to Y_0$. Then we have the following properties.*
(1)) (Uniqueness of the weight vectors) $s=k$ and $Q_i=P_i$ for $i=1,\cdots,k$.
(2) For each $i=1,\cdots,s$, $\Psi_i: Y_i\to Y_0$ gives a good resolution of C_i and the pull backs of the polynomials are written (up to a non-zero constant factor) as

$$(4.7.1)\qquad \Psi_i^*h_\ell(w_i,z_i)=\begin{cases} w_i^{m_i'(h_\ell)}(z_i^{a_{i+1}}+\theta_{i+1}w_i^{b_{i+1}})^{A_{i+2}/A_{\ell+1}}+(\textit{higher terms}), & i<\ell\\ w_i^{m_i'(h_i)}z_i', & i=\ell\end{cases}$$

where z_i' is either z_iU_i with a unit U_i or $c_i((z_i+\eta_iw_i^{\gamma_i})+(\textit{higher terms}))$ with $c_i,\eta_i\in\mathbf{C}^*$ for some integer γ_i, $\gamma_i>b_{i+1}/a_{i+1}$. In particular, putting $\ell=s$, we have

$$(4.7.2)\qquad \Psi_i^*f(w_i,z_i)=\begin{cases} (z_i^{a_{i+1}}+\theta_{i+1}w_i^{b_{i+1}})^{A_{i+2}}+(\textit{higher terms}), & i<s\\ w_s^{m_s'(f)}z_s', & i=s\end{cases}$$

where the multiplicities $m_i'(h_\ell')$ and $m_i'(f)$ of the pull backs $\Psi_i{}^*h_l$ and $\Psi_i{}^*f$ on E_i' satisfy the same inductive equalities:

$$(4.7.3)\qquad \begin{cases} m_i'(h_s)=m_i'(f)/A_{s+1}, & i\leq s\\ m_1'(f)=a_1b_1A_2 & m_i'(f)=a_im_{i-1}'(f)+a_ib_iA_{i+1}\end{cases}$$

Thus we have also the uniqueness of the multiplities: $m_i'(h_s)=m_i(h_s)$ and $m_i'(f)=m_i(f)$.

Proof. We consider the tower $\mathcal{Q}$. Let $\tilde{\alpha}_1=\min(\alpha_1,\beta_1)$ and $\tilde{\beta}_1=\max(\alpha_1,\beta_1)$ and let $n_1=\tilde{\alpha}_1$, $m_1=\tilde{\beta}_1$ and $n_i=\alpha_i$, $m_i=\beta_i+\beta_{i-1}\alpha_i+\cdots+\beta_2\alpha_3\cdots\alpha_i+\tilde{\beta}_1\alpha_2\cdots\alpha_i$ for $i\geq 2$. Then we have shown in Corollary 6.8 of [29] that the Puiseux pairs of C is given by $\{n_i,m_i);\ i=1,\cdots,s\}$, $(\beta_1>1)$ or $\{n_i,m_i);\ i=2,\cdots,s\}$, $(\beta_1=1)$. The same assertion is true for the Tschirnhausen tower $\mathcal{T}$. By the assumption $Q_1=P_1$ and by the uniqueness of the Puiseux pairs, we conclude that $s=k$ and $Q_i=P_i$. The expression (4.7.1) for $\ell>i$ follows easily by the induction on i. In fact, we know that C_ℓ is irreducible and $I(C_\ell,C;O)=\Sigma_{s=1}^{\ell+1}a_sb_sA_{s+1}^2/A_{\ell+1}$. So by Lemma 3.4.2, C_ℓ can not be separated from C on Y_i, $i<\ell$. Thus we have the expression

(4.7.1). As $\Psi_{i-1}^*(h_i(w_i,z_i))$ is non-degenerate, we can write

$$\Psi_i{}^*h_i(w_i,z_i)=\begin{cases} c_i((z_i+\eta_i w_i^{\gamma_i})+(\text{higher terms}), & c_i\eta_i\in \boldsymbol{C}^* \quad \text{or} \\ z_iU_i, \quad U_i:\text{a unit} \end{cases}$$

In the first case, with the formula of Theorem 4.5 we get $I(C_i^{(i)},C^{(i)};\Theta_i)=b_{i+1}A_{i+2}$. So, $\gamma_i\geq b_{i+1}/a_{i+1}$. As b_{i+1}/a_{i+1} is not an integer, we have $\gamma_i>b_{i+1}/a_{i+1}$. Q.E.D.

REMARK 4.8. Theorem 4.7 can be proved without using the uniqueness of the Puiseux pairs by comparison stage by stage of the formulae for the intersections for the two towers.

EXAMPLE 4.9. Put $f(x,y)=(y^4+x^3)^6+x^{17}y^3$. The first toric modification $p_1:X_1\to X_0$ can be defined by the subdivision

$$\Sigma_0^*=\{P_{0,0},\cdots,P_{0,5}\}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

with weight vector $P_1=P_{0,3}$. Let $\sigma_3=\text{Cone}(P_{0,3},P_{0,4})$. On the chart $\boldsymbol{C}_{\sigma_3}^2$, we take $u_1=x_{\sigma_3}$ and $v_1=y_{\sigma_3}+1$. Then $C^{(1)}$ is defined by $\{(u_1,v_1)\in W_1\,;v_1^6+u_1^5+(\text{higher terms})=0\}$. Thus we need one more toric modification $p_2:X_2\to X_1$ and we choose the modification with respect to

$$\Sigma_1^*=\{P_{1,0};,\cdots,P_{1,7}\}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}5\\4\end{pmatrix},\begin{pmatrix}6\\5\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

with weight vector $P_2=P_{1,5}$. The weight vectors of the tower are $P_1={}^t(4,3)$ and $P_2={}^t(6,5)$. By computation, we have $n=24$ and the various Tschirnhausen approximate polynomials are: $H_2(x,y)=y^2$, $H_3(x,y)=y^3$, $H_4(x,y)=h_1(x,y)=y^4+x^3$, $H_6(x,y)=y^6+3/2x^3y^2$, $H_s(x,y)=(y^4+x^3)^2$ and $H_{12}(x,y)=(y^4+x^3)^3$. The intersection multiplicities are given by $I(D_a,C;O)=36,54,77,108,154,231$ respectively for $a=2,3,4,6,8,12$. This example shows that D_a which is different from C_i, $1\leq i\leq k$ is not necessarily irreducible or reduced. The zeta function and the Milnor number are given by Theorem 5.1 in §5: $\zeta(t)=(1-t^{72})(1-t^{462})/(1-t^{24})(1-t^{18})(1-t^{77})$ and $\mu(f)=416$.

5. The zeta function of the monodromy. Let $f(x,y)$ be a monic polynomial in y of degree n and irreducible at the origin. Let $\mathscr{T}=\{X_k\xrightarrow{p_k}X_{k-1}\to\cdots\to X_1\xrightarrow{p_1}X_0\}$ be a Tschirnhausen-good toric resolution tower with the weight vectors $\{P_i={}^t(a_i,b_i);\ i=1,\cdots,k\}$. We will read off the zeta function of the monodromy and Milnor number from the data of the Tschirnhausen-good resolution tower.

Let Σ_i^* be the regular simplicial cone subdivision which is used to construct the modification $p_{i+1}: X_{i+1} \to X_i$ and let $\{P_{i,0}, P_{i,1}, \cdots, P_{i,r_i}, P_{i,r_i+1}\}$ be the vertices of Σ_i^* so that $P_{i,0} = {}^t(1,0)$ and $P_{i,r_i+1} = {}^t(0,1)$. Let $P_{i,j} = {}^t(a_{i,j}, b_{i,j})$. We assume that $P_{i+1} = P_{i,n_i}$ for $i = 0, \cdots, k-1$. Note that, as $\det(P_{i,0}, P_{i,1}) = \det(P_{i,r_i}, P_{i,r_i+1}) = 1$, $P_{i,1}$ and P_{i,r_i} have the forms $P_{i,1} = {}^t(a_{i,1}, 1)$ and $P_{i,r_i} = {}^t(1, b_{i,r_i})$ respectively. This implies that $n_i < r_i$. The configuration of the exceptional divisors $\{\hat{E}(P_{i,j}); j = 1, \cdots, r_i\}$ is a line configuration and $\hat{E}(P_{i,0})$ is nothing but $\hat{E}(P_{i-1,n_{i-1}})$. Thus the exceptional divisors of the resolution $\Phi_k: X_k \to X_0$ is the union of the strict transforms $\{\hat{E}(P_{i,j}); 0 \le i \le k-1, 1 \le j \le r_i\}$. Let $m_{i,j}$ be the multiplicity of the pull-back $\Phi_{i+1}^* f$ along $\hat{E}(P_{i,j})$ and let $\delta_{i,j}$ be the number of irreducible components of the divisor $(\Phi_k^* f)$ which intersect with $\hat{E}(P_{i,j})$. By Theorem 3 of [2], the zeta function $\zeta(t; O)$ of the monodromy of $f(x,y)$ is determined by those $\hat{E}(P_{i,j})$ *with* $\delta_{i,j} \ne 2$. As we have seen in §3, $m_{i,j} = d(P_{i,j}; \Phi_i^* f)$ and

$$\delta_{i,j} = \begin{cases} 3 & j = n_i \\ 1 & j = r_i \quad i \ge 1, \\ 2 & \text{otherwise} \end{cases} \qquad \delta_{0,j} = \begin{cases} 3 & j = n_1 \\ 1 & j = 1 \ \text{ or } \ r_1, \quad i = 0 \\ 2 & \text{otherwise} \end{cases}$$

If $n_0 = 1$, we subdivide $\mathrm{Cone}(P_{0,0}, P_{0,1})$ so that we can assume that $n_0 > 1$. Note that $\delta_{k-1,n_{k-1}} = 3$ as $\hat{E}(P_{k-1,n_{k-1}}) = E(P_k)$ and it intersects with $C^{(k)}$. Recall that the multiplicity m_{i,n_i} is given by $m_{i,n_i} = d(P_{i+1}; \Phi_i^* f) = m_{i+1}(f) = a_{i+1} m_i(f) + a_{i+1} b_{i+1} A_{i+2}$ in the same notation as in §4. Thus we need determine $m_{0,1}$, m_{i,r_i} for $i = 1, \cdots, k$. To determine m_{i,r_i}, we consider the expression by (4.7.2): $\Phi_i^* f(u_i, v_i) = u_i^{m_i(f)} (v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A_{i+2}} + (\text{higher terms})$ for $i < k$. As Σ_i^* is assumed to be admissible for $\Phi_i^* f$, we know that $(m_i(f) + b_{i+1} A_{i+2}, 0) \in \Delta(P_{i,r_i}; \Phi_i^* f)$. This observation and the expression $P_{i,r_i} = {}^t(1, b_{i,r_i})$ implies that $m_{i,r_i} = m_i(f) + b_{i+1} A_{i+2} = m_{i+1}(f) / a_{i+1}$. Finally as $(0,n) \in \Delta(P_{0,1}; f)$, we have that $m_{0,1} = A_1$ by a similar argument as above. Thus applying Theorem 2 of [2], we obtain the first part of

Theorem 5.1. *The zeta function and Milnor number of $f(x,y)$ are*:

$$\zeta(t; O) = \frac{1}{(1 - t^{A_1})} \prod_{i=1}^{k} \frac{(1 - t^{m_i(f)})}{(1 - t^{m_i(f)/a_i})}, \quad \mu(f; O) = 1 - A_1 + \sum_{i=1}^{k} (A_i - 1) b_i A_{i+1}$$

Proof. By the equality $-1 + \mu(f; O) = \deg \zeta(t; O)$, we have

$$\begin{aligned} -1 + \mu(f; O) &= -A_1 + \sum_{i=1}^{k} \left(1 - \frac{1}{a_i}\right) m_i(f) \\ &= -A_1 + \sum_{i=1}^{k} \left(1 - \frac{1}{a_i}\right) \left(\sum_{\ell=1}^{i} a_\ell b_\ell A_{\ell+1}^2\right) / A_{i+1} \end{aligned}$$

$$= -A_1 + \sum_{\ell=1}^{k} a_\ell b_\ell A_{\ell+1}^2 \sum_{i=\ell}^{k} \left(1 - \frac{1}{a_i}\right) / A_{i+1}$$

$$= -A_1 + \sum_{\ell=1}^{k} (A_\ell - 1) b_\ell A_{\ell+1}. \qquad \text{Q.E.D.}$$

6. Conditions implying equi-singularity. Let $f_t(x,y) = f(x,y,t) \in C\{x,t\}[y]$ be an analytic family of monic polynomial in $C\{x\}[y]$ of degree n in y defined for t in an open connected neighborhood U of the origin in C. Let $C(t) := \{f_t(x,y) = 0\}$, $t \in U$, be the corresponding family of germs of curves at the origin. We assume that $C(0)$ is irreducible and reduced at the origin and that $f_t(x,y)$ has an initial expansion

$$f_t(x,y) = (y^{a_1} + \xi_1 x^{b_1})^{A_2} + (\text{higher terms}) \tag{6.1.1}$$

with $\xi_1 \neq 0$ independent of t and $a_1 \geq 2$. Let $X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 = C^2$ be the Tschirnhausen approximate resolution tower of $(C(0), O)$ with the weight vectors $\{P_i = {}^t(a_i, b_i);\ i = 1, \cdots, k\}$. We assume further that the A_{i+1}-th Tschirnhausen approximate polynomials $h_i(x,y,t)$ of $f_t(x,y)$ for $i = 1, \cdots, k-1$ are independent of the parameter t. Note that this is the case if the coefficients of y^j do not depend on t for any $j \geq n - a_1 \cdots a_{k-1}$. Consider the germs of curves $C_i := \{h_i(x,y) := h_i(x,y,t) = 0\}$, $i = 1, \cdots, k-1$. Finally we assume that the local intersection multiplicities satisfy the ineguailities:

$$a_k I(C_{k-1}, C(t); O) \leq I(C(0), C(s); O) < +\infty, \quad \text{for any } t, s, \quad \text{with } s \neq 0. \tag{6.1.2}$$

Theorem 6.2. *Under the above assumptions for the family $f_t(x,y)$, the germs $C(t)$, $t \in U$, are irreducible at the origin and have the same toric tangential direction of depth k', $k' \geq k-1$. The family of germs of plane curves $\{C(t), O);\ t \in U\}$ is an equi-singular family and $\Phi_k : X_k \to X_0$ gives a simultaneous resolution for the family $\{C(t);\ t \in U\}$ where $\Phi_k = p_1 \circ \cdots \circ p_k$. In particular, the Milnor number $\mu(f_t; O)$ is constant and coincides with $\mu(f_0; O)$. Moreover, if equality holds in $a_k I(C_{k-1}, C(t); O) \leq I(C(0), C(s); O)$ for any t,s, with $s \neq 0$, the germs $C(t)$, $t \in U$ do not have the same toric tangential direction of depth k.*

Proof. We fix $\tau \neq 0$, $\tau \in U$. We first assume that $C(\tau)$ is irreducible. The irreducibility will be proved later. Assume that $C(\tau)$ has the same toric tangential direction with $C(0)$ of depth θ, $\theta \leq k$. Then we can write $C^{(j)}(\tau)$ as

$$\begin{cases} C^{(j)}(\tau) = \{(u_j, v_j) \in W_j;\ f_\tau^{(j)}(u_j, v_j) = 0\} \\ f_\tau^{(j)}(u_j, v_j) = (v_j^{a'_{j+1}} + \xi'_{j+1} u_j^{b'_{j+1}})^{A'_{j+2}} + (\text{higher terms}) \end{cases} \tag{6.2.1}$$

where $A'_{j+1}:=a'_{j+1}\cdots a'_{\theta+1}A'_{\theta+2}$ for $j\leq\theta$. Let $P'_{\theta+1}:={}^t(a'_{\theta+1},b'_{\theta+1})$. $P'_{\theta+1}$ is a primitive weight vector and if $P'_{\theta+1}=P_{\theta+1}$, we must have $\xi'_{\theta+1}\neq\xi_{\theta+1}$ by the assumption. Comparing (6.1.1) and (6.2.1) and by the assumption, we have $\xi'_j=\xi_j$, $a'_j=a_j$, $b'_j=b_j$ and $A'_{j+1}=A_{j+1}$ for $j\leq\theta$. Assume first that $\theta\leq k-1$. By Lemma 3.4.2, the local intersection multiplicity is given by

$$(6.2.2)\qquad I(C(\tau),C(0);O)=\sum_{i=1}^{\theta}a_ib_iA_{i+1}^2+I(P_{\theta+1},P'_{\theta+1})A_{\theta+2}A'_{\theta+2}$$
$$\leq\sum_{i=1}^{\theta}b_iA_{i+1}A'_i+b_{\theta+1}A_{\theta+2}A'_{\theta+1}=\sum_{i=1}^{\theta+1}b_ia_iA_{i+1}^2$$

where equality holds if and only if $a'_{\theta+1}b_{\theta+1}\leq a_{\theta+1}b'_{\theta+1}$ or $b'_{\theta+1}=0$. On the other hand, by Theorem 4.5 we have the equality: $a_kI(C_{k-1},C(0);O)=\Sigma_{i=1}^k a_ib_iA_{i+1}^2$. Thus (6.2.2) and the assumption (6.1.2) implies that we must have $\theta=k-1$ and $a'_kb_k\leq a_kb'_k$ or $b'_k=0$ and $I(C(\tau),C(0);O)=\Sigma_{i=1}^k a_ib_iA_{i+1}^2$. We assert furthermore

$$(6.2.3)\qquad b'_k\neq 0,\qquad a'_kb_k=a_kb'_k.$$

In fact, assume first that $b'_k=0$. Then $C(\tau)=a_kC_{k-1}$ and $C(\tau)$ is not reduced. This is a contradiction to the assumption $\dim_{\mathbf{C}}\mathbf{C}\{x,y\}/(f_\tau,h_{k-1})<\infty$. Assume that $b'_k\neq 0$ and $a'_kb_k<a_kb'_k$. Then we get a contradiction:

$$\begin{aligned}a_kI(C(\tau),C_{k-1};O)&=\sum_{i=1}^{k-1}a_ib_iA_{i+1}^2+a_kI(C_{k-1}^{(k-1)},C^{(k-1)}(\tau);O)\\&=\sum_{i=1}^{k-1}a_ib_iA_{i+1}^2+a_kb'_kA'_{k+1}\\&>\sum_{i=1}^{k-1}a_ib_iA_{i+1}^2+a'_kb_kA'_{k+1}=\sum_{i=1}^{k}a_ib_iA_{i+1}^2=I(C(\tau),C(0);O).\end{aligned}$$

Thus we have proved (6.2.3). As $\gcd(a_k,b_k)=\gcd(a'_k,b'_k)=1$, (6.2.3) implies $P'_k=P_k$ and $A'_{k+1}=1$. This also shows that $C^{(k)}(\tau)$ is smooth. Thus under the assumption that $C(\tau)$ is irreducible at the origin, we have proved that $C(\tau)$ is reduced and $\theta\geq k-1$, $P'_k=P_k$. This implies that $\mu(f_\tau;O)=\mu(f_0;O)$ by applying Theorem 4.5 to $C(\tau)$. Note that $\Phi_k\colon X_k\to X_0$ gives a simultaneous resolution of the family $\{C(\tau);\tau\in U\}$. If $\theta=k$, the assertion is obvious and $C^{(k)}(\tau)$ intersects with $C^{(k)}(0)$ at Ξ_k and therefore $I(C(0),C(\tau);O)>\Sigma_{i=1}^k a_ib_iA_{i+1}^2$. This implies that the strict inequality in (6.1.2) must hold.

Irreducibility of $C(\tau)$. Now we prove that $C(\tau)$ is irreducible for any τ. Fix a τ and assume that $C(\tau)$ has s irreducible components at the origin $s\geq 2$. Let $C(\tau;1),\cdots,C(\tau;s)$ be the irreducible components and let $C^{(j)}(\tau;1),\cdots,C^{(j)}(\tau;s)$ be their strict transforms on X_j. We assume that $C(\tau;i)$ has the same toric tangential direction of depth θ_i with $C(0)$. Then we can write

$$\begin{cases} C^{(j)}(\tau;i)=\{(u_j,v_j)\in W_j; f_{\tau,i}^{(j)}(u_j,v_j)=0\} \\ f_{\tau,i}^{(j)}(u_j,v_j)=(v_j^{a_{i,j+1}}+\xi_{i,j+1}u_j^{b_{i,j+1}})^{A_{i,j+2}}+(\text{higher terms}), \quad j\le\theta_i \end{cases}$$

where $A_{i,j}=a_{i,j}\cdots a_{i,\theta_i+1}A_{i,\theta_i+2}$ for $j\le\theta_i$. By the assumption, we have $a_{i,j}=a_j$, $b_{i,j}=b_j$, $\xi_{i,j}=\xi_j$ for $j\le\theta_i$. Put $\theta_0=\min(\theta_1,\cdots,\theta_s)$. Then $C(\tau)$ has the same toric tangential direction of depth θ_0 with $C(0)$ and we can write $C^{(j)}(\tau)$ as:

$$\text{(6.2.4)} \qquad \begin{cases} C^{(j)}(\tau)=\{(u_j,v_j)\in W_j; f_\tau^{(j)}(u_j,v_j)=0\} \\ f_\tau^{(j)}(u_j,v_j)=(v_j^{a'_{j+1}}+\xi'_{j+1}u_j^{b'_{j+1}})^{A'_{j+2}}+(\text{higher terms}), \quad j\le\theta_0 \end{cases}$$

where $A'_{i+1}:=a'_{i+1}\cdots a'_{\theta+1}A'_{\theta+2}$ and by the assumption, we have $a'_i=a_i$, $b'_i=b_i$, $\xi'_i=\xi_i$, $A'_{i+1}=A_{i+1}$ for $i\le\theta_0$. Comparing the defining equations of $C^{(i)}(\tau;1),\cdots,$ $C^{(i)}(\tau;s)$ and $C^{(i)}(\tau)$, we must have

$$f_\tau^{(i)}(x,y)=\prod_{i=1}^{s} f_{\tau,i}^{(i)}(x,y), \qquad A_{1,i}+\cdots+A_{s,i}=A_i, \qquad i\le\theta_0$$

As $A_{i,1}=a_{i,1}\cdots a_{i,\theta_i}A_{i,\theta_i+1}$ and $s\ge 2$, this implies that

$$\text{(6.2.5)} \qquad \theta_i\le k-1 \quad\text{and}\quad A_{i,\theta_i+1}<A_{\theta_i+1}.$$

We use the following notations for simplicity. $\bar{A}_{j,i}:=A_{j,i}$ for $i\le\theta_j+1$ and $\bar{A}_{j,i}:=0$ for $i>\theta_j+1$. Then by (6.2.4) and (6.2.5) we get

$$\text{(6.2.6)} \qquad \sum_{j=1}^{s}\bar{A}_{j,i}=A_i, \quad i\le\theta_0+1 \quad\text{and}\quad \sum_{j=1}^{s}\bar{A}_{j,i}<A_i, \quad i>\theta_0+1$$

By Lemma 3.4.2, we have with $P_{j,\theta_j+1}:=(a_{j,\theta_j+1},b_{j,\theta_j+1})$ that

$$\begin{aligned} I(C(\tau;j),C(0);O)&=\sum_{i=1}^{\theta_j}a_ib_iA_{i+1}A_{j,i+1}+I(P_{\theta_j+1},P_{j,\theta_j+1})A_{\theta_j+2}A_{j,\theta_j+2} \\ &\le\sum_{i=1}^{\theta_j}a_ib_iA_{i+1}A_{j,i+1}+b_{\theta_j+1},A_{\theta_j+2}A_{j,\theta_j+1}=\sum_{i=1}^{k}b_iA_{i+1}\bar{A}_{j,i} \end{aligned}$$

Adding these inequalities for $j=1,\cdots,s$ and using (6.2.5), we get

$$I(C(\tau),C(0);O)\le\sum_{i=1}^{k}b_iA_{i+1}\sum_{j=1}^{s}\bar{A}_{j,i}\le\sum_{i=1}^{k}b_iA_{i+1}A_i,$$

where the right side is equal to $a_kI(C_{k-1},C(0);O)$ by Theorem 4.5. With the assumption (6.1.2), we get $I(C(\tau),C(0);O)=\sum_{i=1}^{k}b_iA_{i+1}A_i$, which is equivalent to the following two equalities:

$$(6.2.7) \qquad I(C(\tau;j),C(0);O)=\sum_{i=1}^{k} b_i A_{i+1}\bar{A}_{j,i}, \qquad j=1,\cdots,s$$

$$(6.2.8) \qquad \sum_{j=1}^{s} \bar{A}_{j,i}=A_i, \quad i=1,\cdots,k$$

By (6.2.6) and (6.2.5), (6.2.8) is equivalent to $\theta_0=k-1$. Therefore (6.2.) and (6.2.8) holds if only if $\theta_i=k-1$ for $i=1,\cdots,s$ and $a_k b_{j,k}\geq a_{j,k}b_k$. But, assuming that $a_k b_{j_0,k}\geq a_{j_0,k}b_k$ for some j_0, we obtain the inequality:

$$a_k I(C_{k-1},C(\tau);O)=\sum_{i=1}^{k-1} b_i A_{i+1}A_i+a_k\sum_{j=1}^{s} b_{j,k}A_{j,k+1}$$
$$>\sum_{i=1}^{k-1} b_i A_{i+1}A_i+\sum_{j=1}^{s} a_{j,k}b_k A_{j,k+1}=\sum_{i=1}^{k} b_i A_{i+1}A_i=I(C(0),C(\tau);O),$$

which contradicts $a_k I(C_{k-1},C(\tau);O)>I(C(0),C(\tau);O)$. So, we must have $a_k b_{j,k}=a_{j,k}b_k$, $j=1,\cdots,s$. As $\gcd(a_k,b_k)=\gcd(a_{j,k},b_{j,k})=1$, this is possible if and only if $a_{j,k}=a_k$ and $b_{j,k}=a_k$. Again this gives a contradiction: $A_k=A_{1,k}+\cdots+A_{s,k}=sA_k$. This proves the irreducibility of $C(\tau)$ and the proof of Theorem 6.2 is now completed.

7. An example of an equi-singular family. We study a typical equi-singular family $f_t(x,y):=f(x,y)+tx^m$, where $f(x,y)$ is a monic polynomial whose Newton diagram $\Delta(f;(x,y))$ is a triangle with the vertices $A=(0,n)$, $B=(b_1A_2,0)$, $C=(m,0)$ with $m>b_1A_2$, having the initial expansion $f(x,y)=(y^{a_1}+\xi_1 x^{b_1})^{A_2}+$(higher terms), $a_1\geq 2$, and defining an irreducible germ of a plane curve $C=\{f(x,y)=0\}$ at the origin. Then the a-th Tschirnhausen approximate polynomial of $f_t(x,y)$ does not depend on t for any $a\,|\,n$ with $1<a$, so we can apply the previous consideration to the family of germs $C(t):=\{(x,y)\in \mathbf{C}^2;f_t(x,y)=0\}$. A similar family is studied by Ephreim [7] using polar invariants. Let $\{P_i={}^t(a_i,b_i);\ i=1,\cdots,k\}$ be the weight vectors of the Tschirnhausen resolution tower. Let h_i be the A_{i+1}-th Tschirnhausen approximate polynomial of $f_t(x,y)$ for $i=1,\cdots,k-1$ and let $C_i=\{(x,y)\in \mathbf{C}^2;$ $h_i(x,y)\}$.

Propositon 7.1. *With the above assumptions and notations, we have $I(C(t),C(s);O)=nm$ for $t\neq s$ and $a_k I(C_{k-1},C(t);O)\leq nm$ for any $t\in \mathbf{C}$.*

Proof. For the proof of the equality, note: $I(C(t),C(s);O)=\dim_{\mathbf{C}}\mathbf{C}\{x,y\}/(f_s,f_t)$ and therefore it is equal to $\dim_{\mathbf{C}}\mathbf{C}\{x,y\}/(f_t,(t-s)x^m)=nm$. To prove the inequality, *we first observe the Newton diagram* $\Delta(h_{k-1})$ is a subset of the triangle Δ' whose vertices are $A'=(0,n/a_k)$, $B'=(b_1A_2/a_k,0)$, $C'=(m/a_k,0)$. Here the Newton diagram $\Delta(h)$ of a polynomial $h(x,y)=\Sigma_{M=(\nu,\mu)}{}^{C}Mx^{\nu}y^{\mu}$ is the convex hull of the lattice point M with $c_M\neq 0$. In the case of $m=n$, the assertion follows from the Bezout theorem in $\boldsymbol{P}^2$: $a_kI(C_{k-1},C(t);O)\leq a_k\bar{C}_{k-1}\cdot\bar{C}(t)=n^2=nm$, where $\bar{C}$ is the projective compactification of $C\subset\boldsymbol{C}^2$ and the right side is the intersection number in $\boldsymbol{P}^2$. In the case $m\neq n$, we need another argument. Choose a small ball B centered at the origin B containing no other intersection than the origin O. Let $f_s'(x,y)=f_s(x,y)+\varepsilon_1$ and $h_{k-1}'(x,y)=h_{k-1}(x,y)+\varepsilon_2$ and let $C(s)'=\{(x,y)\in\boldsymbol{C}^2;f_s'(x,y)=0\}$ and $C_{k-1}'=\{(x,y)\in\boldsymbol{C}^2;h_{k-1}'(x,y)=0\}$. For sufficiently small ε_1, ε_2 the intersection $C(s)'\cap C_{k-1}'$ is a subset of the torus $\boldsymbol{C}^{*2}$ and the number of the points of $C(s)'\cap C_{k-1}'$ in B counted with multiplicity is equal to $I(C_{k-1},C(t);O)$. The Newton diagram $\Delta:=\Delta(f_s';(x,y))$ is the triangle with vertices 0, A and C. The number of intersection points $C(s)'\cdot C_{k-1}'$ in $\boldsymbol{C}^{*2}$ is bounded by the theorem of Bernshtein ([6,30]):

$$C(s)'\cdot C_{k-1}'=2V_2(\Delta(f_s'),\Delta(h_{k-1}'))\leq 2V_2(\Delta,\Delta/a_k)=2\,\mathrm{Vol}(\Delta)/a_k=nm/a_k$$

Here $V_2(\Delta_1,\Delta_2)$ is Minkowski's mixed volume and we have used the monotone increasing property of Minkowski's mixed volume to the inclusion $\Delta(h_{k-1}')\subset\Delta/a_k$. See [6,30]. As $C(s)'\cdot C_{k-1}'\geq I(C_{k-1},C(s))$, the inequality of the proposition follows. Q.E.D.

8. The equi-singularity at infinity and the Abhyankar-Moh-Suzuki theorem. Let $F:\boldsymbol{C}^2\to\boldsymbol{C}$ be a polynomial mapping of degree n. We say that $\tau\in\boldsymbol{C}$ is a *regular value at infinity* if there exits a large number R and a positive number δ so that the restriction $F:E_\infty(R,\delta)\to D_\delta$ is a trivial fibration where

$$D_\delta=\{\eta\in\boldsymbol{C};|\eta-\tau|\leq\delta\},\qquad E_\infty(R,\delta)=\{(x,y);F(x,y)\in D_\delta,\ \sqrt{|x|^2+|y|^2}>R\}$$

Let $C_t=F^{-1}(t)$ and let $\bar{C}_t$ be the projective compactification. The set $\bar{C}_t-C_t=\{\rho_1,\cdots,\rho_\ell\}\subset L_\infty$ does not depend on t. We recall the following result:

Proposition 8.1 ([11]). *A complex number τ is a regular value at infinity if and only if the family of germs of plane curves $\{(\bar{C}_t,\rho_i);t\in\boldsymbol{C}\}$ is topologically stable at $t=\tau$ for any $i=1,\cdots,\ell$.*

We consider hereafter the simplest case that C_0 has one place at infinity, say at $\rho=(1;0;0)$. Namely assume that $\ell=1$ and the germ $(\bar{C}_0,\rho)$ is irreducible. Then $F(x,y)$ is written as

(8.1.1) $$F(x,y)=(y^{a_1}+\xi_1x^{c_1})^{A_2}+(\text{lower terms}),\qquad c_1<a_1,\ n=a_1A_2.$$

for some positive integers a_1, c_1 and A_2 with $a_1\leq 2$. As $\bar{C}_0\cap L_\infty=\{\rho\}$ and $\bar{C}_0$

is assumed to be locally irreducible at ρ, the polynomial $F(x,y)$ has only one outside face and its outside face function has only one factor. See [19] or [30]. The standard affine coordinates $u=Z/X$, $v=Y/X$ are centered at ρ and the curve $\bar{C}_t$ is defined by $\{f_t(u,v)=0\}$ where $f_t(u,v)=f(u,v)-tu^n$ and $f(u,v)=F(1/u,v/u)\times u^n$. In this simplest case, we have the initial expansion

$$f(u,v)=(v^{a_1}+\xi_1 u^{b_1})^{A_2}+\text{higher terms}) \tag{8.1.2}$$

here $b_1=a_1-c_1$. Let $C_t^{\infty}=\{(u,v)\in \boldsymbol{C}^2; f(u,v,t):=f(u,v)-tu^n=0\}$. We can apply Theorem 6.2 using Proposition 7.1 to this family and we obtain:

Theorem 8.2. *For the mapping F and the family $\{(C_t^{\infty},O); t\in \boldsymbol{C}\}$ the following holds:*
(1) *The family of germs of germs of plane curves $\{(C_t^{\infty},O); t\in \boldsymbol{C}\}$ is an equi-singular family of irreducible curves and the Tschirnhausen approximate resolution tower of (C_t^{∞},O) resolves simultaneously each curve of the family $\{(C_t^{\infty},O); t\in \boldsymbol{C}\}$.*
(2) *The mapping $F: \boldsymbol{C}^2\to \boldsymbol{C}$ has no critical point at infinity.*

Ephraim has also obtained a similar result about the equi-singularity using a different method [7]. See also Moh [21].

Before giving applications, we will need the following facts. Let $D\subset \boldsymbol{P}^2$ be a projective curve of degree n and let $q_1,\cdots,q_v$ be the singular points of D. Then by Plücker's formula and by Mayer-Vietoris argument, the topological Euler number of $D-\{q_1,\cdots,q_v\}$ is given by

$$\chi(D-\{q_1,\cdots,q_v\})=2-v-(n-1)(n-2)+\sum_{i=1}^{v}\mu(D;q_i).$$

From this equality follow two equivalences. First, $\mu(D;q_1)=(n-1)(n-2)$ if and only if the curve $D-\{q_1\}$ is smooth and homeomorphic to the line $\boldsymbol{C}$. Second, $\mu(D;q_1)=(n-1)(n-2)-2g$ and $v=1$ if and only if the curve $D-\{q_1\}$ is smooth and homeomorphic to a punctured Riemann surface of genus g. As a first application, we will give an elementary proof of:

Theorem 8.3 (Abhyankar-Moh [5], Suzuki [31]). *Let $F(x,y)$ be a polynomial of two variables of degree n and assume that the plane curve $C=\{(x,y)\in \boldsymbol{C}^2; F(x,y)=0\}$ is smooth and homeomorphic to the complex line $\boldsymbol{C}$. Then there exists another polynomial $G(x,y)$ so that (F,G) is an automorphism of $\boldsymbol{C}^2$.*

Proof. The polynomial $F(x,y)$ has one place at infinity, say at $\rho=(1;0;0)$. To prove the theorem by the induction on $n=$ degree $F(x,y)$, it is enough to show that $c_1=1$ in (8.1.1). In fact, if $c_1=1$, we apply the coordinate change $(X,Y)=(y^{a_1}+\xi_1 x,y)$

and achieve $\deg F((X-Y^{a_1})/\xi_1, Y)<n$. Therefore the assertion is proved by the induction on $\deg F$.

Let C_0^∞ and $\bar{C}_0$ be as above. We have $\mu(C_0^\infty; O)=(n-1)(n-2)$ since the smooth part of the curve $\bar{C}_0$ is homeomorphic to the line $\boldsymbol{C}$. Let us consider the Tschirnhausen approximate resolution tower of $(C_0^\infty, O): \mathscr{T}=\{X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0=\boldsymbol{C}^2\}$ and let $P_i={}^t(a_i,b_i)$, $i=1,\cdots,k$ be the weight vectors of the tower. Then by Theorem 5.1 and $n=A_1$, we have

$$(A_1-1)(A_1-2)=\mu(C_0^\infty; O)=1-A_1+\sum_{i=1}^k (A_i-1)b_iA_{i+1},$$

which leads to

$$\text{(a)} \qquad \sum_{i=1}^k (A_i-1)b_iA_{i+1}=(A_1-1)^2.$$

From Theorem 4.5 and Bezout theorem, we deduce

$$\text{(b)} \qquad \sum_{i=1}^k (A_i-1)a_ib_iA_{i+1}^2 \le A_1^2$$

since $a_kI(C_{k-1}, C(0); \xi_0)=\Sigma_{i=1}^k a_ib_iA_{i+1}^2 \le a_k\bar{C}_{k-1}\cdot\bar{C}(0)=A_1^2$. Now we are ready to show $c_1=1$. We follow the proof of Abhyankar-Moh, Lemma 3.1, [5]. Recall that $c_1=a_1-b_1$. For the case $k\ge 2$, the equality (a) reads $(a_1-1)b_1=(a_1-1)^2$. Thus we get $c_1=a_1-b_1=1$. For the case $k\ge 2$, we rewrite (a) and (b) as

$$\text{(c)} \qquad \sum_{i=2}^k (A_i-1)b_iA_{i+1}=(a_1A_2-1)(c_1A_2-1)$$

$$\text{(d)} \qquad \sum_{i=2}^k a_ib_iA_{i+1}^2 \le c_1a_1A_2^2$$

Thus taking the sum: (c)$\times A_2+$(d)$\times(1-A_2)$, we obtain

$$\sum_{i=2}^k b_iA_{i+1}(A_i-A_2)\ge A_2^2((a_1-1)(c_1-1)-1)+A_2.$$

The left side is obviously negative. The right side is negative only if $c_1=1$, which completes the proof.

Theorem 8.4. *The weight vectors of a good toric resolution of the singularity at infinity of a smooth acyclic curve in $\boldsymbol{C}^2$ satisfy $b_i=a_{i-1}a_i-1$ for each $i=1,\cdots,k$, where $a_0=1$.*

Proof. Substituting $c_1=1$ in (c) and (d), we get

$$\text{(e)} \qquad \sum_{i=3}^{k}(A_i-1)b_iA_{i+1}=(a_2A_3-1)(c_2A_3-1)$$

$$\text{(f)} \qquad \sum_{i=3}^{k}a_ib_iA_{i+1}^2\leq c_2a_2A_3^2$$

where $c_i=a_{i-1}a_i-b_i$ for $i=2,\cdots,k$. Thus again taking the sum: (e)$\times A_3+$(f) $\times(1-A_3)$, we obtain

$$\sum_{i=3}^{k}b_iA_{i+1}(A_i-A_3)\geq A_3\{((a_2-1)(c_2-1)-1)A_3+1\}$$

The left side is obviously negative. The right side is negative only if $c_2=1$. The assertion for $i\geq 2$ can be proved by an easy induction. Q.E.D.

The following example shows that all weight vectors having the property of Theorem 8.4 occur.

EXAMPLE 8.5. Let $a_i\geq 2$, $i=1,\cdots,k$ be given integers, and let $n=a_1\cdots a_k$. Let us consider the sequence of automorphisms:

$$\varphi_i:\begin{pmatrix}x_i\\x_{i+1}\end{pmatrix}\mapsto\begin{pmatrix}x_{i+1}\\x_{i+2}\end{pmatrix},\quad x_{i+2}=x_i+x_{i+1}^{a_{i+1}},\qquad i=0,\cdots,k-1$$

where $x_0=x$ and $x_1=y$. Let $F(x,y)=x_{k+1}(x,y)$. Then $F(x,y)$ obviously satisfies the assumption of Theorem 8.3. Let

$$(\#_1)\qquad\begin{cases}h_0=v\\h_1(u,v)=v^{a_1}+u^{a_1-1}\\h_2(u,v)=h_1(u,v)^{a_2}+h_0(u,v)u^{a_1a_2-1}\\h_i(u,v)=h_{i-1}^{a_i}+h_{i-2}(u,v)u^{a_1\cdots a_{i-2}(a_{i-1}a_i-1)},\qquad 2\leq i\leq k\end{cases}$$

and let $f(u,v)=h_k(u,v)$. Then $(C_0^\infty;O)$ is defined by $C_0^\infty=\{(u,v)\in C^2;f(u,v)=0\}$. It is easy to see that h_i is the A_{i+1}-th Tschirnhausen approximate polynomial of f. By an inductive argument we can prove that the weight vectors of the Tschirnhausen approximate resolution tower are given by

$$P_1=\begin{pmatrix}a_1\\a_1-1\end{pmatrix},\ P_2=\begin{pmatrix}a_2\\a_1a_2-1\end{pmatrix},\cdots,P_k=\begin{pmatrix}a_k\\a_{k-1}a_k-1\end{pmatrix},$$

and the pull-backs of the Tschirnhausen approximate polynomials to X_i are given by

$$(\sharp_{i+1}) \quad \begin{cases} \Phi_i^* h_i(u_i,v_i) = u_i^{m_i(h_i)} v_i \\ \Phi_i^* h_{i+1}(u_i,v_i) = u_i^{m_i(h_{i+1})}(v_i^{a_{i+1}} + \xi_{i+1} u_i^{a_i a_{i+1}-1}) \\ \Phi_i^* h_j(u_i,v_i) = u_i^{m_i(h_j)}(\bar{h}_{j-1}^{a_j} + \xi_{i+1} \bar{h}_{j-2} u_i^{a_i \cdots a_{j-2}(a_{j-1}a_j-1)}) \end{cases}$$

where $\bar{h}_j(u_i,v_i) := \Phi_i^* h_j(u_i,v_i) / u_i^{m_i(h_j)}$ and ξ_{i+1} is a unit in a neighbourhood W_i of Ξ_i. The Milnor number is equal to $(n-1)(n-2)$ by Theorem 5.1. It is convenient to introduce the notation $a_0=1$ and $m_0(h_i)=0$ to understand $(\sharp_1)$ as a special cases of $(\sharp_{i+1})$.

REMARK (8.6). Let $F(x,y)$ be a polynomial of degree n and coefficients in a subfield k of $\boldsymbol{C}$, such that the curve $C=\{F(x,y)=0\} \subset \boldsymbol{C}^2$ is smooth and contractible. Then the completion of C requires one extra point ρ at infinity having its coordinates in k. So, after a linear change of coordinates defined over k, the pencil $L_t=\{y=t\}_{t\in \boldsymbol{C}}$ passes through ρ. Let $(B(t),t)$ be the barycenter, computed in the affine line L_t, of the points of the intersection $L_t \cap C$, weighted by the multiplicity. The automorphism $(x,y) \to (x-B(y),y)$, which is defined over k, moves the curve C to a curve C' of lower degree and having at infinity one Puiseux pair less. In the notation of Theorem 8.3, we can write $B(y) = -y^{a_1}/\xi_1$ $+$(lower terms). The iteration of this procedure constructs an automorphism defined over k, which moves the curve C to a line. Of course, we can apply this procedure to any curve $D=\{G(x,y)=0\}$, as long as the completion of the curve D has only one irreducible singularity at infinity and $c_1=1$. After at most $\log_2(\text{degree}(G))$ automorphism applied to the curve D, either the curve D becomes a line and the equation linear, in which case the curve D was smooth and contractible, or the curve D becomes a curve for which $c_1 \geq 2$. This provides a test for the contractibility and smoothness of the curve D. It is straightforward to make a fast testing procedure with the help of Maple or Mathematica.

We can apply the above remark and argument to get:

Theorem 8.7. *Let $C \subset \boldsymbol{C}^2$ be a smooth curve homeomorphic to a Riemann surface with one puncture of genus g, $g=1$ or 2. Then there exists an automorphism of $\boldsymbol{C}^2$ moving the curve C to a smooth cubic curve which is tangent to the line at infinity with the intersection multiplicity 3 if $g=1$, and to a curve of degree 5 with a cusp singularity at infinity, which is homeomorphic to $v^5+u^3=0$, if $g=2$.*

Proof. Let $\mathscr{T} = \{X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 = \boldsymbol{C}^2\}$ be a Tschirnhausen tower of resolution of the singularity at infinity with the corresponding weight vectors $\{P_i = {}^t(a_i,b_i) ; i=1,\cdots,k\}$ as before. Applying barycentric automorphisms if necessary, we can assume that $c_1 \geq 2$ (see Remark 8.6). The equalities (a) and (c) take the following form.

$$(a_g) \qquad \sum_{i=1}^{k}(A_i-1)b_iA_{i+1}=(A_1-1)^2-2g$$

$$(c_g) \qquad \sum_{i=2}^{k}(A_i-1)b_iA_{i+1}=(a_1A_2-1)(c_1A_2-1)-2g$$

The inequality (b) and (d) are valid as before. Then taking the sum: $(c_g)\times A_2+(\mathrm{d})\times(1-A_2)$, we obtain

$$(e_g) \qquad 0\geq \sum_{i=2}^{k} b_iA_{i+1}(A_i-A_2)\geq A_2^2((a_1-1)(c_1-1)-1)-(2g-1)A_2.$$

Note also that $a_1\geq 3$ by the assumption $c_1\geq 2$.

(1) Assume first $k=1$. Then we have $(a_1-1)(c_1-1)=2g$. So, for the natural number $c_1:=a_1-b_1$ we have $a_1>c_1=1+2g/(a_1-1)$. We conclude that $a_1=3$, if $g=1$ and that $a_1=5$, if $g=2$.
So, for $g=1$, we have $a_1=3$, $c_1=2$. As $b_1=1$, the curve C has no singularity at infinity but C is tangent to the line at infinity with the tangent multiplicity 3. An example of such curve is given by $C=\{y^3+x^2+1=0\}$. For $g=2$ we have $a_1=5$ and $c_1=2$. An example of such curve is given by $C=\{y^5+x^2+1=0\}$. The curve C has a non-degenerate cusp singularity at infinity.

(2) Now we show the case $k\geq 2$ does not occur.
With $a_1>c_1\geq 2$, we deduce from the inequalities (e_g):

$$(*) \qquad A_2\leq \frac{2g-1}{((a_1-1)(c_1-1)-1)}\leq (2g-1)$$

If $g=1$, we get, from $(*)$, $A_2=1$, and hence $k=1$.
If $g=2$, we reduce from $(*)$: $A_2\leq 3$ that $A_2=1,2$ or 3. We first rule out the case $A_2=3$: indeed, from $(*)$ we conclude $k=2$, $a_1=3$, $c_1=2$, $b_1=1$. So, $a_2=A_2=3$, $n=9$ and $b_2=18$ by (a_g). This is not possible since we have assumed $\gcd(a_2,b_2)=1$.
Next, we rule out the case $A_2=2$: from $(*)$, we conclude $k=2$, $a_1=3$, $c_1=2$, $b_1=1$. So, $a_2=A_2=2$, $n=6$, $b_1=1$ and $b_2=11$ by (a_g). Thus the tower has the weight vectors $P_1={}^t(3,1)$ and $P_2={}^t(2,11)$. No easy contradiction yet. However we assert that there is no polynomial $f(u,v)$ of degree 6, irreducible at the origin, whose weight vectors are as above. Indeed, let $f(u,v)=(v^3+u)^2+\Sigma_\nu c_\nu u^{\nu_1}v^{\nu_2}$ where $6<3\nu_1+\nu_2$, $\nu_1+\nu_2\leq 6$. Consider an admissible toric modification $p:X_1\to \boldsymbol{C}^2$. We may assume that $\sigma=\mathrm{Cone}(E_1,P_1)$ is the left toric cone of the divisor $\hat{E}(P_1)$ and let (s,t) be the toric coordinates. Then we have $u=st^3$ and $v=t$. The pull backs can be written as $\pi_\sigma^*(v^3+u)^2=t^6(1+s)^2$ and $\pi_\sigma^*u^{\nu_1}v^{\nu_2}=s^{\nu_1}t^{3\nu_1+\nu_2}$. So, in $t^{-6}\pi_\sigma^*f(s,t)$

the monomial t^{11} does not occur and hence P_2 is not the second weight vector for $f(u,v)$. Thus this case does not occur. So $A_2=1$ proving $k=1$. Q.E.D.

REMARK (8.9). Using (*) and the inequality: $2^{(k-1)} \leq A_2$, we get the following estimate for the length of the tower:

$$k \leq \log_2(2g-1)+1, \quad \text{for} \quad g \geq 1,\ c_1 \geq 2.$$

The classification for $g \geq 3$ is more complicated, as the model is not unique. For example, in the case of $g=3$, we can move C by an automorphism to one of the following.
(a) $k=1$, $P_1={}^t(4,1)$ and $n=4$. The curve is smooth at infinity and tangent to the line at infinity at a single point. An example is given by $y^4+x^3+1=0$.
(b) $k=1$, $P_1={}^t(7,5)$ and $n=7$. The curve has a non-degenerate cusp singularity at infinity. An example is given by $y^7+x^2+1=0$.
(c) $k=2$, $P_1={}^t(3,1)$, $P_2={}^t(2,9)$ and $n=6$. An example is given by $(y^3+x^2)^2+x$.

We thank Professor Walter Neumann for communicating to us the reference of his earlier work [23], in which he obtained the classification of smooth affine curves with one place at infinity for $g \leq 4$. Professor M. Miyanishi recently communicated to us that he gave a new proof of Theorem 8.3 using the classification of surfaces [9]. Also, the paper [34] contains interesting results about contractible affine curves with one isolated singularity.

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